Then A^+ is

$$
\begin{array}{c} \\ \\ A^+ = \\ \\ \end{array}
\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}
\begin{array}{c} \begin{array}{ccccccc} 4 & 5 & 6 & 7 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}.
$$

For simplicity, we assume that $Y = D(y_4y_5y_6y_7y_1y_2y_3)$. Since A is already a form of a fundamental cut-set matrix, we have directed tree $t_0 = \{y_1y_2y_3\}$. Thus from $A \odot A^+ = A^+$

$$T^{v_1}(t_0) = \{y_7y_2y_3\}$$

$$T^{v_2}(t_0) = \{y_4y_1y_3\}$$

$$T^{v_3}(t_0) = \{\{y_5y_1y_2\}\{y_6y_1y_2\}\}.$$

$T^{v_1v_2}(t_0)$ can be obtained by forming $A(12)$ from A as

$$
A(12) = \begin{array}{c} \begin{array}{cccc} 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} -1 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 \end{bmatrix} \end{array}
$$

and

$$
A^+(12) = \begin{array}{c} \begin{array}{cccc} 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}.
$$

Only columns 4 and 7 make a nonzero major determinant of $A^+(12)$, thus

$$T^{v_1v_2}(t_0) = \{y_4y_7y_3\}.$$

Similarly,

$$T^{v_1v_3}(t_0) = \{\{y_6y_7y_2\}\{y_5y_7y_2\}\}$$

$$T^{v_2v_3}(t_0) = \{\{y_4y_5y_1\}\{y_4y_6y_1\}\}$$

and

$$T^{v_1v_2v_3}(t_0) = \{y_7y_4y_6\}.$$

It is easily seen that the procedure becomes simpler if we only need to obtain all trees of a graph. Furthermore, obtaining two-trees corresponding to calculating the cofactors of the coefficient matrix of a node basis equation can be accomplished by using the same procedure with properly modified incidence matrices.

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A Class of Finite Memory Interpolation Filters

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Abstract—Sufficient conditions are given for an interpolation filter to have an impulse response that vanishes outside a finite interval of the time axis, that is to have a finite memory. These conditions are that the transfer function be of the form $G(s)/G(z)$, where $G(s)$ is proper, rational, and has poles limited to the strip $|\text{Im } s| < \pi$; and where $1/G(z)$ is a polynomial. The filters R_{mp} are included in this class, and these are characterized by the fact that their effect is to interpolate an $(m+p-1)$-order polynomial in each interval through p past and m future points. The interpolation filters described can be used to derive digital filters that approximate an arbitrary linear time-invariant continuous-time operator. It is shown that in the case of integration, the R_{mp} filters lead to well-known Lagrangian integration formulas.

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I. Introduction

INTERPOLATION filters—linear operators that produce continuous output signals from discrete input sequences—are of interest to the systems analyst.

This is especially true since the digital computer has been used to simulate and analyze continuous systems. One application is the derivation of numerical operators that approximate continuous operators.[1] This is done by introducing hypothetical samplers and interpolation filters in continuous systems and then using z-transform techniques to derive a sampled-data approximation to the original system. This method is classical in numerical analysis,[2] where, for example, integration formulas are derived by integrating and sampling a polynomial reconstruction of a numerical sequence.

The main purpose of this paper is to derive the transfer functions of a class of interpolation filters with certain desirable properties. These properties are 1) that they are linear time-invariant operators, 2) that they reproduce certain polynomials precisely from their samples, and 3) that they have a finite memory in the sense that their response to a unit sample vanishes outside an interval of the time axis. Special cases of these filters are the familiar zero-order and first-order holds, and the linear point connector. We begin with certain preliminary material concerning digital signals and z-transforms.

II. Preliminaries

Two types of signals, analog and digital, will be considered. Analog signals will be defined as real functions of a continuous time parameter t, and will be assumed to be identically zero for $t < 0$. It will also be assumed that every analog signal $f(t)$ is dominated in magnitude by an exponential function, that is,

$$|f(t)| < Me^{at} \qquad M > 0, t \geq 0,$$

so that the Laplace transform of $f(t)$ and $F(s)$ will be analytic in the half-plane Re $(s) > a$.

