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A CLASS OF CONSISTENT TESTS FOR EXPONENTIALITY BASED ON THE EMPIRICAL LAPLACE TRANSFORM

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Abstract. The Laplace transform $\psi(t) = E[\exp(-tX)]$ of a random variable with exponential density $\lambda\exp(-\lambda x)$, $x \geq 0$, satisfies the differential equation $(\lambda + t)\psi'(t) + \psi(t) = 0$, $t \geq 0$. We study the behaviour of a class of consistent ("omnibus") tests for exponentiality based on a suitably weighted integral of $[(\hat{\lambda}_n + t)\psi_n'(t) + \psi_n(t)]^2$, where $\hat{\lambda}_n$ is the maximum-likelihood-estimate of λ and ψ_n is the empirical Laplace transform, each based on an i.i.d. sample $X_1, \dots, X_n$.

Key words and phrases: Exponential distribution, goodness-of-fit test, empirical Laplace transform, consistency.

1. Introduction

Apart from the normal distribution, the exponential distribution is probably the most widely used probability law in statistical analysis, especially in connection with life testing and reliability theory. Therefore it is not surprising that many tests for exponentiality have been proposed in the literature (see e.g. D'Agostino and Stephens (1986) and Spurrier (1984)). Since the alternatives to the exponential distribution are rarely known in practice and the choice of a test should not be done on the basis of given data, omnibus tests for exponentiality which aim at detecting all distributional departures from exponentiality are of great importance, two prominent members of in this line of work are the tests of Cramér-von Mises and Anderson-Darling (see Davis and Stephens (1989)).

It is clear that each omnibus test for exponentiality must use some characterizing equation (property) of the class of exponential distributions and a distance statistic which measures the deviation from this equation for the empirical distribution of the observed sample.

In this paper, we study a class of omnibus tests for exponentiality based on a differential equation for the Laplace transform, characteristic for the family of exponential distributions. To be specific, let $X, X_1, \dots, X_n$ be independent identically distributed non-negative random variables, and let $\mathrm{Exp}(\lambda)$ denote the exponential distribution with density $\lambda\exp(-\lambda x)$, $x \geq 0$. The problem is then to test, on the basis of $X_1, \dots, X_n$, the composite hypothesis

$$H_0\text{: The law of } X \text{ is } \mathrm{Exp}(\lambda) \text{ for some } \lambda > 0$$

against the general alternative that X is not exponentially distributed. The rationale for the new test is as follows: If the distribution of X is $\mathrm{Exp}(\lambda)$, the Laplace transform $\psi(t) = E[\exp(-tX)]$, $t \geq 0$ of X is given by $\psi(t) = \lambda/(\lambda + t)$ and thus, satisfies the differential equation

$$(1.1) \qquad (\lambda + t)\psi'(t) + \psi(t) = 0, \qquad \forall t \geq 0$$

subject to the boundary condition $\psi(0) = 1$. Since the distribution of a non-negative random variable is determined by its Laplace transform, the equation (1.1) characterizes the exponential distribution $\mathrm{Exp}(\lambda)$. Letting

$$\psi_n(t) = \frac{1}{n}\sum_{j=1}^{n} \exp(-tX_j)$$

denote the empirical Laplace transform of $X_1, \dots, X_n$ and $\hat{\lambda}_n = \bar{X}_n^{-1}$ the maximum-likelihood-estimator of λ, where $\bar{X}_n = (1/n)\sum_{j=1}^{n} X_j$, the test statistic proposed is the weighted integral

$$T_{n,a} = n\int_0^\infty [(\hat{\lambda}_n + t)\psi_n'(t) + \psi_n(t)]^2 \bar{X}_n \exp(-a\bar{X}_n t)dt,$$

where $a > 0$ is a positive constant. It will be seen in Section 3 that a test for exponentiality rejecting the hypothesis H_0 for large values of $T_{n,a}$ leads to a consistent procedure for any positive a. However, the choice of a has a pronounced influence on the power performance of the test (see Section 4).

Some motivations to consider the weight function $\bar{X}_n \exp(-\bar{X}_n at)$ are as follows: Firstly, $T_{n,a}$ may be computed in an easy way (see below) and has the desirable feature of being scale invariant. Secondly, from Tauberian theorems on Laplace transforms (see e.g. Feller (1966), Chapter XIII.5), it is known that the tail behaviour of a probability distribution concentrated on $[0, \infty)$ is reflected by the behaviour of its Laplace transform at zero and vice versa. Consequently, choosing a small value of a and thus, letting the weight function decay slowly, should result in good power properties against alternative distributions having a point mass or infinite density at zero. On the other hand, a large value of a implying that the weight function puts most of its mass near zero should be a safeguard against alternative distributions with great difference in tail behaviour from the exponential distribution.

