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A CHART PRESERVING THE NORMAL VECTOR AND EXTENSIONS OF NORMAL DERIVATIVES IN WEIGHTED FUNCTION SPACES

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Abstract. Given a domain Ω of class $C^{k,1}$, $k \in \mathbb{N}$, we construct a chart that maps normals to the boundary of the half space to normals to the boundary of Ω in the sense that $(\partial/\partial x_n)\alpha(x',0) = -N(x')$ and that still is of class $C^{k,1}$. As an application we prove the existence of a continuous extension operator for all normal derivatives of order 0 to k on domains of class $C^{k,1}$. The construction of this operator is performed in weighted function spaces where the weight function is taken from the class of Muckenhoupt weights.

Keywords: chart, coordinate transformation, normal vector, normal derivative, extension theorem, Muckenhoupt weight

MSC 2010: 47A20, 35A99, 46E35

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a $C^{k,1}$-domain, i.e., its boundary $\partial\Omega$ can locally be expressed as the graph of a $C^{k,1}$-function

$$a\colon V \cap (\mathbb{R}^{n-1} \times \{0\}) \to \mathbb{R}$$

with an appropriate open set $V \subset \mathbb{R}^n$; here $k \in \mathbb{N}$. Then we are looking for a chart $\alpha\colon V \to U \subset \mathbb{R}^n$ of regularity as high as possible such that

$$\frac{\partial}{\partial x_n}\alpha(x',0) = -N(x') \quad \text{whenever} \quad (x',0) \in V, \tag{1.1}$$

where $N(x')$ denotes the unit outer normal vector at $(x', a(x')) \in \partial\Omega$. This means that normals to the boundary of the half space are mapped to normals to $\partial\Omega$. The natural mapping with this property is

$$x = (x', x_n) \mapsto \begin{pmatrix} x' \\ a(x') \end{pmatrix} - x_n \cdot N(x'). \tag{2.2}$$

However, if a is a $C^{k,1}$-function, then, since it includes the outer normal N, the chart (1.2) is only of class $C^{k-1,1}$.

For this reason we introduce a different chart which conserves the $C^{k,1}$-regularity and still has the property (1.1).

Coordinate transforms as in (1.2) are used e.g. by Nečas [13] to prove extension theorems of normal derivatives, see also Chapter 4 of this paper. Moreover, in [12, Chapter 4.1] Giga uses such a coordinate transformation to obtain symbols of pseudo-differential operators of a particular form. In a similar context, according to Abels, the proof of results in [1] can be significantly simplified by the use of a chart with the property (1.1) but which preserves the regularity of a.

In the second part of this paper we present an application of the chart mentioned above. We prove the existence of a continuous operator extending functions defined on the boundary in the following way. Given functions $g_1, \ldots, g_m$ on the boundary we find a function u defined on Ω such that

$$\frac{\partial^j u}{\partial N^j} = g_j \quad \text{on} \quad \partial\Omega,\ 0 \leqslant j \leqslant k.$$

In the context of classical Sobolev spaces this result can be found in [13].

The result of [13] is generalized in two aspects. First, using the particular chart constructed in the first part of this paper, one can deal with more general domains. More precisely, one can permit domains with a boundary regularity that is of one order lower than in the former results. Using this it is possible to show that the results on very weak solutions to the Navier-Stokes equations by Galdi, Simader and Sohr in [10] and by Farwig, Galdi and Sohr in [5] hold not only in $C^{2,1}$-domains but, more generally, in $C^{1,1}$-domains. This can be seen in [14] where a weighted approach to this problem is given.

Secondly, we consider the problem in weighted function spaces. This means, we consider weighted Lebesgue spaces $L^q_w(\Omega)$ and Sobolev spaces $W^{k,q}_w(\Omega)$ which means that we integrate with respect to the measure $w\,\mathrm{d}x$ for an appropriate weight function w, see Section 3 below for the exact definition of these spaces.

