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A class of 3-dimensional almost cosymplectic manifolds

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Abstract: The main interest of the present paper is to classify the almost cosymplectic 3-manifolds that satisfy $\|grad\lambda\| =$ const.($\neq 0$) and $\nabla_\xi h = 2ah\phi$.

Key words: Almost cosymplectic manifold, cosymplectic manifold

1. Preliminaries

Let M be an almost contact metric manifold and let (ϕ, ξ, η, g) be its almost contact metric structure. Thus M is a $(2n+1)$-dimensional differentiable manifold and ϕ is a $(1,1)$ tensor field, ξ is a vector field, and η is a 1-form on M, such that

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(X) = g(X, \xi) \tag{1}$$
$$\phi(\xi) = 0, \quad \eta \circ \phi = 0, \tag{2}$$
$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{3}$$

for any vector fields X, Y on M.

The fundamental 2-form Φ of an almost contact metric manifold (M, ϕ, ξ, η, g) is defined by

$$\Phi(X, Y) = g(X, \phi Y), \tag{4}$$

for any vector fields X, Y on M, and this form satisfies $\eta \wedge \Phi^n \neq 0$. M is said to be almost cosymplectic if the forms η and Φ are closed, that is, $d\eta = 0$ and $d\Phi = 0$.

The theory of an almost cosymplectic manifold was introduced by Goldberg and Yano in [9]. The products of almost Kaehler manifolds and the real $\mathbb{R}$ line or the circle S^1 are the simplest examples of almost cosymplectic manifolds. Topological and geometrical properties of almost cosymplectic manifolds have been studied by many mathematicians (see [4], [11], [5], [9], [15], and [18]).

For M, define $(1,1)$-tensor fields $\tilde{A}$ and h by ([7],[8],[15],[16])

$$\tilde{A}\, X = -\nabla_X \xi, \tag{5}$$

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$$h = \frac{1}{2}\mathcal{L}_{\xi}\phi, \tag{6}$$

where $\mathcal{L}$ indicates the Lie differentiation operator and $\bigtriangledown$ is the Levi–Civita connection determined by g. The tensors $\tilde{A}$ and h are related by

$$h = \tilde{A}\ \phi, \qquad \tilde{A}\ = \phi h. \tag{7}$$

The main algebraic properties of $\tilde{A}$ and h are the following:

$$g(\tilde{A}\ X, Y) = g(\tilde{A}\ Y, X), \quad \tilde{A}\ \phi + \phi\tilde{A}\ = 0, \quad \tilde{A}\ \xi = 0, \quad \eta \circ \tilde{A}\ = 0,$$

$$g(hX, Y) = g(hY, X), \quad h\phi + \phi h = 0, \quad h\tilde{A}\ + \tilde{A}\ h = 0, \quad h\xi = 0, \quad \eta \circ h = 0\ .$$

The curvature tensor R of M is given by $R(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z$ and the Ricci tensor Ric of M are defined by $Ric(X,Y) = TrX \to R(X,Y)Z$ for any vector field X, Y and Z.

In [6], Dacko and Olszak proved the existence of a new class of almost cosymplectic manifolds, which is called (κ, μ, υ)-spaces. This means that the curvature tensor R satisfies the condition

$$\begin{aligned} R(X,Y)\xi &= \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY) \\ &\quad + \upsilon(\eta(Y)\phi hX - \eta(X)\phi hY), \end{aligned} \tag{8}$$

where κ, μ, υ are smooth functions. Contact metric manifolds fulfilling Eq. (8) were investigated in [2], [1], [3], and [12].

This work was inspired by [14] and [13]. We carry on those studies to the 3-dimensional almost cosymplectic manifolds in this paper. The purpose of the present paper is to give a new local classification of 3-dimensional almost cosymplectic manifolds under some conditions. The paper is organized in the following way. Section 2 is devoted to some lemmas related to 3-dimensional almost cosymplectic manifolds for later use. In Section 3, we give our main theorem.

All manifolds considered in this paper are assumed to be connected and of class C^{∞}.

2. Three-dimensional almost cosymplectic manifolds

Now we shall give some essential Lemmas and notations.

