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A CLASS OF ALMOST CONTACT RIEMANNIAN MANIFOLDS

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1. Introduction. Recently S. Tanno has classified connected almost contact Riemannian manifolds whose automorphism groups have the maximum dimension [9]. In his classification table the almost contact Riemannian manifolds are divided into three classes: (1) homogeneous normal contact Riemannian manifolds with constant ϕ-holomorphic sectional curvature if the sectional curvature for 2-planes which contain ξ, say $K(X, \xi), > 0$, (2) global Riemannian products of a line or a circle and a Kaehlerian manifold with constant holomorphic sectional curvature, if $K(X, \xi) = 0$ and (3) a warped product space $L \times_f CE^n$, if $K(X, \xi) < 0$. It is known that the manifold of the class (1) in the above statement is characterized by some tensor equations; it has a Sasakian structure.

The purpose of this paper is to characterize the warped product space $L \times_f CE^n$ by tensor equations (§ 2) and study their properties. From the definition by means of the tensor equations it is easily verified that the structure is normal, but not quasi-Sasakian (and is hence not Sasakian). In § 2, we define a structure closely related to the warped product which is studied by Bishop-O'Neill [1] and prove the local structure theorem. In § 3 we study some properties of the structure. § 4 is devoted to a study of η-Einstein manifolds. In the section 5 we show one of the main theorems in this paper. In the last section we study invariant submanifolds.

We follow here the notations and the terminology of the Volume 1 of Kobayashi-Nomizu [4].

2. Definition and examples. It is well-known that the structure tensors (ϕ, ξ, η, g) of the almost contact Riemannian maifold M satisfy

$$\phi\xi = 0 , \qquad \eta(\phi X) = 0 , \qquad \eta(\xi) = 1 , \tag{2.1}$$

$$\phi\phi X = -X + \eta(X)\xi , \qquad g(X, \xi) = \eta(X) , \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) , \tag{2.3}$$

for any vector fields X and Y on M. It is known that the (ϕ, ξ, η, g)-structure is normal if and only if

$$\phi\nabla_X\phi \cdot Y - \nabla_{\phi X}\phi \cdot Y - (\nabla_X\eta)(Y) \cdot \xi = 0\,, \tag{2.4}$$

where ∇ denotes the Riemannian connection for g [8].

Throughout this paper we study a class of almost contact Riemannian manifolds which satisfy the following two conditions, say (*),:

$$\begin{cases} \nabla_X\phi \cdot Y = -\eta(Y)\phi X - g(X, \phi Y)\,\xi\,, \\ \quad\nabla_X\xi = X - \eta(X)\,\xi\,. \end{cases} \tag{*}$$

REMARK TO (*). S. S. Eum studied the integrability of invariant hypersurfaces immersed in an almost contact Riemannian manifold which satisfies

$$g(\nabla_X\phi \cdot Y, Z) = (\nabla_X\eta)(\eta(Y)\phi Z - \eta(Z)\phi Y)\,. \tag{2.5}$$

If we assume $(*)_2$ in an almost contact Riemannian manifold, then $(*)_1$ is equivalent to (2.5).

From (2.4), (ϕ, ξ, η, g)-structure with (*) is normal and since ξ is not a Killing vector field the structure is not quasi-Sasakian (cf. [2]). Thus we have

PROPOSITION 1. *Let M be an almost contact Riemannian manifold with* (*). *Then M is normal but not quasi-Sasakian and hence not Sasakian.*

Taking the Lie derivative of g, ϕ and η along ξ we see

PROPOSITION 2. *Under the same assumption as Proposition* 1,

$$(\nabla_X\eta)(Y) = g(X, Y) - \eta(X) \cdot \eta(Y)\,, \tag{2.6}$$

$$L(\xi)g = 2(g - \eta \otimes \eta)\,, \tag{2.7}$$

$$L(\xi)\phi = 0\,, \tag{2.8}$$

$$L(\xi)\eta = 0\,, \tag{2.9}$$

where $L(\xi)$ denotes the Lie derivative along ξ.

