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A circular inclusion in a finite domain I. The Dirichlet-Eshelby problem

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Summary. This is the first paper in a series concerned with the precise characterization of the elastic fields due to inclusions embedded in a finite elastic medium. A novel solution procedure has been developed to systematically solve a type of Fredholm integral equations based on symmetry, self-similarity, and invariant group arguments. In this paper, we consider a two-dimensional (2D) circular inclusion within a finite, circular representative volume element (RVE). The RVE is considered isotropic, linear elastic and is subjected to a displacement (Dirichlet) boundary condition. Starting from the 2D plane strain Navier equation and by using our new solution technique, we obtain the exact disturbance displacement and strain fields due to a prescribed constant eigenstrain field within the inclusion. The solution is characterized by the so-called Dirichlet-Eshelby tensor, which is provided in closed form for both the exterior and interior region of the inclusion. Some immediate applications of the Dirichlet-Eshelby tensor are discussed briefly.

1 Introduction

One of the most significant contributions to micro-mechanics in the twentieth century is Eshelby's inclusion solution and the equivalent eigenstrain theory (Eshelby [1], [2], [3], and [14]). It has become an indispensable part of the theoretical foundation of contemporary composite mechanics and materials, and it has many applications in today's nano-science and nano-technologies.

Eshelby's ellipsoidal inclusion solution is derived based on the assumption that an inclusion is embedded in an unbounded ambient space. This is a good approximation only if the size of an inclusion is small compared to the size of the representative volume element (RVE). In real applications, there is no infinite representative volume element, and the size of every RVE is finite. Therefore, certain approximations have to be made in order to utilize Eshelby's solution. This limitation becomes obvious in applications of the Hashin-Shtrikman variational principle [4], [5] because the Hashin-Shtrikman variational principles are essentially developed for elastic composites with a finite volume, and a solution of the inclusion problem in a finite domain is needed (see Weng [19] and Mori and Tanaka [13]). To circumvent this incompatibility, additional approximations have to be made in homogenization procedures (e.g., Willis [20], [21] or Walpole [17]).

Kröner [9], [10], maybe Mazilu [12] as well, are among the first to attempt to study the inclusion problem in a finite domain. Their approach is to seek a Green's function of Navier's equations in a finite domain. The attempt was abandoned, we believe, because of the mathematical difficulty involved in obtaining a closed form solution of Green's function in a finite domain. Another attempt to solve this problem was made by Kirchner and Ni [8], and they were examining the effect of domain influence on the Green's function.

Generally speaking, to study the inclusion problem in a finite domain involves solving a pair of integral equations in the interior of the inclusion as well as in the exterior of the inclusion. To the best of the authors' knowledge, there has never been any exact solution published in the literature, which is related to finding Green's function of Navier's equation in a finite domain.

This paper is the first part of a series systematically studying inclusion problems in a finite elastic domain. Based on symmetry, self-similarity, and invariant group arguments, a new solution has been developed to solve a Fredholm type integral equation. In this paper, we first report an exact and closed form solution of the elastic fields due to a circular inclusion embedded in a two-dimensional (2D), isotropic, finite, circular domain, which is subjected to prescribed displacement boundary conditions. We term the algebraic operator that links the disturbance strain field with the eigenstrain as the Dirichlet-Eshelby tensor. It is shown that the Dirichlet-Eshelby tensor obtained for a circular inclusion within a circular RVE has some remarkable properties.

2 Inclusion problem

Figure 1 shows a circular inclusion Ω_e with radius a embedded at the center of a circular representative volume element Ω with radius H_0. Consider two arbitrary points $\mathbf{x} \in \Omega$ and $\mathbf{y} \in \partial\Omega$ and let $\mathbf{R} = \mathbf{y} - \mathbf{x}$. Each vector $\mathbf{x}$, $\mathbf{y}$, $\mathbf{R}$ can be expressed as its length multiplied by a unit direction vector. We shall denote this as $\mathbf{x} = |\mathbf{x}|\mathbf{r}$, $\mathbf{y} = H_0\mathbf{n}$ and $\mathbf{R} = R\ell$, with $R = |\mathbf{R}|$. Furthermore we define the ratios $\rho = a/|x|$, $\rho_0 = a/H_0$ and $t = |\mathbf{x}|/H_0 = \rho_0/\rho$ to allow for a dimensionless description.

Within the inclusion a constant eigenstrain field is prescribed, i.e.,

$$\epsilon^*_{ij}(\mathbf{x}) = \begin{cases} \epsilon^*_{ij}, & \mathbf{x} \in \Omega_e, \\ 0, & \mathbf{x} \in \Omega/\Omega_e. \end{cases} \tag{1}$$

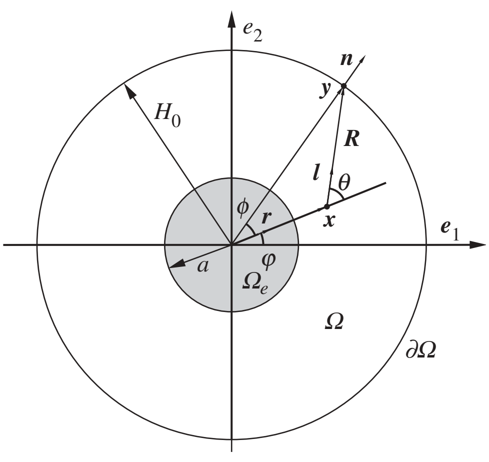

Fig. 1. A circular representative element containing a circular inclusion

On the boundary of the RVE, the following displacement boundary conditions are prescribed:

$$u_i(\mathbf{x}) = \bar{\epsilon}_{ij} x_j, \quad \forall \mathbf{x} \in \partial\Omega, \tag{2}$$

where $\bar{\epsilon}_{ij}$ is a constant strain tensor. Based on Saint-Venant's principle, $\bar{\epsilon}_{ij}$ should be the dominant strain field at the remote region far away from the inclusion, whereas for the region near the inclusion there is an additional disturbance displacement field u_i^d due to the presence of the inclusion. The total infinitesimal strain field may be viewed as the superposition of the remote strain field and the disturbance strain field,

$$\epsilon_{ij} = \bar{\epsilon}_{ij} + \epsilon_{ij}^d. \tag{3}$$

Accordingly, the total displacement field may be viewed as the superposition of the remote displacement field and the disturbance field,

$$u_i(\mathbf{x}) = \bar{\epsilon}_{ij} x_j + u_i^d(\mathbf{x}). \tag{4}$$

The infinitesimal elastic strain equals the total strain subtracting the eigenstrain,

$$e_{ij} = \epsilon_{ij} - \epsilon_{ij}^*. \tag{5}$$

Assume that the medium is linear elastic,

$$\sigma_{ij} = \mathbb{C}_{ijk\ell} e_{k\ell}, \tag{6}$$

where $\mathbb{C}_{ijk\ell}$ is the elasticity tensor and σ_{ij} is the Cauchy stress tensor. The equilibrium equation, $\sigma_{ji,j} = 0$, leads to the following boundary value problem (BVP):

$$\mathbb{C}_{ijk\ell} u_{k,\ell j}^d - \mathbb{C}_{ijk\ell} \epsilon_{k\ell,j}^* = 0, \quad \forall \mathbf{x} \in \Omega, \tag{7}$$

$$u_i^d = 0, \quad \forall \mathbf{x} \in \partial\Omega. \tag{8}$$

Denote Green's function, $G_{mi}^{\infty}(\mathbf{x} - \mathbf{y})$, as the solution of Navier's equation in an infinite elastic domain:

$$\mathbb{C}_{ijk\ell} G_{mk,\ell j}^{\infty}(\mathbf{x} - \mathbf{y}) + \delta_{mi} \delta(\mathbf{x} - \mathbf{y}) = 0, \quad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n, \tag{9}$$

where n is the dimension of the space. In this paper, we are only considering the case $n = 2$. The 2D Green's function of Navier's equation under the plane strain condition is

$$G_{ij}^{\infty}(\mathbf{x} - \mathbf{y}) = \frac{1}{8\pi\mu(1-v)} \left\{ \frac{(x_i - y_i)(x_j - y_j)}{R^2} - (3 - 4v)\delta_{ij} \ln R \right\}, \tag{10}$$

where v is Poisson's ratio, μ is the shear modulus, and $R = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$.

Using Somigliana's identity [16], the disturbance displacement field may be expressed as the integral representation

$$\begin{aligned} u_m^d(\mathbf{x}) &= \oint_{\partial\Omega} \mathbb{C}_{ijk\ell} u_{k,\ell}^d(\mathbf{y}) G_{im}^{\infty}(\mathbf{x} - \mathbf{y}) n_j(\mathbf{y}) dS_y \\ &\quad + \oint_{\partial\Omega} \mathbb{C}_{ijk\ell} u_k^d(\mathbf{y}) G_{im,j}^{\infty}(\mathbf{x} - \mathbf{y}) n_\ell(\mathbf{y}) dS_y - \int_{\Omega} \mathbb{C}_{ijk\ell} G_{im,j}^{\infty}(\mathbf{x} - \mathbf{y}) \epsilon_{k\ell}^*(\mathbf{y}) d\Omega_y \\ &= \oint_{\partial\Omega} \mathbb{C}_{ijk\ell} u_{k,\ell}^d(\mathbf{y}) G_{im}^{\infty}(\mathbf{x} - \mathbf{y}) n_j(\mathbf{y}) dS_y - \int_{\Omega} \mathbb{C}_{ijk\ell} G_{im,j}^{\infty}(\mathbf{x} - \mathbf{y}) \epsilon_{k\ell}^*(\mathbf{y}) d\Omega_y. \end{aligned} \tag{11}$$

In the last line, the boundary condition (8) is used. By considering (1), the disturbance strain field of the BVP (7), (8) can be written as

$$\epsilon^d_{ij}(\mathbf{x}) = -\frac{1}{2}\epsilon^*_{mn}\int_{\Omega_e}\mathbb{C}_{k\ell mn}\Big(G^\infty_{ki,\ell j}(\mathbf{x}-\mathbf{y}) + G^\infty_{kj,\ell i}(\mathbf{x}-\mathbf{y})\Big)d\Omega_y$$
$$+\frac{1}{2}\oint_{\partial\Omega}\mathbb{C}_{k\ell pq}u^d_{p,q}(\mathbf{y})\Big(G^\infty_{ki,j}(\mathbf{x}-\mathbf{y}) + G^\infty_{kj,i}(\mathbf{x}-\mathbf{y})\Big)n_\ell(\mathbf{y})dS_y, \tag{12}$$

where $\mathbf{x} \in \overset{\circ}{\Omega}$. One may note that Eq. (12) becomes a hyper-singular integral equation if $\mathbf{x} \in \partial\Omega$. If Ω is unbounded, i.e., $\Omega = \mathbb{R}^2$, Eshelby [1], [2] found that the disturbance strain field is related to the value of the constant eigenstrain prescribed inside the inclusion,

$$\epsilon^d_{ij}(\mathbf{x}) = \mathbb{S}^\infty_{ijmn}(\mathbf{x})\epsilon^*_{mn}, \tag{13}$$

where $\mathbb{S}^\infty_{ijmn}$ denotes the Eshelby tensor in an infinite domain, which is defined as

$$\mathbb{S}^\infty_{ijmn}(\mathbf{x}) = -\frac{1}{2}\int_{\Omega_e}\mathbb{C}_{k\ell mn}\Big(G^\infty_{ki,\ell j}(\mathbf{x}-\mathbf{y}) + G^\infty_{kj,\ell i}(\mathbf{x}-\mathbf{y})\Big)d\Omega_y. \tag{14}$$

For isotropic linear elastic solids, i.e.,

$$\mathbb{C}_{ijmn} = \lambda\delta_{ij}\delta_{mn} + 2\mu(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}), \tag{15}$$

the tensor, $\mathbb{S}^\infty_{ijmn}(\mathbf{x})$, was given by Eshelby for an ellipsoidal inclusion in an unbounded three-dimensional (3D) space [1], [2].