Digital signals will be defined as real sequences on the non-negative integers; each may or may not represent samples of an analog signal. As in the analog case, every digital signal f_n will be assumed to be dominated in magnitude by

$$|f_n| < MK^n \qquad M > 0, \qquad n \geq 0,$$

so that the z-transform, defined by

$$F(z) = \sum_{n=0}^{\infty} f_n z^{-n},$$

will be analytic in the region $|z| > K$.

In the case that a digital signal, f_n, is obtained from an analog signal, $f(t)$, by sampling it at the time instants $t = nT$, $n \geq 0$, the digital signal f_n can be interpreted as the impulse train

$$\sum_{n=0}^{\infty} f(nT)\ \delta(t - nT) \cdot$$

The Laplace transform of this impulse train,

$$\sum_{n=0}^{\infty} f(nT) e^{-nsT},$$

is a function of e^{sT} and is equal to the z-transform of f_n, provided we make the identification $z = e^{sT}$. The z-transform of f_n is related to the Laplace transform of $f(t)$, when $F(s)$ is proper and rational, by

$$F(z) = Z[F(s)] = \sum_{\substack{\text{poles of} \\ F(s)}} \text{Res} \left[\frac{F(s)}{1 - e^{sT} z^{-1}} \right]. \qquad (1)$$

This operation is also called, by an abuse of notation, the z-transform. In what follows, z will be substituted freely for e^{sT}.

Equation (1) is derived by using the contour integral formula for complex convolution in the frequency domain, and by closing the contour of integration at infinity to the left. It is assumed that $F(s)$ is replaced by $F(s)e^{\epsilon s}$, where ϵ is arbitrarily small, to ensure that the integral on the infinite radius path is zero; this implies that we adopt the convention that the time function is sampled at the times $t = nT + \epsilon$.

A time-invariant linear operator on analog signals will be called an analog filter, and is represented in the time domain by convolution with the impulse response, and in the Laplace transform domain by multiplication by the transfer function. Thus, if $f(t)$ and $g(t)$ denote, respectively, the input and output of an analog filter H,

$$g(t) = \int_{-\infty}^{\infty} h(t - \tau) f(\tau)\, d\tau,$$

and

$$G(s) = H(s)F(s),$$

where $H(s)$ is the Laplace transform of the impulse response $h(t)$. In complete analogy, a time-invariant linear operator on digital signals will be called a digital filter. If f_n and g_n denote the input and output, respectively, of a digital filter H, then

$$g_n = \sum_{k=-\infty}^{\infty} h_{n-k} f_k$$

and

$$G(z) = H(z)F(z),$$

where $H(z)$ is the z-transform of the impulse response h_n.

III. Interpolation Filters

A *reconstruction filter* is defined to be a time-invariant linear operator that accepts digital signals as inputs and produces analog signals as outputs. By "time-invariant" it is meant that a shift in the input sequence of m samples results in a shift in the output of mT. If $h(t)$ is the response of a reconstruction filter H to a unit input at $t = 0$, then the response of H to the digital signal f_n is

$$g(t) = \sum_{n=0}^{\infty} h(t - nT) f_n.$$

The Laplace transform of the output is thus

$$G(s) = \sum_{n=0}^{\infty} H(s)e^{-snT}f_n = H(s)F(z)\cdot$$

An *interpolation filter* is defined to be a reconstruction filter that has the property that sample values of the output at times $t = nT$ coincide with the values of the input digital signal at those times. This means that

$$G(z) = Z[H(s)F(z)] = F(z).$$

Since the z-transform operator commutes with multiplication by a function of z, it follows that for a reconstruction filter to be an interpolation filter, it is necessary and sufficient that $H(z) = 1$.

A *finite memory interpolation filter* is defined to be an interpolation filter that has the property that the response to a unit sample at $t = 0$ is identically zero outside some finite region. This means that the reconstructed signal will depend only on a finite number of neighboring sample values at any time.

If one wishes to reconstruct a function $g(t)$ from its samples g_n, one may use an interpolation filter whose response to a unit input at $t = 0$ has the transform $G(s)/G(z)$. This will result in reconstructing $g(t)$ perfectly, since the response to the digital signal with z-transform $G(z)$ will have a Laplace transform $G(s)$. It is instructive to consider some examples. From now on we assume that the sampling period, T, is one.

