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A CHARACTERIZATION OF TWO WEIGHT NORM INEQUALITIES FOR FRACTIONAL AND POISSON INTEGRALS

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ABSTRACT. For $1 < p \leq q < \infty$ and $w(x), v(x)$ nonnegative functions on $\mathbf{R}^n$, we show that the weighted inequality

$$\left(\int |Tf|^q w\right)^{1/q} \leq C\left(\int f^p v\right)^{1/p}$$

holds for all $f \geq 0$ if and only if both

$$\int [T(\chi_Q v^{1-p'})]^q w \leq C_1 \left(\int_Q v^{1-p'}\right)^{q/p} < \infty$$

and

$$\int [T(\chi_Q w)]^{p'} v^{1-p'} \leq C_2 \left(\int_Q w\right)^{p'/q'} < \infty$$

hold for all dyadic cubes Q. Here T denotes a fractional integral or, more generally, a convolution operator whose kernel K is a positive lower semicontinuous radial function decreasing in $|x|$ and satisfying $K(x) \leq CK(2x)$, $x \in \mathbf{R}^n$. Applications to degenerate elliptic differential operators are indicated.

In addition, a corresponding characterization of those weights v on $\mathbf{R}^n$ and w on $\mathbf{R}^{n+1}_+$ for which the Poisson operator is bounded from $L^p(v)$ to $L^q(w)$ is given.

1. Introduction. Suppose $1 \leq p$, $q \leq \infty$, $v(x)$ and $w(x)$ are nonnegative measurable functions (i.e. weights) on $\mathbf{R}^n$ and $\mathbf{R}^m$ respectively, and that T is an operator taking suitable functions on $\mathbf{R}^n$ into functions on $\mathbf{R}^m$. In his survey article [**26**], B. Muckenhoupt raised the general question of characterizing when the weighted norm inequality,

$$(1.1) \qquad \left(\int_{\mathbf{R}^m} |Tf(x)|^q w(x)\,dx\right)^{1/q} \leq C\left(\int_{\mathbf{R}^n} |f(x)|^p v(x)\,dx\right)^{1/p},$$

holds for all appropriate f. In the case of "one weight", e.g. $p = q$, $m = n$ and $w = v$, and for many classical operators T, inequality (1.1) can be characterized by remarkably simple conditions, most notable being that the A_p condition,

$$(A_p) \qquad \left(\frac{1}{|Q|}\int_Q w\right)\left(\frac{1}{|Q|}\int_Q w^{1-p'}\right)^{p-1} \leq C \quad \text{for all cubes } Q \subset \mathbf{R}^n,$$

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is necessary and sufficient for the Hardy-Littlewood maximal function and Hilbert transform inequalities (see [**19** and **25**]).

The case of different weights has been far less accommodating. Only for the simplest of nontrivial operators, namely the Hardy operator, $Tf(x) = \int_0^x f(t)\,dt$, has a correspondingly simple characterizing condition for (1.1) been obtained (see [**1**, **3**, **25**, **40** and **41**]), namely

$$\left(\int_r^\infty w\right)^{1/q}\left(\int_0^r v^{1-p'}\right)^{1/p'} \le C \quad \text{for all } 0 < r < \infty.$$

In a sense, the next simplest classical operator is the Hardy-Littlewood maximal function,

$$Mf(x) = \sup_{x\in Q \text{ cube}} \frac{1}{|Q|}\int_Q |f|,$$

and in [**31**] it was shown that for $T = M$, (1.1) holds if and only if

$$\left(\int_Q [M(\chi_Q v^{1-p'})]^q w\right)^{1/q} \le C\left(\int_Q v^{1-p'}\right)^{1/p} < \infty \quad \text{for all cubes } Q \subset \mathbf{R}^n.$$

In particular, this says that (1.1), with $T = M$, holds for all f if it holds when tested over functions of the form $f = \chi_Q v^{1-p'}$ (since then $f^p v = f$). This suggested a reasonable conjecture: (1.1) holds for all f provided it holds when tested over functions of the form $\chi_Q v^{1-p'}$, where the sets Q are appropriately related to the geometry of the operator T. While this is born out in the one weight cases considered above and in the two weight inequality for M, it fails, for example, for fractional integrals,

$$Tf(x) = I_\alpha f(x) = \int_{\mathbf{R}^n} |x-y|^{\alpha-n} f(y)\,dy,$$

and for higher dimensional Hardy operators

$$Tf(x_1,\dots,x_n) = \int_0^{x_1}\cdots\int_0^{x_n} f(t_1,\dots,t_n)\,dt_1\cdots dt_n$$

(see [**32**] and [**33**] respectively for counterexamples). The point here, first indicated in the work of B. Muckenhoupt and R. L. Wheeden in [**28**], is that for linear operators, one should also test the inequality dual to (1.1) over appropriate test functions.

