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A CHERN–SIMONS E_8 GAUGE THEORY OF GRAVITY IN $D = 15$, GRAND UNIFICATION AND GENERALIZED GRAVITY IN CLIFFORD SPACES

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A novel Chern–Simons E_8 gauge theory of gravity in $D = 15$ based on an *octic* E_8 invariant expression in $D = 16$ (recently constructed by Cederwall and Palmkvist) is developed. A grand unification model of gravity with the other forces is very plausible within the framework of a supersymmetric extension (to incorporate spacetime fermions) of this Chern–Simons E_8 gauge theory. We review the construction showing why the ordinary $11D$ Chern–Simons gravity theory (based on the Anti de Sitter group) can be embedded into a Clifford-algebra valued gauge theory and that an E_8 Yang–Mills field theory is a small sector of a Clifford (16) algebra gauge theory. An E_8 gauge bundle formulation was instrumental in understanding the topological part of the 11-dim M-theory partition function. The nature of this 11-dim E_8 gauge theory remains unknown. We hope that the Chern–Simons E_8 gauge theory of gravity in $D = 15$ advanced in this work may shed some light into solving this problem after a dimensional reduction.

Keywords: E_8 grand unification; M-theory; Chern–Simons Gravity; Clifford algebras; extended relativity in Clifford spaces.

1. Introduction

Exceptional, Jordan, Division and Clifford algebras are deeply related and essential tools in many aspects of Physics [3, 5, 8, 9, 14–20]. Ever since the discovery [1] that $11D$ supergravity, when dimensionally reduced to an n-dim torus led to maximal supergravity theories with hidden exceptional symmetries E_n for $n \leq 8$, it has prompted intensive research to explain the higher dimensional origins of these hidden exceptional E_n symmetries [2, 6]. More recently, there has been a lot of interest in the infinite-dim hyperbolic Kac–Moody E_{10} and nonlinearly realized E_{11} algebras arising in the asymptotic chaotic oscillatory solutions of supergravity fields close to cosmological singularities [1, 2].

The classification of symmetric spaces associated with the scalars of N extended supergravity theories, emerging from compactifications of $11D$ supergravity to lower dimensions, and the construction of the U-duality groups as spectrum-generating symmetries for four-dimensional BPS black holes [6] also involved exceptional symmetries associated with the exceptional magic Jordan algebras $J_3[R, C, H, O]$. The discovery of the anomaly free 10-dim heterotic string for the algebra $E_8 \times E_8$ was another hallmark of the importance of exceptional Lie groups in Physics.

The E_8 group was proposed long ago [24] as a candidate for a grand unification model building in $D = 4$. An extensive review of the E_6 grand unified models may be found in [26]. The supersymmetric E_8 model has more recently been studied as a fermion family and grand unification model [25] under the assumption that there is a vacuum gluino condensate but this condensate is *not* accompanied by a dynamical generation of a mass gap in the pure E_8 gauge sector. A study of the interplay among exceptional groups, del Pezzo surfaces and the extra massless particles arising from rational double point singularities can be found in [38]. Clifford algebras and E_8 are key ingredients in Smith's $D_4 - D_5 - E_6 - E_7 - E_8$ grand unified model in $D = 8$ [6].

An E_8 gauge bundle was instrumental in the understanding the topological part of the M-theory partition function [27, 32]. A mysterious E_8 bundle which restricts from 12-dim to the 11-dim bulk of M theory can be compatible with 11-dim supersymmetry. The nature of this 11-dim E_8 gauge theory remains unknown. We hope that the Chern–Simons E_8 gauge theory of gravity in $D = 15$ advanced in this work may shed some light into solving this question.

E_8 Yang–Mills theory can naturally be embedded into a Cl(16) algebra gauge theory [33] and the $11D$ Chern–Simons (super) gravity [4] is a very small sector of a more fundamental polyvector-valued gauge theory in Clifford spaces. Polyvector-valued supersymmetries [11] in Clifford-spaces [3] turned out to be more fundamental than the supersymmetries associated with M, F theory superalgebras [7, 10]. For this reason, we believe that Clifford structures may shed some light into the origins behind the hidden E_8 symmetry of $11D$ supergravity and reveal more important features underlying M, F theory.

The main purpose of this work is to develop a Chern–Simons E_8 gauge theory of gravity in $D = 15$ based on an octic E_8 invariant expression in $D = 16$ recently constructed by [23], and to propose a grand unification of gravity with all the other forces within the framework of a supersymmetric extension (to incorporate spacetime fermions) of the Chern–Simons E_8 gauge theory. Our octic E_8 invariant action has 37 terms and contains: (i) the Lanczos–Lovelock gravitational action associated with the 15-dim boundary $\partial\mathcal{M}^{16}$ of the 16-dim manifold; (ii) five terms with the same structure as the Pontryagin $p_4(F^{IJ})$ 16-form associated with the SO(16) spin connection Ω_μ^{IJ} where the indices I, J run from $1, 2, \ldots, 16$; (iii) the fourth power of the standard quadratic E_8 invariant $[I_2]^4$; (iv) plus 30 additional terms involving powers of the E_8-valued $F_{\mu\nu}^{IJ}$ and $F_{\mu\nu}^{\alpha}$ field-strength (two-forms).

In the final section, we explain how a Clifford algebra gauge theory (that includes the Chern–Simons gravity action) can itself be embedded into a more fundamental polyvector-valued gauge theory in Clifford spaces involving tensorial coordinates $x^{\mu_1\mu_2}, x^{\mu_1\mu_2\mu_3}, \ldots$ in addition to antisymmetric tensor gauge fields $A_{\mu_1\mu_2}, A_{\mu_1\mu_2\mu_3}, \ldots$. The polyvector–valued supersymmetric extension of this polyvector valued bosonic gauge theory in Clifford spaces may reveal more important features of a Clifford-algebraic structure underlying M, F, S theory in $D = 11, 12, 13$ dimensions. An overview of the basic features of the extended relativity in Clifford spaces can be found in [3] and a polyvector-valued generalized supersymmetry algebra in Clifford spaces was presented in [11].

2. A Chern–Simons E_8 Gauge Theory of Gravity

2.1. *E_8 Yang–Mills in $D = 4$ and Clifford-algebra-valued gauge theories*

It is well known among the experts that the E_8 algebra admits the SO(16) decomposition $\mathbf{248} \to \mathbf{120} \oplus \mathbf{128}$. The E_8 admits also a SL$(8, R)$ decomposition [6]. Due to the triality property, the SO(8) admits the vector $\mathbf{8}_v$ and spinor representations $\mathbf{8}_s, \mathbf{8}_c$. After a triality rotation, the SO(16) vector and spinor representations decompose as [6]

$$\mathbf{16} \to \mathbf{8}_s \oplus \mathbf{8}_c. \tag{2.1a}$$

$$\mathbf{128}_s \to \mathbf{8}_v \oplus \mathbf{56}_v \oplus \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}_v. \tag{2.1b}$$

$$\mathbf{128}_c \to \mathbf{8}_s \oplus \mathbf{56}_s \oplus \mathbf{8}_c \oplus \mathbf{56}_c. \tag{2.1c}$$

To connect with (real) Clifford algebras [8], i.e. how to fit E_8 into a Clifford structure, start with the 248-dim fundamental representation E_8 that admits a SO(16) decomposition given by the 120-dim bivector representation plus the 128-dim chiral-spinor representations of SO(16). From the modulo eight periodicity of Clifford algebras over the reals one has $\mathrm{Cl}(16) = \mathrm{Cl}(2 \times 8) = \mathrm{Cl}(8) \otimes \mathrm{Cl}(8)$, meaning, roughly, that the $2^{16} = 256 \times 256$ Cl(16)-algebra matrices can be obtained effectively by replacing each one of the entries of the $2^8 = 256 = 16 \times 16$ Cl(8)-algebra matrices by the 16×16 matrices of the second copy of the Cl(8) algebra. In particular, $120 = 1 \times 28 + 8 \times 8 + 28 \times 1$ and $128 = 8 + 56 + 8 + 56$, hence the 248-dim E_8 algebra decomposes into a $120 + 128$ dim structure such that E_8 can be represented indeed within a tensor product of Cl(8) algebras.