Example 1

To reconstruct the Heaviside step function $u(t)$ perfectly, one would use the filter with transfer function

$$\frac{G(s)}{G(z)} = \frac{1}{s}(1 - z^{-1}),$$

since the Laplace transform of $u(t)$ is $1/s$ and the z-transform is $1/(1 - z^{-1})$. This gives rise to the familiar zero-order hold.

Example 2

To reconstruct $tu(t)$ perfectly, one would use

$$\frac{G(s)}{G(z)} = \frac{1}{s^2}\frac{(1 - z^{-1})^2}{z^{-1}}.$$

This is called the linear point connector, since it connects successive input samples with straight lines.

Example 3

To reconstruct $t^2u(t)$ perfectly, one would use

$$\frac{G(s)}{G(z)} = \frac{2}{s^3}\frac{2(1 - z^{-1})^3}{z^{-1}(1 + z^{-1})}.$$

In Example 1 the filter is a finite memory interpolation filter that is physically realizable in the sense that the output does not depend on future input values. In Example 2 the filter is still finite memory but is not physically realizable. In Example 3 the filter is neither finite memory nor physically realizable. In order to simulate a filter on a computer, one does not need physical realizability. The requirement of having a finite memory is of great importance, however, in problems of computation, since the interpolated function can be calculated in terms of a finite number of sample values without truncation error. This is also a sufficient, but not necessary, condition that numerical formulas derived from these filters depend on only a finite number of sample values. For this reason, we shall investigate sufficient conditions to ensure the finite memory property of interpolation filters.

IV. A Sufficient Condition for Finite Memory Interpolation Filters

We need the following:

Lemma 1

Let $G(s)$ be a proper rational function of s with n poles, all within the strip $|\text{Im } s| < \pi$. Then no more than the first $n - 1$ consecutive sample values of $g(k)$ can vanish simultaneously, unless $g(t)$ is identically zero.

Proof: Assume $G(s)$ has p distinct poles $\alpha_1, \alpha_2, \cdots, \alpha_p$ with multiplicities $m_1, m_2, \cdots, m_p$, respectively. Let $n = \sum_{i=1}^{p} m_i$. Then

$$g(k) = \sum_{i=1}^{p} \sum_{j=1}^{m_i} b_{ij}k^{(j-1)}e^{\alpha_i k}.$$

Let $\mathbf{g}_{ij}$ be an n-dimensional vector whose kth coordinate is $k^{j-1}e^{\alpha_i k}$. The output $g(k)$ for the first n sample points can be represented as a vector that is a linear sum of the $\mathbf{g}_{ij}$'s. It can be shown that the $\mathbf{g}_{ij}$'s are linearly independent (see Appendix). Thus for $g(t)$ to vanish for the first n sample points, all the b_{ij}'s must be zero, and hence $g(t)$ itself must be identically zero. Q.E.D.

With this we can state:

Theorem 1

Sufficient conditions for an interpolation filter of the form $G(s)/G(z)$ to be finite memory, having a response that vanishes identically in the ranges $t < -m$ and $t > n - m$, are

1) $G(s)$ be a proper rational function of s with n poles all within the strip $|\text{Im } s| < \pi$;
2) $1/G(z)$ be a finite polynomial in positive and negative powers of z, with highest power term z^m.

Proof. By Condition 2), we may write $G(z)$ in the form

$$G(z) = \frac{z^{-m}}{\sum_{k=0}^{n} a_k z^{-k}} \qquad a_0 \neq 0, \quad 0 \leq m \leq n\cdot \qquad (2)$$

The response of $G(s)/G(z)$ to a unit sample at $n = 0$ is then

$$h(t) = \sum_{k=0}^{n} a_k g(t - k + m)u(t - k + m)\cdot$$

This vanishes for $t < -m$, since the unit step functions all vanish in this range. For $t > n - m$ this becomes

$$h(t) = \sum_{k=0}^{n} a_k g(t - k + m) \qquad t > n - m,$$

since the unit step functions are all unity in this range.