In particular, for spherical inclusions in 3D and circular inclusions in 2D as a special case, the Eshelby tensor can be expressed in elementary forms (e.g., Mura [14], Ju and Sun [7]). For convenience, we list here the plane strain Eshelby tensor for a circular inclusion in an infinite elastic medium:

(i) Interior solution:

$$\mathbb{S}^{I,\infty}_{ijmn}(\mathbf{x}) = \frac{(3-4\nu)}{8(1-\nu)}(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}) + \frac{(4\nu-1)}{8(1-\nu)}\delta_{ij}\delta_{mn}, \quad \mathbf{x} \in \Omega_e, \tag{16}$$

(ii) Exterior solution:

$$\mathbb{S}^{E,\infty}_{ijmn}(\mathbf{x}) = \frac{\rho^2}{8(1-\nu)}\big[(\rho^2 + 4\nu - 2)\delta_{ij}\delta_{mn} + (\rho^2 - 4\nu + 2)$$
$$(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}) + 4(1-\rho^2)\delta_{ij}r_m r_n + 4(1-2\nu-\rho^2)\delta_{mn}r_i r_j$$
$$+ 4(\nu-\rho^2)(\delta_{im}r_j r_n + \delta_{jm}r_i r_n + \delta_{in}r_j r_m + \delta_{jn}r_i r_m)$$
$$+ 8(3\rho^2 - 2)r_i r_j r_m r_n\big], \quad \mathbf{x} \in \mathbb{R}^2/\Omega_e, \tag{17}$$

where $\rho := a/|\mathbf{x}|$ and $|\mathbf{x}| = \sqrt{x_1^2 + x_2^2}$. Note that the superscript (I,∞) denotes the interior Eshelby tensor in an infinite domain and the superscript (E,∞) denotes the exterior Eshelby tensor in an infinite domain.

In Eq. (17), $r_i(\mathbf{x}) := x_i/|\mathbf{x}|, i = 1,2$ are the components of the unit vector in the direction of the position vector, $\mathbf{x}$. It can be easily verified that the following identity holds:

$$\delta_{im}r_j r_n + \delta_{in}r_j r_m + \delta_{jm}r_i r_r + \delta_{jn}r_i r_m = -2\delta_{ij}\delta_{mn} + \delta_{im}\delta_{jn} + \delta_{in}\delta_{jm} + 2\delta_{ij}r_m r_n + 2\delta_{mn}r_i r_j. \tag{18}$$

Therefore it may be observed from Eqs. (16), (17) that for a fixed ρ there are five independent circumference bases, and they may form a non–Abelian finite group[1]. By grouping

[1] In a 3D spherical inclusion, the non–Abelian group has six elements, and for 3D ellipsoidal inclusions, the non–Abelian group may have seven elements (see Li et al. [11]).

them together in a 1D array, we denote $\mathbf{\Theta}_{ijmn}(\mathbf{r})$ as the circumference basis of the Eshelby tensors, i.e.,

$$\mathbf{\Theta}_{ijmn}(\mathbf{r}) := \begin{bmatrix} \delta_{ij}\delta_{mn} \\ \delta_{im}\delta_{jn} + \delta_{in}\delta_{jm} \\ \delta_{ij}r_m r_n \\ \delta_{mn}r_i r_j \\ r_i r_j r_m r_n \end{bmatrix}. \tag{19}$$

Then the components of both the interior and exterior Eshelby tensor may be rewritten in a unified expression as the matrix product of two one-dimensional arrays, i.e.,

$$\mathbb{S}^{I,\infty}_{ijmn}(\mathbf{x}) = \Big\{ S_1^{I,\infty}(t)\delta_{ij}\delta_{mn} + S_2^{I,\infty}(t)(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}) + S_3^{I,\infty}(t)\delta_{ij}r_m r_n$$

$$+ S_4^{I,\infty}(t)\delta_{mn}r_i r_j + S_5^{I,\infty}(t)r_i r_j r_m r_n \Big\} \tag{20}$$

$$= \mathbf{\Theta}^T_{ijmn}(\mathbf{r})\mathbf{S}^{I,\infty}(t), \tag{21}$$

and

$$\mathbb{S}^{E,\infty}_{ijmn}(\mathbf{x}) = \Big\{ S_1^{E,\infty}(t)\delta_{ij}\delta_{mn} + S_2^{E,\infty}(t)(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}) + S_3^{E,\infty}(t)\delta_{ij}r_m r_n$$

$$+ S_4^{E,\infty}(t)\delta_{mn}r_i r_j + S_5^{E,\infty}(t)r_i r_j r_m r_n \Big\}$$

$$= \mathbf{\Theta}^T_{ijmn}(\mathbf{r})\mathbf{S}^{E,\infty}(t), \tag{22}$$

where the vectors, $\mathbf{S}^{I,\infty}(t)$ and $\mathbf{S}^{E,\infty}(t)$, are specified as

$$\mathbf{S}^{I,\infty}(t) = \begin{bmatrix} S_1^{I,\infty}(t) \\ S_2^{I,\infty}(t) \\ S_3^{I,\infty}(t) \\ S_4^{I,\infty}(t) \\ S_5^{I,\infty}(t) \end{bmatrix} = \frac{1}{8(1-\nu)} \begin{bmatrix} 4\nu - 1 \\ 3 - 4\nu \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{23}$$

$$\mathbf{S}^{E,\infty}(t) = \begin{bmatrix} S_1^{E,\infty}(t) \\ S_2^{E,\infty}(t) \\ S_3^{E,\infty}(t) \\ S_4^{E,\infty}(t) \\ S_5^{E,\infty}(t) \end{bmatrix} = \frac{\rho_0^2/t^2}{8(1-\nu)} \begin{bmatrix} 9(\rho_0/t)^2 - 4\nu - 2 \\ -3\rho_0^2/t^2 + 2 \\ -12\rho_0^2/t^2 + 8\nu + 4 \\ -12\rho_0^2/t^2 + 4 \\ 8(3\rho_0^2/t^2 - 2) \end{bmatrix}, \tag{24}$$

where the variable $t = |\mathbf{x}|/H_0 = \rho_0/\rho$, with $\rho = a/|\mathbf{x}|$, and $\rho_0 = a/H_0$.

To find the Eshelby tensor in a finite domain, we consider a circular inclusion embedded in a finite, isotropic, circular RVE. Without loss of generality, we assume that the disturbance strain field can be expressed, analogously to Eq. (13), as

$$\epsilon^{d}_{ij}(\mathbf{x}) := \mathbb{S}^{\bullet,D}_{ijmn}(\mathbf{x})\epsilon^{*}_{mn}, \quad \forall \mathbf{x} \in \Omega, \tag{25}$$

where the superscript D indicates that the Dirichlet boundary condition is prescribed on the boundary of the RVE. Then Eq. (12) yields a tensorial integral equation for the Eshelby tensor in a finite domain,

$$\mathbb{S}^{\bullet,D}_{ijmn}(\mathbf{x}) = \mathbb{S}^{\bullet,\infty}_{ijmn}(\mathbf{x}) + \frac{1}{2}\oint_{\partial\Omega}\Big(G^{\infty}_{ik,j}(\mathbf{x}-\mathbf{y}) + G^{\infty}_{jk,i}(\mathbf{x}-\mathbf{y})\Big) r_{\ell}(\mathbf{y})\mathbb{C}_{k\ell pq}\mathbb{S}^{E,D}_{pqmn}(\mathbf{y})d\Omega_{y}. \tag{26}$$

Note that this integral equation has two different forms, depending on whether $\mathbf{x}$ is inside or outside the inclusion,

$$\mathbb{S}^{I,D}_{ijmn}(\mathbf{x}) = \mathbb{S}^{I,\infty}_{ijmn}(\mathbf{x}) + \frac{1}{2}\oint_{\partial\Omega}\Big(G^{\infty}_{ik,j}(\mathbf{x}-\mathbf{y}) + G^{\infty}_{jk,i}(\mathbf{x}-\mathbf{y})\Big) r_{\ell}(\mathbf{y})\mathbb{C}_{k\ell pq}\mathbb{S}^{E,D}_{pqmn}(\mathbf{y})d\Omega_{y}, \ \forall \mathbf{x} \in \Omega_{e}, \tag{27}$$

$$\mathbb{S}^{E,D}_{ijmn}(\mathbf{x}) = \mathbb{S}^{E,\infty}_{ijmn}(\mathbf{x}) + \frac{1}{2}\oint_{\partial\Omega}\Big(G^{\infty}_{ik,j}(\mathbf{x}-\mathbf{y}) + G^{\infty}_{jk,i}(\mathbf{x}-\mathbf{y})\Big) r_{\ell}(\mathbf{y})\mathbb{C}_{k\ell pq}\mathbb{S}^{E,D}_{pqmn}(\mathbf{y})d\Omega_{y}, \ \forall \mathbf{x} \in \Omega/\Omega_{e}, \tag{28}$$

where the superscript (I, D) denotes the interior Dirichlet-Eshelby tensor for a finite domain and the superscript (E, D) denotes the exterior Dirichlet-Eshelby tensor for a finite domain.

From Eqs. (27), (28), one can see that in order to find the interior Eshelby tensor $\mathbb{S}^{I,D}_{ijmn}(\mathbf{x})$ one has to find the exterior Eshelby tensor $\mathbb{S}^{E,D}_{ijmn}(\mathbf{x})$, and in order to find $\mathbb{S}^{E,D}_{ijmn}(\mathbf{x})$ one has to first solve a Fredholm type integral equation (28).

3 Solution of the integral equations

Before solving the integral equation (28), we first introduce the concept of the so-called radial isotropic tensor. One may find that the interior Eshelby tensor, $\mathbb{S}^{I,\infty}_{ijmn}$, for a circular inclusion inside on unbounded space is an isotropic tensor (see Eq. (16)), which reflects the symmetry property of an isotropic elastic medium, i.e., $\mathbb{C}_{ijmn}$ is isotropic. Nevertheless, the exterior Eshelby tensor, $S^{E,\infty}_{ijmn}$, is not an obviously isotropic tensor (see Eq. (17)), and it depends on the position where it is evaluated. For a given exterior point, $\mathbf{x} \in \mathbb{R}^2/\Omega_e$, if we integrate $\mathbb{S}^{E,\infty}_{ijmn}(\mathbf{x}-\mathbf{y})$ along a circle that is centered at $\mathbf{x}$, and with fixed radius, i.e., $|\mathbf{x}-\mathbf{y}| = \text{const.}$, $\mathbb{S}^{E,\infty}_{ijmn}(\mathbf{x})$ will become an isotropic tensor. We call a tensor that has such a property "*a radial isotropic tensor*".

There is something special about this property, it reflects the symmetric relationship between the inclusion and its ambient space. Moreover, for a concentric circular inclusion and the corresponding circular RVE, this symmetric property should remain. Therefore, the finite Eshelby tensor, $\mathbb{S}^{E,D}_{ijmn}$, should be a radial isotropic tensor as well.