All weight functions that we use are contained in the Muckenhoupt class A_q. This is the class of nonnegative and locally integrable weight functions, for which the expression

$$A_q(w) := \sup_Q \left(\frac{1}{|Q|}\int_Q w\,\mathrm{d}x\right)\left(\frac{1}{|Q|}\int_Q w^{-1/(q-1)}\,\mathrm{d}x\right)^{q-1}$$

is finite, where the supremum is taken over all cubes Q in $\mathbb{R}^n$. As shown in [6], examples of Muckenhoupt weights are $w(x) = (1+|x|)^\alpha$ with $-n < \alpha < n(q-1)$ or $\operatorname{dist}(x, M)^\alpha$, $-(n-k) < \alpha < (n-k)(q-1)$, where M is a compact k-dimensional Lipschitzian manifold.

One reason why the class of Muckenhoupt weights is appropriate for the analysis is that the maximal operator is continuous in weighted L^q-spaces if and only if the weight function is a Muckenhoupt weight. Thus the powerful tools of harmonic analysis may be applied, cf. García-Cuerva and Rubio de Francia [11] and Stein [16]. Moreover, there is a powerful extrapolation theorem by Curbera, García-Cuerva, Martell and Pérez [3] that guarantees estimates in very general Banach function spaces provided the estimates in weighted function spaces are known for all weights from the Muckenhoupt class A_q.

2. Construction of the chart

Let $\Omega \subset \mathbb{R}^n$ be a $C^{k,1}$-domain, $k \in \mathbb{N}$. This means that for every $x_0 \in \partial\Omega$ we can rotate and shift the coordinate system so that its origin is x_0 and so that in a neighborhood $U(x_0)$ of x_0 one has

$$\partial\Omega \cap U(x_0) = \{(x', a(x')) \mid x' \in V(0)\}, \tag{2.1}$$

where $V(0)$ is an appropriate ($(n-1)$-dimensional) neighborhood of 0 and $a\colon V(0) \to \mathbb{R}$ is a $C^{k,1}$-function.

Lemma 2.1. *For $k \in \mathbb{N}$ let $\Omega \subset \mathbb{R}^n$ be a $C^{k,1}$-domain. Then for every $x_0 \in \partial\Omega$ there exists a neighborhood $U \subset \mathbb{R}^n$ of x_0, a neighborhood $V \subset \mathbb{R}^n$ of 0 and a bijective map $\alpha\colon V \to U$ such that*

$$\alpha(0) = x_0, \quad \alpha(V \cap (\mathbb{R}^{n-1} \times \{0\})) = U \cap \partial\Omega, \quad \alpha(V \cap \mathbb{R}^n_+) = U \cap \Omega$$

and with the following properties:
(1) $\alpha \in C^{k,1}(V, U)$,
(2) $(\partial/\partial x_n)\alpha(x', 0) = -N(x')$ *and* $(\partial/\partial x_n)^j \alpha(x', 0) = 0$ *for* $k \leqslant j \geqslant 2$ *even.*
(3) *With the notation of* (2.1) *one has*
  a) $\|\alpha\|_{C^{k,1}(V,U)}$ *can be estimated by* $\|a\|_{C^{k,1}(V\cap(\mathbb{R}^{n-1}\times\{0\}))}$,
  b) *there exists $r > 0$ which only depends on the sets $U(x_0)$, $V(x_0)$ and the size of $\|a\|_{C^{k,1}(V\cap(\mathbb{R}^{n-1}\times\{0\}))}$ such that $B_r(x_0) \subset U$.*

P r o o f. We use the notation $x = (x', x_n)$ with $x' \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$ and $\partial^\gamma = \partial^{\gamma_1}_{x_1} \cdot \ldots \cdot \partial^{\gamma_1}_{x_n}$ for $\gamma \in \mathbb{N}^n$.

After rotating and shifting the coordinate system we may assume that $x_0 = 0$, $(0, a(0)) = 0$ and $\nabla a(0) = 0$.

Let $0 \leqslant \varrho \in C_0^\infty(\mathbb{R}^{n-1})$ be radially symmetric so that

$$\operatorname{supp} \varrho \subset B_1(0) \quad \text{and} \quad \int_{\mathbb{R}^{n-1}} \varrho = 1.$$

For $t \neq 0$ we set $\varrho_t(x') = |t|^{-(n-1)} \varrho(x'/t)$. We define the function α as follows:

$$\alpha(x', x_n) = \begin{cases} \begin{pmatrix} x' \\ a(x') \end{pmatrix} - (x_n \varrho_{x_n} * N)(x') & \text{if } x_n \neq 0, \\ \begin{pmatrix} x' \\ a(x') \end{pmatrix} & \text{if } x_n = 0, \end{cases}$$

where the convolution takes place in $\mathbb{R}^{n-1}$.