At the E_8 Lie algebra level, the E_8 gauge connection decomposes into the SO(16) vector $I, J = 1, 2, \ldots, 16$ and (chiral) spinor $A = 1, 2, \ldots, 128$ indices as follows

$$\begin{gathered} \mathcal{A}_\mu = \mathcal{A}_\mu^{IJ} X_{IJ} + \mathcal{A}_\mu^A Y_A, \quad X_{IJ} = -X_{JI}, \\ I, J = 1, 2, 3, \ldots, 16, \quad A = 1, 2, \ldots, 128, \end{gathered} \tag{2.2}$$

where X_{IJ}, Y_A are the E_8 generators. The Clifford algebra $(\mathrm{Cl}(8) \otimes \mathrm{Cl}(8))$ structure behind the SO(16) decomposition of the E_8 gauge field $\mathcal{A}_\mu^{IJ} X_{IJ} + \mathcal{A}_\mu^A Y_A$ can be deduced from the expansion of the generators X_{IJ}, Y_A in terms of the Cl(16) algebra generators. The Cl(16) bivector basis admits the decomposition

$$X^{IJ} = a_{ij}^{IJ} (\gamma_{ij} \otimes \mathbf{1}) + b_{ij}^{IJ} (\mathbf{1} \otimes \gamma_{ij}) + c_{ij}^{IJ} (\gamma_i \otimes \gamma_j), \tag{2.3}$$

where γ_i, are the Clifford algebra generators of the Cl(8) algebra present in $\mathrm{Cl}(16) = \mathrm{Cl}(8) \otimes \mathrm{Cl}(8)$; $\mathbf{1}$ is the unit Cl(8) algebra element that can be represented by a unit

16×16 diagonal matrix. The tensor products $\otimes$ of the 16×16 Cl(8)-algebra matrices, like $\gamma_i \otimes \mathbf{1}$, $\gamma_i \otimes \gamma_j, \ldots$ furnish a 256×256 Cl(16)-algebra matrix, as expected. Therefore, the decomposition in (2.3) yields the $28+28+8\times 8 = 56+64 = 120$-dim bivector representation of SO(16); i.e. for each fixed values of IJ there are 120 terms in the right-hand side of (2.3), that match the number of independent components of the E_8 generators $X^{IJ} = -X^{JI}$, given by $\frac{1}{2}(16 \times 15) = 120$. The decomposition of Y_A is more subtle. A spinor Ψ in $16D$ has $2^8 = 256$ components and can be decomposed into a 128 component left-handed spinor Ψ^A and a 128 component right-handed spinor $\Psi^{\dot{A}}$; the 256 spinor indices are $\alpha = A, \dot{A}; \beta = B, \dot{B}, \ldots$ with $A, B = 1, 2, \ldots, 128$ and $\dot{A} \cdot \dot{B} = 1, 2, \ldots, 128$, respectively.

Spinors are elements of right (left) ideals of the Cl(16) algebra and admit the expansion $\Psi = \Psi_\alpha \xi^\alpha$ in a 256-dim spinor basis ξ^α which in turn can be expanded as sums of Clifford polyvectors of mixed grade; i.e. into a sum of scalars, vectors, bivectors, trivectors, ... Minimal left/right ideals elements of Clifford algebras may be systematically constructed by means of idempotents $e^2 = e$ such that the geometric product of $\mathrm{Cl}(p, q)e$ generates the ideal [22].

The commutation relations of E_8 are [6]

$$\begin{aligned} &[X^{IJ}, X^{KL}] = 4(\delta^{IK} X^{LJ} - \delta^{IL} X^{KJ} + \delta^{JK} X^{IL} - \delta^{JL} X^{IK}), \\ &\quad [X_{IJ}, Y^\alpha] = -\frac{1}{2}\Gamma^{\alpha\beta}_{IJ} Y_\beta; \quad [Y^\alpha, Y^\beta] = \frac{1}{4}\Gamma^{\alpha\beta}_{IJ} X^{IJ}, \quad \Gamma^{\alpha\beta}_{IJ} = [\Gamma_I, \Gamma_J]^{\alpha\beta}. \end{aligned} \tag{2.4}$$

The combined E_8 indices are denoted by $\mathcal{A} \equiv [IJ]$, α ($120 + 128 = 248$ indices in total) that yield the Killing metric and the structure constants

$$\eta^{\mathcal{A}\mathcal{B}} = \frac{1}{60} Tr T^{\mathcal{A}} T^{\mathcal{B}} = -\frac{1}{60} f^{\mathcal{A}}_{\mathcal{C}\mathcal{D}} f^{\mathcal{B}\mathcal{C}\mathcal{D}}, \tag{2.5a}$$

$$f^{IJ,KL,MN} = -8\delta^{IK}\delta^{LJ}_{MN} + \text{permutations}; \quad f^{IJ}_{\alpha\beta} = -\frac{1}{2}\Gamma^{IJ}_{\alpha\beta}; \tag{2.5b}$$

$$\eta^{IJKL} = -\frac{1}{60} f^{IJ}_{\mathcal{C}\mathcal{D}} f^{KL,\mathcal{C}\mathcal{D}}.$$

We shall proceed with the Cl(16) gauge theory that encodes the exceptional Lie algebra E_8 symmetry from the start. The E_8 gauge theory in $D = 4$ is based on the E_8-valued field strengths

$$F^{IJ}_{\mu\nu} X_{IJ} = (\partial_\mu \mathcal{A}^{IJ}_\nu - \partial_\nu \mathcal{A}^{IJ}_\mu) X_{IJ} + \mathcal{A}^{KL}_\mu \mathcal{A}^{MN}_\nu [X_{KL}, X_{MN}] + \mathcal{A}^\alpha_\mu \mathcal{A}^\beta_\nu [Y_\alpha, Y_\beta], \tag{2.6}$$

$$F^A_{\mu\nu} Y_\alpha = (\partial_\mu \mathcal{A}^\alpha_\nu - \partial_\nu \mathcal{A}^\alpha_\mu) Y_\alpha + \mathcal{A}^\alpha_\mu \mathcal{A}^{IJ}_\nu [Y_\alpha, X_{IJ}]. \tag{2.7}$$

The E_8 actions are

$$\begin{aligned} S_{\text{Topological}}[E_8] &= \int d^4x \frac{1}{60} Tr[F^{\mathcal{A}}_{\mu\nu} F^{\mathcal{B}}_{\rho\tau} T_{\mathcal{A}} T_{\mathcal{B}}] \epsilon^{\mu\nu\rho\tau} = \int d^4x F^{\mathcal{A}}_{\mu\nu} F^{\mathcal{B}}_{\rho\tau} \eta_{\mathcal{A}\mathcal{B}} \epsilon^{\mu\nu\rho\tau} \\ &= \int d^4x [F^{IJ}_{\mu\nu} F^{KL}_{\rho\tau} \eta_{IJKL} + F^\alpha_{\mu\nu} F^\beta_{\rho\tau} \eta_{\alpha\beta} + 2F^{IJ}_{\mu\nu} F^\beta_{\rho\tau} \eta_{IJ\beta}] \epsilon^{\mu\nu\rho\tau}, \end{aligned} \tag{2.8}$$

where $\epsilon^{\mu\nu\rho\tau}$ is the covariantized permutation symbol and

$$S_{YM}[E_8] = \int d^4x\sqrt{g}\frac{1}{60}Tr[F^{\mathcal{A}}_{\mu\nu}F^{\mathcal{B}}_{\rho\tau}T_{\mathcal{A}}T_{\mathcal{B}}]g^{\mu\rho}g^{\nu\tau} = \int d^4x\sqrt{g}F^{\mathcal{A}}_{\mu\nu}F^{\mathcal{B}}_{\rho\tau}\eta_{\mathcal{A}\mathcal{B}}g^{\mu\rho}g^{\nu\tau}$$
$$= \int d^4x\sqrt{g}[F^{IJ}_{\mu\nu}F^{KL}_{\rho\tau}\eta_{IJKL} + F^{\alpha}_{\mu\nu}F^{\beta}_{\rho\tau}\eta_{\alpha\beta} + 2F^{IJ}_{\mu\nu}F^{\beta}_{\rho\tau}\eta_{IJ\beta}]g^{\mu\rho}g^{\nu\tau}. \quad (2.9)$$

The above E_8 actions (are part of) can be embedded onto more general Cl(16) actions with a much larger number of terms given by

$$S_{\text{Topological}}[\text{Cl}(16)] = \int d^4x\langle F^{\mathcal{M}}_{\mu\nu}F^{\mathcal{N}}_{\rho\tau}\Gamma_{\mathcal{M}}\Gamma_{\mathcal{N}}\rangle\epsilon^{\mu\nu\rho\tau}$$
$$= \int d^4x F^{\mathcal{M}}_{\mu\nu}F^{\mathcal{N}}_{\rho\tau}G_{\mathcal{M}\mathcal{N}}\epsilon^{\mu\nu\rho\tau}, \quad (2.10)$$

and

$$S_{YM}[\text{Cl}(16)] = \int d^4x\sqrt{g}\langle F^{\mathcal{M}}_{\mu\nu}F^{\mathcal{N}}_{\rho\tau}\Gamma_{\mathcal{M}}\Gamma_{\mathcal{N}}\rangle g^{\mu\rho}g^{\nu\tau}$$
$$= \int d^4x\sqrt{g}F^{\mathcal{M}}_{\mu\nu}F^{\mathcal{N}}_{\rho\tau}G_{\mathcal{M}\mathcal{N}}g^{\mu\rho}g^{\nu\tau}, \quad (2.11)$$

where $\langle\Gamma_{\mathcal{M}}\Gamma_{\mathcal{N}}\rangle = G_{\mathcal{M}\mathcal{N}}\mathbf{1}$ denotes the scalar part of the Clifford geometric product of the gammas. Notice that there are a total of 65536 terms in

$$F^{\mathcal{M}}_{\mu\nu}F^{\mathcal{N}}_{\rho\tau}G_{\mathcal{M}\mathcal{N}} = F_{\mu\nu}F_{\rho\tau} + F^{I}_{\mu\nu}F^{I}_{\rho\tau} + F^{I_1I_2}_{\mu\nu}F^{I_1I_2}_{\rho\tau}$$
$$+ \cdots + F^{I_1I_2\ldots I_{16}}_{\mu\nu}F^{I_1I_2\ldots I_{16}}_{\rho\tau}, \quad (2.12)$$

where the indices run as $I = 1, 2, \ldots, 16$. The Clifford algebra Cl(16) has the graded structure (scalars, bivectors, trivectors, …, pseudoscalar) given by

$$\begin{matrix} 1\ 16\ 120\ 560\ 1820\ 4368\ 8008\ 11440\ 12870 \\ 11440\ 8008\ 4368\ 1820\ 560\ 120\ 16\ 1 \end{matrix}, \quad (2.13)$$

consistent with the dimension of the Cl(16) algebra $2^{16} = 256 \times 256 = 65536$.