It is convenient at this point to recast (1.1) in a more "natural" form, one that permits the replacement of the functions v and w by positive Borel measures μ and ω, and that leads more naturally to the correct testing functions:

$$(1.2)\quad \left(\int_{\mathbf{R}^m} |T(f\mu)(x)|^q\,d\omega(x)\right)^{1/q} \le C\left(\int_{\mathbf{R}^n} |f(x)|^p\,d\mu(x)\right)^{1/p}, \qquad f \in L^p(\mu).$$

To see that (1.1) is included in (1.2), set $d\omega(x) = w(x)\,dx$, $d\mu(x) = v(x)^{1-p'}\,dx$ and replace f by $fv^{p'-1}$ in (1.2). If T is linear and T^* its dual under the usual pairing, i.e.

$$\int_{\mathbf{R}^m} (Tf)(x)g(x)\,dx = \int_{\mathbf{R}^n} f(y)(T^*g)(y)\,dy \quad \text{for all } f \text{ and } g,$$

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then (1.2) is equivalent to the dual inequality,

(1.3)

$$\left(\int_{\mathbf{R}^n} |T^*(g\omega)(x)|^{p'}\,d\mu(x)\right)^{1/p'} \le C\left(\int_{\mathbf{R}^m} |g(x)|^{q'}\,d\omega(x)\right)^{1/q'}, \qquad g \in L^{q'}(\omega).$$

The new conjecture is that (1.2) holds for all f in $L^p(\mu)$ if and only if both (1.2) and (1.3) hold when tested over characteristic functions of sets Q appropriately related to the geometry of T. In **[33]**, this conjecture was established for the two-dimensional Hardy operator

$$Tf(x_1, x_2) = \int_0^{x_1}\int_0^{x_2} f(t_1, t_2)\,dt_1\,dt_2$$

by showing that it sufficed to test (1.2) over rectangles of the form $[0, a] \times [0, b]$ and to test (1.3) over rectangles of the form $[a, \infty] \times [b, \infty]$.

The purpose of this paper is to establish the conjecture above for fractional integral operators (and some generalizations thereof) along with the Poisson integral operator. For other work on weighted inequalities for these operators, see **[2, 4, 6, 7, 9, 10, 11, 15, 17, 20, 21, 22, 23, 27, 30, 32, 35, 36, 39]** and references given there. Before stating our two theorems, we establish some notation. Given a cube Q and $R > 0$, denote by RQ the cube concentric with Q and with R times the side length. For any measure μ and set E, denote by $|E|_\mu$ the μ-measure of E. Finally, the letter C will be used to denote a positive constant that may change from line to line but will remain independent of the appropriate quantities.

THEOREM 1. *Suppose $1 < p \le q < \infty$, ω and μ are positive Borel measures on $\mathbf{R}^n$, and $Tf = K * f$ where $K(x)$ is a positive lower semicontinuous radial function decreasing in $|x|$ and satisfying the growth condition $K(x) \le CK(2x)$, $x \in \mathbf{R}^n$.*

Then the weighted inequality

$$\left(\int [T(f\mu)]^q\,d\omega\right)^{1/q} \le C\left(\int f^p\,d\mu\right)^{1/p} \quad \text{for all } f \ge 0 \tag{1.4}$$

holds if and only if both

$$\left(\int [T(\chi_Q\mu)]^q\,d\omega\right)^{1/q} \le C_1|Q|_\mu^{1/p} < \infty \quad \text{for all dyadic cubes } Q \tag{1.5}$$

and

$$\left(\int [T(\chi_Q\omega)]^{p'}\,d\mu\right)^{1/p'} \le C_2|Q|_\omega^{1/q'} < \infty \quad \text{for all dyadic cubes } Q. \tag{1.6}$$

THEOREM 2. *Suppose $1 < p \le q < \infty$, ω and μ are positive Borel measures on $\mathbf{R}^{n+1}_+$ and $\mathbf{R}^n$ respectively, and*

$$P(f\mu)(x,t) = \int_{\mathbf{R}^n} P_t(x-y)f(y)\,d\mu(y),$$

$$P^*(g\omega)(y) = \int_{\mathbf{R}^{n+1}_+} P_t(y-x)g(x,t)\,d\omega(x,t),$$

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denote the Poisson and dual Poisson operators. Then the weighted inequality

$$(1.7)\qquad \left(\int_{\mathbf{R}_+^{n+1}} [P(f\mu)]^q \, d\omega\right)^{1/q} \le C \left(\int_{\mathbf{R}^n} f^p \, d\mu\right)^{1/p} \quad \text{for all } f \ge 0$$

holds if and only if

$$(1.8)\qquad \left(\int_{\mathbf{R}_+^{n+1}} [P(\chi_Q \mu)]^q \, d\omega\right)^{1/q} \le C|Q|_\mu^{1/p} < \infty \quad \text{for all dyadic cubes } Q \subset \mathbf{R}^n$$

and

$$(1.9)\qquad \left[\int_{\mathbf{R}^n} |P^*[\chi_{\hat{Q}}(x,t) t^{q-1} \, d\omega(x,t)]|^{p'} \, d\mu\right]^{1/p'} \le C \left(\int_{\hat{Q}} t^q \, d\omega(x,t)\right)^{1/q'} < \infty \quad \text{for all dyadic cubes } Q \subset \mathbf{R}^n$$

where $\hat{Q}$ *denotes the cube in* $\mathbf{R}_+^{n+1}$ *having* Q *as a face.*