Straightforward manipulation of integrals gives the computationally simple form

$$T_{n,a} = \frac{1}{n}\sum_{i,j=1}^{n}\left[\frac{(1-Y_i)(1-Y_j)}{Y_i+Y_j+a} - \frac{Y_i+Y_j}{(Y_i+Y_j+a)^2} + \frac{2Y_iY_j}{(Y_i+Y_j+a)^2} + \frac{2Y_iY_j}{(Y_i+Y_j+a)^3}\right],$$

where $Y_j = X_j/\bar{X}_n$, $1 \le j \le n$. This shows that $T_{n,a} = nV_n(\hat{\lambda}_n)$, where

$$V_n(\lambda) = \frac{1}{n^2}\sum_{i,j=1}^{n} h_a(X_i,\, X_j;\, \lambda)$$

is a degree-two V-statistic with kernel

$$\begin{aligned} h_a(x,\, y;\, \lambda) = {} & \frac{(1-\lambda x)(1-\lambda y)}{\lambda(x+y)+a} - \frac{\lambda(x+y)}{(\lambda(x+y)+a)^2} \\ & + \frac{2\lambda^2 xy}{(\lambda(x+y)+a)^2} + \frac{2\lambda^2 xy}{(\lambda(x+y)+a)^3} \end{aligned}$$

(see, e.g. Serfling (1980)). Moreover, the distribution of $T_{n,a}$ under H_0 is seen to be independent of the underlying parameter λ. An alternative expression of $T_{n,a}$ is

$$T_{n,a} = \int_0^1 Z_n^2(u) u^{a-1} du, \tag{1.2}$$

where

$$Z_n(u) = n^{-1/2} \sum_{j=1}^{n} u^{Y_j}[1 - (1-\log(u))Y_j], \qquad 0 \le u \le 1. \tag{1.3}$$

The representation (1.2) in terms of a functional of a stochastic process is particularly useful for deriving the asymptotic null distribution of $T_{n,a}$ as $n \to \infty$. This will be done in the next section.

2. The limiting null distribution of the test statistic

The stochastic process Z_n introduced in (1.3) may be regarded as a random element in $C[0,\, 1]$, the Banach space of real valued continuous functions on the unit interval, endowed with the supremum norm $\sup_{0 \le u \le 1} |x(u)|$, $x \in C[0,\, 1]$. Obviously, $T_{n,a}$ is a continuous function of Z_n. We shall prove that, under the hypothesis H_0 of exponentiality, Z_n tends in distribution to a zero-mean Gaussian process $Z = \{Z(u),\, 0 \le u \le 1\}$ with continuous sample paths. Consequently, the limiting null distribution of $T_{n,a}$ is the same as that of

$$T_a = \int_0^1 Z(u)^2 u^{a-1} du.$$

It is well known that the distribution of T_a is that of $\sum_{j \ge 1} \gamma_j N_j^2$, where $N_1,\, N_2, \ldots$ are independent unit normal random variables, and γ_j, $j \ge 1$, are the eigenvalues of the integral operator associated with the kernel $k(u,\, v) = \mathrm{Cov}(Z(u),\, Z(v))$, $0 \le u,\, v \le 1$, i.e.

$$\gamma_j \varphi_j(u) = \int_0^1 k(u,\, v)\varphi_j(v) v^{a-1} dv, \qquad 0 \le u \le 1.$$

The eigenfunction φ_j corresponding to γ_j is square integrable with respect to the measure $d\nu_a(u) = u^{a-1}du$ on the unit interval. The kernel $k(u, v)$ turns out to be

$$(2.1)\quad k(u, v) = \frac{(1-\log u)(1-\log v) + (\log u)(\log v)}{(1-\log(vu))^3} - \frac{1}{(1-\log u)^2(1-\log v)^2}$$

$(0 \le u, v \le 1)$. In what follows, we may assume that the random variables X_j are exponentially distributed with the parameter $\lambda = 1$. The stochastic process $W_n = \{W_n(u), 0 \le u \le 1\}$, where

$$W_n(u) = n^{-1/2}\sum_{j=1}^{n}\left[u^{X_j}(1-(1-\log u)X_j) + \frac{X_j - 1}{(1-\log u)^2}\right], \qquad 0 \le u \le 1,$$

can also be regarded as a random element of $C[0, 1]$. We first show that W_n converges in distribution to Z. By applying the multivariate central limit theorem we see that the finite dimensional distributions of W_n converge weakly to multivariate normal distributions with zero means and covariance matrices determined by the kernel $k(\,\cdot\,, \,\cdot\,)$ given in (2.1). Now,