Lemma 2.1 *[10] Let M be a smooth manifold $f: M \to \mathbb{R}$ be a smooth real function. Let V_1 and V_2 be open sets of M defined by*

$$\begin{aligned} V_1 &= \{m \in M \mid f(m) \neq 0 \textit{ in a neighborhood of } m\}, \\ V_2 &= \{m \in M \mid f(m) = 0 \textit{ in a neighborhood of } m\}. \end{aligned}$$

Then $V_1 \cup\ V_2$ is open and dense in M.

Let (M, ϕ, ξ, η, g) be an almost cosymplectic 3-manifold. Let

$$\begin{aligned} U &= \{p \in M \mid h(p) \neq 0 \textit{ in a neighborhood of } p\} \subset M, \\ U_0 &= \{p \in M \mid h(p) = 0 \textit{ in a neighborhood of } p\} \subset M \end{aligned}$$

be open sets of M. Using Lemma 2.1, we can say that $U \cup U_0$ is an open and dense subset of M, and so any property satisfied in $U_0 \cup U$ is also satisfied in M. For any point $p \in U \cup U_0$, there exists a local orthonormal basis $\{e, \phi e, \xi\}$ of smooth eigenvectors of h in a neighborhood of p (this we call a ϕ-basis).

On U, we put $he = \lambda e$, $h\phi e = -\lambda \phi e$, where λ is a nonvanishing smooth function assumed to be positive.

Lemma 2.2 *[17] On the open set U we have*

$$\begin{aligned} \nabla_\xi e &= -a\phi e, \qquad \nabla_e e = b\phi e, \qquad \nabla_{\phi e} e = -c\phi e + \lambda\xi, && (9) \\ \nabla_\xi \phi e &= ae, \qquad \nabla_e \phi e = -be + \lambda\xi, \qquad \nabla_{\phi e}\phi e = ce, && (10) \\ \nabla_\xi \xi &= 0, \qquad \nabla_e \xi = -\lambda\phi e, \qquad \nabla_{\phi e}\xi = -\lambda e, && (11) \\ \nabla_\xi h &= 2ah\phi + \xi(\lambda)s, && (12) \end{aligned}$$

where a is a smooth function,

$$\begin{aligned} b &= \frac{1}{2\lambda}((\phi e)(\lambda) + A) \quad \textit{with} \quad A = \sigma(e) = Ric(e, \xi), && (13) \\ c &= \frac{1}{2\lambda}(e(\lambda) + B) \quad \textit{with} \quad B = \sigma(\phi e) = Ric(\phi e, \xi), && (14) \end{aligned}$$

and s is the type $(1,1)$ tensor field defined by $s\xi = 0$, $se = e$, and $s\phi e = -\phi e$, and Ric is Ricci tensor field.

By Lemma 2.2, we can prove that

$$\begin{aligned} [e, \phi e] &= \nabla_e \phi e - \nabla_{\phi e} e = -be + c\phi e, && (15) \\ [e, \xi] &= \nabla_e \xi - \nabla_\xi e = (a - \lambda)\phi e, && (16) \\ [\phi e, \xi] &= \nabla_{\phi e}\xi - \nabla_\xi \phi e = -(a + \lambda)e. && (17) \end{aligned}$$

If we adapt Theorem 7 of [17] to a 3-dimensional almost cosymplectic manifolds, we get the following:

Lemma 2.3 *[17] Let (M, ϕ, ξ, η, g) be a 3-dimensional almost cosymplectic manifold. If $\sigma \equiv 0$, then the (κ, μ, ν)-structure always exists on every open and dense subset of M. This means that the Riemannian curvature tensor R of M satisfies*

$$\begin{aligned} R(X, Y)\xi = &-\lambda^2(\eta(Y)X - \eta(X)Y) \\ &+2a(\eta(Y)hX - \eta(X)hY) \\ &+\frac{\xi(\lambda)}{\lambda}(\eta(Y)\phi hX - \eta(X)\phi hY), \end{aligned}$$

for all vector fields X and Y on M.

3. Main theorem and proof

In this section, we will give our main theorem and prove it.