Applications of Theorem 1 to regularity and eigenvalve estimates for degenerate elliptic differential operators are readily suggested by the following observation: Suppose f has support in, or mean zero on, a cube Q. Then (see [**12**])

$$(1.10)\qquad |f(x)| \le C I_1(\chi_Q |\nabla f|)(x), \qquad x \in Q,$$

where I_1 denotes the fractional integral of order 1 given by

$$I_1 f(x) = \int_{\mathbf{R}^n} |x-y|^{1-n} f(y) \, dy.$$

Thus if $d\mu = \chi_Q(x) v(x)^{1-p'} \, dx$ and $d\omega = \chi_Q(x) w(x) \, dx$ satisfy conditions (1.5) and (1.6) with $T = I_1$, then we have the two weight Poincaré-Sobolev inequality

$$(1.11)\qquad \int_Q |f(x)|^2 w(x) \, dx \le C_Q \int_Q |\nabla f(x)|^2 v(x) \, dx$$

for all f with either $\operatorname{supp} f \subset \overset{\circ}{Q}$ or $\int_Q f = 0$, and where C_Q is a fixed multiple of $C_1 + C_2$. This inequality has been used by S. Chanillo and R. L. Wheeden [**8**] to study the local behaviour of solutions to degenerate elliptic operators $P = \nabla \cdot A \nabla$ where the weights $w(x)$ and $v(x)$ are given by the largest and smallest eigenvalues of $A(x)$ (see also [**12**, **18** and **38**]). Inequality (1.11) is also the crucial ingredient in estimating eigenvalues for degenerate Schrödinger operators $-\nabla \cdot A\nabla + V$; see S. Chanillo and R. L. Wheeden [**7**] (see also [**5**, **14**, **21** and **22**] for earlier related results).

REMARK 1. Neither (1.5) nor (1.6) alone is in general sufficient for (1.4) as shown by the example in [**32**]. Note however that (1.5) and (1.6) coincide when $\omega = \mu$ and $q = p'$.

REMARK 2. A modification of the proof of Theorem 1 given below shows that in conditions (1.5) and (1.6), the integrations on the left need be taken only over the cubes Q and $12Q$ respectively. We do not know if it is possible to restrict both integrations to Q.

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2. Proof of Theorem 1. Assume (1.4) holds. Then so does its dual inequality

$$\left(\int [T(g\omega)]^{p'} d\mu\right)^{1/p'} \leq C \left(\int g^{q'} d\omega\right)^{1/q'} \quad \text{for all } g \geq 0. \tag{2.1}$$

With $f = \chi_Q$ in (1.4) and $g = \chi_Q$ in (2.1) we obtain (1.5) and (1.6).

Conversely, suppose (1.5) and (1.6) hold and, without loss of generality, that f is nonnegative and bounded with compact support. Now $T(f\mu)$ is lower semicontinuous and so for each k, we can write the open set $\Omega_k = \{T(f\mu) > 2^k\}$ as $\dot{\bigcup}_j Q_j^k$ where the Q_j^k are the dyadic cubes maximal among those dyadic cubes Q satisfying $RQ \subset \Omega_k$. Choosing $R \geq 3$ sufficiently large, depending only on the dimension n, we obtain

(2.2) (i) (disjoint cover) $\Omega_k = \bigcup_j Q_j^k$ and $Q_j^k \cap Q_i^k = \varnothing$ for $i \neq j$,

(ii) (Whitney condition) $RQ_j^k \subset \Omega_k$ and $3RQ_j^k \cap \Omega_k^c \neq \varnothing$ for all k, j,

(iii) (finite overlap) $\sum_j \chi_{3Q_j^k} \leq C\chi_{\Omega_k}$ for all k,

(iv) (crowd control) The number of cubes Q_s^k intersecting a fixed cube $3Q_j^k$ is at most C,

(v) (nested property) $Q_j^k \subset Q_i^l$ implies $k \geq l$.

In fact, (i) and (v) are obvious, (ii) follows as in Theorem 2.1 of [**16**], and (iii) and (iv) are a consequence of (ii) and a geometric packing argument on p. 16 of [**13**].

We now claim the following maximum principle holds:

$$T(\chi_{(3Q_j^k)^c} f\mu)(x) \leq C2^k, \qquad x \in Q_j^k, \tag{2.3}$$

for all (k, j) where C is a constant. To see this, momentarily fix (k, j) and choose $z \in 3RQ_j^k \cap \Omega_k^c$, which is possible by the Whitney condition (2.2)(ii). From the growth assumption on $K(x)$, we conclude there is a constant C such that $K(x - y) \leq CK(z - y)$ for $x \in Q_j^k$, $y \in (3Q_j^k)^c$. Multiplying this inequality by $f(y)$ and then integrating over $(3Q_j^k)^c$ with respect to $d\mu(y)$ yields $T(\chi_{(3Q_j^k)^c} f\mu)(x) \leq CT(f\mu)(z) \leq C2^k$ since $z \notin \Omega_k$. This proves (2.3).