The possibility that one can accommodate another copy of the E_8 algebra within the Cl(16) algebraic structure warrants further investigation by working with the duals of the bivectors X_{IJ} and recurring to the remaining $Y_{\dot{A}}$ generators. The motivation is to understand the full symmetry of the $E_8 \times E_8$ heterotic string from this Clifford algebraic perspective. A clear embedding is, of course, the following

$$E_8 \times E_8 \subset \text{Cl}(8) \otimes \text{Cl}(8) \otimes \text{Cl}(8) \otimes \text{Cl}(8) \subset \text{Cl}(16) \otimes \text{Cl}(16) = \text{Cl}(32), \quad (2.14)$$

where SO(32) $\subset$ Cl(32) and SO(32) is also an anomaly free group of the heterotic string that has the same dimension and rank as $E_8 \times E_8$.

2.2. *An E_8 gauge theory of gravity based on an octic invariant*

The action that defines a Chern–Simons E_8 gauge theory of gravity in 15-dim is

$$\begin{aligned} S &= \int_{\mathcal{M}^{16}} \langle FF \dots F \rangle_{E_8} \\ &= \int_{\mathcal{M}^{16}} (F^{M_1} \wedge F^{M_2} \wedge \cdots \wedge F^{M_8}) \Upsilon_{M_1 M_2 M_3 \cdots M_8} \\ &= \int_{\partial \mathcal{M}^{16}} \mathcal{L}^{(15)}_{\mathrm{CS}}(\mathbf{A}, \mathbf{F}). \end{aligned} \tag{2.15}$$

The E_8 Lie-algebra-valued 16-form $\langle F^8 \rangle$ is *closed* : $d(\langle F^{M_1} T_{M_1} \wedge F^{M_2} T_{M_2} \wedge \cdots \wedge F^{M_8} T_{M_8} \rangle) = 0$ and locally can always be written as an exact form in terms of an E_8-valued Chern–Simons 15-form as $I_{16} = d\mathcal{L}^{(15)}_{\mathrm{CS}}(\mathbf{A}, \mathbf{F})$. For instance, when $\mathcal{M}^{16} = S^{16}$ the 15-dim boundary integral (2.15) is evaluated in the two coordinate patches of the equator $S^{15} = \partial \mathcal{M}^{16}$ of S^{16} leading to the integral of $tr(\mathbf{g}^{-1} d\mathbf{g})^{15}$ (up to numerical factors) when the gauge potential $\mathbf{A}$ is written locally as $\mathbf{A} = \mathbf{g}^{-1} d\mathbf{g}$ and $\mathbf{g}$ belongs to the E_8 Lie-algebra. The integral is characterized by the elements of the homotopy group $\pi_{15}(E_8)$. S^{16} can also be represented in terms of quaternionic and octonionic projectives spaces as HP^4, OP^2, respectively.

In order to evaluate the operation $\langle \cdots \rangle_{E_8}$ in the action it involves the existence of an octic E_8 group invariant tensor $\Upsilon_{M_1 M_2 \dots M_8}$ that was recently constructed by Cederwall and Palmkvist [23] using the Mathematica package GAMMA based on the full machinery of the Fierz identities. The entire octic E_8 invariant contains powers of the SO(16) bivector X^{IJ} and spinorial Y^α generators $X^8, X^6 Y^2, X^4 Y^4, X^2 Y^6, Y^8$. The corresponding number of terms is $6, 11, 12, 5, 2$, respectively, giving a total of **36** terms for the octic E_8 invariant involving **36** numerical coefficients multiplying the corresponding powers of the E_8 generators. There is an extra term (giving a total of **37** terms) with an arbitrary constant multiplying the fourth power of the quadratic invariant $I_2 = -\frac{1}{2} tr[(F^{IJ}_{\mu\nu} X_J)^2 + (F^\alpha_{\mu\nu} Y_\alpha)^2]$.

The Euler-density in $16D$ corresponds to the Pfaffian associated with the 16×16 antisymmetric matrix F^{IJ} where the components F^{IJ} can be read from Eq. (2.6). The Euler (Born–Infled) action density is

$$\mathrm{Pfaffian}(\mathbf{F}) \equiv \sqrt{\det \mathbf{F}} = L_{\mathrm{Euler}} = F^{I_1 J_1} F^{I_2 J_2} F^{I_3 J_3} \dots F^{I_8 J_8} \epsilon_{I_1 J_1 I_2 J_2 \dots I_8 J_8}, \tag{2.16}$$

such that the exterior derivative of the gravitational 15-dim Lanczos–Lovelock (LL) action $\mathcal{L}^{(15)}_{LL}$ corresponding to the 15-dim boundary $\Sigma = \partial \mathcal{M}^{16}$ yields the Euler-density 16-form $d\mathcal{L}_{LL} = L_{Euler}$. Upon inserting the spacetime indices $\mu_1, \mu_2, \dots, \mu_{16}$, the Euler characteristic class invariant $e(\mathcal{TM})$ of the SO(16) tangent bundle associated with $\mathcal{M}^{16}$ is given by

$$\begin{aligned} S_{\text{Euler}} &= \int_{\mathcal{M}^{16}} \epsilon^{\mu_1\mu_2\ldots\mu_{16}} F^{I_1J_1}_{\mu_1\mu_2} F^{I_2J_2}_{\mu_3\mu_4} \ldots F^{I_8J_8}_{\mu_{15}\mu_{16}} \epsilon_{I_1J_1I_2J_2\ldots I_8J_8} \\ &= \int_{\mathcal{M}^{16}} d\mathcal{L}^{(15)}_{LL} \\ &= \int_{\Sigma\partial\mathcal{M}^{16}} \mathcal{L}^{(15)}_{LL} \end{aligned} \tag{2.17}$$

Despite the higher powers of the curvature (after eliminating the spin connection ω^{ab}_μ in terms of the e^a_μ field) the $\mathcal{L}^{(15)}_{\text{Lovelock}}$ furnishes equations of motion for the e^a_μ field containing at most derivatives of second order, and not higher, due to the Topological property of the Lovelock terms

$$\begin{aligned} d(\mathcal{L}^{(15)}_{\text{Lovelock}}) &= \epsilon_{a_1a_2\ldots a_{16}} \left(R^{a_1a_2} + \frac{e^{a_1}e^{a_2}}{l^2}\right) \ldots \left(R^{a_{13}a_{14}} + \frac{e^{a_{13}}e^{a_{14}}}{l^2}\right) T^{a_{15}} \\ &= \text{Euler density in } 16D. \end{aligned} \tag{2.18}$$

The exterior derivative of the Lovelock terms can be rewritten compactly as

$$d(\mathcal{L}^{15}_{\text{Lovelock}}) = \epsilon_{I_1I_2\ldots I_{16}} F^{I_1I_2} \ldots F^{I_{15}I_{16}}, \tag{2.19}$$

where $F^{I_1I_2}$ is the curvature field strength associated with the $\mathrm{SO}(14,2)$ connection $\Omega^{I_1I_2}_\mu$ in $16D$ and which can be decomposed in terms of the fields $e^a_\mu, \omega^{ab}_\mu, a, b = 1, 2, \ldots, 15$ by identifying $\Omega^{aD}_\mu = \frac{1}{l}e^a_\mu$ and $\Omega^{ab}_\mu = \omega^{ab}_\mu$ so that the Torsion and Lorenz curvature two-forms are

$$\begin{aligned} T^a(\omega, e) = F^{aD} &= d\Omega^{aD} + \Omega^a_b \wedge \Omega^{bD} = \frac{1}{l}(de^a - \omega^a_b \wedge e^b), \\ F^{ab} &= (d\Omega^{ab} + \Omega^a_c \wedge \Omega^{cb}) + (\Omega^a_D \wedge \Omega^{Db}) = R^{ab}(\omega) + \frac{1}{l^2} e^a \wedge e^b, \\ R^{ab}(\omega) &= d\omega^{ab} + \omega^a_c \wedge \omega^{cb}, \end{aligned} \tag{2.20}$$

where a length parameter l must be introduced to match dimensions since the connection has units of $1/l$. This l parameter is related to the cosmological constant.