We may write in general

$$g(t) = \sum_{i=1}^{p} \sum_{j=1}^{m_i} b_{ij} t^{j-1} e^{\alpha_i t} \qquad t \geq 0.$$

Hence

$$h(t) = \sum_{k=0}^{n} a_k \sum_{i=1}^{p} \sum_{j=1}^{m_i} b_{ij}(t - k + m)^{j-1} e^{\alpha_i(t-k+m)} \qquad t > n - m.$$

The function $h(t + n - m)u(t)$ vanishes at all sample points because $h(t)$ is the response of an interpolation filter. But this function has a proper rational Laplace transform, with poles α_i. By Lemma 1 this is impossible unless $h(t + n - m)u(t) \equiv 0$. Hence $h(t)$ vanishes identically for $t > n - m$. Q.E.D.

Examples 1 and 2 illustrate this theorem for the cases $n = 1$, $m = 0$; and $n = 2$, $m = 1$, respectively.

The filters $G(s)/G(z)$ are analogous to least-mean-square reconstruction filters where $G(s)$ and $G(z)$ are spectral densities. Theorem 1, in fact, corresponds to the condition of an all-pole sampled spectral density, and this theorem is similar to that given in a statistical context.[3]

V. The Class R_{mp}

Most of the interpolation formulas used in numerical analysis employ polynomials as the interpolating functions. This corresponds to a $G(s)$ with poles only at $s = 0$ and to a $G(z)$ with poles only at $z = 1$. The most general such functions that satisfy the conditions of the preceding lemma correspond to a choice of

$$G(z) = z^{-m}(1 - z^{-1})^{-(m+p)}$$

$$m \geq 0, \qquad p \geq 0, \qquad m + p > 0, \tag{3}$$

and the resulting finite memory interpolation filters will henceforth be called R_{mp}. Of interest is the corresponding time function $g(t)$.

Lemma 2

With $G(z)$ given by (3) an $G(s)$ a rational function of s, we have

$$g(t) = \frac{(t + p - 1)(t + p - 2) \cdots (t - m + 1)}{(m + p - 1)(m + p - 2) \cdots (1)} u(t)$$

$$= \frac{(t + p - 1)!\, u(t)}{(t - m)!\,(m + p - 1)!} \tag{4}$$

where $x!$ is defined formally to be $\Gamma(x + 1)$ if x is not an integer.

Proof: Since $g(n) = 0$ for $n < m$

$$G(z) = \sum_{n=m}^{\infty} g(n) z^{-n} = \sum_{n=m}^{\infty} \frac{(n + p - 1)!\, z^{-n}}{(n - m)!\,(m + p - 1)!}$$

$$= z^{-m} \sum_{k=0}^{\infty} \frac{(k + m + p - 1)!\, z^{-k}}{k!\,(m + p - 1)!}$$

$$= z^{-m}(1 - z^{-1})^{-(m+p)}. \qquad \text{Q.E.D.}$$

The central property of the interpolation filters R_{mp} is expressed in the following.

Theorem 2

Consider the digital signal with samples f_n. If this is used as an input to the interpolation filter R_{mp}, the output in the time interval $j < t < j + 1$ is the unique polynomial of order $m + p - 1$ through the $m + p$ points $f_{j-p+1}, f_{j-p+2}, \cdots, f_{j+m}$.

Proof: For R_{mp} filters

$$\frac{z^{-m}}{(1 - z^{-1})^{m+p}} = \frac{z^{-m}}{\sum_{i=0}^{m+p} (-1)^i \binom{m+p}{i} (z^{-1})^i},$$

and the impulse response

$$h(t) = \sum_{i=0}^{m+p} (-1)^i \binom{m+p}{i} \cdot \frac{(t + p + m - 1 - i)!}{(t - i)!(m + p - 1)!} u(t + m - i).$$

For $t < -m$ and for $t > p$, $h(t)$ is identically zero, by Theorem 1. We shall show that for $-m \leq j \leq p - 1$, in the time interval $j < t < j + 1$, $h(t)$ is the unique polynomial $P(t)$, of order $m + p - 1$ with

$$P(t) = \begin{cases} 1 & t = 0 \\ 0 & t = j - p + 1, \quad j - p + 2, \cdots, j + m \\ & \text{except} \quad t = 0. \end{cases}$$

The theorem then follows trivially by linearity. For $j = -m$

$$h(t) = \frac{1}{(m + p - 1)!} \prod_{\substack{s=-p-m+1 \\ s \neq 0}}^{0} (t - s)$$

$$\text{for} \quad -m < t < -m + 1$$

as claimed. Assume for $j = -m + l$, $0 \leq l \leq m + p - 1$,

$$h(t) = \frac{(-1)^l}{(m + p - l - 1)!\, l!} \left[\prod_{\substack{s=-p-m+l+1 \\ s \neq 0}}^{l} (t - s) \right] \qquad \text{for} \quad j < t < j + 1. \tag{5}$$