Since the average of a *radial isotropic tensor* over a circle is an isotropic tensor, it can only be the combination of homogeneous functions of $r_i(\mathbf{x})$ of the zeroth order, second order and fourth order. This implies that the circumference basis of $\mathbb{S}^{E,D}_{ijmn}(\mathbf{x})$ should be similar to the circumference basis of $\mathbb{S}^{E,\infty}_{ijmn}(\mathbf{x})$, which is the hallmark that represents the concentric property of the circular inclusion and the circular RVE. Therefore, we assume that the exterior Eshelby tensor in the finite domain can only have the following form:

$$\begin{aligned}\mathbb{S}^{E,D}_{ijmn}(\mathbf{x}) &= S^{E,D}_1(t)\delta_{ij}\delta_{mn} + S^{E,D}_2(t)(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}) + S^{E,D}_3(t)\delta_{ij}r_m r_n \\ &\quad + S^{E,D}_4(t)\delta_{mn}r_i r_j + S^{E,D}_5(t) r_i r_j r_m r_n \\ &= \mathbf{\Theta}^T_{ijmn}(\mathbf{r})\mathbf{S}^{E,D}(t),\end{aligned} \tag{29}$$

where $r_i = x_i/|\mathbf{x}|$, and the 1D array, $\mathbf{S}^{E,D}(t)$, is defined as

$$\mathbf{S}^{E,D}(t) := \Big[S^{E,D}_1(t), S^{E,D}_2(t), S^{E,D}_3(t), S^{E,D}_4(t), S^{E,D}_5(t)\Big]^T, \tag{30}$$

where $S^{E,D}_i(t)$, $i = 1,2,3,4,5$, are unknown functions of a nondimensional radial variable $t = |\mathbf{x}|/H_0$ that are to be determined, and $\mathbf{\Theta}_{ijmn}(\mathbf{r})$ is the circumference tensorial basis that is known based on the symmetry argument. Note that the radial basis array $\mathbf{S}^{E,D}(t)$ does not depend on subscripts $ijmn$. In particular, for $\mathbf{x} \to \partial\Omega$ it follows that $|\mathbf{x}| \to H_0$, $t \to 1$, $\mathbf{r} \to \mathbf{n}$ and $\rho \to \rho_0$. Then the array $\mathbf{S}^{E,D}(t) = \mathbf{S}^{E,D}(1)$ becomes constant and

$$\mathbb{S}^{E,\infty}_{ijmn}(\mathbf{x}) \to \mathbf{\Theta}^T_{ijmn}(\mathbf{n})\mathbf{S}^{E,\infty}(1), \quad \forall \mathbf{x} \in \partial\Omega. \tag{31}$$

The basic hypothesis of our approach is that the circumference basis of the Eshelby tensor of an isotropic finite domain, $\mathbb{S}^{E,D}_{ijmn}(\mathbf{x})$, is similar to that of the Eshelby tensor in an isotropic infinite domain. This postulate is true only if the circumference basis is invariant under the following boundary integration operator, i.e.,

$$\begin{aligned}\mathbb{S}^{B,D}_{ijmn}(\mathbf{x}) &= \frac{1}{2}\int_{\partial\Omega} \mathbb{C}_{k\ell st}\Big(G^{\infty}_{ik,j}(\mathbf{x}-\mathbf{y}) + G^{\infty}_{jk,i}(\mathbf{x}-\mathbf{y})\Big) n_\ell(\mathbf{y})\mathbb{S}^{E,D}_{stmn}(\mathbf{y}) dS_y \\ &= \int_{\partial\Omega} \mathbb{G}_{ijmn}(\mathbf{x},\mathbf{y}) dS_y = K_{ijst}\big[\mathbb{S}^{E,D}_{stmn}(\partial\Omega)\big] \end{aligned} \tag{32}$$

$$\begin{aligned} &= S^{B,D}_1(t)\delta_{ij}\delta_{mn} + S^{B,D}_2(t)(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}) + S^{B,D}_3(t)\delta_{ij}r_m r_n \\ &\quad + S^{B,D}_4(t)\delta_{mn}r_i r_j + S^{B,D}_5(t) r_i r_j r_m r_n \\ &= \mathbf{\Theta}^T_{ijmn}(\mathbf{r})\mathbf{S}^{B,D}(t),\end{aligned} \tag{33}$$

where the superscript (B,D) refers to the fact that $\mathbb{S}^{B,D}_{ijmn}$ is the contribution coming from the Dirichlet boundary. Furthermore we have denoted the integrand as

$$\mathbb{G}_{ijmn}(\mathbf{x},\mathbf{y}) = \frac{1}{2}\mathbb{C}_{k\ell st}\Big(G^{\infty}_{ik,j}(\mathbf{x}-\mathbf{y}) + G^{\infty}_{jk,i}(\mathbf{x}-\mathbf{y})\Big) n_\ell(\mathbf{y})\mathbb{S}^{E,D}_{stmn}(\mathbf{y}). \tag{34}$$

By explicit integration, we shall show that the circumference basis remains invariant under the integration operator K_{ijst}, or the integral operator K_{ijst} has the invariant property (33). After a few algebraic manipulations, we find that

$$\begin{aligned}\mathbb{G}_{ijmn} = \frac{-1}{8\pi(1-\nu)\mu R}\Big[& T_1 \ell_k n_k \delta_{ij}\delta_{mn} + T_1(2\nu - 1)(\ell_i n_j + \ell_j n_i)\delta_{mn} \\ & + T_2(\ell_m n_n + \ell_n n_m)\delta_{ij} + T_2(2\nu-1)(\delta_{im}\ell_j n_n + \delta_{in}\ell_j n_m \\ & + \delta_{jm}\ell_i n_n + \delta_{jn}\ell_i n_m) - 2T_1 \ell_i \ell_j \ell_k n_k \delta_{mn} + T_3 \ell_k n_k n_m n_n \delta_{ij} \\ & - 2T_2(\ell_i \ell_j \ell_m n_n + \ell_i \ell_j \ell_n n_m) + T_3(2\nu - 1)(\ell_i n_j n_m n_n + \ell_j n_i n_m n_n) \\ & - 2T_3 \ell_i \ell_j \ell_k n_k n_m n_n\Big],\end{aligned} \tag{35}$$

where $\ell_i = (y_i - x_i)/R$ and $R = |\mathbf{y} - \mathbf{x}|$. For a circular RVE $n_i(\mathbf{y}) = y_i/|\mathbf{y}|$. Denoting

$$\mathbf{T} = [T_1, T_2, T_3]^T, \tag{36}$$

we identify

$$\mathbf{T} = \mathbf{K}_1\mathbf{S}^{E,D}(1), \quad \mathbf{K}_1 = \mu\begin{bmatrix} \frac{2}{1-2\nu} & \frac{4\nu}{1-2\nu} & 0 & \frac{2(1-\nu)}{1-2\nu} & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{1-2\nu} & 0 & \frac{2(1-\nu)}{1-2\nu} \end{bmatrix}. \tag{37}$$

Integral (32) consists of the following independent integrals, which are evaluated in the Appendix:

$$\text{(I)} \int_{\partial\Omega} \frac{1}{R}\ell_k n_k(\mathbf{y})\, dS_y = 2\pi, \tag{38}$$

$$\text{(II)} \int_{\partial\Omega} \frac{1}{R}\ell_i n_j(\mathbf{y})\, dS_y = \pi\delta_{ij}, \tag{39}$$

$$\text{(III)} \int_{\partial\Omega} \frac{1}{R}\ell_i\ell_j\ell_k n_k(\mathbf{y})\, dS_y = \pi\delta_{ij}, \tag{40}$$

$$\text{(IV)} \int_{\partial\Omega} \frac{1}{R}\ell_k n_k(\mathbf{y}) n_m(\mathbf{y}) n_n(\mathbf{y})\, dS_y = \frac{\pi}{2}\left(2 - t^2\right)\delta_{mn} + \pi t^2 r_m(\mathbf{x}) r_n(\mathbf{x}), \tag{41}$$

$$\text{(V)} \int_{\partial\Omega} \frac{1}{R}\ell_i\ell_j(\ell_m n_n(\mathbf{y}) + \ell_n n_m)\, dS_y = \frac{\pi}{2}(\delta_{ij}\delta_{mn} + \delta_{im}\delta_{jn} + \delta_{jm}\delta_{in}), \tag{42}$$

$$\text{(VI)} \int_{\partial\Omega} \frac{1}{R}(\ell_i n_j(\mathbf{y}) + \ell_j n_i(\mathbf{y})) n_m(\mathbf{y}) n_n(\mathbf{y})\, dS_y = \frac{\pi}{2}(\delta_{im}\delta_{jn}$$
$$+\delta_{jm}\delta_{in}) + \frac{\pi}{2}\left(1 - t^2\right)\delta_{ij}\delta_{mn} + \pi t^2 \delta_{ij} r_m(\mathbf{x}) r_n(\mathbf{x}), \tag{43}$$

$$\text{(VII)} \int_{\partial\Omega} \frac{1}{R}\ell_i\ell_j\ell_k n_k(\mathbf{y}) n_m(\mathbf{y}) n_n(\mathbf{y})\, dS_y = \frac{\pi}{4}\left(1 - t^2\right)(\delta_{im}\delta_{jn}$$
$$+\delta_{jm}\delta_{in}) + \frac{\pi}{4}\delta_{ij}\delta_{mn} + \frac{\pi}{2} t^2 \delta_{ij} r_m(\mathbf{x}) r_n(\mathbf{x}). \tag{44}$$

Thus the integral (32) can be evaluated exactly. Substituting Eqs. (38)–(44) into Eq. (32) gives

$$\begin{aligned} \mathbb{S}^{B,D}_{ijmn}(\mathbf{x}) &= S^{B,D}_1(t)\delta_{ij}\delta_{mn} + S^{B,D}_2(t)(\delta_{im}\delta_{jn} + \delta_{in}\delta_{im}) + S^{B,D}_3(t)\delta_{ij} r_m r_n \\ &\quad + S^{B,D}_4(t) r_i r_j \delta_{mn} + S^{B,D}_5(t) r_i r_j r_m r_n \\ &= \mathbf{\Theta}^T_{ijmn}(\mathbf{r})\mathbf{S}^{B,D}(t), \end{aligned} \tag{45}$$

where

$$\mathbf{S}^{B,D}(t) = \mathbf{K}_2(t)\mathbf{K}_1\mathbf{S}^{E,D}(1), \tag{46}$$

and

$$\mathbf{K}_2(t) = \frac{-1}{8(1-\nu)\mu}\begin{bmatrix} 2(2\nu - 1) & 1 & \nu\left(1 - t^2\right) \\ 0 & 4\nu - 3 & \frac{2\nu-1}{2} - \frac{1-t^2}{2} \\ 0 & 0 & (2\nu - 1)t^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{47}$$