Since the proof of Proposition 2 follows by a routine calculation, we shall omit it. We give here examples of almost contact Riemannian manifolds which satisfy the condition (*). These examples are closely related to the warped product space defined by Bishop-O'Neill [1]: Let B and F be Riemannian manifolds and $f > 0$ a differentiable function on B. Consider the product manifold $B \times F$ with its projection $p\colon B \times F \to B$ and $\pi\colon B \times F \to F$. The warped product $M = B \times_f F$ is the manifold $B \times F$ furnished with the Riemannian structure such that

$$\| X \|^2 = \| p_* X \|^2 + f^2(px) \| \pi_* X \|^2$$

for every tangent vector $X \in T_x(M)$. We have

PROPOSITION 3. *Let F be a Kaehlerian manifold and c a nonzero constant. Let $f(t) = ce^t$ be a function on a line L. Then the warped product space $M = L \times_f F$ have an almost contact metric structure which satisfies* (*).

PROOF. (G, J) denotes the Kaehlerian structure of F and D denotes the Riemannian connection for the Kaehlerian metric G. Let $(t, x_1, \cdots, x_{2n})$ be a local coordinates of M where t and $(x_1, \cdots, x_{2n})$ denotes the local coordinates of L and F, respectively. We define a Riemannian metric tensor g, a vector field ξ and a 1-form η on M as follows:

$$g_{(t,x)} = \begin{pmatrix} 1 & 0 \\ 0 & f^2(t) G_{(x)} \end{pmatrix}, \tag{2.10}$$

$$\xi = \left(\frac{d}{dt} \right), \quad \eta(X) = g(X, \xi) . \tag{2.11}$$

By a direct calculation or Lemma 7.3 of [1] we have easily $(*)_2$ because of $\xi(f) = f$. By $(*)_2$ we see

$$L(\xi)\eta = 0 . \tag{2.12}$$

A (1, 1)-tensor field ϕ is defined ϕ by $\phi_{(t,x)} = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{\phi}_{(t,x)} \end{pmatrix}$, where

$$\tilde{\phi}_{(t,x)} = (\exp(t\xi))_* J_x (\exp(-t\xi))_* . \tag{2.13}$$

(2.13) is well-defined by (2.12). We can easily verify that (ϕ, ξ, η, g) defines an almost contact metric structure on $M = L \times_f F$ by (2.10) $\sim$ (2.13). By (2.13) we see

$$(\exp s\xi)_* \phi = \phi (\exp s\xi)_* . \tag{2.14}$$

Making use of $(*)_2$ and (2.14), we have easily

$$L(\xi) g = 2 (g - \eta \otimes \eta) , \tag{2.15}$$

$$L(\xi)\phi = 0 . \tag{2.16}$$

By virtue of $(*)_2$ and (2.16) we have

$$\nabla_\xi \phi = L(\xi)\phi = 0 . \tag{2.17}$$

By (2.10), we have

$$\nabla_{X_0} Y_0 = D_{X_0} Y_0 - g(X_0, Y_0)\xi , \tag{2.18}$$

where X_0 and Y_0 are vector fields with $\eta(X_0) = 0$ and $\eta(Y_0) = 0$, respectively. We see the almost contact structure in consideration satisfies $(*)_1$. Let X_0 and Y_0 denote the F-components of X and Y. Then we have

$$\begin{aligned}\nabla_X\phi \cdot Y &= \nabla_{X_0+\eta(X)\xi}(\phi Y_0) - \phi\nabla_{X_0+\eta(X)\xi}(Y_0 + \eta(Y)\xi) \\ &= D_{X_0}(JY_0) - g(X_0, \phi Y_0)\xi + \eta(X)\nabla_\xi(\phi Y_0) \\ &\quad - \phi\{D_{X_0}Y_0 + \eta(Y)X_0 + \eta(X)\nabla_\xi Y_0\} \\ &\qquad\qquad\qquad\qquad \text{(because of (2.18) and } (*)_2) \\ &= D_{X_0}(JY_0) - \phi D_{X_0}Y_0 - g(X_0, \phi Y_0)\xi - \eta(Y)\phi X_0 \\ &\qquad\qquad\qquad\qquad \text{(because of (2.17))} \\ &= -g(X, \phi Y)\xi - \eta(Y)\phi X\,,\end{aligned}$$

since $\exp t\xi$ is a homothety with respect to the distribution $\eta = 0$ and $DJ = 0$. q.e.d.