Now fix an integer $m \geq 2$ satisfying $2^{m-2} > C$ where C is the constant appearing in (2.3). Define $E_j^k = Q_j^k \cap (\Omega_{k+m-1} - \Omega_{k+m})$ for all (k, j). For $x \in E_j^k \subset \Omega_{k+m-1}$, the maximum principle (2.3) yields

$$\begin{aligned} T(\chi_{3Q_j^k} f\mu)(x) &= T(f\mu)(x) - T(\chi_{(3Q_j^k)^c} f\mu)(x) \\ &> 2^{k+m-1} - C2^k > 2^{k+m-1} - 2^{k+m-2} = 2^{k+m-2} \geq 2^k, \end{aligned}$$

and so

$$\begin{aligned} |E_j^k|_\omega &\leq 2^{-k} \int_{E_j^k} T(\chi_{3Q_j^k} f\mu) d\omega = 2^{-k} \int_{3Q_j^k} fT(\chi_{E_j^k}\omega)\, d\mu \\ &= 2^{-k} \left[\int_{3Q_j^k - \Omega_{k+m}} fT(\chi_{E_j^k}\omega)\, d\mu + \int_{3Q_j^k \cap \Omega_{k+m}} fT(\chi_{E_j^k}\omega)\, d\mu\right] \\ &= 2^{-k}[\sigma_j^k + \tau_j^k]. \end{aligned}$$

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We now estimate the left side of (1.4) by

$$\begin{aligned}\int [T(f\mu)]^q \,d\omega &\le \sum_k (2^{k+m})^q |\Omega_{k+m-1} - \Omega_{k+m}|_\omega \\ &\le C \sum_{k,j} |E_j^k|_\omega 2^{kq} \\ &= C\left(\sum_{(k,j)\in E} + \sum_{(k,j)\in F} + \sum_{(k,j)\in G}\right) |E_j^k|_\omega 2^{kq} \\ &= C(\mathrm{I} + \mathrm{II} + \mathrm{III})\end{aligned} \tag{2.4}$$

where

$$\begin{aligned}E &= \{(k,j)\colon |E_j^k|_\omega \le \beta |Q_j^k|_\omega\}, \\ F &= \{(k,j)\colon |E_j^k|_\omega > \beta |Q_j^k|_\omega \text{ and } \sigma_j^k > \tau_j^k\}, \\ G &= \{(k,j)\colon |E_j^k|_\omega > \beta |Q_j^k|_\omega \text{ and } \sigma_j^k \le \tau_j^k\},\end{aligned}$$

and where β, to be chosen later, satisfies $0 < \beta < 1$. We have

$$\begin{aligned}\mathrm{I} = \sum_{(k,j)\in E} |E_j^k|_\omega 2^{kq} &\le \beta \sum_{k,j} |Q_j^k|_\omega 2^{kq} \\ &\le \beta \sum_k 2^{kq} |\{T(f\mu) > 2^k\}|_\omega \quad \text{by (2.2)(i)} \\ &\le \beta \int \left(\sum_k 2^{kq} \chi_{\{Tf\mu > 2^k\}}\right) d\mu \\ &\le C\beta \int [T(f\mu)]^q \,d\omega;\end{aligned} \tag{2.5}$$

$$\begin{aligned}\mathrm{II} &= \sum_{(k,j)\in F} |E_j^k|_\omega 2^{kq} \le \sum_{(k,j)\in F} |E_j^k|_\omega \left[\frac{2\sigma_j^k}{|E_j^k|_\omega}\right]^q \\ &\le C\beta^{-q} \sum_{k,j} |E_j^k|_\omega \left[\frac{1}{|Q_j^k|_\omega} \int_{3Q_j^k - \Omega_{k+m}} fT(\chi_{E_j^k}\omega)\,d\mu\right]^q \\ &\le C\beta^{-q} \sum_{k,j} \frac{|E_j^k|_\omega}{|Q_j^k|_\omega^q} \left[\int_{3Q_j^k} [T(\chi_{Q_j^k}\omega)]^{p'} \,d\mu\right]^{q/p'} \left[\int_{3Q_j^k - \Omega_{k+m}} f^p \,d\mu\right]^{q/p} \\ &\le C(C_2)^q \beta^{-q} \sum_{k,j} \left(\int_{3Q_j^k - \Omega_{k+m}} f^p \,d\mu\right)^{q/p} \quad \text{by (1.6)} \\ &\le C\beta^{-q} \left(\sum_{k,j} \int_{3Q_j^k - \Omega_{k+m}} f^p \,d\mu\right)^{q/p} \quad \text{since } p \le q \\ &\le C\beta^{-q} \left(\int f^p \,d\mu\right)^{q/p}\end{aligned} \tag{2.6}$$

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since

$$\sum_{k,j} \chi_{3Q_j^k - \Omega_{k+m}} \le C \sum_k \chi_{\Omega_k - \Omega_{k+m}} \le C(m+1) \quad \text{by (2.2)(iii).}$$