The integral equations (27) and (28) then reduce to a pair of algebraic evolution equations,

$$\mathbf{\Theta}^T_{ijmn}(\mathbf{r})\mathbf{S}^{I,D}(t) = \mathbf{\Theta}^T_{ijmn}(\mathbf{r})\big(\mathbf{S}^{I,\infty}(t) + \mathbf{K}(t)\mathbf{S}^{E,D}(1)\big), \tag{48}$$

$$\mathbf{\Theta}^T_{ijmn}(\mathbf{r})\mathbf{S}^{E,D}(t) = \mathbf{\Theta}^T_{ijmn}(\mathbf{r})\big(\mathbf{S}^{E,\infty}(t) + \mathbf{K}(t)\mathbf{S}^{E,D}(1)\big), \tag{49}$$

where $\mathbf{K}(t) = \mathbf{K}_2(t)\mathbf{K}_1$ is given by

$$\mathbf{K}(t) = \frac{1}{8(1-v)}\begin{bmatrix} 4 & 8v-2 & \frac{2v(t^2-1)}{1-2v} & 4-4v & \frac{2v(t^2-1)(1-v)}{1-2v} \\ 0 & 2(3-4v) & \frac{2-t^2-2v}{1-2v} & 0 & \frac{(2-t^2-2v)(1-v)}{1-2v} \\ 0 & 0 & 2t^2 & 0 & 2(1-v)t^2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{50}$$

Eliminating the circumference basis from Eqs. (48) and (49), we have

$$\mathbf{S}^{I,D}(t) = \mathbf{S}^{I,\infty}(t) + \mathbf{S}^{B,D}(t), \quad 0 \le t < \rho_0, \tag{51}$$

$$\mathbf{S}^{E,D}(t) = \mathbf{S}^{E,\infty}(t) + \mathbf{S}^{B,D}(t), \quad \rho_0 \le t < 1, \tag{52}$$

where $\mathbf{S}^{B,D}(t) = \mathbf{K}(t)\mathbf{S}^{E,D}(1)$. It is reasonable to assume that Eqs. (51) and (52) continuously depend on t. Let $t \to 1$ in (51) and (52), one can find $\mathbf{S}^{E,D}(1)$ by solving the equation

$$\mathbf{S}^{E,D}(1) = (\mathbf{1} - \mathbf{K}(1))^{-1}\mathbf{S}^{E,\infty}(1). \tag{53}$$

The solution is

$$\mathbf{S}^{E,D}(1) = \begin{bmatrix} \dfrac{4v\rho_0^2 - 3\rho_0^4}{(4v-3)} \\ \dfrac{4v\rho_0^2 - 3\rho_0^4}{2(3-4v)} \\ \dfrac{4v\rho_0^2 - 3\rho_0^4}{3-4v} \\ \dfrac{\rho_0^2 - 3\rho_0^3}{2(1-v)} \\ \dfrac{2\rho_0^2 - 3\rho_0^4}{(v-1)} \end{bmatrix}. \tag{54}$$

We comment in passing that the algebraic equations (51) and (52) are obtained from the integral equations (27) and (28) under the condition $t \neq 1$. By the continuity property of the radial basis $\mathbf{S}^{E,D}(t)$ we obtain Eq. (53). By doing so, we circumvent the difficulties in solving a hyper-singular integral equation.

Substituting Eq. (54) back into Eqs. (51) and (52), one can solve for both $\mathbf{S}^{I,D}(t)$ and $\mathbf{S}^{E,D}(t)$, which are the radial basis for the Dirichlet–Eshelby tensor in a finite domain,

$$\mathbf{S}^{I,D}(t) = \frac{1-\rho_0^2}{8(1-v)}\begin{bmatrix} 4v-1 \\ 3-4v \\ 0 \\ 0 \\ 0 \end{bmatrix} + \frac{3(\rho_0^2-\rho_0^4)}{8(1-v)(3-4v)}\begin{bmatrix} 1-4vt^2 \\ 2t^2-1 \\ 4t^2(2v-1) \\ 0 \\ 0 \end{bmatrix}, \tag{55}$$

and

$$\mathbf{S}^{E,D}(t) = \frac{\rho_0^2/t^2}{8(1-\nu)}\begin{bmatrix} 9\rho_0^2/t^2 - 4\nu - 2 - t^2(4\nu-1) \\ -3\rho_0^2/t^2 + 2 - t^2(3-4\nu) \\ -12\rho_0^2/t^2 + 8\nu + 4 \\ -12\rho_0^2/t^2 + 4 \\ 8(3\rho_0^2/t^2 - 2) \end{bmatrix} + \frac{3(\rho_0^2 - \rho_0^4)}{8(1-\nu)(3-4\nu)}\begin{bmatrix} 1 - 4\nu t^2 \\ 2t^2 - 1 \\ 4t^2(2\nu - 1) \\ 0 \\ 0 \end{bmatrix}. \tag{56}$$

The exact expressions for the Eshelby tensors for a circular inclusion embedded in a circular RVE under the prescribed displacement boundary condition then become

$$\begin{aligned}\mathbb{S}^{I,D}_{ijmn}(\mathbf{x}) = \frac{1}{8(1-\nu)}\Bigg\{&\left[(4\nu-1)\left(1-\rho_0^2\right) + \frac{3\rho_0^2\left(\rho_0^2-1\right)\left(1-4\nu t^2\right)}{(4\nu-3)}\right]\delta_{ij}\delta_{mn} \\ &+\left[(3-4\nu)\left(1-\rho_0^2\right) + \frac{3\rho_0^2\left(\rho_0^2-1\right)\left(2t^2-1\right)}{(4\nu-3)}\right](\delta_{im}\delta_{jn}+\delta_{in}\delta_{jm}) \\ &+\left[\frac{12(2\nu-1)\rho_0^2\left(\rho_0^2-1\right)}{(4\nu-3)}\right]t^2\delta_{ij}r_m r_n\Bigg\}, \quad \forall \mathbf{x} \in \Omega_e,\end{aligned} \tag{57}$$

and

$$\begin{aligned}\mathbb{S}^{E,D}_{ijmn}(\mathbf{x}) = \frac{\rho_0^2/t^2}{8(1-\nu)}\Bigg\{&\Bigg[9\rho_0^2/t^2 + 4\nu - 2 - t^2(4\nu-1) \\ &+\frac{3t^2\left(\rho_0^2-1\right)\left(1-4\nu t^2\right)}{(4\nu-3)}\Bigg]\delta_{ij}\delta_{mn} + \Bigg[-3\rho_0^2/t^2 + 2 - t^2(3-4\nu) \\ &+\frac{3t^2\left(\rho_0^2-1\right)\left(2t^2-1\right)}{(4\nu-3)}\Bigg](\delta_{im}\delta_{jn}+\delta_{in}\delta_{jm}) + \Bigg[-12\rho_0^2/t^2 + 8\nu + 4 \\ &+\frac{12t^2(2\nu-1)\left(\rho_0^2-1\right)}{(4\nu-3)}\Bigg]\delta_{ij}r_m r_n + \left[-12\rho_0^2/t^2 + 4\right]\delta_{mn}r_i r_j \\ &+\left[8\left(3\rho_0^2/t^2-2\right)\right]r_i r_j r_m r_n\Bigg\}, \quad \forall \mathbf{x} \in \Omega/\Omega_e.\end{aligned} \tag{58}$$

If we let $H_0 \to \infty$, $\rho_0/t = a/|\mathbf{x}|$, and $\rho_0, t \to 0$, Eqs. (57) and (58) recover the classical Eshelby solutions (16) and (17).

To visualize the Eshelby tensors, the coefficients $\mathbf{S}^{\bullet,\infty}$, $\mathbf{S}^{B,D}$ and $\mathbf{S}^{\bullet,D}$, where $\bullet = I$ or E, are plotted as functions of the radial distance t (see Fig. 2). Here Poisson's ratio is chosen as $\nu = 0.3$ and the inclusion size as $\rho_0 = 0.2$. Note that all coefficients have a jump at $t = \rho_0$, which implies discontinuities of the strain components across the inclusion/matrix interface.

From Fig. 2, one may observe that all coefficients of $\mathbf{S}^{B,D}$, i.e., the difference between $\mathbf{S}^{\bullet,D}$ and $\mathbf{S}^{\bullet,\infty}$, are actually small, if H_0 is large. This difference will increase if ρ_0 is increased.

4 Displacement field

To show that the prescribed Dirichlet boundary condition is indeed satisfied by the above solution, we examine the corresponding disturbance displacement field. Substituting $u^d_{k,\ell}(\mathbf{y}) = S^{E,D}_{k\ell pq}(1)\epsilon^*_{pq}, \mathbf{y} \in \partial\Omega$, into Eq. (11), we obtain

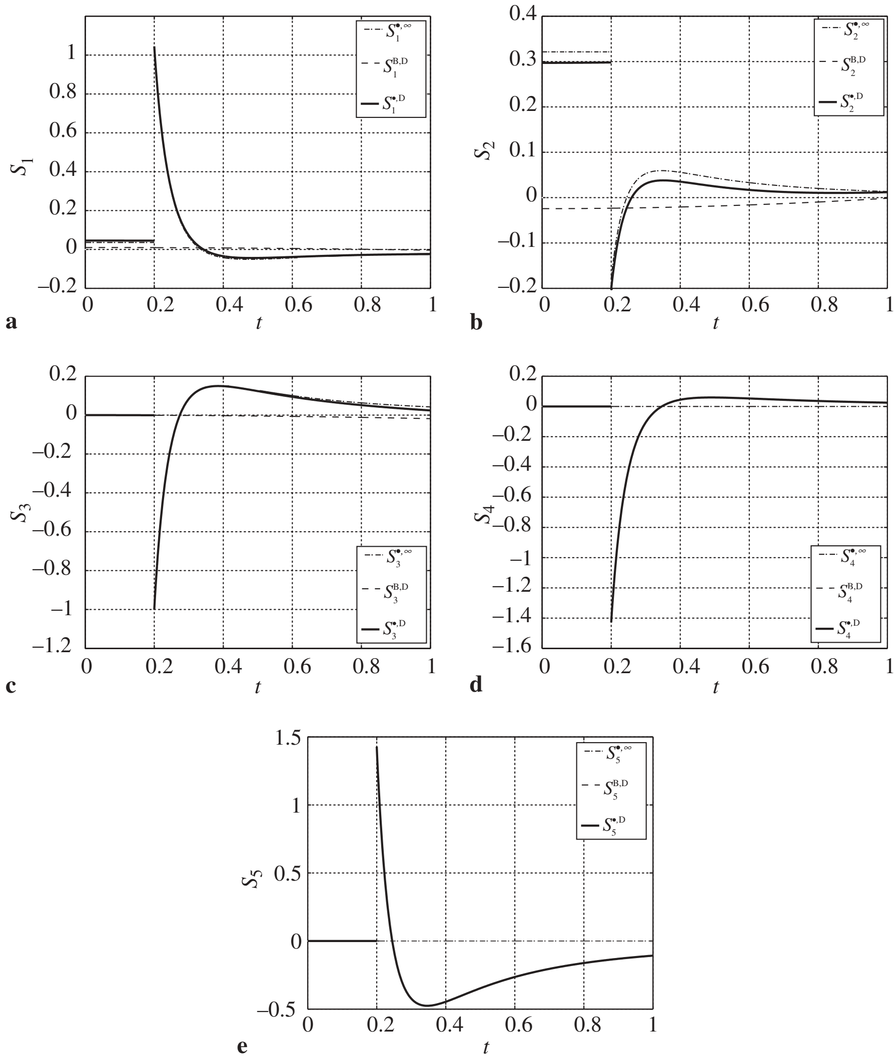

Fig. 2. Eshelby coefficients $S_1(t)$, $S_2(t)$, $S_3(t)$, $S_4(t)$ and $S_5(t)$

$$
\begin{aligned}
u_i^d(\mathbf{x}) &= -\int_{\Omega_e} \mathbb{C}_{pqk\ell} G^{\infty}_{pi,q}(\mathbf{x}-\mathbf{y})\epsilon^*_{k\ell} d\Omega_y + \oint_{\partial\Omega} \mathbb{C}_{pqk\ell} S^{E,D}_{k\ell st}(1)\epsilon^*_{st} G^{\infty}_{pi}(\mathbf{x}-\mathbf{y}) n_q dS_y \\
&= \epsilon^*_{mn} \boldsymbol{\Xi}^T_{imn}(\mathbf{r}) \mathbf{U}^{\bullet,D}(t),
\end{aligned}
\tag{59}
$$

where the 1D array

$$
\mathbf{U}^{\bullet,D}(t) = \left[U_1^{\bullet,D}(t), U_2^{\bullet,D}(t), U_3^{\bullet,D}(t)\right]^T
$$

is the radial basis for the displacement field and $\boldsymbol{\Xi}_{imn}(\mathbf{r})$ is the circumference basis for the displacement field,

$$
\boldsymbol{\Xi}_{imn}(\mathbf{r}) = \begin{bmatrix} r_i\delta_{mn} \\ r_m\delta_{in} + r_n\delta_{im} \\ r_i r_n r_m \end{bmatrix}. \tag{60}
$$