Obviously, if $V \subset \mathbb{R}^n$ is small enough, then $\alpha(x', 0) \in \partial\Omega$ for $(x', 0) \in V$. Moreover, since $\partial\Omega$ is at least of class $C^{1,1}$ it follows easily from the construction of α that $\alpha(x', x_n) \in \Omega$ and $\alpha(x', -x_n) \notin \Omega$, if $x_n > 0$ is small.

Next we show (1): For every multi-index $\gamma = (\gamma', \gamma_n) \in \mathbb{N}_0^n$, with $|\gamma| \leqslant k$ and $\gamma_n \neq 0$ one has for $x_n \neq 0$

$$\begin{aligned} \partial^\gamma (x_n \varrho_{x_n} * N)(x') &= \gamma_n (-1)^{\gamma_n - 1} \int \varrho(\xi) \nabla^{\gamma_n - 1} \partial^{\gamma'} N(x' - x_n \xi) \underbrace{(\xi, \ldots, \xi)}_{\gamma_n - 1} \, \mathrm{d}\xi \\ &\quad + x_n \frac{\partial}{\partial x_n} \Big((-1)^{\gamma_n - 1} \int \varrho(\xi) \nabla^{\gamma_n - 1} \partial^{\gamma'} N(x' - x_n \xi) \underbrace{(\xi, \ldots, \xi)}_{\gamma_n - 1} \, \mathrm{d}\xi \Big). \end{aligned}$$

Then using change of variables an the fact that the map $\nabla^{\gamma_n - 1} \partial^{\gamma'} N(x' - \xi)$ is $(\gamma_n - 1)$-linear, the second summand is equal to

$$\begin{aligned} &(-1)^{\gamma_n - 1} x_n \frac{\partial}{\partial x_n} \int \frac{1}{|x_n|^{n + \gamma_n - 2}} \varrho\Big(\frac{\xi}{x_n}\Big) \nabla^{\gamma_n - 1} \partial^{\gamma'} N(x' - \xi)(\xi, \ldots, \xi) \, \mathrm{d}\xi \\ &\quad = (-1)^{\gamma_n - 1} \int ((-n - \gamma_n + 2) \varrho(\xi) - \nabla \varrho(\xi) \cdot \xi) \, \nabla^{\gamma_n - 1} \partial^{\gamma'} N(x' - \xi x_n) \underbrace{(\xi, \ldots, \xi)}_{\gamma_n - 1} \, \mathrm{d}\xi. \end{aligned}$$

Hence

$$\begin{aligned} \partial^\gamma (x_n \varrho_{x_n} * N)(x') = (-1)^{\gamma_n - 1} \int &((-n + 2) \varrho(\xi) - \nabla \varrho(\xi) \cdot \xi) \\ &\times \nabla^{\gamma_n - 1} \partial^{\gamma'} N(x' - \xi x_n) \underbrace{(\xi, \ldots, \xi)}_{\gamma_n - 1} \, \mathrm{d}\xi. \end{aligned} \tag{2.2}$$

Still we have to consider the case $\gamma_n = 0$ in which the situation is easier. Integration by parts yields

$$\partial^\gamma (x_n \varrho_{x_n} * N)(x') = \int \partial^{\beta_1}\varrho(\xi)\partial^{\beta_2} N(x' - x_n\xi)\,\mathrm{d}\xi, \tag{2.3}$$

where $\gamma = \beta_1 + \beta_2$ and $|\beta_1| = 1$.