Then for $j = -m + l + 1$,

$$h(t) = \frac{(-1)^l}{(m+p-l-1)!\,l!}\left[\prod_{\substack{s=-p-m+l+1 \\ s\neq 0}}^{l} (t-s)\right] + \frac{(-1)^{l+1}}{(m+p-1)!}\binom{m+p}{l+1}\prod_{s=-p-m+l+2}^{l}(t-s)$$

$$= \left[\prod_{\substack{s=-p-m+l+2 \\ s\neq 0}}^{l}(t-s)\right]\left[\frac{(-1)^l(t+p+m-l-1)}{(m+p-l-1)!\,l!} + \frac{(-1)^{l+1}(m+p)!\,t}{(m+p-1)!\,(l+1)!\,(m+p-l-1)!}\right]$$

$$= \frac{(-1)^{l+1}}{(m+p-l-2)!\,(l+1)!}\left[\prod_{\substack{s=-p-m+l+2 \\ s\neq 0}}^{l}(t-s)\right]\left[\frac{-(l+1)(t+p+m-l-1)}{(m+p-l-1)} + \frac{(m+p)t}{(m+p-l-1)}\right]$$

$$= \frac{(-1)^{l+1}}{(m+p-l-2)!\,(l+1)!}\left[\prod_{\substack{s=-p-m+l+2 \\ s\neq 0}}^{l}(t-s)\right][t-(l+1)]$$

$$= \frac{(-1)^{l+1}}{(m+p-l-2)!\,(l+1)!}\left[\prod_{\substack{s=-p-m+l+2 \\ s\neq 0}}^{l+1}(t-s)\right]. \tag{6}$$

But (6) is (5) with l replaced by $l + 1$ and thus the claim follows by induction on l. Q.E.D.

From the explicit expression (4) for the functions $g_{mp}(t)$, there follow the useful recursion relations involving the Laplace transforms $G_{mp}(s)$:

$$G_{m,p+1}(s) = \frac{1}{m+p}\left[-\frac{d}{ds} + p\right] G_{mp}(s) \tag{7}$$

$$G_{m+1,p}(s) = \frac{1}{m+p}\left[-\frac{d}{ds} - m\right] G_{mp}(s), \tag{8}$$

and in the symmetrical case when $m = p$

$$G_{m+1,m+1}(s) = \frac{1}{(2m)(2m+1)}\left[\frac{d^2}{ds^2} - m^2\right] G_{mm}(s). \tag{9}$$

To derive the $G_{mm}(s)$ for example, we start with

$$G_{11}(s) = \frac{1}{s^2}$$

and find from (9)

$$G_{22}(s) = \frac{1}{6}\left[\frac{6}{s^4} - \frac{1}{s^2}\right]$$

$$G_{33}(s) = \frac{1}{20}\left[\frac{20}{s^6} - \frac{5}{s^4} + \frac{4}{s^2}\right] \cdots \text{etc.},$$

and these together with (3) give R_{mm}.

The filters $R_{0p}(s)$ correspond to realizable polynomial extrapolators, such as the zero- and first-order holds.

VI. Derivation of Numerical Operators Using Filters R_{mp}

An immediate application of the filters R_{mp} is the derivation of numerical formulas for simulating continuous operations on a digital computer.[1] Suppose we wish to simulate a linear operator with transfer function $G(s)$. Introduce a hypothetical sampler and interpolation filter R_{mp} before $G(s)$, and then sample the output. The overall effect is equivalent to a digital filter with transfer function $Z[R_{mp}(s)G(s)]$.