Let us denote

$$\boldsymbol{\Xi}^T_{imn}(\mathbf{r})\mathbf{U}^{B,D}(t) = \oint_{\partial\Omega} \mathbb{C}_{pqk\ell}\mathbb{S}^{E,D}_{k\ell mn}(1)G^{\infty}_{pi}(\mathbf{x}-\mathbf{y})n_q dS_y, \tag{61}$$

$$\boldsymbol{\Xi}^T_{imn}(\mathbf{r})\mathbf{U}^{I,\infty}(t) = -\int_{\Omega_e} \mathbb{C}_{pqmn}G^{\infty}_{pi,q}(\mathbf{x}-\mathbf{y})d\Omega_y, \quad \mathbf{x}\in\Omega_e, \tag{62}$$

and

$$\boldsymbol{\Xi}^T_{imn}(\mathbf{r})\mathbf{U}^{I,\infty}(t) = -\int_{\Omega_e} \mathbb{C}_{pqmn}G^{\infty}_{pi,q}(\mathbf{x}-\mathbf{y})d\Omega_y, \quad \forall\mathbf{x}\in\Omega/\Omega_e. \tag{63}$$

Substituting Eqs. (61), (62), and (63) into (59), one may find the radial bases for the displacement field,

$$\mathbf{U}^{I,D}(t) = \mathbf{U}^{I,\infty}(t) + \mathbf{U}^{B,D}(t), \quad 0 \le t < \rho_0, \tag{64}$$

$$\mathbf{U}^{E,D}(t) = \mathbf{U}^{E,\infty}(t) + \mathbf{U}^{B,D}(t), \quad \rho_0 \le t \le 1, \tag{65}$$

where

$$\mathbf{U}^{I,\infty}(t) = \frac{H_0 t}{8(1-v)}\begin{bmatrix} 4v-1 \\ 3-4v \\ 0 \end{bmatrix}, \tag{66}$$

$$\mathbf{U}^{E,\infty}(t) = \frac{\rho_0^2 H_0}{8t(1-v)}\begin{bmatrix} 4v-2+\dfrac{\rho_0^2}{t^2} \\ 2-4v+\dfrac{\rho_0^2}{t^2} \\ 4\left(1-\dfrac{\rho_0^2}{t^2}\right) \end{bmatrix}, \tag{67}$$

and

$$\mathbf{U}^{B,D}(t) = \frac{\rho_0^2 H_0 t}{8(1-v)}\begin{bmatrix} 1-4v+\left(1-\rho_0^2\right)\frac{3-4t^2v}{3-4v} \\ 4v-3+\left(1-\rho_0^2\right)\frac{6t^2-3-4vt^2}{3-4v} \\ -4\left(1-\rho_0^2\right)t^2 \end{bmatrix}. \tag{68}$$

It is readily verified that when $t=1$,

$$\mathbf{U}^{E,D}(1) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow u^d_i(\mathbf{y}) = \epsilon^*_{mn}\boldsymbol{\Xi}^T_{imn}(\mathbf{n})\mathbf{U}^{E,D}(1) = 0, \quad \forall\mathbf{y}\in\partial\Omega. \tag{69}$$

Note that when $\mathbf{x}\in\partial\Omega \Rightarrow \mathbf{r}=\mathbf{n}$ for a circular RVE. To this end, we have shown that the prescribed displacement boundary condition is satisfied automatically and the above solution is indeed the solution of the BVP (7), (8).

5 Traction field

Consider an arbitrary point $\mathbf{x}\in\Omega$. The disturbance traction vector on a plane tangential to the normal direction is

$$t^d_i = \sigma^d_{ji} r_j, \tag{70}$$

where the disturbance stress is given by

$$\sigma^d_{ij}(\mathbf{x}) = \begin{cases} \mathbb{C}_{ijk\ell}\big(\epsilon^d_{k\ell}(\mathbf{x}) - \epsilon^*_{k\ell}\big), & \forall \mathbf{x} \in \Omega_e, \\ \mathbb{C}_{ijk\ell}\epsilon^d_{k\ell}(\mathbf{x}), & \forall \mathbf{x} \in \Omega/\Omega_e. \end{cases} \tag{71}$$

Combining the above equations we find

$$t^d_i(\mathbf{x}) = \begin{cases} r_j(\mathbf{x})\mathbb{C}_{ijk\ell}\Big(\mathbb{S}^{I,D}_{k\ell mn}(\mathbf{x}) - \mathbb{I}^{(4s)}_{k\ell mn}\Big)\epsilon^*_{mn}, & \forall \mathbf{x} \in \Omega_e, \\ r_j(\mathbf{x})\mathbb{C}_{ijk\ell}\mathbb{S}^{E,D}_{k\ell mn}(\mathbf{x})\epsilon^*_{mn}, & \forall \mathbf{x} \in \Omega/\Omega_e, \end{cases} \tag{72}$$

where $\mathbb{I}^{(4s)}_{k\ell mn}$ is the fourth-order symmetric identity tensor. It is noted that it also falls into our new category of a *fourth-order radial isotropic tensor*.

Analogously to the displacement field, we write the traction field as

$$t^d_i = \boldsymbol{\Xi}^T_{imn}(\mathbf{r})\mathbf{T}^{\bullet,D}(t)\epsilon^*_{mn}. \tag{73}$$

Careful analysis reveals the traction coefficients

$$\mathbf{T}^{I,D}(t) = \mathbf{K}_1\big(\mathbf{S}^{I,D}(t) - \mathbf{I}^{4s}\big), \quad 0 \le t < \rho_0 \tag{74}$$

$$\mathbf{T}^{E,D}(t) = \mathbf{K}_1\mathbf{S}^{E,D}(t), \quad \rho_0 \le t \le 1, \tag{75}$$

where $\mathbf{I}^{4s} = [0\ \tfrac{1}{2}\ 0\ 0\ 0]^T$ are the coefficients of $\mathbb{I}^{(4s)}_{k\ell mn}$. Similar to before, we write

$$\mathbf{T}^{I,D}(t) = \mathbf{T}^{I,\infty}(t) + \mathbf{T}^{B,D}(t), \quad 0 \le t < \rho_0, \tag{76}$$

$$\mathbf{T}^{E,D}(t) = \mathbf{T}^{E,\infty}(t) + \mathbf{T}^{B,D}(t), \quad \rho_0 \le t \le 1, \tag{77}$$

with

$$\mathbf{T}^{I,\infty}(t) = -\frac{\mu}{4(1-\nu)}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \tag{78}$$

$$\mathbf{T}^{E,\infty}(t) = \frac{\mu\rho_0^2/t^2}{4(1-\nu)}\begin{bmatrix} -3\rho_0^2/t^2 + 2 \\ -3\rho_0^2/t^2 + 2 \\ 12\big(\rho_0^2/t^2 - 1\big) \end{bmatrix}, \tag{79}$$

and

$$\mathbf{T}^{B,D}(t) = \frac{\mu\rho_0^2}{4(1-\nu)}\begin{bmatrix} (8\nu^2 - 10\nu + 1)/(2\nu - 1) \\ 3 - 4\nu \\ 0 \end{bmatrix} + \frac{3\mu\big(\rho_0^2 - \rho_0^4\big)}{4(1-\nu)(3-4\nu)}\begin{bmatrix} 1 \\ 2t^2 - 1 \\ -4t^2 \end{bmatrix}. \tag{80}$$

In Fig. 3, the three displacement coefficients $\mathbf{U}^{\bullet,\infty}$, $\mathbf{U}^{B,D}$ and $\mathbf{U}^{\bullet,D}$ are plotted, and they are juxtaposed with the three traction coefficients $\mathbf{T}^{\bullet,\infty}$, $\mathbf{T}^{B,D}$ and $\mathbf{T}^{\bullet,D}$ over the normalized radial distance $t, (0 \le t \le 1)$. In the figure, we choose $\nu = 0.3$ and $\rho_0 = 0.2$. Note that all coefficients are continuous and $\mathbf{U}^{E,D} = \mathbf{0}$ at $t = 1$. Contrary to the variation of the Eshelby coefficients or strain fields, the boundary contribution to the displacement field, $\mathbf{U}^{B,D} = \mathbf{U}^{\bullet,D} - \mathbf{U}^{\bullet,\infty}$, is significant, and it changes the nature of the displacement field. Furthermore one can see that the traction components are decaying along the radial direction.

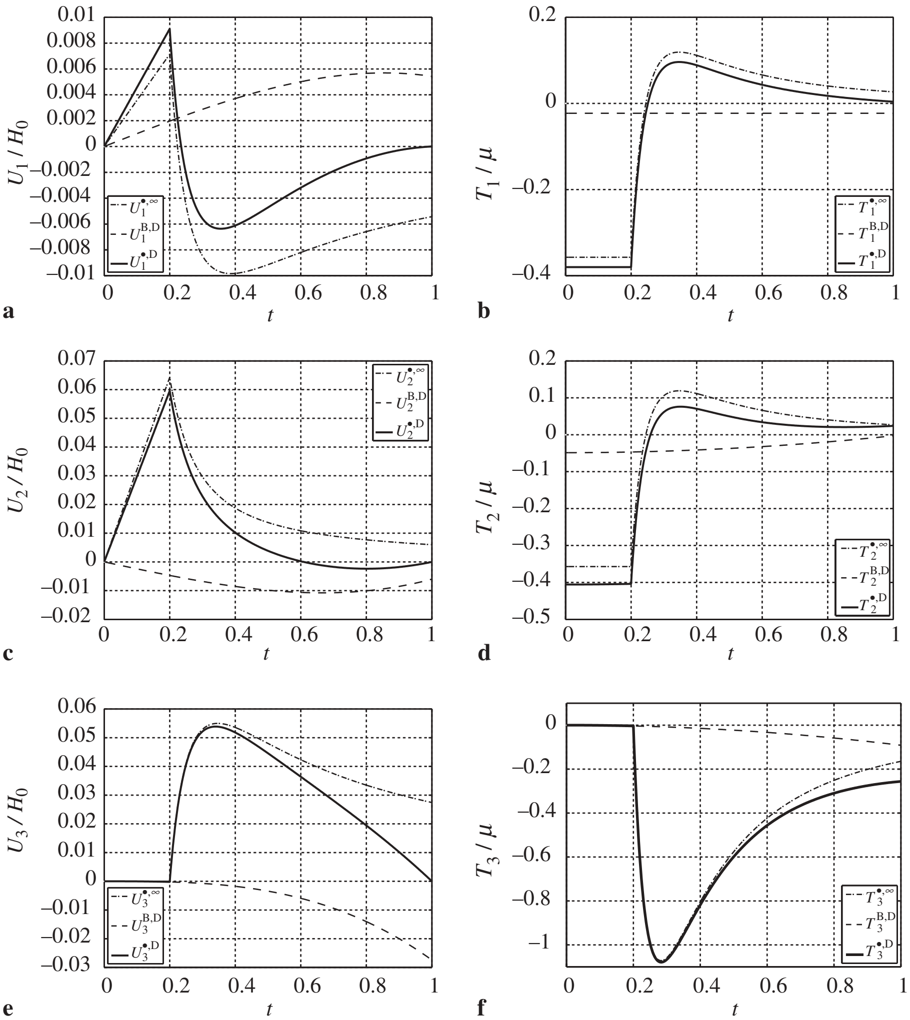

Fig. 3. Displacement and traction coefficients $U_i(t)$ $i = 1, 2, 3$ (**a, c, e**); $T_i(t)$, $i = 1, 2, 3$ (**b, d, f**)