To estimate term III in (2.4) we will need the following easy variant of the maximal theorem in [**37**]. For ν a positive Borel measure on $\mathbf{R}^n$, define

$$M_\nu f(x) = \sup_{\substack{x\in Q \text{ dyadic} \\ \text{cube}}} \left[\frac{1}{|Q|_\nu} \int_Q |f|\, d\nu\right], \qquad f \in L^1_{\text{loc}}(\nu).$$

MAXIMAL THEOREM. *For $1 < r \le \infty$, and ν a positive Borel measure on $\mathbf{R}^n$,*

$$\int (M_\nu f)^r\, d\nu \le C_r \int |f|^r\, d\nu \quad \textit{for all } f \in L^r(\nu). \tag{2.7}$$

Inequality (2.7) follows by interpolation from the weak type $(1,1)$ and type (∞,∞) estimates for M_ν (both with constant 1) as in [**37**].

Let $H_j^k = \{i\colon Q_i^{k+m} \cap 3Q_j^k \ne \varnothing\}$ so that $3Q_j^k \cap \Omega_{k+m} \subset \bigcup_{i\in H_j^k} Q_i^{k+m}$. In order to estimate τ_j^k we observe that the growth condition imposed on the kernel $K(x)$ implies that for $y \notin 3Q_i^{k+m}$,

$$\max_{x\in Q_i^{k+m}} K(x-y) \le C \min_{x\in Q_i^{k+m}} K(x-y)$$

which in turn yields

$$\max_{x\in Q_i^{k+m}} T(\chi_{E_j^k}\omega)(x) \le C \min_{x\in Q_i^{k+m}} T(\chi_{E_j^k}\omega)(x), \qquad i \in H_j^k,$$

since $3Q_i^{k+m} \subset \Omega_{k+m}$ (see (2.2)(ii)) and Ω_{k+m} does not intersect E_j^k. It follows that

$$\begin{aligned} \tau_j^k &= \int_{3Q_j^k\cap\Omega_{k+m}} fT(\chi_{E_j^k}\omega)\, d\mu \le C \sum_{i\in H_j^k} \left[\min_{x\in Q_i^{k+m}} T(\chi_{E_j^k}\omega)(x)\right] \int_{Q_i^{k+m}} f\, d\mu \\ &\le C \sum_{i\in H_j^k} \left[\int_{Q_i^{k+m}} T(\chi_{E_j^k}\omega)\, d\mu\right] \left[\frac{1}{|Q_i^{k+m}|_\mu} \int_{Q_i^{k+m}} f\, d\mu\right]. \end{aligned} \tag{2.8}$$

For notational convenience, set

$$A_j^k = \frac{1}{|Q_j^k|_\mu} \int_{Q_j^k} f\, d\mu$$

and let $L_j^k = \{s\colon Q_s^k \cap 3Q_j^k \ne \varnothing\}$. Then we have

$$\begin{aligned} \tau_j^k &\le C \sum_{i\in H_j^k} \left[\int_{Q_i^{k+m}} T(\chi_{E_j^k}\omega)\, d\mu\right] A_i^{k+m} \\ &\le C \sum_{s\in L_j^k} \left[\sum_{i\colon Q_i^{k+m}\subset Q_s^k} \left[\int_{Q_i^{k+m}} T(\chi_{E_j^k}\omega)\, d\mu\right] A_i^{k+m}\right]. \end{aligned} \tag{2.9}$$

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For future reference, note that the cardinality of L_j^k is at most C by (2.2)(iv). We now claim that

$$\sum_{\substack{(k,j)\in G\\ k\geq N\\ k\equiv M(\operatorname{mod} m)}} |E_j^k|_\omega 2^{kq} \leq C\left(\int f^p\,d\mu\right)^{q/p} \tag{2.10}$$

with a constant C independent of the integers N and M where $-\infty < N < \infty$, $0 \leq M < m$. Fix such integers N and M and introduce the convention, in force until the proof of (2.10) is completed, that all indices (k,j) are understood to be restricted to $k \geq N$, $k \equiv M \pmod m$.

With this convention we introduce "principal" cubes as in B. Muckenhoupt and R. L. Wheeden [**29**, p. 804]. Let G_0 consist of those indices (k,j) for which Q_j^k is maximal. If G_n has been defined, let G_{n+1} consist of those (k,j) for which there is $(t,u) \in G_n$ with $Q_j^k \subset Q_u^t$ and

(i) $A_j^k > 2A_u^t$,

(ii) $A_i^l \leq 2A_u^t$ whenever $Q_j^k \subsetneqq Q_i^l \subset Q_u^t$.