Theorem 3.1 *(Main theorem) Let $M(\phi, \xi, \eta, g)$ be a 3-dimensional almost cosymplectic manifold with $\|grad\ \lambda\| = 1$ and $\nabla_\xi h = 2ah\phi$. Then at any point $p \in M$ there exists a chart $(U, (x, y, z))$ such that $\lambda = f(z) \neq 0$ and*

$A=0,\ B=F(y,z)$ or $A=F(y,z),\ B=0$. In the first case $(A=Ric(e,\xi)=0,\ B=Ric(\phi e,\xi)=F(y,z))$, the following are valid:

$$\xi=\frac{\partial}{\partial x},\quad \phi e=\frac{\partial}{\partial y}\quad \text{and } e=k_1\frac{\partial}{\partial x}+k_2\frac{\partial}{\partial y}+k_3\frac{\partial}{\partial z},\quad k_3\neq 0.$$

In the second case $(A=Ric(e,\xi)=F(y,z),\ B=Ric(\phi e,\xi)=0)$, the following are valid:

$$\xi=\frac{\partial}{\partial x}, e=\frac{\partial}{\partial y}\ \text{ and }\ \phi e=k_1'\frac{\partial}{\partial x}+k_2'\frac{\partial}{\partial y}+k_3'\frac{\partial}{\partial z}\ ,\quad k_3'\neq 0,$$

where

$$k_1(x,y,z)=r(z)=k_1'(x,y,z),$$

$$k_2(x,y,z)=k_2'(x,y,z)=2xf(z)-\frac{(H(y,z)+y)}{2f(z)}+\beta(z),$$

$$k_3(x,y,z)=k_3'(x,y,z)=t(z)+\delta,\qquad \frac{\partial H(y,z)}{\partial y}=F(y,z),$$

and r,β are smooth functions of z and δ is constant. Furthermore, $f(z)=\int\frac{1}{k_3(z)}dz$.

Proof. By virtue of Lemma 2.2, it can be easily proven that the assumption $\nabla_\xi h=2ah\phi$ is equivalent to $\xi(\lambda)=0$. From the definition of a gradient of a differentiable function, we get

$$\begin{aligned} grad\lambda &= e(\lambda)e+(\phi e)(\lambda)\phi e+\xi(\lambda)\xi \\ &= e(\lambda)e+(\phi e)(\lambda)\phi e. \end{aligned} \tag{18}$$

Using Eq. (18) and $\|grad\ \lambda\|=1$ we have

$$(e(\lambda))^2+((\phi e)(\lambda))^2=1. \tag{19}$$

Differentiating (19) with respect to ξ and using Eqs. (16) and (17) and $\xi(\lambda)=0$, we obtain

$$\begin{aligned} \xi(e(\lambda))e(\lambda)+\xi((\phi e)(\lambda))(\phi e)(\lambda) &= 0 \\ ([\xi,e](\lambda))\,e(\lambda)+([\xi,\phi e](\lambda))\,(\phi e)\lambda &= 0 \\ \lambda e(\lambda)(\phi e)(\lambda) &= 0 \end{aligned}$$

and since $\lambda\neq 0$,

$$e(\lambda)(\phi e)(\lambda)=0. \tag{20}$$

To study this system, we consider the open subsets of U:

$$\begin{aligned} U' &= \{p\in U\mid e(\lambda)(p)\neq 0,\ \textit{in a neighborhood of } p\}, \\ U'' &= \{p\in U\mid (\phi e)(\lambda)p\neq 0, \textit{in a neighborhood of } p\}. \end{aligned}$$

From Lemma 2.1 we have that $U'\cup U''$ is open and dense in the closure of U. We distinguish 2 cases.

Case 1: We suppose that $p \in U'$. By virtue of Eqs. (19) and (20), we have $(\phi e)(\lambda) = 0$, and $e(\lambda) = \mp 1$. Changing to the basis $(\xi, -e, -\phi e)$ if necessary, we can assume that $e(\lambda) = 1$. The Eqs. (15), (16), (17), and (13), Eq. (14) reduces to

$$[e, \phi e] = -be + c\phi e \tag{21}$$

$$[e, \xi] = -2\lambda \phi e \tag{22}$$

$$[\phi e, \xi] = 0, \qquad \lambda = -a \tag{23}$$

$$b = \frac{A}{2\lambda}, \quad c = \frac{B+1}{2\lambda}, \quad a = -\lambda, \tag{24}$$

respectively.

Since $[\phi e, \xi] = 0$, the distribution that is spanned by ϕe and ξ is integrable, and so for any $p \in U'$ there exists a chart $\{V,\ (x, y, z)\}$ at p, such that

$$\xi = \frac{\partial}{\partial x}, \quad \phi e = \frac{\partial}{\partial y}, \quad e = k_1 \frac{\partial}{\partial x} + k_2 \frac{\partial}{\partial y} + k_3 \frac{\partial}{\partial z} \tag{25}$$

where k_1, k_2, k_3 are smooth functions on V. Since ξ, e, ϕe are linearly independent we have $k_3 \neq 0$ at any point of V.