Conversely we have the following structure theorem.

THEOREM 4. *Let M be an almost contact Riemannian manifold with $(*)$. Then, for any $p \in M$, some neighborhood $U(p)$ of $p \in M$ is identified with a warped product space $(-\varepsilon, +\varepsilon) \times_f V$ such that $(-\varepsilon, +\varepsilon)$ is an open interval, $f(t) = ce^t$ and V is a Kaehlerian manifold.*

PROOF. We define a distribution $\mathfrak{d}$ by $\eta = 0$. It is completely integrable by (2.6). Let $M(p)$ be the maximal integral submanifold through p. $M(p)$ is a totally umbilical hypersurface of M because of $(*)_2$. J and G denote the restriction of ϕ and g to $M(p)$ respectively. Then $M(p)$ is an almost Hermitian manifold for (J, G).

Moreover, by $(*)_1$, $M(p)$ is a Kaehlerian manifold. By virtue of Proposition 2, $\exp t\xi$ leaves ϕ and η invariant for each t and $\exp t\xi$ are homotheties on $\mathfrak{d}$, whose propotional factor is monotonously increasing as t. Thus the metric are written by

$$g_{(t,x)} = \begin{pmatrix} 1 & 0 \\ 0 & f^2(t)G_x \end{pmatrix}.$$

From (2.9) the differential equation for f is $f' - f = 0$. We have $f(t) = ce^t$ and M is locally a warped product space. q.e.d.

3. Some Properties. In the theory of Sasakian manifolds the following result is well-known: $K(X, \xi) = 1$ and if a Sasakian manifold is locally symmetric, then it is of constant positive curvature $+1$. On the other hand an almost contact Riemannian manifold with $(*)$ is not compact because of $\operatorname{div}\xi = 2n$ and we get

PROPOSITION 5. *Under the same assumption as Proposition* 1,

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X\,, \tag{3.1}$$

$$K(X, \xi) = -1\,, \tag{3.2}$$

$$(\nabla_Z R)(X, Y; \xi) = g(Z, X)Y - g(Z, Y)X - R(X, Y)Z\,. \tag{3.3}$$

PROOF. (3.1) follows directly from $(*)_2$, (2.6) and the definition of R. (3.2) is a result of (3.1). By virtue of $(*)_2$, (2.6) and (3.1) we get (3.3):

$$\begin{aligned}(\nabla_Z R)(X, Y; \xi) &= \nabla_Z(R(X, Y)\,\xi) - R(\nabla_Z X, Y) - R(X, \nabla_Z Y)\\ &\quad -R(X, Y)(\nabla_Z \xi)\\ &= g(Z, X)Y - g(Z, Y)X - R(X, Y)Z\,.\end{aligned}$$

q.e.d.

COROLLARY 6. *If M is locally symmetric, then it is of constant negative curvature* -1.

PROOF. Corollary 6 follows from (3.3).

We can generalize Corollary 6 slightly as follows:

PROPOSITION 7. *Under the same assumption as Proposition* 1, *if M satisfies the Nomizu's condition, i.e.,* $R(X, Y)R = 0$, *then it is of constant negative curvature* -1.

Since the proof of this Proposition is done by the same method as M. Okumura proved the Theorem 3.2 in [7], we shall omit it.

4. **η-Einstein manifold.** In an almost contact Riemannian manifold, if the Ricci tensor R_1 satisfies $R_1 = ag + b\,\eta\otimes\eta$, where a and b are scalar functions, then it is called an η-Einstein manifold. If a Sasakian manifold is η-Einsteinian and the dimension > 3, then a and b are constant.