6 Applications

Using the tensorial bases

$$\mathbb{E}^{(1)}_{ijmn} = \frac{1}{2}\delta_{ij}\delta_{mn}, \tag{81}$$

$$\mathbb{E}^{(2)}_{ijmn} = \frac{1}{2}(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm} - \delta_{ij}\delta_{mn}), \tag{82}$$

the interior Eshelby tensor can be rewritten in terms of the $\mathbb{E}$ basis as

$$\mathbb{S}^{I,D}_{ijmn}(\mathbf{x}) = \frac{1-\rho_0^2}{2(1-v)}\mathbb{E}^{(1)}_{ijmn} + \frac{3t^2\rho_0^2(1-\rho_0^2)(1-2v)}{2(1-v)(3-4v)}\left(\mathbb{E}^{(1)}_{ijmn} - \delta_{ij}r_m r_n\right) + \frac{1-\rho_0^2}{4(1-v)}\left((3-4v) - \frac{3\rho_0^2(2t^2-1)}{(3-4v)}\right)\mathbb{E}^{(2)}_{ijmn}. \tag{83}$$

Consider the averaging

$$\langle t^2 \rangle_{\Omega_e} = \frac{1}{\pi a^2} \int_0^{2\pi} \int_0^a \frac{r^2}{H_0^3} dr d\varphi = \frac{1}{2} \left(\frac{a^2}{H_0^2} \right), \tag{84}$$

$$\langle t^2 r_m r_n \rangle_{\Omega_e} = \frac{1}{\pi a^2} \int_0^{2\pi} \int_0^a \frac{r^3}{H_0^2} r_m r_n dr d\varphi = \frac{1}{4} \left(\frac{a^2}{H_0^2} \right) \delta_{mn}, \tag{85}$$

and recognize that the ratio

$$f = \frac{a^2}{H_0^2} = \rho_0^2 \tag{86}$$

is the volume fraction of the inclusion in an RVE. Then the average interior Eshelby tensor has the following form:

$$\langle \mathbb{S}^{I,D}_{ijmn} \rangle_{\Omega_e} = s_1 \mathbb{E}^{(1)}_{ijmn} + s_2 \mathbb{E}^{(2)}_{ijmn}, \tag{87}$$

with

$$s_1 = \frac{1-f}{2(1-\nu)} \text{ and } s_2 = \frac{1-f}{4(1-\nu)} \left((3-4\nu) - \frac{3f(1-f)}{3-4\nu} \right). \tag{88}$$

One may compare the above result with the interior Eshelby tensor of an infinite space:

$$\mathbb{S}^{I,\infty}_{ijmn} = s_1^0 \mathbb{E}^{(1)}_{ijmn} + s_2^0 \mathbb{E}^{(2)}_{ijnm}, \text{ with } s_1^0 = \frac{1}{2(1-\nu)}, \, s_2^0 = \frac{(3-4\nu)}{4(1-\nu)}. \tag{89}$$

It is both sensible and remarkable that the finite Eshelby tensor depends on the volume fraction of the inclusion, which distinguishes it from the Eshelby tensor for an infinite ambient space. Moreover one can verify that:

(i) when $f = 0$, $s_1 = s_1^0$, and $s_2 = s_2^0$;
(ii) when $f = 1$, $s_1 = 0$, and $s_2 = 0$.

The newly derived Dirichlet-Eshelby tensor is used in homogenization of a two-phase composite material. The commonly used dilute homogenization method (e.g., Nemat-Nasser and Hori [15]) is employed in calculating the effective material constants K_{eff}, μ_{eff} and ν_{eff}. The homogenization results using $\mathbb{S}^{I,\infty}_{ijmn}$ and $\mathbb{S}^{I,D}_{ijmn}$ are depicted in Fig. 4. The inclusion properties are chosen as $K_e = 10K$ and $\mu_e = 5\mu$, thus $\nu_e = 3\nu$, where K, μ and ν are the matrix properties. It is common knowledge (which one can also observe from Fig. 4) that in the homogenization of dilute inclusion distribution using $\mathbb{S}^{I,\infty}_{ijmn}$ the estimate of the effective material constants becomes poor when the volume fraction of the second phase, f, becomes large. The results obtained by using the Dirichlet-Eshelby tensor converge at both ends, $f = 0$ and $f = 1$, which comes as a pleasant surprise. The reason for this improvement is that $\mathbb{S}^{I,\infty}_{ijmn}$ loses its physical meaning when $f \to 1$, whereas $\mathbb{S}^{I,D}_{ijmn}$ is always valid for the whole range of the volume fraction of the inhomogeneity, $f \in [0, 1]$.

Finally we include the following remarks: First, we have observed that the Dirichlet-Eshelby tensor derived in this paper is independent from the prescribed boundary data. This indicates that it is a good quantity that will serve well in material homogenization and characterization.

Secondly, we calculate the sum $S^{I,D}_{iijj}(\mathbf{x})$,

$$\begin{aligned} S^{I,D}_{iijj}(\mathbf{x}) &= \mathbb{S}^{I,D}_{1111}(\mathbf{x}) + \mathbb{S}^{I,D}_{2222}(\mathbf{x}) + \mathbb{S}^{I,D}_{1122}(\mathbf{x}) + \mathbb{S}^{I,D}_{2211}(\mathbf{x}) \\ &= 4S^{I,D}_1(t) + 4S^{I,D}_2(t) + 2S^{I,D}_3(t) + 2S^{I,D}_4(t) + S^{I,D}_5(t) \\ &= \frac{1-f}{1-\nu} = 2s_1. \end{aligned} \tag{90}$$

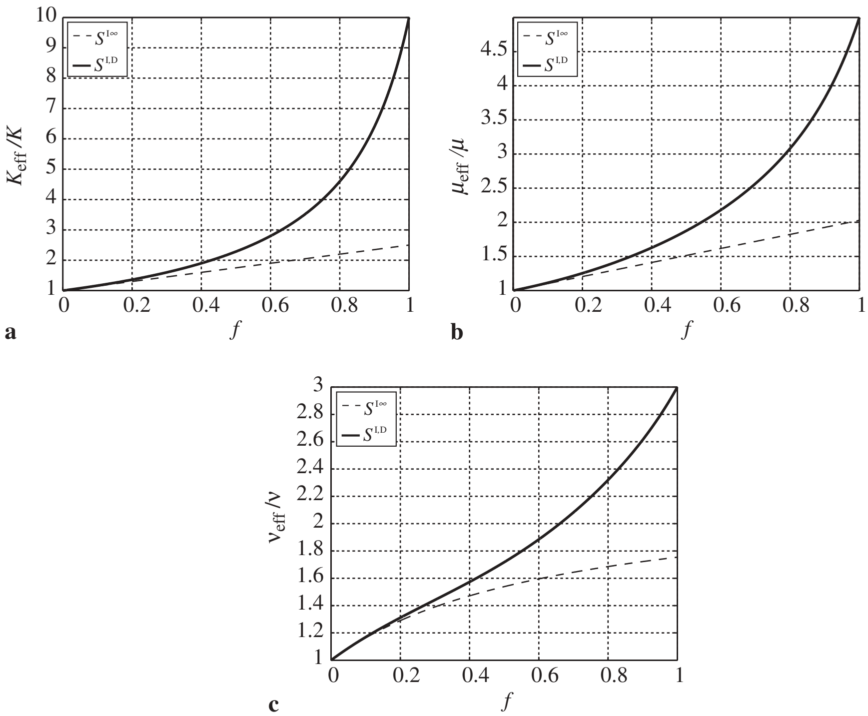

Fig. 4. Comparison of dilute homogenization results using $\mathbb{S}^{I,\infty}_{ijmn}$ and $\mathbb{S}^{I,D}_{ijmn}$

This indicates that if the prescribed eigenstrain field is bi-axially uniform, i.e., $\epsilon^*_{11} = \epsilon^*_{22}$, the induced disturbance strain field will be uniform as well. It means that even though the individual components are position dependent, under uniform bi-axial loading, the volumetric disturbance field is constant within the interior of the inclusion.

Moreover, one can calculate the sum of the exterior Eshelby tensor under uniform bi-axial loading,

$$\begin{aligned} S^{E,D}_{iijj}(\mathbf{x}) &= \mathbb{S}^{E,D}_{1111}(\mathbf{x}) + \mathbb{S}^{E,D}_{2222}(\mathbf{x}) + \mathbb{S}^{E,D}_{1122}(\mathbf{x}) + \mathbb{S}^{E,D}_{2211}(\mathbf{x}) \\ &= 4S^{E,D}_1(t) + 4S^{E,D}_2(t) + 2S^{E,D}_3(t) + 2S^{E,D}_4(t) + S^{E,D}_5(t) \\ &= \frac{-f}{1-\nu}. \end{aligned} \tag{91}$$

It is also interesting to note that

$$\mathbb{S}^{I,D}_{iijj} - \mathbb{S}^{E,D}_{iijj} = \mathbb{S}^{I,\infty}_{iijj} = \frac{1}{1-\nu} = 2s^0_1. \tag{92}$$

7 Conclusions

In this paper, the problem of a 2D plane strain circular inclusion in a finite representative volume element is studied. It has been argued that if a 2D circular inclusion is placed concentrically within a circular RVE, the tensorial circumference basis for the finite Eshelby tensor is the same as that for the infinite Eshelby tensor.

By utilizing this property, we have solved a pair of Fredholm type integral equations, and we have obtained, for the first time, the closed form exact solutions for both the interior and exterior Eshelby tensors for a finite RVE.

By applying the present results to evaluate the effective material properties, the methods employing the finite Eshelby tensor show a remarkable accuracy in simple homogenization procedures.

The dual, traction boundary problem is reported in a sequential paper [18]. Further we have obtained the results for the three-dimensional case. Those results and their application to homogenization are reported in separate papers.