An important case is that of integration. Consider first the linear point connector of Example 2. The digital integration filter becomes

$$Z\left[R_{11}(s)\frac{1}{s}\right] = \frac{1}{2}\frac{1+z^{-1}}{1-z^{-1}},$$

and corresponds to the following pattern of weights for definite integration

$$\tfrac{1}{2}, 1, 1, \cdots, 1, \tfrac{1}{2},$$

which is the trapezoidal rule.[4]

The filter R_{22} leads similarly to the digital integration filter

$$Z\left[R_{22}(s)\frac{1}{s}\right] = \frac{-z + 13 + 13z^{-1} - z^{-2}}{24(1-z^{-1})}$$

and the weights

$$-\frac{1}{24}, \frac{1}{2}, \frac{25}{24}, 1, 1, \cdots, 1, 1, \frac{25}{24}, \frac{1}{2}, -\frac{1}{24}.$$

This uses one point outside the integration interval and is the modified trapezoidal rule.[4]

Similarly, R_{33} yields the digital integration filter

$$Z\left[R_{33}(s)\frac{1}{s}\right] = \frac{11z^2 - 93z + 802 + 802z^{-1} - 93z^{-2} + 11z^{-3}}{1440(1-z^{-1})}$$

and the weights

$$\frac{11}{1440}, \frac{-82}{1440}, \frac{1}{2}, \frac{1522}{1440}, \frac{1429}{1440}, 1, 1, \cdots, 1, 1, \frac{1429}{1440}, \frac{1522}{1440}, \frac{1}{2}, \frac{-82}{1440}, \frac{11}{1440}$$

using two points outside the interval of integration. The numerator coefficients of the last two integration filters can be found in a table of Lagrangian integration coefficients.[5]

VII. Summary

We have described a class of finite memory interpolation filters characterized by the transfer functions $G(s)/G(z)$, where $G(s)$ and $G(z)$ have the restrictions stated in Theorem 1. Of particular interest is the class R_{mp}, which represents interpolation filters whose effect in each interval is to interpolate an $(m + p - 1)$-order polynomial through p points in the past and m future points. Such filters are useful in deriving digital filters that approximate arbitrary time-invariant linear operators. In order to approximate an operator with transfer function $H(s)$, the digital filter with transfer function $Z[R_{mp}(s)H(s)]$ is used. In the simple case of integration, this yields the well-known Lagrangian integration formulas. Such an approach provides a link between sampled-data theory and classical numerical analysis.

Appendix

The nth-order Vandermonde determinant is

$$\begin{vmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-1} \\ \vdots \\ 1 & a_n & a_n^2 & \cdots & a_n^{n-1} \end{vmatrix}.$$

This determinant arises in discrete linear systems with distinct poles. For discrete linear systems with nondistinct poles, determinants similar to Vandermonde's determinant appear. We shall call these determinants generalized Vandermonde determinants.

Let

$$D_n \begin{bmatrix} a_1 & a_2 & \cdots & a_r \\ m_1 & m_2 & \cdots & m_r \end{bmatrix}$$

be the generalized Vandermonde determinant.

$$\begin{vmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\ 0 & a_1 & 2a_1^2 & \cdots & (n-1)a_1^{n-1} \\ \vdots \\ 0 & a_1 & 2^{m_1-1}a_1^2 & \cdots & (n-1)^{m_1-1}a_1^{n-1} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-1} \\ \vdots \\ 0 & a_r & 2^{m_r-1}a_r^2 & \cdots & (n-1)^{m_r-1}a_r^{n-1} \end{vmatrix}.$$

where

$$\sum_{i=1}^{r} m_i = n, \quad m_i \geq 1$$

Theorem 3

$$D_n \begin{bmatrix} a_1 & a_2 & \cdots & a_r \\ m_1 & m_2 & \cdots & m_r \end{bmatrix} = \prod_{i=1}^{r} \left\{ \prod_{k=1}^{m_i} \left[(m_i - k)! \right] a_i^{[m_i(m_i-1)]/2} \prod_{j<i} (a_i - a_j)^{m_i m_j} \right\}.$$

Proof: The proof is by induction on r. For the special case $r = n$, $m_i = 1$ for all m_i, and we get the usual Vandermonde determinant. Note

$$D_n \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ 1 & 1 & \cdots & 1 \end{bmatrix} = \prod_{i=1}^{n} \prod_{j<i} (a_i - a_j)$$

which is indeed the value of the Vandermonde determinant.