$$e(u, v) = |(1-\log u)^{-1/4} - (1-\log v)^{-1/4}|, \qquad 0 \le u, v \le 1,$$

defines a continuous metric e on the unit interval satisfying the metric entropy condition

$$\int_0^1 (\log N(u))^{1/2} du < \infty.$$

Here, for each $u > 0$, $N(u)$ is the smallest positive integer m such that the unit interval can be covered by m subsets, each having a diameter at most $2u$ with respect to e. Letting

$$S(u) = u^{X_1}(1-(1-\log u)X_1) + \frac{X_1 - 1}{(1-\log u)^2}, \qquad 0 \le u \le 1,$$

the second mean value theorem implies that there is a positive constant c such that

$$|S(u) - S(v)| \le c\max(X_1^2, X_1^{-1/4})e(u, v), \qquad 0 \le u, v \le 1.$$

Since $E[\max(X_1^2, X_1^{-1/4})^2] < \infty$, the sequence of distributions of W_n in $C[0, 1]$ is tight and converges weakly to the distribution of a zero-mean Gaussian process with continuous sample paths and covariance function $k(\,\cdot\,, \,\cdot\,)$ (see, e.g. Araujo and Giné (1980)).

Introducing for $\xi > 0$, the process $M_n(\xi) = \{M_n(\xi, u), 0 \le u \le 1\}$, where

$$M_n(\xi, u) = n^{-1/2}\sum_{j=1}^{n} u^{\xi X_j}(1-(1-\log u)\xi X_j), \qquad 0 \le u \le 1,$$

and denoting by L_n and $\hat{L}_n$ the processes

$$L_n(u) = (1 - \log u)^{-2} n^{-1/2} \sum_{j=1}^{n} (X_j - 1), \qquad 0 \leq u \leq 1$$

and

$$\hat{L}_n(u) = -(1 - \log u)^{-2} n^{1/2} (\hat{\lambda}_n - 1), \qquad 0 \leq u \leq 1,$$

we have that

$$\|Z_n - W_n\|_a = \|M_n(\hat{\lambda}_n) - M_n(1) - L_n\|_a,$$

where, generically,

$$\|f\|_a^2 = \int_0^1 f^2 d\nu_a, \qquad f \text{ square integrable with respect to } d\nu_a.$$

Using a Taylor expansion of $M_n(\xi, 1)$ in a neighbourhood of $\xi = 1$ we see that $\|M_n(\hat{\lambda}_n) - M_n(1) - \hat{L}_n\|_a = o_P(1)$. Since $\|L_n - \hat{L}_n\|_a = o_P(1)$ it follows that $\|Z_n - W_n\|_a = o_P(1)$. Summarizing, we have the following result.

THEOREM 2.1. *The limiting null distribution of the test statistic $T_{n,a}$ is that of $\sum_{j \geq 1} \gamma_j N_j^2$, where $N_1, N_2, \ldots$ are independent unit normal random variables, and $\gamma_1, \gamma_2, \ldots$ are the eigenvalues of the integral operator associated with the kernel $k(\cdot, \cdot)$ given in* (2.1).

It should be remarked that a different method of proof of the result stated above is provided by the work of De Wet and Randles (1987).

3. Consistency

For a given level of significance $\alpha \in (0, 1)$, let $t_{n,a}(\alpha)$ be the $(1 - \alpha)$-quantile of $T_{n,a}$ when the hypothesis H_0 is true.

THEOREM 3.1. *The test rejecting the hypothesis of exponentiality if $T_{n,a} > t_{n,a}(\alpha)$ is consistent against any fixed non-exponential distribution having finite positive first moment.*

PROOF. Let X_1 have a distribution with finite expectation $\lambda > 0$ and Laplace-transform $\phi(t)$, $t \geq 0$. Then $n^{-1} T_{n,a}$ tends to

$$\lambda \int_0^\infty ((\lambda^{-1} + t)\phi'(t) + \phi(t))^2 \exp(-a\lambda t) dt$$

in probability. This stochastic limit is zero if and only if ϕ is the Laplace-transform of an exponential distribution. Thus, for non-exponential distributions, $\lim_{n \to \infty} P(T_{n,a} \leq t_{n,a}(\alpha)) = 0$.

4. Power results

The main justification to propose a new test is that it provides a higher power than the presently used procedures. To compare the power of the proposed test with some of the prominent competitive procedures, especially the omnibus tests of Cramér-von Mises and Anderson-Darling, a Monte Carlo simulation study was done.

The following procedures were compared.