Appendix

Integration formulas

In this Appendix, we document the detailed integration procedures in evaluating the seven integrals listed in Eqs. (38)–(44). According to Fig. 1, we define the vectors

$$\mathbf{n} = \begin{bmatrix} \cos(\varphi + \phi) \\ \sin(\varphi + \phi) \end{bmatrix} \quad \text{and } \mathbf{y} = y n_i \mathbf{e}_i, \tag{93}$$

$$\mathbf{r} = \begin{bmatrix} \cos\varphi \\ \sin\varphi \end{bmatrix} \quad \text{and } \mathbf{x} = x r_i \mathbf{e}_i, \tag{94}$$

$$\ell = \begin{bmatrix} \cos(\varphi + \theta) \\ \sin(\varphi + \theta) \end{bmatrix} \quad \text{and } \mathbf{R} = \mathbf{y} - \mathbf{x} = R\ell_i \mathbf{e}_i. \tag{95}$$

Considering $\mathbf{x} + R\ell = \mathbf{y}$, we have

$$\ell_i = \frac{H_0}{R}(n_i - tr_i), \text{ or } n_i = \frac{R}{H_0}\ell_i + t\,r_i, \tag{96}$$

where

$$t = \frac{|\mathbf{x}|}{H_0}, \tag{97}$$

$$\frac{H_0}{R} = \frac{1}{\sqrt{1 - 2t\cos\phi + t^2}}. \tag{98}$$

Elemental integrals

Instead of integrating along ϕ we pick θ as our integration variable.

From Fig. 5, we obtain the relation

$$dS = \frac{R}{\cos\psi} d\theta. \tag{99}$$

Applying the law of cosine on the triangle **(0xy)** in Fig. 5 one can find that

$$\cos\psi = \sqrt{1 - t^2 \sin^2\theta}. \tag{100}$$

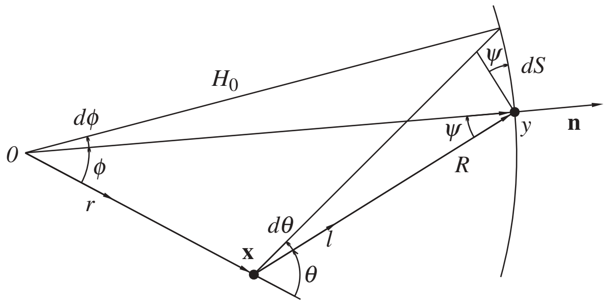

Fig. 5. Relation between dS, $d\phi$ and $d\theta$

Furthermore we express Eq. (98) in terms of θ giving

$$R = H_0\left(-t\cos\theta + \sqrt{1 - t^2\sin^2\theta}\right). \tag{101}$$

Considering an odd function $\mathscr{L}^o$ of ℓ we find by applying Eqs. (99), (100) and (101) that

$$\int_{\partial\Omega} \frac{1}{R}\mathscr{L}^o(\ell)\,dS = \int_0^{2\pi} H_0 \frac{\mathscr{L}^o(\hat{\mathbf{n}})}{\sqrt{1 - t^2\sin^2\theta}}\,d\theta = 0, \tag{102}$$

where $\hat{\mathbf{n}} = [\cos\theta\ \sin\theta]^T$. For an even function $\mathscr{L}^e$ of ℓ it follows that

$$\int_{\partial\Omega} \mathscr{L}^e(\ell)\,dS = \int_0^{2\pi} H_0\left(1 - \frac{t\cos\theta}{\sqrt{1 - t^2\sin^2\theta}}\right)\mathscr{L}^e(\hat{\mathbf{n}})\,d\theta = \int_0^{2\pi} H_0\mathscr{L}^e(\hat{\mathbf{n}})\,d\theta. \tag{103}$$

In both equations above we have used the fact that an odd function of $\hat{\mathbf{n}}$ integrates to zero over the range $[0, 2\pi]$.

We shall break integrals (38)–(44) into the following seven elemental integrals, which are easily integrated using Eqs. (103) and (102). The seven elemental integrals are

$$\text{(i)} \quad \int_0^{2\pi} d\phi = 2\pi, \tag{104}$$

$$\text{(ii)} \quad \int_0^{2\pi} \frac{H_0}{R}\ell_i d\phi = 0, \tag{105}$$

$$\text{(iii)} \quad \int_0^{2\pi} \ell_i\ell_j d\phi = \int_0^{2\pi} \hat{n}_i\hat{n}_j d\theta = \pi\delta_{ij}, \tag{106}$$

$$\text{(iv)} \quad \int_0^{2\pi} \frac{H_0}{R}\ell_i\ell_j\ell_m d\phi = 0, \tag{107}$$

$$\text{(v)} \quad \int_0^{2\pi} \ell_i\ell_j\ell_m\ell_n d\phi = \int_0^{2\pi} \hat{n}_i\hat{n}_j\hat{n}_m\hat{n}_n d\theta = \frac{\pi}{4}(\delta_{ij}\delta_{mn} + \delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}), \tag{108}$$

$$\text{(vi)} \quad \int_0^{2\pi} \frac{H_0}{R}\ell_i\ell_j\ell_m\ell_n\ell_r d\phi = 0, \tag{109}$$

$$\begin{aligned} \text{(vii)} \quad \int_0^{2\pi} \ell_i\ell_j\ell_m\ell_n\ell_r\ell_s d\phi &= \int_0^{2\pi} \hat{n}_i\hat{n}_j\hat{n}_m\hat{n}_n\hat{n}_r\hat{n}_s d\theta \\ &= \frac{\pi}{24}\Big(\delta_{ij}\delta_{mn}\delta_{rs} + \delta_{im}\delta_{jn}\delta_{rs} + \delta_{in}\delta_{jm}\delta_{rs} + \delta_{ir}\delta_{mn}\delta_{js} + \delta_{is}\delta_{mn}\delta_{jr} \\ &\quad \delta_{ij}\delta_{mr}\delta_{ns} + \delta_{im}\delta_{jr}\delta_{ns} + \delta_{in}\delta_{jr}\delta_{ms} + \delta_{ir}\delta_{mj}\delta_{ns} + \delta_{is}\delta_{mj}\delta_{nr} \\ &\quad \delta_{ij}\delta_{ms}\delta_{nr} + \delta_{im}\delta_{js}\delta_{nr} + \delta_{in}\delta_{js}\delta_{mr} + \delta_{ir}\delta_{ms}\delta_{nj} + \delta_{is}\delta_{mr}\delta_{nj}\Big). \end{aligned} \tag{110}$$

Utilizing (96), one can find the following identities:

$$\int_0^{2\pi} \frac{R}{H_0}\ell_i d\phi = \int_0^{2\pi} \frac{R}{H_0}\ell_i\ell_s\ell_s d\phi = -2t\ r_s \int_0^{2\pi} \frac{H_0}{R}\ell_i n_s d\phi = -2t\ r_s \int_0^{2\pi} \ell_i\ell_s d\phi,$$

$$\int_0^{2\pi} \frac{R}{H_0}\ell_i\ell_j\ell_m d\phi = -2t\, r_s \int_0^{2\pi} \ell_i\ell_j\ell_m\ell_s d\phi, \tag{111}$$

$$\int_0^{2\pi} \frac{R}{H_0}\ell_i\ell_j\ell_m\ell_n\ell_r d\phi = -2t\ r_s \int_0^{2\pi} \ell_i\ell_j\ell_m\ell_n\ell_r\ell_s d\phi. \tag{112}$$

The seven integrals

(i) Consider integration (38), we have

$$I_I = \int\limits_{\partial\Omega} \frac{1}{R}\ell_k n_k dS_y = \int_0^{2\pi} \frac{H_0}{R}\ell_k n_k d\phi\ = \int_0^{2\pi} d\phi + t\ r_k \int_0^{2\pi} \frac{H_0}{R}\ell_k d\phi = 2\pi. \tag{113}$$

(ii) Likewise, by using the elemental integrals, we can show that

$$I_{II} = \int_{\partial\Omega} \frac{1}{R}\ell_i n_j dS_y = \int_0^{2\pi} \frac{H_0}{R}\ell_i n_j d\phi\ = \int_0^{2\pi} \ell_i\ell_j d\phi + t\ r_j \int_0^{2\pi} \frac{H_0}{R}\ell_i d\phi = \pi\delta_{ij}. \tag{114}$$

(iii) It can also be readily shown that

$$I_{III} = \int\limits_0^{2\pi} \frac{H_0}{R}\ell_i\ell_j\ell_k n_k d\phi = \int_0^{2\pi} \ell_i\ell_j d\phi + t\ r_k \int_0^{2\pi} \frac{H_0}{R}\ell_i\ell_j\ell_k d\phi = \pi\delta_{ij}. \tag{115}$$

(iv) First break I_{IV} into the following pieces:

$$\begin{aligned} I_{IV} = \int_{\partial\Omega} \frac{1}{R}\ell_k n_k n_m n_n dS_y &= \int_0^{2\pi} \ell_m\ell_n d\phi - t\ r_k \int_0^{2\pi} \ell_m\ell_n n_k d\phi \\ &+ t^2 r_m r_n \int_0^{2\pi} \frac{H_0^2}{R^2} d\phi - t^2 r_k \left(r_m \int_0^{2\pi} n_k\ell_n d\phi + r_n \int_0^{2\pi} \frac{H_0}{R} n_k\ell_m d\phi \right) \\ &- t^3 r_k r_m r_n \int_0^{2\pi} \frac{H_0^2}{R^2} d\phi. \end{aligned} \tag{116}$$

Considering the following identities:

- $$\begin{aligned} t\ r_k \int_0^{2\pi} \ell_m\ell_n n_k d\phi &= t\ r_k \int_0^{2\pi} \frac{R}{H_0}\ell_m\ell_n\ell_k d\phi + t^2 \int_0^{2\pi} \ell_m\ell_n d\phi \\ &= -2t^2 r_k r_s \int_0^{2\pi} \ell_m\ell_n\ell_k\ell_s d\phi + t^2 \int_0^{2\pi} \ell_m\ell_n d\phi, \end{aligned} \tag{117}$$

- $$\int_0^{2\pi} \frac{H_0^2}{R^2} d\phi = \frac{1}{1-t^2} \int_0^{2\pi} d\phi, \tag{118}$$

- $$\int_0^{2\pi} \frac{H_0}{R} n_k \ell_m d\phi = \pi \delta_{km}, \tag{119}$$

- $$\int_0^{2\pi} \frac{H_0^2}{R^2} n_k d\phi = t^4 \int_0^{2\pi} \frac{R_0^2}{H^2} d\phi = \frac{t^4}{1-t^2} \int_0^{2\pi} d\phi, \tag{120}$$

one may obtain

$$\begin{aligned} I_{IV} &= t^2 r_m r_n \int_0^{2\pi} d\phi + (1-t^2) \int_0^{2\pi} \ell_m \ell_n d\phi \\ &= -t^2 r_k \left(r_m \int_0^{2\pi} \ell_k \ell_n d\phi + r_n \int_0^{2\pi} \ell_k \ell_m d\phi \right) + 2t^2 r_k r_s \int_0^{2\pi} \ell_m \ell_n \ell_k \ell_s d\phi \\ &= \frac{\pi}{2}(2-t^2)\delta_{mn} + \pi t^2 r_m r_n. \end{aligned} \tag{121}$$

(v) Integral I_V is shown straightforwardly as

$$\begin{aligned} I_V &= \int_0^{2\pi} \frac{H_0}{R} \ell_i \ell_j (\ell_m n_n + n_m \ell_n) d\phi \\ &= 2 \int_0^{2\pi} \ell_i \ell_j \ell_m \ell_n d\phi + t \left(r_m \int_0^{2\pi} \frac{H_0}{R} \ell_i \ell_j \ell_n d\phi + r_n \int_0^{2\pi} \frac{H_0}{R} \ell_i \ell_j \ell_m d\phi \right) \\ &= \frac{\pi}{2}(\delta_{ij}\delta_{mn} + \delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}). \end{aligned} \tag{122}$$