Using Eqs. (21), (22) and (25), we get the following partial differential equations:

$$\frac{\partial k_1}{\partial y} = \frac{A}{2\lambda} k_1, \quad \frac{\partial k_2}{\partial y} = \frac{1}{2\lambda}\left[A k_2 - B - 1\right], \quad \frac{\partial k_3}{\partial y} = \frac{A}{2\lambda} k_3, \tag{26}$$

$$\frac{\partial k_1}{\partial x} = 0, \quad \frac{\partial k_2}{\partial x} = 2\lambda, \quad \frac{\partial k_3}{\partial x} = 0. \tag{27}$$

Moreover, we know that

$$\frac{\partial \lambda}{\partial x} = 0, \quad \frac{\partial \lambda}{\partial y} = 0. \tag{28}$$

Differentiating the equation $\frac{\partial k_3}{\partial x} = 0$ with respect to $\frac{\partial}{\partial y}$, and using $\frac{\partial k_3}{\partial y} = \frac{A}{2\lambda} k_3$, we find

$$0 = \frac{\partial^2 k_3}{\partial y \partial x} = \frac{\partial^2 k_3}{\partial x \partial y} = \frac{1}{2\lambda} \frac{\partial A}{\partial x} k_3 + \frac{1}{2\lambda} A \frac{\partial k_3}{\partial x} = \frac{1}{2\lambda} \frac{\partial A}{\partial x} k_3.$$

So,

$$\frac{\partial A}{\partial x} = 0. \tag{29}$$

Differentiating $\frac{\partial k_2}{\partial x} = 2\lambda$ with respect to $\frac{\partial}{\partial y}$, and using $\frac{\partial k_2}{\partial y} = \frac{1}{2\lambda}\left[A k_2 - B - 1\right]$ and Eq. (29), we prove that

$$\frac{\partial^2 k_2}{\partial y \partial x} = 0 = \frac{\partial^2 k_2}{\partial x \partial y} = \frac{1}{2\lambda}\left[\frac{\partial A}{\partial x} k_2 + A \frac{\partial k_2}{\partial x} - \frac{\partial B}{\partial x}\right].$$

So,

$$\frac{\partial B}{\partial x} = 2\lambda A. \tag{30}$$

From Eq. (28) we have the following solution:

$$\lambda(z) = f(z) + d, \tag{31}$$

where d is constant. For the sake of shortness, we will use $\tilde{f}(z)$ instead of $f(z) + d$. Using $e(\lambda) = k_1 \frac{\partial \lambda}{\partial x} + k_2 \frac{\partial \lambda}{\partial y} + k_3 \frac{\partial \lambda}{\partial z} = 1$ and Eq. (28), we get

$$\frac{\partial \lambda}{\partial z} = \frac{1}{k_3}, \quad k_3 \neq 0. \tag{32}$$

If we differentiate Eq. (32) with respect to $\frac{\partial}{\partial y}$ because of the equation $\frac{\partial \lambda}{\partial y} = 0,$ we obtain

$$0 = \frac{\partial^2 \lambda}{\partial z \partial y} = \frac{\partial^2 \lambda}{\partial y \partial z} = -\frac{1}{k_3^2} \frac{\partial k_3}{\partial y}. \tag{33}$$

Since $k_3 \neq 0,$ Eq. (33) reduces and then we obtain

$$\frac{\partial k_3}{\partial y} = 0. \tag{34}$$

Combining Eqs. (26) and (34), we deduced that

$$A = 0. \tag{35}$$

Using Eqs. (30) and (35), we have

$$\frac{\partial B}{\partial x} = 0. \tag{36}$$

It follows from Eq. (36) that

$$B = F(y, z). \tag{37}$$

By virtue of Eqs. (35) , (26), and (27), we easily see that

$$k_1 = r(z), \tag{38}$$

where $r(z)$ is an integration function.

Combining Eqs. (27) and (34), we get

$$k_3 = t(z) + \delta, \tag{39}$$

where δ is constant.