The map $x \mapsto \binom{x'}{a(x')}$ is of class $C^{k,1}$ because a is. It remains to show that $\partial^\gamma(x_n\varrho_{x_n} * N(x'))$ is Lipschitz continuous for every $\gamma \in \mathbb{N}^n$, $|\gamma| \leqslant k$. This is an easy consequence of the representations (2.2) and (2.3) and of $N \in C^{k-1,1}$, e.g.,

$$\begin{aligned} &|\partial^\gamma(x_n\varrho_{x_n} * N(x')) - \partial^\gamma(y_n\varrho_{y_n} * N(y'))| \\ &\quad\leqslant \int\limits_{B_1(0)} |c_n\varrho(\xi) - \nabla\varrho(\xi)\cdot\xi||\nabla^{\gamma_n-1}\partial^{\gamma'}N(x'-\xi x_n) - \nabla^{\gamma_n-1}\partial^{\gamma'}N(y'-\xi y_n)|\,\mathrm{d}\xi \\ &\quad\leqslant cL_\gamma|x-y|. \end{aligned}$$

A similar calculation shows that the right-hand side of (2.3) is Lipschitz continuous.

It remains to show (2): From (2.2) we have for $j > 1$ even

$$\begin{aligned} \Big(\frac{\partial}{\partial x_n}\Big)^j \alpha(x',0) &= (-1)^{j-1}\int((-n+2)\varrho(\xi) - \nabla\varrho(\xi)\cdot\xi)\nabla^{j-1}N(x')(\xi,\dots,\xi)\,\mathrm{d}\xi \\ &= (-1)^{j-1}\Big(\int(-n+2)\varrho(\xi)\nabla^{j-1}N(x')(\xi,\dots,\xi)\,\mathrm{d}\xi \\ &\quad+\sum_{k=1}^{n-1}\int\varrho(\xi)[\nabla^{j-1}N(x')(\xi,\dots,\xi) + (j-1)\xi_k\nabla^{j-1}N(x')(e_k,\xi,\dots,\xi)]\,\mathrm{d}\xi\Big) \\ &= (-1)^{j-1}j\int\varrho(\xi)\nabla^{j-1}N(x')(\xi,\dots,\xi)\,\mathrm{d}\xi = 0, \end{aligned}$$

since ϱ is assumed to be rotationally symmetric and $\xi \mapsto \nabla^{j-1}N(x')(\xi,\dots,\xi)$ is an odd function for $j-1$ odd. Similarly,

$$\frac{\partial}{\partial x_n}\alpha(x',0) = -N(x')\Big((2-n)\int\varrho(\xi)\,\mathrm{d}\xi - \sum_{i=1}^{n-1}\int\partial_i\varrho(\xi)\xi_i\,\mathrm{d}\xi\Big) = -N(x').$$

It remains to show (3) b).

By (2.2) and (2.3) one has, since $\nabla a(0) = 0$ and $N(0) = -e_n$,

$$\nabla\alpha(0) = \nabla\begin{pmatrix} x' \\ a(x')\end{pmatrix} - \nabla(x_n\varrho_{x_n} * N(x'))\big|_{x=0} = \mathrm{id}\,.$$

Since $\nabla\alpha$ is Lipschitz continuous with a constant K, we get for $x, y \in \overline{B_r(0)}$, $r < (2K)^{-1}$ that

$$
\begin{aligned}
|\alpha(x) - \alpha(y)| &= \sup_{|v|=1} |v \cdot \nabla\alpha(\xi_v)(x-y)|, \quad \xi_v \in \{(1-t)x + ty | t \in (0,1)\} \\
&\geqslant \inf_{\xi \in B_r(0)} \Big(\frac{|x-y|^2}{|x-y|} - \Big|(x-y)(\nabla\alpha(\xi) - \nabla\alpha(0))\frac{x-y}{|x-y|}\Big|\Big) > \frac{1}{2}|x-y|.
\end{aligned}
$$

This inequality immediately implies that α is injective on $B_r(0)$.

Moreover, it is easily seen that $B_{\frac{r}{2}}(x_0) \subset \alpha B_r(0)$. Indeed, for $x \in \partial B_r(0)$ one has $|\alpha(x) - x_0| > \frac{1}{2}|x - 0|$. Since $\nabla\alpha(x)$ is invertible for every $x \in B_r(0)$ it follows from the Inverse Function Theorem that $\alpha(B_r(0))$ is open. Together with the continuity of α we obtain

$$
B_{\frac{r}{2}}(x_0) \cap \partial\alpha(B_r(0)) = B_{\frac{r}{2}}(x_0) \cap \alpha(\partial B_r(0)) = \emptyset.
$$

Assume now that $y \in B_{\frac{r}{2}}(x_0) \setminus \alpha(B_r(0))$. Then the straight line from y to x_0 intersects $\partial\alpha(B_r(0))$. Thus this intersection point is contained in the intersection which we have shown to be empty. This is a contradiction.