PROPOSITION 8. *Let M be an almost contact Riemannian manifold with* $(*)$ *of dimension* $(2n+1)$. *If M is η-Einsteinian, we have*

$$a + b = -2n\,, \tag{4.1}$$

$$Z(b) + 2b\,\eta(Z) = 0\,, \quad \text{if} \quad n > 1\,, \quad \text{for any vector field } Z \text{ on } M. \tag{4.2}$$

PROOF. (4.1) follows from $R_1(X, \xi) = -2n\,\eta(X)$ which is derived from (3.1) and $R_1(X, Y) =$ the trace of the map $[W \rightarrow R(W, X)Y]$. As M is an η-Einstein manifold, the scalar curvature S is $2n(a-1)$. We define a (1, 1)-tensor field R^1 as follows: $g(R^1(X), Y) = R_1(X, Y)$. By the identity $\nabla_Y S = 2$ (trace of the map $[X \rightarrow (\nabla_X R^1)Y]$), we have

$$Z(a) + \xi(b)\eta(Z) + 2nb\,\eta(Z) = nZ(a)\,.$$

Setting $Z = \xi$, we get $\xi(b) = -2b$. Therefore we have $Z(b) + 2b\eta(Z) = 0$ if $n > 1$. q.e.d.

COROLLARY 9. *Under the same assumption as the Proposition* 8, *if* $b=$ *constant* (*or* $a=$ *constant*), *then* M *is an Einstein one.*

PROOF. Corollary 9 is a direct consequence of (4.2).

5. Curvature tensor. At first we shall prove

PROPOSITION 10. *Let* R *be the Riemannian curvature tensor of* M *with* (*). *Then*

$$R(X, Y)\phi Z-\phi R(X, Y)Z=g(Y, Z)\phi X-g(X, Z)\phi Y + g(X, \phi Z)Y-g(Y, \phi Z)X, \tag{5.1}$$

$$R(\phi X, \phi Y)Z=R(X, Y)Z+g(Y, Z)X-g(X, Z)Y + g(Y, \phi Z)\phi X-g(X, \phi Z)\phi Y. \tag{5.2}$$

PROOF. (5.1) follows from (*) and the Ricci's identity:

$$\nabla_X\nabla_Y\phi-\nabla_Y\nabla_X\phi-\nabla_{[X,Y]}\phi=R(X, Y)\phi-\phi R(X, Y).$$

We verify (5.2): By (5.1), we have

$$\begin{aligned} & g(R(X, Y)\phi Z, \phi W)-g(\phi R(X, Y)Z, \phi W) \\ &= g(Y, Z)\, g(\phi X, \phi W)-g(X, Z)\, g(\phi Y, \phi W) \\ &\quad + g(X, \phi Z)g(Y, \phi W)-g(Y, \phi Z)g(X, \phi W). \end{aligned}$$

Using $\eta(R(X, Y)Z)=\eta(Y)g(X, Z)-\eta(X)g(X, Z)$, the above formula is

$$\begin{aligned} g(R(\phi Z, \phi W)X, Y) &= g(R(Z, W)X, Y)+g(Y, Z)g(X, W) \\ &\quad - g(X, Z)g(Y, W)+g(X, \phi Z)g(Y, \phi W) \\ &\quad - g(Y, \phi Z)g(X, \phi W). \end{aligned}$$

q.e.d.

As an application of Proposition 10, we show

PROPOSITION 11. *Let* M *be an almost contact Riemannian manifold with* (*) *of dimension* >3. *If* M *is conformally flat, then* M *is a space of constant negative curvature* -1.

PROOF. Since M is conformally flat, the Riemannian curvature tensor of M is written by

$$\begin{aligned} & R(X, Y)Z \\ &= \frac{1}{(2n-1)}\{R_1(Y, Z)X-R_1(X, Z)Y+g(Y, Z)R^1(X)-g(X, Z)R^1(Y)\} \\ &\quad + \frac{S}{(2n)(2n-1)}\{g(X, Z)Y-g(Y, Z)X\}. \end{aligned} \tag{5.3}$$