(i) The new test based on $T_{n,a}$ for $a = 0.1$, $a = 1$ and $a = 10$ which is indicated as $T(.1)$, $T(1)$ and $T(10)$ in Tables 4 and 5. Critical points for $T_{n,a}$ may be obtained from Table 1 ($a = 0.1$), Table 2 ($a = 1$) and Table 3 ($a = 10$). The entries in Tables 1–3 represent 20%-trimmed means of 10 Monte Carlo estimates, each based on 10000 replications. The results indicate rapid convergence of the true quantiles to their limiting values as $n \to \infty$.

Table 1. Empirical percentage points for $T_{n,a}$, $a = 0.1$.

	$1-\alpha$					
n	0.5	0.75	0.9	0.95	0.975	0.99
5	0.284	0.534	0.832	1.071	1.529	2.465
6	0.290	0.543	0.854	1.091	1.569	2.437
7	0.293	0.546	0.871	1.130	1.585	2.414
8	0.293	0.548	0.887	1.181	1.601	2.397
9	0.294	0.550	0.908	1.212	1.614	2.386
10	0.294	0.553	0.923	1.231	1.624	2.376
11	0.294	0.556	0.936	1.246	1.632	2.367
12	0.294	0.560	0.946	1.258	1.639	2.359
13	0.294	0.563	0.952	1.268	1.646	2.351
14	0.294	0.565	0.957	1.277	1.653	2.344
15	0.294	0.567	0.961	1.284	1.659	2.336
16	0.294	0.569	0.964	1.290	1.666	2.330
17	0.294	0.571	0.967	1.295	1.672	2.325
18	0.294	0.572	0.970	1.300	1.678	2.320
19	0.294	0.573	0.973	1.305	1.684	2.316
20	0.294	0.574	0.975	1.309	1.689	2.312
25	0.294	0.576	0.986	1.327	1.707	2.305
30	0.294	0.578	0.994	1.339	1.722	2.301
35	0.294	0.579	1.001	1.346	1.731	2.298
40	0.294	0.580	1.003	1.351	1.735	2.295
45	0.294	0.580	1.005	1.354	1.737	2.292
50	0.294	0.580	1.006	1.356	1.739	2.289

Table 2. Empirical percentage points for $T_{n,a}$, $a = 1$.

	$1-\alpha$					
n	0.5	0.75	0.9	0.95	0.975	0.99
5	0.050	0.118	0.205	0.262	0.308	0.378
6	0.049	0.117	0.209	0.272	0.329	0.407
7	0.049	0.117	0.211	0.276	0.338	0.425
8	0.049	0.117	0.213	0.282	0.352	0.446
9	0.048	0.117	0.214	0.286	0.359	0.464
10	0.048	0.117	0.215	0.289	0.366	0.469
11	0.048	0.116	0.216	0.291	0.369	0.476
12	0.048	0.116	0.217	0.294	0.372	0.482
13	0.047	0.116	0.217	0.295	0.374	0.486
14	0.047	0.116	0.217	0.296	0.376	0.490
15	0.047	0.116	0.217	0.297	0.378	0.492
16	0.047	0.116	0.217	0.298	0.380	0.495
17	0.047	0.116	0.218	0.299	0.381	0.497
18	0.047	0.115	0.218	0.300	0.383	0.499
19	0.047	0.115	0.219	0.302	0.386	0.501
20	0.047	0.115	0.220	0.303	0.389	0.506
25	0.047	0.115	0.220	0.304	0.392	0.512
30	0.047	0.115	0.221	0.305	0.393	0.513
35	0.047	0.115	0.221	0.306	0.396	0.515
40	0.047	0.114	0.222	0.308	0.398	0.521
45	0.047	0.114	0.223	0.309	0.399	0.523
50	0.047	0.114	0.223	0.310	0.401	0.526

(ii) The tests of Cramér-von Mises and Anderson-Darling. These are based on measures of discrepancy between the empirical distribution function

$$G_n(u) = \frac{1}{n}\sum_{j=1}^{n} I\{W_{(j)} \le u\}$$

of $W_{(j)} = 1 - \exp(-Y_{(j)})$, $1 \le j \le n$, and the uniform distribution function on the unit interval. Here and in what follows, $Y_{(1)} \le \cdots \le Y_{(n)}$ denote the order statistics of $Y_1, \ldots, Y_n$, and $I\{A\}$ is the indicator function of an event A. The Cramér-von Mises statistic is

$$\begin{aligned} C^2 &= n\int_0^1 (G_n(u) - u)^2 du \\ &= \sum_{j=1}^{n}\left(W_{(j)} - \frac{2j-1}{2n}\right)^2 + \frac{1}{12n}, \end{aligned}$$

Table 3. Empirical percentage points for $T_{n,a}$, $a = 10$.