This argument completes the proof. □

3. Weighted function spaces

In Section 4 we want to prove an extension theorem that requires low boundary regularity. Since this is done in weighted function spaces, in this section we collect the basic definitions of weight functions and function spaces which are needed in the sequel.

Definition 3.1. Let A_q, $1 < q < \infty$, the set of Muckenhoupt weights, be given by all $0 \leqslant w \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ for which

$$
A_q(w) := \sup_Q \left(\frac{1}{|Q|}\int_Q w \,\mathrm{d}x\right)\left(\frac{1}{|Q|}\int_Q w^{-1/(q-1)} \,\mathrm{d}x\right)^{q-1} < \infty. \tag{3.1}
$$

The supremum is taken over all cubes $Q \subset \mathbb{R}^n$ and $|Q|$ denotes the Lebesgue measure of Q. To avoid trivial cases, we exclude the case where w vanishes almost everywhere.

We introduce some function spaces. First, by $C_0^\infty(\Omega)$ we denote the space of smooth functions with compact support in Ω. For $1 < q < \infty$, $w \in A_q$ and an open set Ω we define the weighted Lebesgue space by

$$
L^q_w(\Omega) := \left\{ f \in L^1_{\mathrm{loc}}(\overline{\Omega}) \mid \|f\|_{q,w} := \left(\int_\Omega |f|^q w \,\mathrm{d}x\right)^{1/q} < \infty \right\}.
$$

For $k \in \mathbb{N}_0$, the set of nonnegative integers, the weighted Sobolev spaces are defined by

$$W_w^{k,q}(\Omega) = \Big\{u \in L_w^q(\Omega) \mid \|u\|_{k,q,w} := \sum_{|\alpha|\leqslant k} \|D^\alpha u\|_{q,w} < \infty\Big\}.$$

Finally, for $k \in \mathbb{N}$ we define the space $T_w^{k,q}(\partial\Omega) := (W_w^{k,q}(\Omega))|_{\partial\Omega}$ equipped with the norm $\|\cdot\|_{T_w^{k,q}} = \|\cdot\|_{T_w^{k,q}(\partial\Omega)}$ of the factor space, i.e.,

$$\|g\|_{T_w^{k,q}(\partial\Omega)} := \inf\{\|u\|_{W_w^{k,q}(\Omega)} \mid u \in W_w^{k,q}(\Omega) \text{ and } u|_{\partial\Omega} = g\}.$$

By [7], [8] and [2] the spaces $L_w^q(\Omega)$, $W_w^{k,q}(\Omega)$ and $T_w^{k,q}(\partial\Omega)$ are reflexive Banach spaces in which $C_0^\infty(\overline{\Omega})$ or $C_0^\infty(\Omega)$ or $C_0^\infty(\overline{\Omega})|_{\partial\Omega}$, respectively, are dense.

Note that by Slobodeckiĭ [15] and Nečas [13, Chapitre 2, §5], in the unweighted case one has

$$T_1^{k,q}(\partial\Omega) = W^{k-1/q,q}(\partial\Omega).$$

However, in the setting of Muckenhoupt weights such a characterization of the spaces by an intrinsic norm is known only for few examples of weight functions.

For weighted function spaces change of variables is possible in the following sense.

Lemma 3.2. *Let Ω and $\mathcal{O}$ be two domains in $\mathbb{R}^n$ and let*

$$\alpha\colon \overline{\mathcal{O}} \to \overline{\Omega}$$

be a $C^{k-1,1}$-diffeomorphism, $k \geqslant 1$.

1. *The operator*

$$T\colon\ u \mapsto u \circ \alpha\colon\ W_w^{k,q}(\Omega) \to W_{w\circ\alpha}^{k,q}(\mathcal{O})$$

is continuous.