If we use Eqs. (27), (31), (35), and (37) in Eq. (26),

$$\frac{\partial k_2}{\partial x} = 2\tilde{f}(z), \quad \frac{\partial k_2}{\partial y} = \frac{-(B+1)}{2\lambda} = \frac{-(F(y,z)+1)}{2\check{f}(z)}. \tag{40}$$

It follows from this last partial differential equation that

$$k_2 = 2x\tilde{f}(z) - \frac{(H(y,z)+y)}{2\check{f}(z)} + \beta(z), \tag{41}$$

where

$$\frac{\partial H(y,z)}{\partial y} = F(y,z). \tag{42}$$

Because of Eq. (32), there is a relation between $\lambda(z) = \tilde{f}(z)$ and $k_3(z)$ such that $\tilde{f}(z) = \int \frac{1}{k_3(z)} dz$. We will calculate the tensor fields η, ϕ, g with respect to the basis $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$. For the components g_{ij} of the Riemannian metric g, using Eq. (25) we have

$$\begin{aligned} g_{11} &= 1, \quad g_{22} = 1, \quad g_{12} = g_{21} = 0, \quad g_{13} = g_{31} = \frac{-k_1}{k_3}, \\ g_{23} &= g_{32} = \frac{-k_2}{k_3}, \qquad g_{33} = \frac{1 + k_1^2 + k_2^2}{k_3^2}. \end{aligned}$$

The components of the tensor field ϕ are immediate consequences of

$$\begin{aligned} \phi(\xi) &= \phi(\frac{\partial}{\partial x}) = 0, \quad \phi(\frac{\partial}{\partial y}) = -k_1\frac{\partial}{\partial x} - k_2\frac{\partial}{\partial y} - k_3\frac{\partial}{\partial z}\ , \\ \phi(\frac{\partial}{\partial z}) &= \frac{k_1 k_2}{k_3}\frac{\partial}{\partial x} + \frac{1+k_2^2}{k_3}\frac{\partial}{\partial y} + k_2\frac{\partial}{\partial z}. \end{aligned}$$

The expression of the 1-form η immediately follows from $\eta(\xi) = 1, \eta(e) = \eta(\phi e) = 0$.

$$\eta = dx - \frac{k_1}{k_3} dz.$$

Now we calculate the components of tensor field h with respect to the basis $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$.

$$\begin{aligned} h(\xi) &= h(\frac{\partial}{\partial x}) = 0, \quad h(\frac{\partial}{\partial y}) = -\lambda\frac{\partial}{\partial y}, \\ h(\frac{\partial}{\partial z}) &= \lambda\frac{k_1}{k_3}\frac{\partial}{\partial x} + 2\lambda\frac{k_2}{k_3}\frac{\partial}{\partial y} + \lambda\frac{\partial}{\partial z}. \end{aligned}$$

Case 2: Now we suppose that $p \in U^{''}$. As in Case 1, we can assume that $(\phi e)(\lambda) = 1$. The Eqs. (15), (16) ,(17), and (13), Eq. (14) reduces to

$$\begin{aligned} [e, \phi e] &= -be + c\phi e, && (43) \\ [e, \xi] &= 0, && (44) \\ [\phi e, \xi] &= -2\lambda e, && (45) \end{aligned}$$

$$b = \frac{A+1}{2\lambda}, \qquad c = \frac{B}{2\lambda}, \qquad a = \lambda, \tag{46}$$

respectively. Because of Eq. (44), we find that there exists a chart $\{V', (x,y,z)\}$ at $p \in U^{''}$ such that

$$\xi = \frac{\partial}{\partial x}, \quad \phi e = k_1'\frac{\partial}{\partial x} + k_2'\frac{\partial}{\partial y} + k_3'\frac{\partial}{\partial z}, \quad e = \frac{\partial}{\partial y}, \tag{47}$$

where k_1', k_2', and k_3' $(k_3' \neq 0)$, are smooth functions on V'.