	$1-\alpha$					
n	0.5	0.75	0.9	0.95	0.975	0.99
5	0.0010	0.0022	0.0035	0.0043	0.0050	0.0073
6	0.0010	0.0023	0.0038	0.0047	0.0056	0.0089
7	0.0010	0.0024	0.0040	0.0050	0.0062	0.0100
8	0.0010	0.0025	0.0041	0.0053	0.0067	0.0109
9	0.0010	0.0025	0.0043	0.0056	0.0072	0.0121
10	0.0010	0.0025	0.0044	0.0058	0.0076	0.0124
11	0.0010	0.0026	0.0045	0.0060	0.0079	0.0127
12	0.0010	0.0026	0.0046	0.0062	0.0081	0.0129
13	0.0010	0.0026	0.0047	0.0063	0.0084	0.0131
14	0.0010	0.0026	0.0048	0.0065	0.0086	0.0134
15	0.0010	0.0027	0.0049	0.0066	0.0088	0.0139
16	0.0010	0.0027	0.0049	0.0067	0.0090	0.0141
17	0.0011	0.0027	0.0050	0.0068	0.0091	0.0143
18	0.0011	0.0027	0.0050	0.0069	0.0093	0.0145
19	0.0011	0.0028	0.0051	0.0070	0.0095	0.0147
20	0.0011	0.0028	0.0052	0.0072	0.0097	0.0149
25	0.0011	0.0028	0.0054	0.0074	0.0100	0.0152
30	0.0011	0.0029	0.0055	0.0077	0.0103	0.0153
35	0.0011	0.0029	0.0056	0.0079	0.0105	0.0153
40	0.0011	0.0029	0.0057	0.0080	0.0106	0.0154
45	0.0011	0.0030	0.0058	0.0081	0.0107	0.0154
50	0.0011	0.0030	0.0058	0.0082	0.0108	0.0155

whereas the Anderson-Darling procedure is based on

$$\begin{aligned} A^2 &= n \int_0^1 \frac{(G_n(u) - u)^2}{u(1-u)} du \\ &= -n - \frac{1}{n} \sum_{j=1}^{n} (2j-1)[\log(W_{(j)}) + \log(1 - W_{(n-j+1)})]. \end{aligned}$$

The tests were carried out using the modifications and percentage points given in Table 4.11 of D'Agostino and Stephens (1986). The Anderson-Darling test should be used with caution due to severe effects of recording errors close to zero.

(iii) The test of Moran (Moran (1951)). This procedure is based on the statistic

$$M = -2 \sum_{j=1}^{n} \log(Y_j).$$

Table 4. Percentage of 5000 Monte Carlo samples declared significant by the various tests of exponentiality; test size $\alpha = 0.05$; sample size $n = 20$.

Alternative	$T(.1)$	$T(1)$	$T(10)$	M	Q_1	W^*	C^2	A^2	S
Gamma(0.4)	91	82	67	92	40	52	75	89	18
Gamma(0.6)	53	39	31	52	16	20	32	48	9
Gamma(0.8)	17	12	11	16	8	8	10	15	6
Gamma(1.0)	5	5	5	5	5	5	5	5	6
Gamma(1.4)	11	16	10	17	6	13	15	13	5
Gamma(1.6)	19	26	17	28	7	21	24	21	6
Gamma(1.8)	32	41	27	43	12	32	36	34	8
Gamma(2.0)	44	53	38	57	15	43	49	46	7
Gamma(2.4)	67	75	56	78	27	61	69	68	10
Gamma(3.0)	89	93	80	94	46	82	89	88	12
Weibull(0.4)	*	99	97	*	83	92	98	*	60
Weibull(0.6)	83	76	67	82	40	54	69	81	25
Weibull(0.8)	29	25	24	27	12	18	20	28	11
Weibull(1.2)	10	14	10	14	6	13	14	12	6
Weibull(1.4)	26	38	28	38	10	33	34	31	8
Weibull(1.6)	51	67	54	65	20	61	62	59	14
Weibull(2.0)	88	96	92	95	47	93	93	92	26
Uniform(0,1)	33	60	66	45	27	77	67	63	79
Half-Normal	11	20	17	18	9	23	21	17	11
Half-Cauchy	58	65	70	55	53	67	64	64	57
Log-Normal(0.5)	*	99	91	*	87	91	99	99	9
Log-Normal(0.7)	67	61	37	70	30	39	61	62	9
Log-Normal(0.8)	37	34	18	42	16	21	34	35	11
Log-Normal(1.0)	9	12	17	8	11	17	16	15	18
Log-Normal(1.5)	56	64	67	48	44	60	61	63	43
χ_1^2	76	62	49	76	25	34	53	71	13
Power(0.5)	99	*	*	*	87	*	*	*	98
Power(0.8)	67	88	91	77	43	96	91	90	90
Power(1.2)	16	32	38	21	20	50	42	38	65
Power(1.4)	11	14	18	12	18	27	24	24	51
Power(2.0)	45	14	3	40	21	3	19	41	23
Power(3.0)	90	65	28	90	38	11	63	88	9
Power(4.0)	99	91	64	99	58	39	89	98	9
JSHAPE(0.5)	39	45	51	36	29	44	41	44	33
JSHAPE(1.0)	79	83	84	77	65	79	80	82	62
JSHAPE(1.5)	95	95	95	94	85	92	94	95	79
LIFR(1.0)	10	18	13	15	8	19	18	14	9
LIFR(2.0)	17	28	23	25	11	30	29	25	13
LIFR(4.0)	26	42	36	37	14	44	44	37	15
LIFR(6.0)	31	49	43	43	17	52	49	44	18
LIFR(10.0)	40	59	53	51	21	62	58	54	20