(vi) Taking (95) for n_i, n_j, n_m and n_n in the sixth integral I_{VI} and removing zero parts gives

$$\begin{aligned} I_{VI} &= \int_0^{2\pi} \frac{R_0}{H} (\ell_i n_j + n_i \ell_j) n_m n_n \, d\phi \\ &= 2 \int_0^{2\pi} \ell_i \ell_j n_m n_n \, d\phi + t \int_0^{2\pi} \frac{R_0}{H} (\ell_i r_j + r_i \ell_j) n_m n_n \, d\phi \\ &= 2 \int_0^{2\pi} \frac{H^2}{R_0^2} \ell_i \ell_j \ell_m \ell_n \, d\phi + 2t \int_0^{2\pi} \frac{H}{R_0} \ell_i \ell_j (\ell_m r_n + r_m \ell_n) \, d\phi \\ &\quad + 2t^2 r_m r_n \int_0^{2\pi} \ell_i \ell_j \, d\phi + t \int_0^{2\pi} \frac{H}{R_0} (\ell_i r_j + r_i \ell_j) \ell_m \ell_n \, d\phi \\ &\quad + t^2 \int_0^{2\pi} (\ell_i r_j + r_i \ell_j)(\ell_m r_n + r_m \ell_n) \, d\phi. \end{aligned} \tag{123}$$

For the individual pieces we write

- $$2\int_0^{2\pi}\frac{H^2}{R_0^2}\ell_i\ell_j\ell_m\ell_n\,d\phi = 2\int_0^{2\pi}\frac{H^2}{R_0^2}\ell_i\ell_j\ell_m\ell_n\ell_s\ell_s\,d\phi$$

$$= 2(1+t^2)\int_0^{2\pi}\ell_i\ell_j\ell_m\ell_n\,d\phi - 4t\int_0^{2\pi}\ell_i\ell_j\ell_m\ell_n n_s r_s\,d\phi$$

$$= 2(1-t^2)\int_0^{2\pi}\ell_i\ell_j\ell_m\ell_n\,d\phi - 4t\int_0^{2\pi}\frac{H}{R_0}\ell_i\ell_j\ell_m\ell_n\ell_s r_s\,d\phi$$

$$= 2(1-t^2)\int_0^{2\pi}\ell_i\ell_j\ell_m\ell_n\,d\phi + 8t^2 r_r r_s\int_0^{2\pi}\frac{R_0}{H}\ell_i\ell_j\ell_m\ell_n\ell_s n_r\,d\phi$$

$$= 2(1-t^2)\int_0^{2\pi}\ell_i\ell_j\ell_m\ell_n\,d\phi + 8t^2 r_r r_s\int_0^{2\pi}\ell_i\ell_j\ell_m\ell_n\ell_r\ell_s\,d\phi, \tag{124}$$

- $$2t\int_0^{2\pi}\frac{H}{R_0}\ell_i\ell_j(\ell_m r_n + r_m\ell_n)\,d\phi = 2t\int_0^{2\pi}\frac{H}{R_0}\ell_i\ell_j(\ell_m r_n + r_m\ell_n)\ell_s\ell_s\,d\phi$$

$$= -4t^2\int_0^{2\pi}\frac{R_0}{H}\ell_i\ell_j(\ell_m r_n + r_m\ell_n)n_s r_s\,d\phi$$

$$= -4t^2\int_0^{2\pi}\ell_i\ell_j(\ell_m r_n + r_m\ell_n)\ell_s r_s\,d\phi$$

$$= -4t^2 r_m r_s\int_0^{2\pi}\ell_i\ell_j\ell_n\ell_s\,d\phi - 4t^2 r_n r_s\int_0^{2\pi}\ell_i\ell_j\ell_m\ell_s\,d\phi, \tag{125}$$

- $$t\int_0^{2\pi}\frac{H}{R_0}(\ell_i r_j + r_i\ell_j)\ell_m\ell_n\,d\phi = -2t^2 r_i r_s\int_0^{2\pi}\ell_j\ell_n\ell_m\ell_s\,d\phi - 2t^2 r_j r_s\int_0^{2\pi}\ell_i\ell_m\ell_n\ell_s\,d\phi, \tag{126}$$

- $$t^2\int_0^{2\pi}(\ell_i r_j + r_i\ell_j)(\ell_m r_n + r_m\ell_n)\,d\phi = t^2 r_i r_m\int_0^{2\pi}\ell_j\ell_n\,d\phi + t^2 r_i r_n\int_0^{2\pi}\ell_j\ell_m\,d\phi$$

$$+ t^2 r_j r_m\int_0^{2\pi}\ell_i\ell_n\,d\phi + t^2 r_j r_n\int_0^{2\pi}\ell_i\ell_m\,d\phi. \tag{127}$$

Thus the sixth integral can be written in the elemental integrals as

$$I_{\mathrm{VI}} = 2t^2 r_m r_n\int_0^{2\pi}\ell_i\ell_j\,d\phi + t^2 r_i r_m\int_0^{2\pi}\ell_j\ell_n\,d\phi + t^2 r_i r_n\int_0^{2\pi}\ell_j\ell_m\,d\phi$$

$$+ t^2 r_j r_m\int_0^{2\pi}\ell_i\ell_n\,d\phi + t^2 r_j r_n\int_0^{2\pi}\ell_i\ell_m\,d\phi$$

$$- 4t^2 r_m r_s\int_0^{2\pi}\ell_i\ell_j\ell_n\ell_s\,d\phi - 4t^2 r_n r_s\int_0^{2\pi}\ell_i\ell_j\ell_m\ell_s\,d\phi$$

$$- 2t^2 r_i r_s\int_0^{2\pi}\ell_j\ell_n\ell_m\ell_s\,d\phi - 2t^2 r_j r_s\int_0^{2\pi}\ell_i\ell_m\ell_n\ell_s\,d\phi$$

$$+ 2(1-t^2)\int_0^{2\pi}\ell_i\ell_j\ell_m\ell_n\,d\phi + 8t^2 r_r r_s\int_0^{2\pi}\ell_i\ell_j\ell_m\ell_n\ell_r\ell_s\,d\phi. \tag{128}$$

We evaluate I_{VI} by using

$$r_r r_s \int_0^{2\pi} \ell_i \ell_j \ell_r \ell_s \, d\phi = \frac{\pi}{4}(\delta_{ij} + 2 r_i r_j),$$

$$r_r r_s \int_0^{2\pi} \ell_i \ell_j \ell_m \ell_n \ell_r \ell_s \, d\phi = \frac{\pi}{24}(\delta_{ij}\delta_{mn} + \delta_{im}\delta_{jn} + \delta_{in}\delta_{jm} + 2\delta_{ij} r_m r_n + 2\delta_{mn} r_i r_j + 2\delta_{im} r_j r_n + 2\delta_{in} r_j r_m + 2\delta_{jm} r_i r_n + 2\delta_{jn} r_i r_n), \tag{129}$$

which gives

$$I_{VI} = (3 - t^2)\frac{\pi}{6}(\delta_{ij}\delta_{mn} + \delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}) + \frac{2\pi t^2}{3}\delta_{ij} r_m r_n - \frac{\pi t^2}{3}\delta_{mn} r_i r_j + \frac{\pi t^2}{6}(\delta_{im} r_j r_n + \delta_{in} r_j r_m + \delta_{jm} r_i r_n + \delta_{jn} r_i r_n). \tag{130}$$

By applying the identity

$$\delta_{im} r_j r_n + \delta_{in} r_j r_m + \delta_{jm} r_i r_n + \delta_{jn} r_i r_m = -2\,\delta_{ij}\delta_{mn} + \delta_{im}\delta_{jn} + \delta_{in}\delta_{jm} + 2\,\delta_{ij} r_m r_n + 2\,\delta_{mn} r_i r_j \tag{131}$$

we finally get:

$$I_{VI} = \frac{\pi}{2}(\delta_{ij}\delta_{mn} + \delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}) - \frac{\pi t^2}{2}\delta_{ij}\delta_{mn} + \pi t^2 \delta_{ij} r_m r_n. \tag{132}$$

(vii) Substituting n_s, n_m and n_n into the final integral I_{VII} gives

$$\begin{aligned} I_{VII} &= \int_0^{2\pi} \frac{R_0}{H} \ell_i \ell_j \ell_s n_s n_m n_n d\phi \\ &= \int_0^{2\pi} \frac{H^2}{R_0^2} \ell_i \ell_j \ell_m \ell_n \, d\phi + t \int_0^{2\pi} \frac{H}{R_0} \ell_i \ell_j (\ell_m r_n + r_m \ell_n) \, d\phi \\ &\quad + t^2 r_m r_n \int_0^{2\pi} \ell_i \ell_j \, d\phi + t \int_0^{2\pi} \frac{H}{R_0} \ell_i \ell_j \ell_m \ell_n \ell_s r_s \, d\phi + t^2 \int_0^{2\pi} \ell_i \ell_j (\ell_m r_n + r_m \ell_n) \ell_s r_s \, d\phi. \end{aligned} \tag{133}$$

The first two parts are given by $\frac{1}{2}$ (124) and $\frac{1}{2}$ (125). For the remaining complicated part we write

$$\begin{aligned} t \int_0^{2\pi} \frac{H}{R_0} \ell_i \ell_j \ell_m \ell_n \ell_s r_s \, d\phi &= t \int_0^{2\pi} \frac{H}{R_0} \ell_i \ell_j \ell_m \ell_n \ell_s r_s \ell_r \ell_r \, d\phi \\ &= -2t^2 r_r r_s \int_0^{2\pi} \frac{R_0}{H} \ell_i \ell_j \ell_m \ell_n \ell_s n_r \, d\phi \\ &= -2t^2 r_r r_s \int_0^{2\pi} \ell_i \ell_j \ell_m \ell_n \ell_r \ell_s \, d\phi. \end{aligned} \tag{134}$$

Thus we break the seventh integral into the elemental pieces as

$$I_{VII} = t^2 r_m r_n \int_0^{2\pi} \ell_i \ell_j \, d\phi + (1 - t^2) \int_0^{2\pi} \ell_i \ell_j \ell_m \ell_n \, d\phi - t^2 r_m r_s \int_0^{2\pi} \ell_i \ell_j \ell_n \ell_s \, d\phi - t^2 r_n r_s \int_0^{2\pi} \ell_i \ell_j \ell_m \ell_s \, d\phi + 2t^2 r_r r_s \int_0^{2\pi} \ell_i \ell_j \ell_m \ell_n \ell_r \ell_s \, d\phi. \tag{135}$$

Evaluation gives

$$I_{VII} = \left(3 - 2t^2\right) \frac{\pi}{12} (\delta_{ij}\delta_{mn} + \delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}) + \frac{2\pi t^2}{3} \delta_{ij} r_m r_n$$
$$+ \frac{\pi t^2}{6} \delta_{mn} r_i r_j - \frac{\pi t^2}{12} (\delta_{im} r_j r_n + \delta_{in} r_j r_m + \delta_{jm} r_i r_n + \delta_{jn} r_i r_n), \tag{136}$$

and by using (131) we prove

$$I_{VII} = (1 - t^2) \frac{\pi}{4} (\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}) + \frac{\pi}{4} \delta_{ij}\delta_{mn} + \frac{\pi t^2}{2} \delta_{ij} r_m r_n. \tag{137}$$

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