2. *The same is true for the operator*

$$S\colon\ g \mapsto g \circ \alpha\colon\ T_w^{k,q}(\partial\Omega) \to T_{w\circ\alpha}^{k,q}(\partial\mathcal{O}).$$

Proof. The first assertion follows immediately from the change of variables formula, the second follows from the first using the definition of $T_w^{k,q}(\partial\Omega)$. □

By [9] the following weighted analogue of the Poincaré inequality holds: there exists a constant $c = c(q,w) > 0$ such that

$$\|u\|_{q,w} \leqslant c\|\nabla u\|_{q,w} \quad \text{for every } \ u \in W_{w,0}^{1,q}(\Omega). \tag{3.2}$$

Moreover, solvability of the following Laplace resolvent problem continues to hold in weighted function spaces.

Theorem 3.3 (Regularity of the Dirichlet Problem). *Let $1 < q < \infty$, $k \in \mathbb{Z}$, $k \geqslant -1$ and let $f \in W_w^{k,q}(\mathbb{R}_+^n)$. Then there exists a unique weak solution $u \in W_w^{k+2,q}(\mathbb{R}_+^n)$ to the Dirichlet Problem*

$$(1 - \Delta)u = f \quad \text{and} \quad u|_{\mathbb{R}^{n-1}} = 0.$$

It fulfils the estimate $\|u\|_{k+2,q,w} \leqslant c\|f\|_{k,q,w}$, where $c = c(k, q, w)$.

The same is true for the solution u of $(1-\Delta)u = 0$, $u|_{\mathbb{R}^{n-1}} = g$, if $g \in T_w^{k+2,q}(\mathbb{R}^{n-1})$, i.e., it fulfils the estimate

$$\|u\|_{k+2,q,w} \leqslant c\|g\|_{T_w^{k+2,q}}.$$

Proof. For $k = -1$ the first assertion has been proved by Fröhlich in [7]. Using this, one obtains regularity of this boundary value problem as in the classical unweighted case which can be found e.g. in Evans [4].

For the second assertion let $v \in W_w^{k+2,q}(\mathbb{R}_+^n)$ be an extension of g. Then we find a unique $u \in W_w^{k+2,q}(\mathbb{R}_+^n)$ with $(\mathrm{id} - \Delta)u = (\mathrm{id} - \Delta)v$ and $u|_{\mathbb{R}^{n-1}} = 0$. Thus $v - u$ solves the problem and by the first assertion it fulfils the estimate. □

4. Extensions of functions on the boundary

Our next objective is to construct a linear extension operator that maps functions defined on the boundary $\partial\Omega$ to a function defined on the domain Ω whose boundary values or normal derivatives are given preimages.

We start with the half space.

Theorem 4.1. *Let $1 < q < \infty$, $w \in A_q$ and $k \in \mathbb{N}$. Then there exists a continuous linear operator*

$$T\colon \prod_{j=0}^{k-1} T_w^{k-j,q}(\mathbb{R}^{n-1}) \to W_w^{k,q}(\mathbb{R}_+^n)$$

such that $(\partial^j/\partial x_n^j)T(g_0, \ldots, g_{k-1})|_{x_n=0} = g_j$, $j = 0, \ldots, k-1$.

Proof. It suffices to show that for every $g \in T_w^{k-j,q}(\mathbb{R}^{n-1})$, $j = 0, \ldots, k-1$, there exists $u \in W_w^{k,q}(\mathbb{R}_+^n)$ depending continuously and linearly on g such that $(\partial^j/\partial x_n^j)u = g$ and $(\partial^i/\partial x_n^i)u = 0$ for every $i = 0, \ldots, j-1$. To see this assume that for every $j = 0, \ldots, k-1$ there exists a continuous linear operator

$$T_j\colon T_w^{k-j,q}(\mathbb{R}^{n-1}) \to W_w^{k,q}(\mathbb{R}_+^n), \quad \frac{\partial^i}{\partial x_n^i} T_j(h)\big|_{x_n=0} = \begin{cases} 0, & \text{if } i < j, \\ h, & \text{if } i = j. \end{cases}$$

For $g = (g_0, \ldots, g_{k-1})$ we can define $S_0(g) := T_0(g)$ and

$$S_{j+1}(g) := S_j(g) + T_{j+1}\Big(g_{j+1} - \frac{\partial^{j+1}}{\partial x_n^{j+1}} S_j(g)\Big).$$

Then $T = S_{k-1}$ solves our problem.