Table 5. Percentage of 5000 Monte Carlo samples declared significant by the various tests of exponentiality; test size $\alpha = 0.05$; sample size $n = 50$.

Alternative	$T(.1)$	$T(1)$	$T(10)$	M	Q_1	W^*	C^2	A^2	S
Gamma(0.4)	*	99	96	*	64	87	99	*	29
Gamma(0.6)	84	75	59	86	23	42	66	80	12
Gamma(0.8)	28	21	17	27	9	11	17	24	7
Gamma(1.0)	5	5	5	5	5	5	4	5	5
Gamma(1.4)	34	37	24	41	9	26	32	32	4
Gamma(1.6)	62	64	44	69	17	45	56	58	6
Gamma(1.8)	83	84	65	88	27	64	77	80	7
Gamma(2.0)	94	94	81	96	41	79	91	93	8
Gamma(2.4)	*	99	96	*	65	94	99	99	12
Gamma(3.0)	*	*	*	*	89	99	*	*	16
Weibull(0.6)	99	99	96	99	65	90	98	99	47
Weibull(0.8)	54	50	45	54	16	33	43	51	15
Weibull(1.2)	25	32	24	32	8	26	28	27	5
Weibull(1.4)	73	82	73	81	23	73	75	76	11
Weibull(1.6)	96	98	96	98	47	96	96	97	22
Uniform(0,1)	79	96	99	80	52	*	98	99	*
Half-Normal	30	49	50	37	13	56	48	44	17
Half-Cauchy	88	93	95	86	83	94	93	92	88
Log-Normal(0.7)	*	96	63	99	83	57	98	99	13
Log-Normal(0.8)	94	67	30	81	52	29	76	85	18
Log-Normal(1.0)	34	18	28	15	23	29	30	34	34
Log-Normal(1.2)	38	52	66	21	36	62	57	55	52
Log-Normal(1.5)	89	94	95	84	70	92	94	93	74
χ^2_1	97	94	82	98	39	66	90	96	18
Power(0.8)	99	*	*	99	79	*	*	*	*
Power(1.2)	42	72	88	39	35	94	86	86	99
Power(1.4)	22	34	57	13	30	72	61	65	96
Power(2.0)	74	25	3	64	34	6	49	76	68
Power(3.0)	*	95	48	*	64	15	96	*	27
JSHAPE(0.1)	9	11	16	8	7	13	9	10	12
JSHAPE(0.2)	19	25	33	18	13	29	22	23	23
JSHAPE(0.5)	69	79	82	68	51	78	75	76	60
JSHAPE(1.0)	98	99	99	98	92	99	99	99	92
LIFR(1.0)	26	43	42	34	11	47	41	38	13
LIFR(2.0)	46	66	66	53	18	71	64	61	21
LIFR(4.0)	68	85	86	72	27	89	83	81	30
LIFR(6.0)	78	92	92	81	34	94	91	89	34
LIFR(8.0)	84	95	95	86	40	96	93	93	37
LIFR(10.0)	87	96	97	88	41	97	96	95	39

A two-sided test, based on M, is a uniformly most powerful unbiased test against Gamma alternatives (see also Shorack (1972)). Bartholomew (1957) showed it to be a strong test against Weibull alternatives. For our simulation study, critical values for M (two-sided rejection region) were found for sample sizes of $n = 20$ and $n = 50$ by extensive simulations (10^6 replications). A severe deficiency of the test based on M, is the effect on M of inaccurate measurements of the values of X_j close to zero.