Let us calculate $R(\xi, Y)\xi$ by the above formula. Using (3.1) and

$$R_1(X, \xi) = -2n\,\eta(X) ,$$

we get

$$2nR_1 = (S + 2n)\,g - (S+4n^2+2n)\eta \otimes \eta . \tag{5.4}$$

By virtue of (5.1), (5.3) and (5.4), we have

$$\begin{aligned}(S+4n^2+2n)\{g(Y, \phi Z)X - g(X, \phi Z)Y + g(X, Z)\phi Y \\ - g(Y, Z)\phi X + \eta(X)g(Y, \phi Z)\,\xi - \eta(Y)g(X, \phi Z)\,\xi \\ - \eta(Y)\,\eta(Z)\phi X + \eta(X)\,\eta(Z)\phi Y\} = 0 .\end{aligned} \tag{5.5}$$

Let $\{\xi, E_1, \phi E_1, \cdots, E_n, \phi E_n\}$ be an orthonormal basis of $T_x(M)$, $x \in M$. Setting $X = E_1$, $Y = E_2$ and $Z = \phi E_2$ in (5.5), we see $S = -2n(2n+1)$. Thus we have $R_1 = -2ng$. Proposition 11 follows from (5.3). q.e.d.

In a Sasakian manifold with constant ϕ-holomorphic sectional curvature, say H, the curvature tensor has a special feature [6]: The necessary and sufficient condition for a Sasakian manifold to have constant ϕ-holomorphic sectional curvature H is

$$\begin{aligned}4R(X, Y)Z = (H+3)\,(g(Y, Z)X - g(X, Z)Y) + (H-1)(\eta(X)\,\eta(Z)Y \\ - \eta(Y)\,\eta(Z)X + \eta(Y)g(X, Z)\,\xi - \eta(X)g(Y, Z)\,\xi \\ + g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z) .\end{aligned}$$

In our case we have

PROPOSITION 12. *Let M be an almost contact Riemannian manifold with (*). The necessary and sufficient condition for M to have constant ϕ-holomorphic sectional curvature H is*

$$\begin{aligned}4R(X, Y)Z = (H-3)(g(Y, Z)X - g(X, Z)Y) + (H+1)(\eta(X)\eta(Z)Y \\ - \eta(Y)\,\eta(Z)X + \eta(Y)g(X, Z)\,\xi - \eta(X)g(Y, Z)\,\xi \\ + g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z) .\end{aligned} \tag{5.6}$$

PROOF. For any vector fields X and $Y \in \mathfrak{d}$, we have

$$g(R(X, \phi X)X, \phi X) = -Hg(X, X)^2 . \tag{5.7}$$

By (5.1) we get

$$\begin{aligned}g(R(X, \phi Y)X, \phi Y) = g(R(X, \phi Y)Y, \phi X) - g(X, \phi Y)^2 - g(X, Y)^2 \\ + g(X, X)g(Y, Y) ,\end{aligned} \tag{5.8}$$

$$g(R(X, \phi X)Y, \phi X) = g(R(X, \phi X)X, \phi Y) , \qquad \text{for } X, Y \in \mathfrak{d} . \tag{5.9}$$

Substituting $X + Y$ in (5.7), we see

$$-H(2g(X,Y)^2 + 2g(X,X)g(X,Y) + 2g(X, Y)g(Y,Y) + g(X,X)g(Y,Y))$$
$$= \frac{1}{2} g(R(X+Y, \phi X+\phi Y)(X+Y), \phi X+\phi Y) + \frac{1}{2} H(g(X, X)^2 + g(Y, Y)^2)$$
$$= g(R(Y, \phi X)X, \phi X) + g(R(X, \phi X)X, \phi Y) + g(R(Y, \phi Y)X, \phi X)$$
$$+ g(R(Y, \phi Y)Y, \phi X) + g(R(X, \phi Y)Y, \phi X) + g(R(X, \phi Y)Y, \phi Y)$$
$$+ g(R(X, \phi Y)X, \phi Y) \qquad \text{(because of (5.1))}$$
$$= 2g(R(X, \phi X)X, \phi Y) + 2g(R(Y, \phi Y)Y, \phi X) - g(R(\phi Y, X)Y, \phi X)$$
$$- g(R(X, Y)\phi Y, \phi X) + g(R(X, \phi Y)Y, \phi X) + g(R(X, \phi Y)X, \phi Y),$$