Next we show the weaker assertion. For $g \in T_w^{k-j,q}(\mathbb{R}^{n-1})$ let $v \in W_w^{k-j,q}(\mathbb{R}^n_+)$ with $(1-\Delta)v = 0$ and $v|_{\mathbb{R}^{n-1}} = g$ which is uniquely defined by Theorem 3.3. Let $\zeta \in C^\infty(\mathbb{R}_+)$ be a cut-off function with $\zeta(t) = 1$ for $t < 1$ and $\zeta(t) = 0$ for $t > 2$. We set

$$\varphi(x) = \varphi(x_n) = \frac{1}{j!}\, x_n^j \cdot \zeta(x_n) \quad \text{and} \quad u(x) = \varphi(x)v(x). \tag{4.1}$$

We show that φu solves the problem. More precisely, we prove the following claim:

If $\varphi \in C^\infty(\overline{\mathbb{R}^n_+})$ with $\varphi(x) = \varphi(x_n)$, $\operatorname{supp}\varphi \subset \mathbb{R}^{n-1} \times [0,2]$ and $(\partial/\partial x_n)^m \varphi|_{x_n=0} = 0$ for $m = 0, \ldots, l$ and $v \in W_w^{k,q}(\mathbb{R}^n_+)$ with $(1-\Delta)v = 0$ then $\varphi v \in W_w^{k+l,q}(\mathbb{R}^n_+)$ with $\|\varphi v\|_{k+l,q,w} \leqslant c\|v\|_{k,q,w}$.

To prove this we use mathematical induction with respect to l and assume that we already know the assertion is true for $l-1$, $l-2$ and all k.

Since $(1-\Delta)v = 0$ we obtain

$$(1-\Delta)(\varphi v) = -\Delta\varphi v - 2\nabla v \cdot \nabla\varphi. \tag{4.2}$$

As $(\partial/\partial x_n)^m \Delta\varphi|_{x_n=0} = 0$ for $m = 0, \ldots, l-2$, $(\partial/\partial x_n)^m \nabla\varphi|_{x_n=0} = 0$ for $m = 0, \ldots, l-1$ and $(1-\Delta)\nabla v = 0$, (4.2) and the induction hypothesis yield $(1-\Delta)(\varphi v) \in W_w^{k+l-2,q}(\mathbb{R}^n_+)$. Thus and since $\varphi v|_{\mathbb{R}^{n-1}} = 0$, one has $\varphi v \in W_w^{k+l,q}(\mathbb{R}^n_+)$ by the regularity of the Laplace resolvent problem. Moreover,

$$\|\varphi v\|_{k+l,q,w} \leqslant c\|(\Delta\varphi)v + 2\nabla v \nabla\varphi\|_{k+l-2,q,w} \leqslant c(\|v\|_{k,q,w} + \|\nabla v\|_{k-1,q,w}) \leqslant c\|v\|_{k,q,w}.$$

For the start of induction we need the cases $l = 0$ and $l = 1$. The case $l = 0$ is trivial, the case $l = 1$ is proved in the same way as the induction step.

If one applies the above claim to u given by (4.1) one gets $u \in W_w^{k,q}(\Omega)$. Moreover,

$$\frac{\partial^l}{\partial x_n^l} u(x',0) = \sum_{\nu=0}^{l} \binom{l}{\nu} \frac{\partial^\nu}{\partial x_n^\nu} v \frac{\partial^{l-\nu}}{\partial x_n^{l-\nu}} \varphi(x',0) = \begin{cases} 0 & \text{if } l < j, \\ g(x') & \text{if } l = j. \end{cases}$$

This shows the assertion about the boundary values. $\square$

Theorem 4.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded $C^{k-1,1}$-domain, $k \geqslant 1$. Then there exists a continuous linear operator*

$$L\colon \prod_{j=0}^{k-1} T_w^{k-j,q}(\partial\Omega) \to W_w^{k,q}(\Omega)$$

such that $(\partial^j/\partial N^j)L(g)|_{\partial\Omega} = (-1)^j g_j$, $0 \leqslant j \leqslant k-1$, where $g = (g_0, \dots, g_{k-1})$.