Define $\Gamma = \bigcup_{n=0}^\infty G_n$ and for each (k,j), define $P(Q_j^k)$ to be the smallest cube Q_u^t containing Q_j^k and with $(t,u) \in \Gamma$. Then we have

$$\begin{aligned} &\text{(i)} \quad P(Q_j^k) = Q_u^t \text{ implies } A_j^k \leq 2A_u^t,\\ &\text{(ii)} \quad Q_j^k \subsetneqq Q_u^t, (k,j) \text{ and } (t,u) \in \Gamma \text{ imply } A_j^k > 2A_u^t. \end{aligned} \tag{2.11}$$

Using (2.9) and the fact that the cardinality of L_j^k is at most C, we obtain
(2.12)

$$\begin{aligned}
\sum_{(k,j)\in G} |E_j^k|_\omega 2^{kq} &\leq \sum_{(k,j)\in G} |E_j^k|_\omega \left[\frac{2\tau_j^k}{|E_j^k|_\omega}\right]\\
&\leq C\beta^{-q}\sum_{k,j}\frac{|E_j^k|_\omega}{|Q_j^k|_\omega^q}[\tau_j^k]^q\\
&\leq C\beta^{-q}\sum_{k,j}\sum_{s\in L_j^k}\frac{|E_j^k|_\omega}{|Q_j^k|_\omega^q}\left[\sum_{\substack{i:\ P(Q_i^{k+m})=P(Q_s^k)\\ Q_i^{k+m}\subset Q_s^k}}\left[\int_{Q_i^{k+m}} T(\chi_{Q_j^k}\omega)\,d\mu\right]A_i^{k+m}\right]^q\\
&\quad + C\beta^{-q}\sum_{k,j}\frac{|E_j^k|_\omega}{|Q_j^k|_\omega^q}\left[\sum_{i\in H_j^k:\ (k+m,i)\in\Gamma}\left[\int_{Q_i^{k+m}} T(\chi_{Q_j^k}\omega)\,d\mu\right]A_i^{k+m}\right]^q\\
&\qquad\qquad \text{(since if } Q_i^{k+m}\subset Q_s^k \text{ and } (k+m,i)\notin\Gamma,\\
&\qquad\qquad\qquad \text{then } P(Q_i^{k+m}) = P(Q_s^k))\\
&= \mathrm{IV} + \mathrm{V}.
\end{aligned}$$

We will use conditions (1.5) and (1.6) to estimate terms IV and V respectively.

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First, note that for a fixed $(t,u) \in \Gamma$,

$$\begin{aligned}
&\sum_{k,j} \sum_{s \in L_j^k :\ P(Q_s^k) = Q_u^t} \frac{|E_j^k|_\omega}{|Q_j^k|_\omega^q} \left[\sum_{\substack{i:\ P(Q_i^{k+m}) = P(Q_s^k) \\ Q_i^{k+m} \subset Q_s^k}} \left[\int_{Q_i^{k+m}} T(\chi_{Q_j^k}\omega)\, d\mu \right] A_i^{k+m} \right]^q \\
&\le \sum_{k,j} \sum_{s \in L_j^k :\ P(Q_s^k) = Q_u^t} |E_j^k|_\omega \left[\frac{1}{|Q_j^k|_\omega} \int_{Q_s^k} T(\chi_{Q_j^k}\omega)\, d\mu \right]^q (2A_u^t)^q \quad \text{by (2.11)(i)} \\
&\le (2A_u^t)^q \sum_{k,j} \sum_{s \in L_j^k :\ P(Q_s^k) = Q_u^t} |E_j^k|_\omega \left[\frac{1}{|Q_j^k|_\omega} \int_{Q_j^k} T(\chi_{Q_u^t}\mu)\, d\omega \right]^q \\
&\le C(A_u^t)^q \int [M_\omega[T(\chi_{Q_u^t}\mu)]]^q\, d\omega \quad \text{since cardinality of } L_j^k \le C(n,R) \\
&\le C(A_u^t)^q \int [T(\chi_{Q_u^t}\mu)]^q\, d\omega \quad \text{by the maximal theorem} \\
&\le C(A_u^t)^q (C_1)^q |Q_u^t|_\mu^{q/p} \quad \text{by (1.5).}
\end{aligned} \tag{2.13}$$

Summing (2.13) over $(t,u) \in \Gamma$ yields

$$\mathrm{IV} \le C\beta^{-q} \sum_{(t,u)\in\Gamma} |Q_u^t|_\mu^{q/p} (A_u^t)^q \le C\beta^{-q} \left[\sum_{(t,u)\in\Gamma} |Q_u^t|_\mu (A_u^t)^p \right]^{q/p} \quad \text{since } p \le q. \tag{2.14}$$

To obtain the corresponding estimate for V, we note that for a fixed (k,j), Hölder's inequality yields

$$\begin{aligned}
&\frac{|E_j^k|_\omega}{|Q_j^k|_\omega^q} \left[\sum_{i \in H_j^k :\ (k+m,i)\in\Gamma} \left[\int_{Q_i^{k+m}} T(\chi_{Q_j^k}\omega)\, d\mu \right] A_i^{k+m} \right]^q \\
&\le \frac{|E_j^k|_\omega}{|Q_j^k|_\omega^q} \left[\sum_{i \in H_j^k} |Q_i^{k+m}|_\mu^{-p'/p} \left[\int_{Q_i^{k+m}} T(\chi_{Q_j^k}\omega)\, d\mu \right]^{p'} \right]^{q/p'} \\
&\quad \times \left[\sum_{i \in H_j^k :\ (k+m,i)\in\Gamma} |Q_i^{k+m}|_\mu (A_i^{k+m})^p \right]^{q/p} \\
&\le \frac{|E_j^k|_\omega}{|Q_j^k|_\omega^q} \left[\sum_{i \in H_j^k} \int_{Q_i^{k+m}} [T(\chi_{Q_j^k}\omega)]^{p'}\, d\mu \right]^{q/p'} \left[\sum_{i \in H_j^k :\ (k+m,i)\in\Gamma} |Q_i^{k+m}|_\mu (A_i^{k+m})^p \right]^{q/p} \\
&\le (C_2)^q \left[\sum_{i \in H_j^k :\ (k+m,i)\in\Gamma} |Q_i^{k+m}|_\mu (A_i^{k+m})^p \right]^{q/p} \quad \text{by (1.6).}
\end{aligned} \tag{2.15}$$