Another invariant is the $\mathcal{L}^{15}_{\text{CS}}(\Omega^{IJ}_\mu)$ Chern–Simons 15-form associated with the $\mathrm{SO}(16)$ spin connection whose exterior derivative

$$\begin{aligned} d(\mathcal{L}_{\text{CS}})(\Omega^{IJ}_\mu) &= F^{I_1}_{I_2} F^{I_2}_{I_3} \ldots F^{I_7}_{I_8} F^{I_8}_{I_1} \\ &\Rightarrow \int_{\partial\mathcal{M}^{16}} (\mathcal{L}_{\text{CS}})(\Omega^{IJ}_\mu) \\ &= \int_{\mathcal{M}^{16}} F^{I_1}_{I_2} F^{I_2}_{I_3} \ldots F^{I_7}_{I_8} F^{I_8}_{I_1} \end{aligned} \tag{2.21}$$

is one of the five terms contained in the definition of the Pontryagin $p_4(F^{IJ})$ invariant 16-form (up to numerical factors) for the $\mathrm{SO}(14,2)$ gauge connection in $16D$. As mentioned above, the $\mathrm{SO}(14,2)$ connection Ω^{IJ}_μ can be broken into the e^a_μ field which gauges translations along the 15-dim boundary ∂M^{16} and the SO$(14,1)$ spin connection ω_μ^{ab} which gauges the Lorentz group SO$(14,1)$ associated with the tangent space of the 15-dim boundary $\partial\mathcal{M}^{16}$ and such that the net number of components is $15+\frac{1}{2}(15\times 14)=120=\frac{1}{2}(16\times 15)$.

The relevant five terms contained in the octic E_8 invariant found by [23] and related to the five terms comprising the Pontryagin $p_4(F^{IJ})$ invariant 16-form (but with different numerical factors) are of the form

$$tr[(F^{IJ}X_{IJ})^8] \Rightarrow \epsilon^{\mu_1\mu_2\ldots\mu_{16}} F^{I_1I_2}_{\mu_1\mu_2} F^{I_2I_3}_{\mu_3\mu_4} F^{I_3I_4}_{\mu_5\mu_6} F^{I_4I_5}_{\mu_7\mu_8} F^{I_5I_6}_{\mu_9\mu_{10}} F^{I_6I_7}_{\mu_{11}\mu_{12}} F^{I_7I_8}_{\mu_{13}\mu_{14}} F^{I_8I_1}_{\mu_{15}\mu_{16}}, \tag{2.22}$$

which is the same term as (2.21), plus the other terms of the Pontryagin $p_4(F^{IJ})$ invariant 16-form given by

$$tr[(F^{IJ}X_{IJ})^2]^4 \Rightarrow \epsilon^{\mu_1\mu_2\ldots\mu_{16}} \left(F^{I_1I_2}_{\mu_1\mu_2} F^{I_2I_1}_{\mu_3\mu_4}\right)\left(F^{J_1J_2}_{\mu_5\mu_6} F^{J_2J_1}_{\mu_7\mu_8}\right) \times \left(F^{K_1K_2}_{\mu_9\mu_{10}} F^{K_2K_1}_{\mu_{11}\mu_{12}}\right)\left(F^{L_1L_2}_{\mu_{13}\mu_{14}} F^{L_2L_1}_{\mu_{15}\mu_{16}}\right), \tag{2.23}$$

$$tr[(F^{IJ}X_{IJ})^4]^2 \Rightarrow \epsilon^{\mu_1\mu_2\ldots\mu_{16}} \left(F^{I_1I_2}_{\mu_1\mu_2} F^{I_2I_3}_{\mu_3\mu_4} F^{I_3I_4}_{\mu_5\mu_6} F^{I_4I_1}_{\mu_7\mu_8}\right) \times \left(F^{J_1J_2}_{\mu_9\mu_{10}} F^{J_2J_3}_{\mu_{11}\mu_{12}} F^{J_3J_4}_{\mu_{13}\mu_{14}} F^{J_4J_1}_{\mu_{15}\mu_{16}}\right), \tag{2.24}$$

and similar expressions for the remaining two terms

$$tr[(F^{IJ}X_{IJ})^6]tr[(F^{IJ}X_{IJ})^2], \quad tr[(F^{IJ}X_{IJ})^4]tr[(F^{IJ}X_{IJ})^2]^2.$$

The terms involving the fermionic generators $F^\alpha_{\mu\nu}$ (where the components $F^\alpha_{\mu\nu}$ are given by Eq. (2.7)) in the octic E_8 invariant are

$$tr[(F^\alpha Y_\alpha)^8] \Rightarrow \epsilon^{\mu_1\mu_2\ldots\mu_{16}} \epsilon^{I_1I_2\ldots I_{16}} \left(F^{\alpha_1}_{\mu_1\mu_2} \Gamma^{\alpha_1\beta_1}_{I_1I_2I_3I_4} F^{\beta_1}_{\mu_3\mu_4}\right) \ldots \left(F^{\alpha_4}_{\mu_{13}\mu_{14}} \Gamma^{\alpha_4\beta_4}_{I_{13}I_{14}I_{15}I_{16}} F^{\beta_4}_{\mu_{15}\mu_{16}}\right), \tag{2.25}$$

$$tr[(F^\alpha Y_\alpha)^2]^4 \Rightarrow \epsilon^{\mu_1\mu_2\ldots\mu_{16}} \left(F^{\alpha_1}_{\mu_1\mu_2} F^{\alpha_1}_{\mu_3\mu_4}\right) \ldots \left(F^{\alpha_4}_{\mu_{13}\mu_{14}} F^{\alpha_4}_{\mu_{15}\mu_{16}}\right), \ldots \tag{2.26}$$

The terms involving both fermionic and bivector generators in the octic E_8 invariant are

$$tr[(F^{IJ}X_{IJ})^6(F^\alpha Y_\alpha)^2] \Rightarrow \epsilon^{\mu_1\mu_2\ldots\mu_{16}} \left(F^{I_1J_1}_{\mu_1\mu_2} F^{I_2J_2}_{\mu_3\mu_4} \ldots F^{I_6J_6}_{\mu_{11}\mu_{12}}\right) \times \left(F^\alpha_{\mu_{13}\mu_{14}} \Gamma^{\alpha\beta}_{I_1J_1I_2J_2\ldots I_6J_6} F^\beta_{\mu_{15}\mu_{16}}\right). \tag{2.27}$$

$$tr[(F^{IJ}X_{IJ})^4(F^\alpha Y_\alpha)^4] \Rightarrow \epsilon^{\mu_1\mu_2\ldots\mu_{16}} \left(F^{I_1J_1}_{\mu_1\mu_2} F^{I_2J_2}_{\mu_3\mu_4} F^{I_3J_4}_{\mu_5\mu_6} F^{I_4J_4}_{\mu_7\mu_8}\right) \times\left(F^{\alpha_1}_{\mu_9\mu_{10}} \Gamma^{\alpha_1\beta_1}_{I_1J_1I_2J_2} F^{\beta_1}_{\mu_{11}\mu_{12}}\right)\left(F^{\alpha_2}_{\mu_{13}\mu_{14}} \Gamma^{\alpha_2\beta_2}_{I_3J_3I_4J_4} F^{\beta_2}_{\mu_{15}\mu_{16}}\right); \tag{2.28}$$

$$\epsilon^{\mu_1\mu_2\ldots\mu_{16}} \left(F^{I_1J_1}_{\mu_1\mu_2} F^{I_2J_2}_{\mu_3\mu_4} F^{I_3J_4}_{\mu_5\mu_6} F^{I_4J_4}_{\mu_7\mu_8}\right) \left(F^{\alpha_1}_{\mu_9\mu_{10}} \Gamma^{\alpha_1\beta_1}_{I_1J_1I_2J_2I_3J_3I_4J_4} F^{\beta_1}_{\mu_{11}\mu_{12}}\right)\left(F^{\alpha_2}_{\mu_{13}\mu_{14}} F^{\alpha_2}_{\mu_{15}\mu_{16}}\right); \ldots \tag{2.29}$$

$$tr[(F^{IJ}X_{IJ})^6]tr[(F^{\alpha}Y_{\alpha})^2] \Rightarrow \epsilon^{\mu_1\mu_2\ldots\mu_{16}}\left(F^{I_1I_2}_{\mu_1\mu_2}F^{I_2I_3}_{\mu_3\mu_4}F^{I_3I_4}_{\mu_5\mu_6}F^{I_4I_5}_{\mu_7\mu_8}F^{I_5I_6}_{\mu_9\mu_{10}}F^{I_6I_1}_{\mu_{11}\mu_{12}}\right) \times \left(F^{\alpha_1}_{\mu_{13}\mu_{14}}F^{\alpha_1}_{\mu_{15}\mu_{16}}\right). \tag{2.30}$$