P r o o f. We start with the case $k = 0$. Then in the unweighted case the result is known and can be found in [13]. Since for a Lipschitz-mapping α and an A_q-weight w the concatenation $w \circ \alpha$ is again contained in A_q, in this case the proof of the result without weight can be transferred to the weighted case without change.

Form now on we assume $k \geqslant 1$. As in the proof of Theorem 4.1 we construct an operator

$$L_j\colon T_w^{k-j,q}(\partial\Omega) \to W_w^{k,q}(\Omega), \quad \frac{\partial^k}{\partial N^j} L_j(g) = \begin{cases} (-1)^j g & \text{if } k = j, \\ 0 & \text{if } k < j. \end{cases}$$

Then the general case follows as in the proof of Theorem 4.1.

We choose the collection of charts $(\alpha_i, V_i, U_i)_{i=1}^m$ according to Lemma 2.1 and a decomposition of unity $(\varphi_i)_{i=1}^m$ subordinate to the covering $\{U_i\}$.

To simplify the notation we fix i and set $\gamma = \alpha_i$, $U = U_i$, $V = V_i$ and $\varphi = \varphi_i$. Moreover, for $g \in T_w^{k-j,q}(\partial\Omega)$ we set $\tilde{g} = (g \cdot \varphi) \circ \gamma$. By Lemma 3.2 we know $\tilde{g}_j \in T_{w\circ\gamma}^{k-j,q}(\mathbb{R}^{n-1})$. Thus we may apply the operator T from Theorem 4.1 and set

$$v := v_i := L_{i,j}(g) := (\psi_i T(0, \dots, 0, \tilde{g}, 0, \dots, 0)) \circ \gamma^{-1},$$

meaning that the j'th component of $(0, \dots, 0, \tilde{g}, 0, \dots, 0)$ is $\tilde{g}$.

Moreover, $(\psi_i)_i \subset C_0^\infty(\overline{\mathbb{R}}_+^n)$ with $\psi_i = 1$ in a neighborhood of $\operatorname{supp} \tilde{g}$ and $\operatorname{supp} \psi_i \subset V_i$. Here ψ_i can be chosen such that $(\partial^k/\partial x_n^k)\psi_i(x', 0) = 0$ for every $k \in \mathbb{N}$.

Then we have by the choice of γ according to Lemma 2.1 for every $k \leqslant j$

$$\begin{aligned}(-1)^k \delta_{j,k} \tilde{g}(x') &= \frac{\partial^k}{\partial x_n^k} T(\dots 0, \tilde{g}, 0 \dots)(x', 0) = \Big(\frac{\partial^k}{\partial x_n^k}(v \circ \gamma)\Big)(x', 0) \\ &= (\nabla^k v \circ \gamma) \cdot (\partial_n \gamma, \dots, \partial_n \gamma)(x', 0) + \text{terms containing } \nabla^i v \circ \gamma(x', 0),\ i < j \\ &= (\nabla^k v(\gamma(x', 0)))\underbrace{(-N(x'), \dots, -N(x'))}_{k} = (-1)^k \Big(\frac{\partial^k}{\partial^k N} v\Big)(\gamma(x', 0)).\end{aligned}$$

The terms containing $\nabla^i v \circ \gamma(x', 0)$ vanish for $i < j$, since

$$\nabla^i (v \circ \gamma)(x', 0) = \nabla^i (\psi_i T(0, \dots, 0, \tilde{g}, 0, \dots, 0))(x', 0) = 0$$

for $i = 1, \dots, j-1$ by the definition of T.

Finally, we set $L_j(g) = \sum\limits_{i=1}^{m} L_{i,j}(g)$ and obtain

$$\frac{\partial^k}{\partial N^k} L_j(g)|_{\partial\Omega} = \sum_{i=1}^{m} \frac{\partial^k}{\partial N^k} L_{i,j}(g)|_{\partial\Omega} = \begin{cases} g & \text{if } k = j, \\ 0 & \text{if } k < j. \end{cases}$$

The continuity of L_j follows from Lemma 3.2 and the continuity of T in Theorem 4.1. □

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