Using Eqs.(43), (45), and (47), we get the following partial differential equations:

$$\frac{\partial k_1'}{\partial y} = \frac{B}{2\lambda}k_1', \quad \frac{\partial k_2'}{\partial y} = \frac{1}{2\lambda}\left[Bk_2' - A - 1\right], \quad \frac{\partial k_3'}{\partial y} = \frac{B}{2\lambda}k_3', \tag{48}$$

$$\frac{\partial k_1'}{\partial x} = 0, \quad \frac{\partial k_2'}{\partial x} = 2\lambda, \quad \frac{\partial k_3'}{\partial x} = 0.$$

Moreover, we know that

$$\frac{\partial \lambda}{\partial x} = 0, \quad \frac{\partial \lambda}{\partial y} = 0. \tag{49}$$

As in Case 1, if we solve the partial differential equations Eq. (48) and Eq. (49), then we find

$$B = 0, \quad A = F'(y,z) \tag{50}$$

$$\lambda(z) = f'(z) + d' = \tilde{f}'(z), \quad k_1' = r'(z), \quad k_3' = t'(z) + \delta' \tag{51}$$

$$k_2' = 2x\tilde{f}'(z) - \frac{(H'(y,z) + y)}{2f(z)} + \beta'(z) \tag{52}$$

$$\frac{\partial H'(y,z)}{\partial y} = F'(y,z) \tag{53}$$

where d' and δ' are constants.

By the help of Eq. (51), the equation $(\phi e)(\lambda) = 1$ implies

$$\lambda(z) = \tilde{f}'(z) = \int \frac{1}{k_3'(z)} dz.$$

As in Case1, we can directly calculate the tensor fields g, ϕ, η, and h with respect to the basis $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$.

$$g = \begin{pmatrix} 1 & 0 & -\frac{k_1'}{k_3'} \\ 0 & 1 & -\frac{k_2'}{k_3'} \\ -\frac{k_1'}{k_3'} & -\frac{k_2'}{k_3'} & \frac{1+k_1'^2+k_2'^2}{k_3'^2} \end{pmatrix}, \quad \phi = \begin{pmatrix} 0 & k_1' & -\frac{k_1'k_2'}{k_3'} \\ 0 & k_2' & -\frac{1+k_2'^2}{k_3'} \\ 0 & k_3' & -k_2' \end{pmatrix},$$

$$\eta = dx - \frac{k_1'}{k_3'}dz \quad \text{and} \quad h = \begin{pmatrix} 0 & 0 & -\lambda\frac{k_1'}{k_3'} \\ 0 & \lambda & -2\lambda\frac{k_2'}{k_3'} \\ 0 & 0 & -\lambda \end{pmatrix}$$

□

Example 3.2

$$M = \{(x,y,z) \in R^3, z \neq 0\}$$

and the vector fields

$$\xi = \frac{\partial}{\partial x}, \quad e = \frac{\partial}{\partial y}, \quad \phi e = z\frac{\partial}{\partial x} + (2xz - 1)\frac{\partial}{\partial y} + \frac{\partial}{\partial z}.$$

The 1-form $\eta = dx - zdz$ *is closed and the characteristic vector field is* $\xi = \frac{\partial}{\partial x}$. *Let* g, ϕ *be the Riemannian metric and the* $(1,1)$*-tensor field given by*

$$\begin{aligned} g &= \begin{pmatrix} 1 & 0 & -a_1 \\ 0 & 1 & a_2 \\ -a_1 & a_2 & 1+a_1^2+(a_2)^2 \end{pmatrix}, \ \phi = \begin{pmatrix} 0 & a_1 & a_1a_2 \\ 0 & -a_2 & -(1+a_2^2) \\ 0 & 1 & a_2 \end{pmatrix}, \\ h &= \begin{pmatrix} 0 & 0 & -\lambda a_1 \\ 0 & \lambda & 2\lambda a_2 \\ 0 & 0 & -\lambda \end{pmatrix}, \quad \lambda = z, \end{aligned}$$

with respect to the basis $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$, *where* $a_1 = z$ *and* $a_2 = 1 - 2xz$.

$$\begin{aligned} \eta &= dx - zdz \ , \ d\eta = 0, \\ \Phi &= -dy \wedge dz \ , \quad d\Phi = 0. \end{aligned}$$

By a straightforward calculation, we obtain

$$\nabla_\xi h = 2zh\phi, F(y,z) = -1, \|grad\ \lambda\| = 1.$$

Remark 3.3 *Let* $M(\phi,\xi,\eta,g)$ *be an almost cosymplectic manifold. A* D_α*-homothetic transformation [19] is the transformation*

$$\bar{\eta} = \alpha\eta, \quad \bar{\xi} = \frac{1}{\alpha}\xi, \quad \bar{\phi} = \phi, \quad \bar{g} = \alpha g + \alpha(\alpha-1)\eta\otimes\eta \tag{54}$$

of the structure tensors, where α *is a positive constant. It is well known [19] that* $M(\bar{\phi},\bar{\xi},\bar{\eta},\bar{g})$ *is also an almost cosymplectic manifold. When 2 contact structures* (ϕ,ξ,η,g) *and* $(\bar{\phi},\bar{\xi},\bar{\eta},\bar{g})$ *are related by Eq. (54), we will say that they are* D_α*-homothetic.* *We can easily show that* $\bar{h} = \frac{1}{\alpha}h$ *so* $\bar{\lambda} = \frac{1}{\alpha}\lambda$.