$$tr[(F^{IJ}X_{IJ})^4]tr[(F^{\alpha}Y_{\alpha})^4] \Rightarrow \epsilon^{\mu_1\mu_2\ldots\mu_{16}}\left(F^{I_1I_2}_{\mu_1\mu_2}F^{I_2I_3}_{\mu_3\mu_4}F^{I_3I_4}_{\mu_5\mu_6}F^{I_4I_1}_{\mu_7\mu_8}\right) \times\left(F^{\alpha_1}_{\mu_9\mu_{10}}\Gamma^{\alpha_1\beta_1}_{J_1J_2J_3J_4}F^{\beta_1}_{\mu_{11}\mu_{12}}\right)\left(F^{\alpha_2}_{\mu_{13}\mu_{14}}\Gamma^{\alpha_2\beta_2}_{J_3J_4J_1J_2}F^{\beta_2}_{\mu_{15}\mu_{16}}\right). \tag{2.31}$$

$$tr[(F^{IJ}X_{IJ})^2]tr[(F^{\alpha}Y_{\alpha})^6] \Rightarrow \epsilon^{\mu_1\mu_2\ldots\mu_{16}}\left(F^{I_1I_2}_{\mu_1\mu_2}F^{I_2I_1}_{\mu_3\mu_4}\right)\left(F^{\alpha_1}_{\mu_5\mu_6}\Gamma^{\alpha_1\beta_1}_{J_1J_2J_3J_4}F^{\beta_1}_{\mu_7\mu_8}\right) \times\left(F^{\alpha_2}_{\mu_9\mu_{10}}\Gamma^{\alpha_2\beta_2}_{J_3J_4J_5J_6}F^{\beta_2}_{\mu_{11}\mu_{12}}\right) \times\left(F^{\alpha_3}_{\mu_{13}\mu_{14}}\Gamma^{\alpha_3\beta_3}_{J_5J_6J_1J_2}F^{\beta_3}_{\mu_{15}\mu_{16}}\right)\ldots \tag{2.32}$$

Therefore, the E_8 invariant octic action in $16D$ given by Eq. (2.15) with $36 + 1 = 37$ terms contains: (i) the Lanczos–Lovelock gravitational action (2.17), (2.18) associated with the 15-dim boundary $\partial\mathcal{M}^{16}$; (ii) five terms with the same structure as the Pontryagin $p_4(F^{IJ})$ 16-form associated with the SO(16) spin connection Ω^{IJ}_μ; (iii) the fourth power of the quadratic invariant $[I_2]^4$; (iv) plus 30 additional terms involving powers of the E_8-valued $F^{IJ}_{\mu\nu}$ and $F^{\alpha}_{\mu\nu}$ field-strength (two-forms) as shown in Eqs. (2.22)–(2.32).

The impending project is the supersymmetric version of the octic E_8 invariant action (2.15). A vector supermultiplet [24, 25] involves A^m_μ, λ^m with 248 spacetime fermions λ^m in the fundamental 248-dim representation of E_8 ($m = 1, 2, \ldots, 248$) and 248 spacetime vectors (gluons) A^m_μ in the 248-dim adjoint representation. The fermions are the gluinos in this very special case because the 248-dim fundamental and 248-dim adjoint representations of the exceptional E_8 group coincide. The exceptional group E_8 is unique in this respect. In ordinary supersymmetric Yang–Mills the superpartners of the fermions are scalars, however, in the supersymmetric E_8 Yang–Mills case, the fermions λ^m (gluinos) and the vectors A^m_μ (gluons) comprise the vector supermultiplet. For a thorough discussion of the unique phenomenological features of the E_8 group as a candidate for a (supersymmetric) grand unification model of all fermion families in $D = 4$ see [24, 25]. An extensive review of the E_6 grand unified models may be found in [26].

A generalized Yang–Mills action in $D = 16$ involving the E_8-valued two-form field strength $\mathbf{F} = F^{IJ}X_{IJ} + F^{\alpha}Y_{\alpha}$ is

$$S_{GYM}(E_8) = \int_{\mathcal{M}^{16}} tr[(\mathbf{F}\wedge\mathbf{F}\wedge\mathbf{F}\wedge\mathbf{F}) \wedge^* (\mathbf{F}\wedge\mathbf{F}\wedge\mathbf{F}\wedge\mathbf{F})]. \tag{2.33}$$

The analog of a theta term in $D = 16$ is

$$S_{\text{theta}}(E_8) = \int_{\mathcal{M}^{16}} tr[\mathbf{F}^8]. \tag{2.34}$$

Self dual configurations, E_8 instantons in $D = 16$ obey $G_{(8)} =^* G_{(8)}$ and turn the action (2.33) into (2.34) when the self dual eight-form is defined by $G_{(8)} = \mathbf{F}\wedge\mathbf{F}\wedge\mathbf{F}\wedge\mathbf{F}$.

Related to the construction of instantons in higher dimensions, a SO(8) $\times$ SO(7) $\subset$ SO(16) invariant self-duality equation for a three-form in $D = 16$ was studied by [29] who built Topological QFT on 8-dim manifolds with holonomy group smaller than or equal to $Spin(7)$ after a dimensional reduction from $D = 16$ to $D = 8$. A further dimensional reduction to $D = 4$ furnished new supersymmetric theories in $D = 4$. The inclusion of gravitational interactions in $D = 8$ allowed the construction of a $D = 8$ topological gravity and its correspondence with supergravity via an octonionic self duality equation for the spin connection [29].

A topologically nontrivial gauging of $N = 16$ supergravity in $D = 3$ based on an $N = 16$ supersymmetric 3-dim nonlinear sigma model valued on the exceptional coset E_8/SO(16) (128-dimensional) including a combination of a BF and Chern–Simons term for an SO(16) gauge field was provided by [30]. It remains an open problem to see if the supersymmetric version of the octic E_8 invariant action (2.15) upon dimensional reduction to $D = 3$ bears a relationship to the topological gauging of $N = 16$ supergravity in $D = 3$. The 128 scalars parametrizing the coset E_8/SO(16) fit into 16 copies of 128 scalars resulting from the decomposition of the E_8-valued gauge field $A^\alpha_\mu Y_\alpha$, $\mu = 1, 2, \ldots, 16$ and $\alpha = 1, 2, \ldots, 128$ where Y_α are the the SO(16) chiral spinorial generators of the E_8 algebra.

Another dimensional reduction that is warranted to study is from $D = 16$ to $D = 11$ because $D = 11$ supergravity with a local SO(16) invariance permits the bosonic fields to be assigned to a representation of E_8 [31]. The $D = 11$ supergravity four-form determines an E_8 gauge bundle which was instrumental in understanding the topological part of the M-theory partition function [27, 32]. A mysterious E_8 bundle which restricts from 12-dim to 11-dim bulk of M-theory can be compatible with 11-dim supersymmetry. When M-theory is compactified on a manifold with boundary the anomalies caused by the chiral gauginos and gravitinos on each 10-dim boundary component cancels the anomalies in the 11-dim bulk if each 10-dim boundary component supports 248 vector multiplets transforming in the adjoint representation of E_8. The Casimir effect between the M-theory analog of a D-brane/anti-D-brane system exhibiting an $E_8 \times E_8$ symmetry living at the 10-dim boundaries of the 11-dim bulk has been studied by [28]. The nature of this bulk 11-dim E_8 gauge theory remains unknown. We hope that the Chern–Simons E_8 gauge theory of gravity in $D = 15$ advanced in this work may shed some light into solving this question. Another interpretation is to view the 10-dim boundary component of the 11-dim bulk of M-theory as a topological defect in 12-dimensions.

The action for $D = 4$ Einstein gravity has been attained from a generalized dimensional reduction of a Chern–Simons gravity action in higher $D = 2n + 1$ dimensions by Nastase [34]. This occurs after imposing a very strong constraint which in the Schwarzschild space time case is tantamount of setting the ADM mass to zero [37]. Hence, we may follow such generalized dimensional reduction of our $D = 15$ Lanczos–Lovelock gravitational action (2.17), (2.18) to lower dimensions. For example, the reduction of the $D = 6$ action (integral of the Euler density in $D = 6$)

$$\int_{\mathcal{M}^6} d(\mathcal{L}^{(5)}_{\text{Lovelock}}) = \int_{\mathcal{M}^6} \epsilon_{a_1 a_2 \dots a_6} \left(R^{a_1 a_2} + \frac{e^{a_1} e^{a_2}}{l^2}\right)\left(R^{a_3 a_4} + \frac{e^{a_3} e^{a_4}}{l^2}\right) T^{a_5}, \quad (2.35)$$

to $D = 4$ leads to the standard action for Einstein gravity with the cosmological constant $(1/l^2)$ plus the Gauss–Bonnet topological invariant in $D = 4$ that coincides with the MacDowell–Mansouri–Chamseddine–West (anti de Sitter group) SO$(3, 2)$ gauge formulation of gravity:

$$\int_{\mathcal{M}^4} \epsilon_{a_1 a_2 a_3 a_4} \left(R^{a_1 a_2} + \frac{e^{a_1} e^{a_2}}{l^2}\right) \left(R^{a_3 a_4} + \frac{e^{a_3} e^{a_4}}{l^2}\right). \quad (2.36)$$

The so-called Born–Infield gravity in Eq. (2.41) is not invariant under SO$(3, 2)$ unless one imposes the torsionless condition (the action is not off-shell invariant) [37].