because of (5.9) and the Bianchi identity. It then turns to

$$= 2g(R(X, \phi X)X, \phi Y) + 2g(R(Y, \phi Y)Y, \phi X) + 2g(R(X, \phi Y)Y, \phi X)$$
$$+ g(R(\phi X, \phi Y)X, Y) + g(R(X, \phi Y)X, \phi Y)$$
$$= 2g(R(X, \phi X)X, \phi Y) + 2g(R(Y, \phi Y)Y, \phi X) + 3g(R(X, \phi Y)Y, \phi X)$$
$$+ g(R(X, Y)X, Y),$$

because of (5.2) and (5.8). Thus we get

$$\begin{aligned} (5.10) \qquad & 2g(R(X, \phi X)X, \phi Y) + 2g(R(Y, \phi Y)Y, \phi X) \\ & + 3g(R(X, \phi Y)Y, \phi X) + g(R(X, Y)X, Y) \\ & = -H(2g(X, Y)^2 + 2g(X, X)g(X, Y) + 2g(X, Y)g(Y, Y) \\ & + g(X, X)g(Y, Y)). \end{aligned}$$

Replacing Y by $-Y$ in (5.10) and summing it to (5.10) we have

$$\begin{aligned} (5.11) \qquad & 3g(R(X, \phi Y)Y, \phi X) + g(R(X, Y)X, Y) \\ & = -H(2g(X, Y)^2 + g(X, X)g(Y, Y)). \end{aligned}$$

By virtue of (5.11) we see

$$\begin{aligned} (5.12) \qquad & 4g(R(X, Y)X, Y) \\ & = (H-3)(g(X, Y)^2 - g(X, X)g(Y, Y)) - 3(H+1)g(X, \phi Y)^2. \end{aligned}$$

We verify (5.12):

$$-H(2g(X, \phi Y)^2 + g(X, X)g(\phi Y, \phi Y))$$
$$= -3g(R(X, Y)\phi Y, \phi X) + g(R(X, \phi Y)X, \phi Y)$$
$$= 3g(R(\phi X, \phi Y)X, Y) + g(R(X, \phi Y)X, \phi Y)$$
$$= 3g(R(X, Y)X, Y) + g(R(X, \phi Y)Y, \phi X) + 2g(X, Y)^2 - 2g(X, X)g(Y, Y)$$
$$+ 2g(X, \phi Y)^2 \qquad \text{(because of (5.2) and (5.8))}$$
$$= 3g(R(X, Y)X, Y) - \frac{1}{3} g(R(X,Y)X,Y) - \frac{H}{3}(2g(X, Y)^2 + g(X,X)g(Y,Y))$$
$$+ 2g(X, Y)^2 - 2g(X, X)g(Y, Y) + 2g(X, \phi Y)^2 \qquad \text{(because of (5.11))}.$$

After simplication (5.12) follows. Therefore by a standard calculation we have

$$(5.13)\quad 4R(X, Y)Z = (H-3)(g(Y, Z)X - g(X, Z)Y) + (H+1)(g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z)\ ,$$

where X, Y and $Z \in \mathfrak{d}$.

We verify (5.13) for completeness: We calculate $g(R(X+Z, Y+W)(X+Z), Y+W)$. Using (5.12) we see

$$\begin{aligned}(5.14)\quad & 4g(R(X, Y)Z, W) + 4g(R(X, W)Z, Y)\\ &= (H-3)(g(X, Y)g(Z, W) + g(X, W)g(Y, Z) - 2g(X, Z)g(Y, W))\\ &\quad - 3(H+1)(g(X, \phi Y)g(Z, \phi W) + g(X, \phi W)g(Z, \phi Y))\end{aligned}$$

and we have

$$\begin{aligned}(5.14)'\quad & -4g(R(X, Z)Y, W) - 4g(R(X, W)Y, Z)\\ &= -(H-3)(g(X, Z)g(Y, W) + g(X, W)g(Y, Z) - 2g(X, Y)g(Z, W))\\ &\quad + 3(H+1)(g(X, \phi Z)g(Y, \phi W) + g(X, \phi W)g(Y, \phi Z))\ .\end{aligned}$$

Making (5.14) + (5.14)′ we get by virtue of the Bianchi identity

$$\begin{aligned}4g(R(X, W)Z, Y) = {}& (H-3)(g(X, Y)g(Z, W) - g(X, Z)g(Y, W))\\ & -(H+1)(g(X, \phi Y)g(Z, \phi W) - g(X, \phi Z)g(Y, \phi W)\\ & + 2g(X, \phi W)g(Z, \phi Y))\ ,\end{aligned}$$

where X, Y, Z and $W \in \mathfrak{d}$. For any vector fields X, Y, Z, using (3.1), we get (5.6). q.e.d.