(iv) The W^*-test. This test was originally proposed by Shapiro and Wilk (1972) to test the more general hypothesis

$$\tilde{H}_0\colon\ P(X \geq t) = \exp(-\lambda(t-\theta)), \qquad t \geq \theta, \text{ for some } \lambda, \theta\ (\lambda > 0).$$

It was modified by Stephens (1978) to test the hypothesis H_0 of exponentiality with origin ($= 0$) known. The test statistic is

$$W^* = \frac{\left(\sum_{j=1}^n X_j\right)^2}{n\left[(n+1)\sum_{j=1}^n X_j^2 - \left(\sum_{j=1}^n X_j\right)^2\right]}.$$

As a general test for exponentiality a two-sided rejection region must be used. Since

$$\begin{aligned}\frac{1}{W^*} &= 1 + (n+1)\frac{S_n^2}{\bar{X}_n^2}\\ &= 1 + \frac{n+1}{n}\sum_{j=1}^n (Y_j - 1)^2\end{aligned}$$

with $S_n^2 = n^{-1}\sum_{j=1}^n (X_j - \bar{X}_n)^2$, we see that the test merely aims at investigating the first two moments of the underlying distribution and thus, it is not an omnibus procedure. Since W^* has the same null distribution as the statistic W_E of Shapiro and Wilk (1972) for a sample of size $n+1$, their tables may be used to obtain critical points.

(v) The Q_1-test. This test was recently proposed by Patwardhan (1988) as a (purportedly) omnibus procedure for assessing exponentiality. It rejects the hypothesis H_0 of exponentiality for large values of

$$Q_1 = (Y_{()} - \delta)'\Sigma_{()}^{-}(Y_{()} - \delta) + (n+1), \tag{4.1}$$

where

$$Y_{()} = (Y_{(1)}, \ldots, Y_{(n)})', \quad Y_{(j)} = X_{(j)}/\bar{X}_n \quad (1 \leq j \leq n), \tag{4.2}$$

$$\delta = (\delta_1, \ldots, \delta_n)', \quad \delta_j = \sum_{k=1}^{j} \frac{1}{n-k+1} \tag{4.3}$$

and $\Sigma_{()}^{-}$ is a generalized inverse of the covariance matrix of $Y_{()}$ for which an explicit expression is given by Patwardhan (1988). In the notation given above, c'

generically denotes the transpose of a column vector c. Since $E[Y_{()}] = \delta$, the quadratic form occuring in (4.1) represents a standardized deviation of a plot of $Y_{(j)}$ versus δ_j $(j = 1, \ldots, n)$ from a straight line. An alternative expression for Q_1 is (Patwardhan (1988))

$$Q_1 = \frac{n+1}{n} \sum_{j=1}^{n} U_{n,j}^2,$$

where $U_{n,j}$ denotes the *normalized scaled spacing*

$$U_{n,j} = (n - j + 1)(Y_{(j)} - Y_{(j-1)}) \qquad (j = 1, \ldots, n,\ Y_{(0)} := 0).$$

Although a test for exponentiality based on Q_1 has an extreme poor power compared to the other tests under discussion (and thus should not be recommended at all for testing H_0), its consistency against a large class of alternative distributions may be proved (this was conjectured by Patwardhan (1988)). The reasoning is as follows: Let X_1 have distribution function F and density function f, where $F(0) = 0$ and $E[X_1] = 1$. From Renyi's representation of order statistics from a uniform distribution and Taylor expansion, we have

$$\begin{aligned}\frac{Q_1}{n+1} &\overset{\mathcal{D}}{=} \frac{1}{n} \sum_{j=1}^{n} (n - j + 1)^2 \\ &\quad \cdot \left(F^{-1}\left(\frac{E_1 + \cdots + E_j}{(n+1)\bar{E}_{n+1}} \right) - F^{-1}\left(\frac{E_1 + \cdots + E_{j-1}}{(n+1)\bar{E}_{n+1}} \right) \right)^2 \\ &= \frac{1}{n\bar{E}_{n+1}} \sum_{j=1}^{n} \left(1 - \frac{j}{n+1} \right)^2 \frac{E_j^2}{f^2(F^{-1}(W_{j,n}))},\end{aligned}$$

where $E_1, \ldots, E_{n+1}$ are i.i.d. unit exponential variates with arithmetic mean $\bar{E}_{n+1}$, and

$$\frac{E_1 + \cdots + E_{j-1}}{(n+1)\bar{E}_{n+1}} \le W_{j,n} \le \frac{E_1 + \cdots + E_j}{(n+1)\bar{E}_{n+1}}.$$