$D = 4$ Einstein gravity was shown by [35] to arise from a 6-dim gauge theory of the conformal group SO$(4, 2)$ where the 4-dim spacetime was interpreted as a 4-dim topological defect in $D = 6$ and obtained from a topological dimensional reduction of the Euler density in $D = 6$. In view of these latest findings of how to perform generalized and topological dimensional reductions [34, 35], it is no longer implausible to propose a grand unification of gravity with all the other forces within the framework of a supersymmetric extension (to incorporate the 248 spacetime fermions λ^m) of our Chern–Simons E_8 gauge theory in $D = 15$ based on the octic E_8 invariant action (2.15) after a judicious dimensional reduction. Working in particular with S^{16} and whose equator is S^{15} is very appealing since it allows to accommodate quaternions and octonions into the picture $HP^4 \sim OP^2 \sim S^{16}$; $HP^2 \sim OP^1 \sim S^8$ and $HP^1 \sim S^4$. The four nonassociative (not Lie) superconformal algebras with $N = 5, 6, 7, 8$ supersymmetries all share interesting properties with the Cayley (octonions), covariant derivation of spinors on round and squashed S^7 and torsion on supercoset manifolds [36].

To finalize this section, we simply recall that in odd dimensions $D = 2n-1$, the Lanczos–Lovelock gravitational Lagrangian is

$$\mathcal{L}^D_{\text{Lovelock}} = \sum_{p=0}^{n-1} a_p L_p(D), \quad a_p = \kappa \frac{(\pm 1)^{p+1} l^{2p-D}}{(D-2p)} C^{n-1}_p, \quad p = 1, 2, \dots, n-1 \quad (2.37)$$

C^{n-1}_p is the binomial coefficient. The constants κ, l are related to the Newton's constant G and to the cosmological constant Λ through $\kappa^{-1} = 2(D-2)\Omega_{D-2} G$ where Ω_{D-2} is the area of the $D-2$-dim unit sphere and $\Lambda = \pm(D-1)(D-2)/2l^2$ for de Sitter (anti de Sitter) spaces [4].

The terms inside the summand of (2.42) are

$$L_p(D) = \epsilon_{a_1 a_2 \dots a_D} R^{a_1 a_2} R^{a_3 a_4} \dots R^{a_{2p-1} a_{2p}} e^{a_{2p+1}} \dots e^{a_D}, \quad (2.38)$$

where we have omitted the space-time indices $\mu_1, \mu_2, \dots$. Despite the higher powers of the curvature (after eliminating the spin connection ω^{ab}_μ in terms of the

e^a_μ field) the $\mathcal{L}^D_{\text{Lovelock}}$ furnishes equations of motion for the e^a_μ field containing at most derivatives of second order, and not higher, due to the topological property of the Lovelock terms

$$\begin{aligned} d(\mathcal{L}^{2n-1}_{\text{Lovelock}}) &= \epsilon_{a_1 a_2 \ldots a_{2n}} \left(R^{a_1 a_2} + \frac{e^{a_1} e^{a_2}}{l^2} \right) \\ &\quad \ldots \left(R^{a_{2n-3} a_{2n-2}} + \frac{e^{a_{2n-3}} e^{a_{2n-2}}}{l^2} \right) T^{a_{2n-1}} \\ &= \text{Euler density.} \end{aligned} \tag{2.39}$$

Therefore, the exterior derivative of the Lovelock terms can be rewritten compactly as

$$d(\mathcal{L}^{2n-1}_{\text{Lovelock}}) = \epsilon_{I_1 I_2 \ldots I_{2n}} F^{I_1 I_2} \ldots F^{I_{2n-1} I_{2n}}, \tag{2.40}$$

where $F^{I_1 I_2}$ is the curvature field strength associated with the $\mathrm{SO}(2n-2,2)$ connection $\Omega^{I_1 I_2}_\mu$ in $2n$-dim and which can be decomposed in terms of the fields $e^a_\mu, \omega^{ab}_\mu, a, b = 1, 2, \ldots, 2n-1$ as shown in Eqs. (2.19), (2.20). Gauge theories based on the Anti de Sitter group allowed us to derive the vacuum energy density of Anti de Sitter space (de Sitter) as the geometric mean between an upper and lower scale [17] based on a BF–Chern–Simons–Higgs theory. Upon setting the lower scale to the Planck scale L_P and the upper scale to the Hubble radius (today) R_H, it yields the observed value of the cosmological constant $\rho = L_P^{-2} R_H^{-2} = L_P^{-4} (L_P/R_H)^2 \sim 10^{-120} M^4_{\text{Planck}}$.

3. On Chern–Simons–Clifford Gravity

We end this work by reviewing Chern–Simons gravitational actions in Clifford spaces [33] in order to point its relevance to future research related to E_8 gauge theories of gravity. The $11D$ Chern–Simons supergravity action is based on the smallest Anti de Sitter $OSp(32|1)$ superalgebra. The Anti de Sitter group $\mathrm{SO}(10,2)$ must be embedded into a larger group $Sp(32, R)$ to accommodate the fermionic degrees of freedom associated with the superalgebra $OSp(32|1)$. The bosonic sector involves the connection [4]:

$$\mathbf{A}_\mu = A^a_\mu \Gamma_a + A^{ab}_\mu \Gamma_{ab} + A^{a_1 a_2 \ldots a_5}_\mu \Gamma_{a_1 a_2 \ldots a_5} = e^a_\mu \Gamma_a + \omega^{ab}_\mu \Gamma_{ab} + A^{a_1 a_2 \ldots a_5}_\mu \Gamma_{a_1 a_2 \ldots a_5} \tag{3.1}$$

with $11 + 55 + 462 = 528$ generators. A Hermitian complex 32×32 matrix has a total of $32 + 2(\frac{32 \times 31}{2}) = 992 + 32 = 1024 = 32^2 = 2^{10}$ independent real components (parameters), the same number as the real parameters of the anti-symmetric and symmetric real 32×32 matrices, respectively, $496 + 528 = 1024$. The dimension of $Sp(32) = (1/2)(32 \times 33) = 528$. Notice that $2^{10} = 1024$ is also the number of independent generators of the $\mathrm{Cl}(11)$ algebra since out of the 2^{11} generators, only half of them 2^{10}, are truly independent due to the duality conditions valid in *odd* dimensions only:

$$\epsilon^{a_1 a_2 \ldots a_{2n+1}} \Gamma_{a_1} \wedge \Gamma_{a_2} \wedge \cdots \wedge \Gamma_{a_p} \sim \Gamma^{a_{p+1}} \wedge \Gamma^{a_{p+2}} \wedge \cdots \wedge \Gamma^{a_{2n+1}}. \tag{3.2}$$

This counting of components is the underlying reason why the Cl(11) algebra appears in this section. The generators of the Cl(11) algebra $\{\Gamma^a, \Gamma^b\} = 2\eta^{ab}\mathbf{1}$ and the unit element $\mathbf{1}$ generate the Clifford polyvectors (including a scalar, pseudoscalar) of different grading

$$\Gamma^A = \mathbf{1}, \Gamma^a, \Gamma^{a_1} \wedge \Gamma^{a_2}, \Gamma^{a_1} \wedge \Gamma^{a_2} \wedge \Gamma^{a_3}, \ldots, \Gamma^{a_1} \wedge \Gamma^{a_2} \wedge \cdots \wedge \Gamma^{a_{11}}. \tag{3.3}$$

obeying the conditions (3.2). The commutation relations (see Eq. (3.4) below) involving the generators $\Gamma_a, \Gamma_{ab}, \Gamma_{a_1 a_2 \ldots a_5}$ do in fact close due to the duality conditions (3.2). The Cl(11) algebra commutators, up to numerical factors, are

$$[\Gamma^a, \Gamma^b] = \Gamma^{ab}, \quad [\Gamma^a, \Gamma^{bc}] = 2\eta^{ab}\Gamma^c - 2\eta^{ac}\Gamma^b, \tag{3.4a}$$

$$[\Gamma^{a_1 a_2}, \Gamma^{b_1 b_2}] = -\eta^{a_1 b_1}\Gamma^{a_2 b_2} + \eta^{a_1 b_2}\Gamma^{a_2 b_1} - \cdots, \tag{3.4b}$$

$$[\Gamma^{a_1 a_2 a_3}, \Gamma^{b_1 b_2 b_3}] = \Gamma^{a_1 a_2 a_3 b_1 b_2 b_3} - (\eta^{a_1 b_1 a_2 b_2}\Gamma^{a_3 b_3} + \cdots), \tag{3.4c}$$

$$[\Gamma^{a_1 a_2 a_3 a_4}, \Gamma^{b_1 b_2 b_3 b_4}] = -(\eta^{a_1 b_1}\Gamma^{a_2 a_3 a_4 b_2 b_3 b_4} + \cdots) - (\eta^{a_1 b_1 a_2 b_2 a_3 b_3}\Gamma^{a_4 b_4} + \cdots), \tag{3.4d}$$

$$[\Gamma^{a_1 a_2}, \Gamma^{b_1 b_2 b_3 b_4}] = -\eta^{a_1 b_1}\Gamma^{a_2 b_2 b_3 b_4} + \cdots, \tag{3.4e}$$