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Summing (2.15) over (k,j), using the fact that $p \leq q$, and then noting that any fixed Q_i^{k+m} occurs at most C times in the resulting sum (by (2.2)(iii)), we obtain

$$V \leq C\beta^{-q} \left[\sum_{(t,u)\in\Gamma} |Q_u^t|_\mu (A_u^t)^p \right]^{q/p}. \tag{2.16}$$

Combining (2.12), (2.14) and (2.16) shows that the left side of (2.10) is bounded by

$$\begin{aligned} &C\beta^{-q} \left[\sum_{(t,u)\in\Gamma} |Q_u^t|_\mu (A_u^t)^p \right]^{q/p} \\ &\leq C\beta^{-q} \left[\int \left[\sum_{(t,u)\in\Gamma} (A_u^t)^p \chi_{Q_u^t}(x) \right] d\mu(x) \right]^{q/p} \\ &\leq C\beta^{-q} \left[\int (M_\mu f)^p \, d\mu \right]^{q/p} \end{aligned} \tag{2.17}$$

since (2.11)(ii) implies that for any fixed x,

$$\sum_{(t,u)\in\Gamma} (A_u^t)^p \chi_{Q_u^t}(x) \leq 2^p \sup_{x\in Q_u^t} (A_u^t)^p \leq 2^p M_\mu f(x)^p.$$

From (2.17) and the maximal theorem we obtain (2.10). Now let $N \to -\infty$ in (2.10) and then sum over $M = 0, 1, 2, \dots, m-1$ to obtain

$$\mathrm{III} \leq C\beta^{-q} \left(\int f^p \, d\mu \right)^{q/p}. \tag{2.18}$$

Combining (2.4), (2.5), (2.6) and (2.18) we have

$$\int [T(f\mu)]^q \, d\omega \leq C\beta \int [T(f\mu)]^q \, d\omega + C\beta^{-q} \left(\int f^p \, d\mu \right)^{q/p}. \tag{2.19}$$

Now chose β so small that $C\beta < \frac{1}{2}$ and then subtract the first term on the right side of (2.19) from both sides (it is finite by (1.5) and our assumptions on f) to obtain (1.4) for $f \geq 0$ bounded with compact support, and hence for arbitrary $f \geq 0$ by the monotone convergence theorem. This completes the proof of Theorem 1.

3. Proof of Theorem 2. The proof of Theorem 2 follows very closely the line of argument used in Theorem 1, but applied to the dual Poisson operator P^* rather than P. In order to minimize confusion in referring to the proof of Theorem 1, we set $T = P^*$, i.e. $T(f\mu)(x) = \int P_t(x-y) f(y,t) \, d\mu(y,t)$, interchange the roles of ω and μ, q and p', and consider instead the inequality,

$$\left(\int_{\mathbf{R}^n} [T(f\mu)]^q \, d\omega \right)^{1/q} \leq C \left(\int_{\mathbf{R}_+^{n+1}} f^p \, d\mu \right)^{1/p} \quad \text{for all } f \geq 0. \tag{3.1}$$

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We will show that (3.1) holds if and only if both

$$\left[\int_{\mathbf{R}^n} (T[\chi_{\hat{Q}}(x,t)t^{p'-1}\,d\mu(x,t)])^q\,d\omega\right]^{1/q} \le C\left[\int_{\hat{Q}} t^{p'}\,d\mu(x,t)\right]^{1/p} < \infty, \tag{3.2}$$

$$\left[\int_{\mathbf{R}^{n+1}_+} (P\chi_Q\omega)^{p'}\,d\mu\right]^{1/p'} \le C\left(\int_Q d\omega\right)^{1/q'} < \infty \tag{3.3}$$

hold for all dyadic cubes $Q \subset \mathbf{R}^n$.

Condition (3.2) follows by setting $f(x,t) = \chi_{\hat{Q}}(x,t)t^{p'-1}$ in (3.1), and (3.3) follows by testing the inequality dual to (3.1) with χ_Q.

Conversely, we begin exactly as in the proof of Theorem 1 but with the maximum principle in (2.3) replaced by

$$T(\chi_{(3\hat{Q}^k_j)^c} f\mu)(x) \le C2^k, \qquad x \in Q^k_j. \tag{3.4}$$

To see (3.4), choose $z \in 3RQ^k_j \cap \Omega^c_k$ as before by the Whitney condition. Inequality (3.4) now follows from the inequality

$$P_t(x-y) \le CP_t(z-y), \qquad x \in Q^k_j,\ (y,t) \notin 3\hat{Q}^k_j,$$

after multiplying by $f(y,t)$ and then integrating over $(3\hat{Q}^k_j)^c$ with respect to $d\mu(y,t)$.