The symbol "$\overset{\mathcal{D}}{=}$" denotes equality in distribution. Under suitable regularity conditions on f, $Q_1/(n+1)$ is stochastically equivalent to

$$\frac{1}{n} \sum_{j=1}^{n} \left(1 - \frac{j}{n+1} \right)^2 \frac{E_j^2}{f^2\left(F^{-1}\left(\frac{j}{n+1} \right) \right)}$$

which converges stochastically to

$$2 \int_0^\infty \frac{(1 - F(t))^2}{f(t)} dt \tag{4.4}$$

as $n \to \infty$ provided that $\int_0^\infty [(1 - F(t))/f(t)]^4 f(t) dt < \infty$. Since the asymptotic null distribution of $n^{1/2}(Q_1/2n - 1)$ is standard normal (see Patwardhan (1988)) and the quantity occuring in (4.4) has a minimum value (= 2) if and only if $X_1 \sim \text{Exp}(1)$ (use Jensen's inequality), the consistency of a test based on Q_1 follows.

(vi) The test of Sarkadi (Sarkadi (1975)). Sarkadi (1975) proved that a test for exponentiality rejecting the hypothesis H_0 for small values of

$$S = \frac{\left(\sum_{i=1}^{n} \delta_i Y_{(i)} - n\right)^2}{\sum_{i=1}^{n} (Y_{(i)} - 1)^2},$$

where $Y_{(i)}$ and δ_i are given in (4.2) and (4.3) respectively, is consistent against general alternatives. Note that, apart from a constant factor, S is the empirical correlation coefficient of $(\delta_i, Y_{(i)})$, $i = 1, \ldots, n$. It will be seen that although being consistent, the test based on S shows poor power performance and thus, should not be recommended.

Among the alternative distributions considered are the Gamma, the Weibull and the Lognormal family of distributions with scale parameter 1 and shape parameter θ as well as the Uniform, the Half-Normal, the Half-Cauchy and the χ_1^2 distribution. Other families included are the Power distributions (density $\theta^{-1}x^{(1-\theta)/\theta}$, $0 \leq x \leq 1$), the LIFR (linear increasing failure rate) distributions (density $(1 + \theta x)\exp(-(x + (\theta/2)x^2)))$ and the JSHAPE family of distributions (density $(1 + \theta x)^{-(\theta+1)/\theta}$).

These distributions include widely used, more complex alternatives to the exponential model so as to satisfy the analyst's interest to detect the existence of such a situation. Apart from distributions with increasing and decreasing hazard rates, models with U-shaped (Power(θ) for $\theta > 1$) and inverted U-shaped hazard functions (Lognormal(θ)) have been included. The JSHAPE(θ) family has J-shaped densities with heavier tails than the exponential distribution which arises as limiting case as $\theta \to 0$.

Estimates of powers are shown in Tables 4 and 5. Each number represents the percentage of 5000 Monte Carlo samples declared to be significant by the various tests under discussion, rounded to the next integer. An asterisk denotes power 100%. The level of significance is 5%, and the sample size is $n = 20$ for Table 4 and $n = 50$ for Table 5. All simulations were run on an IBM PS/2 personal computer. Using a linear congruential method to generate uniform random numbers, pseudorandom numbers of all distributions given above were generated using standard techniques (acceptance-rejection method, polar method or direct inversion).

The main conclusions that can be drawn from the simulation results are the following:

1) The tests of Patwardhan (1988) and Sarkadi (1975) have poor power over the whole range of alternatives in comparison with the other procedures under discussion and thus should not be recommended as omnibus tests for exponentiality.

2) The new test based on $T(1)$ is slightly less powerful than M but slightly more powerful than both C^2 and A^2 for Gamma alternatives with $\theta > 1$. In the case of these alternatives it clearly dominates W^*.

3) For Weibull alternatives with $\theta < 1$, $T(.1)$ provides the best results, followed by M. In this case $T(1)$ is slightly less powerful, but comparable to C^2 and A^2. For Weibull alternatives with $\theta > 1$, $T(1)$ is slightly better than all the other tests.

4) For the Lognormal family, $T(1)$ is comparable in power to the omnibus tests C^2 and A^2. The same holds for LIFR distributions. For the JSHAPE family it slightly dominates over C^2 and A^2.

5) For the Power(θ) family the performance of the various procedures depends markedly on the value of θ.

6) Of the three new tests under discussion, $T(.1)$ provides the best results for some alternatives having infinite density at zero (Weibull(θ) for $\theta < 1$, Lognormal(θ) for small θ, χ_1^2 and Power(θ) for large θ). $T(10)$ works best for some alternative distributions with markedly different tail behaviour compared with the exponential distribution (Half-Cauchy, Uniform$(0,1)$ and Lognormal(θ) for large θ).

Over the whole range of alternative distributions considered, $T(1)$ constitutes a serious competitor to the omnibus tests of Cramér-von Mises and Anderson-Darling, both based on the empirical distribution function.

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