$$[\Gamma^{a_1}, \Gamma^{b_1 b_2 b_3}] = \Gamma^{a_1 b_1 b_2 b_3}, \quad [\Gamma^{a_1 a_2}, \Gamma^{b_1 b_2 b_3}] = -2\eta^{a_1 b_1}\Gamma^{a_2 b_2 b_3} + \cdots, \tag{3.4f}$$

$$[\Gamma^{a_1}, \Gamma^{b_1 b_2 b_3 b_4}] = -\eta^{a_1 b_1}\Gamma^{b_2 b_3 b_4} + \cdots, \tag{3.4g}$$

$$\begin{aligned}
[\Gamma^{a_1 a_2 \ldots a_5}, \Gamma^{b_1 b_2 \ldots b_5}] &= \Gamma^{a_1 a_2 \ldots a_5 b_1 b_2 \ldots b_5} + (\eta^{a_1 b_1 a_2 b_2}\Gamma^{a_3 a_4 a_5 b_3 b_4 b_5} + \cdots) \\
&\quad + (\eta^{a_1 b_1 a_2 b_2 a_3 b_3 a_4 b_4}\Gamma^{a_5 b_5} + \cdots) \\
&= \epsilon^{a_1 a_2 \ldots a_5 b_1 b_2 \ldots b_5 c}\Gamma_c \\
&\quad + (\eta^{a_1 b_1 a_2 b_2}\epsilon^{a_3 a_4 a_5 b_3 b_4 b_5 c_1 c_2 \ldots c_5}\Gamma_{c_1 c_2 \ldots c_5} + \cdots) \\
&\quad + (\eta^{a_1 b_1 a_2 b_2 a_3 b_3 a_4 b_4}\Gamma^{a_5 b_5} + \cdots), \text{ etc}
\end{aligned} \tag{3.4h}$$

with

$$\eta_{a_1 b_1 a_2 b_2} = \eta_{a_1 b_1}\eta_{a_2 b_2} - \eta_{a_2 b_1}\eta_{a_1 b_2}, \tag{3.5a}$$

$$\eta_{a_1 b_1 a_2 b_2 a_3 b_3} = \eta_{a_1 b_1}\eta_{a_2 b_2}\eta_{a_3 b_3} - \eta_{a_1 b_2}\eta_{a_2 b_1}\eta_{a_3 b_3} + \cdots, \tag{3.5b}$$

$$\eta_{a_1 b_1 a_2 b_2 \ldots a_n b_n} = \frac{1}{n!}\epsilon_{i_1 i_2 \ldots i_n}\epsilon_{j_1 j_2 \ldots j_n}\eta_{a_{i_1} b_{j_1}}\eta_{a_{i_2} b_{j_2}} \cdots \eta_{a_{i_n} b_{j_n}}. \tag{3.5c}$$

The Cl(11) algebra gauge field is

$$\begin{aligned}
\mathbf{A}_\mu = \mathcal{A}_\mu^A &= \mathcal{A}_\mu \mathbf{1} + \mathcal{A}_\mu^a \Gamma_a + \mathcal{A}_\mu^{a_1 a_2}\Gamma_{a_1 a_2} + \mathcal{A}_\mu^{a_1 a_2 a_3}\Gamma_{a_1 a_2 a_3} \\
&\quad + \cdots + \mathcal{A}_\mu^{a_1 a_2 \ldots a_{11}}\Gamma_{a_1 a_2 \ldots a_{11}}.
\end{aligned} \tag{3.6}$$

and the Cl(11)-algebra-valued field strength

$$\begin{aligned}\mathcal{F}^A_{\mu\nu}\Gamma_A = {} & \partial_{[\mu}A_{\nu]}\mathbf{1} + [\partial_{[\mu}A^a_{\nu]} + A^{b_2}_{[\mu}A^{b_1 a}_{\nu]}\eta_{b_1 b_2} + \cdots]\Gamma_a + [\partial_{[\mu}A^{ab}_{\nu]} \\ & + A^a_{[\mu}A^b_{\nu]} - A^{a_1 a}_{[\mu}A^{b_1 b}_{\nu]}\eta_{a_1 b_1} - A^{a_1 a_2 a}_{[\mu}A^{b_1 b_2 b}_{\nu]}\eta_{a_1 b_1 a_2 b_2} \\ & - A^{a_1 a_2 a_3 a}_{[\mu}A^{a_1 b_2 b_3 b}_{\nu]}\eta_{a_1 b_1 a_2 b_2 a_3 b_3} + \cdots]\Gamma_{ab} + [\partial_{[\mu}A^{abc}_{\nu]} \\ & + A^{a_1 a}_{[\mu}A^{b_1 bc}_{\nu]}\eta_{a_1 b_1} + \cdots]\Gamma_{abc} + [\partial_{[\mu}A^{abcd}_{\nu]} \\ & - A^{a_1 a}_{[\mu}A^{b_1 bcd}_{\nu]}\eta_{a_1 b_1} + \cdots]\Gamma_{abcd} + \cdots [\partial_{[\mu}A^{a_1 a_2 \ldots a_5 b_1 b_2 \ldots b_5}_{\nu]} \\ & + A^{a_1 a_2 \ldots a_5}_{[\mu}A^{b_1 b_2 \ldots b_5}_{\nu]} + \cdots]\Gamma_{a_1 a_2 \ldots a_5 b_1 b_2 \ldots b_5} + \cdots \end{aligned} \tag{3.7}$$

The Chern–Simons actions corresponding to the Clifford group rely on Stokes theorem

$$\int_{M^{12}} d(\mathcal{L}_{\text{Clifford}}) = \int_{\partial M^{12}=\Sigma^{11}} (\mathcal{L}_{\text{Clifford}}), \tag{3.8}$$

which in our case reads

$$d(\mathcal{L}_{\text{Clifford}}) = \langle \mathcal{F}\wedge\mathcal{F}\wedge\cdots\wedge\mathcal{F}\rangle = \langle \mathcal{F}^{A_1}\wedge\mathcal{F}^{A_2}\wedge\cdots\wedge\mathcal{F}^{A_6}\Gamma_{A_1}\Gamma_{A_2}\ldots\Gamma_{A_6}\rangle, \tag{3.9}$$

where the bracket $\langle\cdots\rangle$ means taking the scalar part of the Clifford geometric product among the gammas. It involves products of the d_{ABC}, f_{ABC} structure constants corresponding to the (anti) commutators $\{\Gamma_A, \Gamma_B\} = d_{ABC}\Gamma^C$ and $[\Gamma_A, \Gamma_B] = f_{ABC}\Gamma^C$.

One of the main results of [33] was that the Cl(11) algebra-based action (3.9) contains a vast number of terms among which is the Chern–Simons action of [4] $\mathcal{L}^{11}_{\text{CS}}(e, \omega, A_5)$

$$\mathcal{L}_{\text{Clifford}}(\mathcal{A}^A_\mu\Gamma_A) = \mathcal{L}^{11}_{\text{CS}}(\omega, e, A_5) + \text{Extra Terms}. \tag{3.10}$$

$$S_{\text{CS}}(\omega, e, A_5) = \int_{\partial M^{12}} \mathcal{L}^{11}_{\text{CS}} = \int_{\Sigma^{11}} \mathcal{L}^{11}_{\text{CS}}. \tag{3.11}$$

The Cl(11) algebra-based action (3.9), (3.10) can in turn be embedded into a more general expression in C-space (Clifford space) which is a generalized tensorial spacetime of coordinates $\mathbf{X} = \sigma, x^\mu, x^{\mu\nu}, x^{\mu\nu\rho}, \ldots$ [3] involving a scalar $\Phi(\mathbf{X})$ and antisymmetric tensor gauge fields $A_\mu(\mathbf{X}), A_{\mu\nu}(\mathbf{X}), A_{\mu\nu\rho}(\mathbf{X}), \ldots$ of higher rank (higher spin theories) [13]. The most general action onto which the action (3.9), (3.10) itself can be embedded requires a tensorial gauge field theory [13] (generalized Yang–Mills theories) and an integration with respect to all the Clifford-valued coordinates $\mathbf{X} = X^M\Gamma_M$ corresponding to the 2^D-dim C-space associated with the underlying Cl($2n$)-algebra in $D = 2n$ dimensions

$$S = \int [d^{2^n}X]\langle(\mathcal{F}\wedge\mathcal{F}\wedge\cdots\wedge\mathcal{F})\rangle, \quad [d^{2^n}X] = (d\sigma)(dx^\mu)(dx^{\mu\nu})(dx^{\mu\nu\rho})\ldots. \tag{3.12}$$

A different sort of generalized Yang–Mills theories have been studied by [12] without the Clifford algebraic structure. Given a Lie algebra $\mathbf{G}$ like E_8 with generators T_a for