(a) As a result, an almost cosymplectic manifold with $\|grad\ \lambda\|_g = d \neq 0$ *(const.) is* D_α*-deformed in another almost cosymplectic manifold with* $\|grad\bar{\lambda}\|_{\bar{g}} = d\alpha^{-\frac{3}{2}}$ *and choosing* $\alpha = d^{\frac{2}{3}}$, *it is enough to study those almost cosymplectic manifolds with* $\|grad\ \lambda\| = 1$.

(b) If $d = 0$, *then* λ *is constant. As a result, if* $\lambda = 0$, *then* M *is a cosymplectic manifold.*

Remark 3.4 *There are no compact 3-dimensional almost cosymplectic manifolds with* $\|grad\ \lambda\| = const \neq 0$. *In fact, if such a manifold is compact, then the smooth function* λ *will attain a maximum value at some point* p *of M. Then grad* λ *vanishes at* p, *contrary to the requirement that grad* λ *is a nonzero constant.*

Remark 3.5 *Using Theorem 3.1, we can produce infinitely many possible examples about 3-dimensional almost cosymplectic manifolds. If we add the condition* $F(y,z) = 0$ *to Theorem 3.1, we have* $A = 0$ *and* $B = 0$. *Thus, by Lemma 2.3, we can state that a 3-dimensional almost cosymplectic manifold under the same conditions of Theorem 3.1 is a 3-dimensional almost cosymplectic* (κ,μ) *manifold.*

Now we will give an example satisfying Remark 3.5.

Example 3.6 *We consider the 3-dimensional manifold*

$$M = \{(x, y, z) \in R^3 \mid z > 0\}$$

and the vector fields

$$\xi = \frac{\partial}{\partial x}, \quad \phi e = \frac{\partial}{\partial y}, \quad e = z^2 \frac{\partial}{\partial x} + (2xz - \frac{z+y}{2z})\frac{\partial}{\partial y} + \frac{\partial}{\partial z}.$$

The 1*-form* $\eta = dx - z^2 dz$ *is closed and the characteristic vector field is* $\xi = \frac{\partial}{\partial x}$. *Let* g, ϕ *be the Riemannian metric and the* $(1,1)$*-tensor field given by*

$$g = \begin{pmatrix} 1 & 0 & -\frac{a_1}{a_3} \\ 0 & 1 & -\frac{a_2}{a_3} \\ -\frac{a_1}{a_3} & -\frac{a_2}{a_3} & \frac{1+a_1^2+a_2^2}{a_3^2} \end{pmatrix}, \quad \phi = \begin{pmatrix} 0 & -a_1 & \frac{a_1 a_2}{a_3} \\ 0 & -a_2 & \frac{1+a_2^2}{a_3} \\ 0 & -a_3 & a_2 \end{pmatrix},$$

$$\eta = dx - \frac{a_1}{a_3} dz, \quad \text{and} \quad h = \begin{pmatrix} 0 & 0 & \lambda\frac{a_1}{a_3} \\ 0 & -\lambda & 2\lambda\frac{a_2}{a_3} \\ 0 & 0 & \lambda \end{pmatrix}$$

with respect to the basis $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$, *where* $a_1 = z^2$, $a_2 = 2xz - \frac{z+y}{2z}$, $a_3 = 1$, $\lambda = z$.

$$\begin{aligned} \eta &= dx - z^2 dz, \quad d\eta = 0, \\ \Phi &= dy \wedge dz, \quad d\Phi = 0. \end{aligned}$$

By direct computations, we get

$$\|grad\ \lambda\| = 1, \nabla_\xi h = -2zh\phi\ , F(y,z) = 0$$

and

$$R(X,Y)\xi = (-z^2)(\eta(Y)X - \eta(X)Y) - 2z(\eta(Y)hX - \eta(X)hY)$$

for any vector field X, Y *on* M.

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