THEOREM 13. *Let M be an almost contact Riemannian manifold with (*). If M is a space of constant ϕ-holomorphic sectional curvature H, then M is a space of constant curvature and $H = -1$.*

PROOF. By virtue of Proposition 12, M is an η-Einstein space:

$$(5.15)\quad R_1 = \frac{1}{2}(n(H-3)+H+1)g - \frac{1}{2}(n+1)(H+1)\,\eta\otimes\eta\ .$$

Since the coefficients of R_1 is constant on M, we see $H = -1$ by Corollary 9. q.e.d.

OBSERVATION 14. *Let $F[k]$ be a Kaehlerian manifold with constant holomorphic sectional curvature. Then the curvature tensor of the warped product space $L \times_f F[k]$, where $f(t) = ce^t$, is expressed by*

$$(5.16)\quad R(X, Y)Z = H_1(t)(g(Y, W)X - g(X, Z)Y) + (H_1(t) + 1)(\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + \eta(Y)g(X, Z)\xi - \eta(X)g(Y, Z)\xi + g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z) .$$

PROOF. (5.16) follows directly from Lemma 7.4 in [1].

REMARK. From the Tanno's Theorem [9], the maximum dimension of the automorphism group of $L \times {}_f F[k]$, where $F[k]$ is connected, is attained if and only if $F[k] = CE^n$ (and hence $H_1(t) = -1$).

6. Invariant submanifold. Invariant submanifolds in a Sasakian manifold are also Sasakian and minimal. In this section we study invariant submanifolds in an almost contact Riemannian manifold M with (*). Let N be an almost contact manifold and (ϕ_0, η_0, ξ_0) denote its structure tensor. An invariant immersion, say i, of N into M is an immersion which satisfies

$$(6.1)\qquad i_*\phi_0 = \phi i_* , \qquad i_*\xi_0 = \xi .$$

Then we can easily see that i is a minimal immersion for the induced metric g_0 and $(\phi_0, \xi_0, \eta_0, g_0)$ is an almost contact metric structure with (*) on N. Moreover by the local structure theorem 4, it is easy to show that

PROPOSITION 15. *Let $F[c]$ be a complex projective space CP^{n+1} with a Fubini-Study metric or a complex Euclidean space CE^{n+1} or an open ball CD^{n+1} with a homogeneous Kaehlerian structure of negative constant holomorphic sectional curvature, and let N be an invariant submanifold of codimension 2 in $M = L \times_f F[c]$. If N is an η-Einstein manifold for the induced metric, then N is totally geodesic or N is locally isometric to $L \times_f Q^n$, where Q^n is a hypersphere in $CP^{n+1} (n \geq 2)$.*

PROOF. Since N is an invariant submanifold of M, the distribution defined by $\eta_0 = 0$ is completely integrable. Let $N(p)$ be the maximal integral submanifold through $p \in N$. By Theorem 4, $N(p)$ is a Kaehlerian hypersurface in M and an Einstein manifold for the restricted metric since N is an η-Einstein one. Therefore $N(p)$ is totally geodesic or locally holomorphically isometric to Q^n(see [5]). Thus N is totally geodesic or locally isometric to $L \times_f Q^n$. q.e.d.

Let $\bar{N}$ be an almost complex manifold with an almost complex structure J. When an immersion j of $\bar{N}$ into M satisfies $j_*J = \phi j_*$ and $j^*\eta = 0$, we call $j(\bar{N})$ an invariant hypersurface. Such an immersion is studied by S. Eum [3], etc. If $j(\bar{N})$ is an invariant hypersurface of M, $j(\bar{N})$ is umbilical by $(*)_2$.

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