$a = 1, 2, 3, \ldots, \dim \mathbf{G}$, it has for commutators $[T_a, T_b] = f^c_{ab} T_c$ and whose structure constants f_{abc} are fully antisymmetric in their indices. The Lie-algebra-valued one-form is $\mathbf{A} = (A^a_M(\mathbf{X})T_a)dX^M$ and its generalized Lie-algebra valued field strength

$$\begin{aligned}\mathbf{F} &= [F^c_{MN}(X)T_c]dX^M \wedge dX^N \\ &= [\partial_{[M}A^c_{N]}(X)T_c + gA^a_M(X)A^b_N(X)f^c_{ab}T_c]dX^M \wedge dX^N\end{aligned} \tag{3.13}$$

has for components

$$\begin{aligned}F^c_{[[\mu_1\mu_2\ldots\mu_m][\nu_1\nu_2\ldots\nu_n]]} &= \partial_{x^{[\mu_1\mu_2\ldots\mu_m]}} A^c_{[\nu_1\nu_2\ldots\nu_n]} - \partial_{x^{[\nu_1\nu_2\ldots\nu_n]}} A^c_{[\mu_1\mu_2\ldots\mu_m]} \\ &\quad + gA^a_{[\mu_1\mu_2\ldots\mu_m]}A^b_{[\nu_1\nu_2\ldots\nu_n]}f^c_{ab}.\end{aligned} \tag{3.14}$$

The remaining components are of the form

$$F^c_{[0N]} = F^c_{[0[\nu_1\nu_2\ldots\nu_n]]} = \partial_\sigma A^c_{[\nu_1\nu_2\ldots\nu_n]} - \partial_{x^{[\nu_1\nu_2\ldots\nu_n]}} A^c_0 + gA^a_0 A^b_{[\nu_1\nu_2\ldots\nu_n]} f^c_{ab}. \tag{3.15}$$

where A^c_0 is the Clifford-scalar part $\Phi(\mathbf{X})$ of the Lie-algebra-valued Clifford-polyvector, and in general, we must consider the $m = n$ and $m \neq n$ cases resulting from the mixing of different grades (ranks). The antisymmetry with respect to the collective indices MN is explicit.

In order to raise, lower and contract polyvector indices in C-space it requires a generalized metric G^{MN}. In flat C-space it is defined by the components:

$$G^{\mu\nu} = \eta^{\mu\nu}, G^{\mu_1\mu_2\nu_1\nu_2} = \eta^{\mu_1\nu_1}\eta^{\mu_2\nu_2} - \eta^{\mu_1\nu_2}\eta^{\mu_2\nu_1} \quad \text{etc.}, \tag{3.16a}$$

in addition to the scalar–scalar component $G^{\sigma\sigma} = 1$. It can be recast as

$$G^{\mu_1\mu_2\ldots\mu_m\nu_1\nu_2\ldots\nu_m} = det\ \mathbf{G}^{\mu_I\nu_J} = \frac{1}{m!}\epsilon_{i_1i_2\ldots i_m}\epsilon_{j_1j_2\ldots j_m}\eta^{\mu_{i_1}\nu_{j_1}}\eta^{\mu_{i_2}\nu_{j_2}}\ldots\eta^{\mu_{i_m}\nu_{j_m}}, \tag{3.16b}$$

where $\mathbf{G}^{\mu_I\nu_J}$ is an $m \times m$ matrix whose entries are $\eta^{\mu_i\nu_j}$ for $i, j = 1, 2, 3, \ldots, m \leq D$ and $\mu, \nu = 1, 2, 3, \ldots, D$.

As a result of the expression for the flat C-space metric, given by sums of antisymmetrized products of $\eta^{\mu\nu}$, the Clifford-space generalized Yang–Mills action is of the form

$$\begin{aligned}S_{YM} = &-\frac{1}{2}\int[\mathcal{D}X]\sum \text{trace}[F^a_{[[\mu_1\mu_2\ldots\mu_m][\nu_1\nu_2\ldots\nu_m]]}F^{[[\mu_1\mu_2\ldots\mu_m][\nu_1\nu_2\ldots.\nu_m]]b}T_aT_b] \\ &-\frac{1}{2}\int[\mathcal{D}X]\sum \text{trace}[F^a_{[0[\nu_1\nu_2\ldots\nu_m]]}F^{[[0[\nu_1\nu_2\ldots\nu_m]]b}T_aT_b],\end{aligned} \tag{3.17}$$

where the C-space 2^D-dim measure associated with a Clifford algebra in D-dim is

$$[\mathcal{D}X] = [d\sigma][\mathbf{\Pi}dx^\mu][\mathbf{\Pi}dx^{\mu_1\mu_2}][\mathbf{\Pi}dx^{\mu_1\mu_2\mu_3}]\ldots[dx^{\mu_1\mu_2\ldots\mu_d}] \tag{3.18}$$

and the indices are ordered as $\mu_1 < \mu_2 < \mu_3 \cdots < \mu_m$, etc.

The action (3.17) is invariant under the infinitesimal gauge transformations

$$\delta_\xi A^c_M = \partial_M \xi^c + g f^c_{ab} A^a_M \xi^b; \quad \delta_\xi A^c_{\mu_1\mu_2\ldots\mu_n} = \partial_{x_{\mu_1\mu_2\ldots\mu_n}} \xi^c + g f^c_{ab} A^a_{\mu_1\mu_2\ldots\mu_n} \xi^b. \tag{3.19}$$

associated with a Lie-algebra-valued Clifford-scalar parameter $\xi(\mathbf{X}) = \xi^a(\mathbf{X}) T_a$.

In [3] it was explained why another alternative to define the transformations in C-space was by writing the generators of polyrotations as $R = \exp(\Omega^{AB}[E_A, E_B])$ where the commutator $[E_A, E_B] = F^C_{AB} E_C$ is the C-space analog of the $i[\gamma_\mu, \gamma_\nu]$ commutator which is the generator of the Lorentz algebra, and the parameters Ω^{AB} are the C-space analogs of the rotation/boots parameters. This last alternative seems to be more physical because a polyrotation should map the E_A direction into the E_B direction in C-spaces, hence the meaning of the generator $[E_A, E_B]$ which is the generalization of the ordinary $i[\gamma_\mu, \gamma_\nu]$Lorentz generator.

Therefore, when we recast the generators of polyrotations as $\mathcal{J}_{AB} = [\Gamma_A, \Gamma_B]$, an action of the form

$$S(C_{\text{space}}) = \int [\mathcal{D}X] F^{A_1 B_1}_{M_1 N_1} F^{A_2 B_2}_{M_2 N_2} \ldots F^{A_{2d-1} B_{2d-1}}_{M_{2d-1} N_{2d-1}} \times \epsilon_{A_1 B_1 A_2 B_2 \ldots A_{2d-1} B_{2d-1}} \epsilon^{M_1 N_1 M_2 N_2 \ldots M_{2d-1} N_{2d-1}} \tag{3.20}$$

is the natural generalization of the Euler density types of the D-dim ($D = 2n$) actions in C-space. In particular, when $D = 16$, the action (3.20) is the C-space generalization of the action (2.22). This action $S(C_{\text{space}})$ (3.20) is more general than the action $S_{\text{Clifford}}(\mathcal{A}^A_\mu \Gamma_A)$ of Eq. (3.10), and which in turn, is more general than the Chern–Simons gravitational action $S_{\text{CS}}(\omega, e, A_5)$ given in [4]. Therefore, we have the inclusions

$$S_{\text{CS}}(\omega, e, A_5) \subset S_{\text{Cl}(11)}[\mathcal{A}^A_\mu(x^\mu)\Gamma_A] \subset S(C_{\text{space}})[\mathcal{A}^{AB}_M(\sigma, x^\mu, x^{\mu_1\mu_2}, x^{\mu_1\mu_2\mu_3}, \ldots)\mathcal{J}_{AB}] \tag{3.21}$$

and similarly one would expect the Cl(16) algebra gauge theory case in C-spaces to includes the E_8 Chern–Simons gauge theory formulated in the previous section

$$S_{\text{CS}}(\mathbf{A}, \mathbf{F}) \subset S_{\text{Cl}(16)}[\mathcal{A}^A_\mu(x^\mu)\Gamma_A] \subset S(C_{\text{space}})[\mathcal{A}^{AB}_M(\sigma, x^\mu, x^{\mu_1\mu_2}, x^{\mu_1\mu_2\mu_3}, \ldots)\mathcal{J}_{AB}], \tag{3.22}$$

which should be very relevant in future developments of M, F theory upon the introduction of polyvector-valued supersymmetries in C-spaces [11]. These generalized supersymmetries deserve to be investigated further since they are more fundamental than the supersymmetries associated with M, F theory superalgebras and also span well beyond the N-extended supersymmetric field theories involving superalgebras, like $OSp(32|N)$ for example, which are related to a SO(N) gauge theory coupled to matter fermions (besides the gravitinos). It is these polyvector-valued supersymmetries in C-spaces [11] that will permit the supersymmetrization of the most general action in C-spaces $S(C_{\text{space}})$ given by (3.20).

Finally, the results of this work may shed some light into the origins behind the hidden E_8 symmetry of $11D$ supergravity, the hyperbolic Kac–Moody algebra E_{10} and the nonlinearly realized E_{11} algebra related to chaos in M theory and oscillatory solutions close to cosmological singularities [1, 2, 6].

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