From this point up to the inequality in (2.8), the proof of Theorem 1 can be applied verbatim provided that in the context of the measure space $(\mathbf{R}^{n+1}_+, d\mu)$, cubes Q are replaced by $\hat{Q}$ and the sets Ω_k are replaced by $\hat{\Omega}_k = \bigcup_j \hat{Q}^k_j$. The only new development in this proof arises now: $T^*(\chi_{E^k_j}\omega)$ is no longer roughly constant on any $\hat{Q}^{k+m}_i$, but merely roughly constant on level planes of $\hat{Q}^{k+m}_i$ (see (3.6) below). The substitute for (2.8) is

$$\begin{aligned}\tau^k_j &= \int_{3\hat{Q}^k_j \cap \hat{\Omega}_{k+m}} fT^*(\chi_{E^k_j}\omega)\,d\mu = \sum_{i\in H^k_j} \int_{\hat{Q}^{k+m}_i} fT^*(\chi_{E^k_j}\omega)\,d\mu \\ &\le C\sum_{i\in H^k_j}\left[\int_{\hat{Q}^{k+m}_i} T^*(\chi_{E^k_j}\omega)t^{-1}d\tilde{\mu}\right]\left[\frac{1}{|\hat{Q}^{k+m}_i|_{\tilde{\mu}}}\int_{\hat{Q}^{k+m}_i} ft^{1-p'}\,d\tilde{\mu}\right]\end{aligned} \tag{3.5}$$

where $d\tilde{\mu}(y,t) = t^{p'}\,d\mu(y,t)$. To see the inequality in (3.5), observe that if x^{k+m}_i is the centre of Q^{k+m}_i, then for $(x,t) \in \hat{Q}^{k+m}_i$,

$$T^*(\chi_{E^k_j}\omega)(x,t) \approx T^*(\chi_{E^k_j}\omega)(x^{k+m}_i, t) \approx \frac{t}{d^k_j}T^*(\chi_{E^k_j}\omega)(x^{k+m}_i, d^k_j) \tag{3.6}$$

since $3Q^{k+m}_i \cap E^k_j = \varnothing$. Here d^k_j is the side length of Q^k_j and the symbol of approximate equality means the ratio of the sides is bounded between absolute

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positive constants. From (3.6) we have

$$\int_{\hat{Q}_i^{k+m}} fT^*(\chi_{E_j^k}\omega)\,d\mu \approx \frac{1}{d_j^k}T^*(\chi_{E_j^k}\omega)(x_i^{k+m}, d_j^k)\int_{\hat{Q}_i^{k+m}} f(x,t)t\,d\mu(x,t)$$
$$= \left[\frac{\int_{\hat{Q}_i^{k+m}} \frac{t}{d_j^k}T^*(\chi_{E_j^k}\omega)(x_i^{k+m}, d_j^k)t^{p'-1}\,d\mu(x,t)}{\int_{\hat{Q}_i^{k+m}} t^{p'}\,d\mu(x,t)}\right]\int_{\hat{Q}_i^{k+m}} f(x,t)t\,d\mu(x,t)$$
$$\approx \left[\frac{\int_{\hat{Q}_i^{k+m}} T^*(\chi_{E_j^k}\omega)(x,t)t^{-1}\,d\tilde{\mu}(x,t)}{\int_{\hat{Q}_i^{k+m}} d\tilde{\mu}}\right]\int_{\hat{Q}_i^{k+m}} ft^{1-p'}\,d\tilde{\mu}$$

which yields (3.5) upon summing over $i \in H_j^k$.

In view of (3.5), it is now appropriate to define the averages A_j^k by $(1/|\hat{Q}_j^k|_{\tilde{\mu}})\int_{\hat{Q}_j^k} ft^{1-p'}\,d\tilde{\mu}$. With these changes, the argument in the proof of Theorem 1 now leads to the conclusion, as in (2.17), that the left side of (2.10) is dominated by $C\beta^{-q}(\int(M_{\tilde{\mu}}g)^p\,d\tilde{\mu})^{q/p}$ where $g(x,t) = t^{1-p'}f(x,t)$ and

$$M_{\tilde{\mu}}g(x,t) = \sup_{\substack{(x,t)\in Q \text{ dyadic} \\ \text{cube in } R_+^{n+1}}} \left[\frac{1}{|Q|_{\tilde{\mu}}}\int_Q |g|\,d\tilde{\mu}\right].$$

The maximal theorem now shows that

$$\int (M_{\tilde{\mu}})^p\,d\tilde{\mu} \le C\int g^p\,d\tilde{\mu} = C\int f^p t^{(1-p')p}t^{p'}\,d\mu$$
$$= C\int f^p\,d\mu$$

and the proof of Theorem 2 is now completed as in the proof of Theorem 1.

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