Now assume the theorem true for all D_n with $n \geq r > p$ and show that this implies that the theorem is true for all D_n with $r = p$.

$$D_n \begin{bmatrix} a_1 & a_2 & \cdots & a_p \\ m_1 & m_2 & \cdots & m_p + 1 \end{bmatrix} = \left(a_{p+1} \frac{\partial}{\partial a_{p+1}} \right)^{m_p} D_n \begin{pmatrix} a_1 & a_2 & \cdots & a_p & a_{p+1} \\ m_1 & m_2 & \cdots & m_p & 1 \end{pmatrix} \Bigg|^{*}_{a_{p+1} \to a_p}.$$

Thus by the induction hypothesis

$$D_n \begin{bmatrix} a_1 & a_2 & \cdots & a_p \\ m_1 & m_2 & \cdots & m_p + 1 \end{bmatrix} = \left(a_{p+1} \frac{\partial}{\partial a_{p+1}} \right)^{m_p} \prod_{i=1}^{p+1} \left\{ \prod_{k=1}^{m_i} [(m_i - k)!]\, a_i^{[m_i(m_i-1)]/2} \prod_{j<i} (a_i - a_j)^{m_i m_j} \right\} \Bigg|_{a_{p+1} \to a_p}$$

$$= \prod_{i=1}^{p} \left\{ \prod_{k=1}^{m_i} [(m_i - k)!]\, a_i^{[m_i(m_i-1)]/2} \prod_{j<i} (a_i - a_j)^{m_i m_j} \right\}$$

$$\cdot \left(a_{p+1} \frac{\partial}{\partial a_{p+1}} \right)^{m_p} \left\{ \prod_{k=1}^{m_{p+1}} [(m_{p+1} - k)!]\, a_{p+1}^{[m_{p+1}(m_{p+1}-1)/2]} \prod_{j<p+1} (a_{p+1} - a_j)^{m_{p+1} m_j} \right\} \Bigg|_{a_{p+1} \to a_p}.$$

Note that $m_{p+1} = 1$. Thus

$$D_n \begin{bmatrix} a_1 & a_2 & \cdots & a_p \\ m_1 & m_2 & \cdots & m_p + 1 \end{bmatrix}$$

$$= \prod_{i=1}^{p} \left\{ \prod_{k=1}^{m_i} [(m_i - k)!]\, a_i^{[m_i(m_i-1)]/2} \prod_{j<i} (a_i - a_j)^{m_i m_j} \right\} \left(a_{p+1} \frac{\partial}{\partial a_{p+1}} \right)^{m_p} \left\{ \prod_{j<p+1} (a_{p+1} - a_j)^{m_j} \right\} \Bigg|_{a_{p+1} \to a_p}.$$

* Note that interchanging the order of the parameters leaves the formula unchanged. Thus without loss of generality we consider only the case where the multiplicity of a_p is greater than 1.

$$D_n\begin{bmatrix} a_1 & a_2 & \cdots & a_p \\ m_1 & m_2 & \cdots & m_p + 1\end{bmatrix} = \prod_{i=1}^{p}\left\{\prod_{k=1}^{m_i}[(m_i - k)!]\, a_i^{[m_i(m_i-1)]/2} \prod_{j<i}(a_i - a_j)^{m_i m_j}\right\} a_p^{m_p} m_p! \prod_{j<p}(a_p - a_j)^{m_j}$$

$$= \prod_{i=1}^{p-1}\left\{\prod_{k=1}^{m_i}[(m_i - k)!]\, a_i^{[m_i(m_i-1)]/2} \prod_{j<i}(a_i - a_j)^{m_i m_j}\right\}$$

$$\cdot \prod_{k=1}^{m_p}[(m_p - k)!]\, a_p^{[m_p(m_p-1)]/2} \prod_{j<p}(a_p - a_j)^{m_p m_j} a_p^{m_p} m_p! \prod_{j<p}(a_p - a_j)^{m_j}$$

$$= \prod_{i=1}^{p-1}\left\{\prod_{k=1}^{m_i}[(m_i - k)!]\, a_i^{[m_i(m_i-1)]/2} \prod_{j<i}(a_i - a_j)^{m_i m_j}\right\}$$

$$\cdot \prod_{k=1}^{m_p+1}[(m_p + 1 - k)!]\, a_p^{[(m_p+1)(m_p)]/2} \prod_{j<p}(a_p - a_j)^{(m_p+1)m_j}. \qquad \text{Q.E.D.}$$

Now $(a_{p+1}\,\partial/\partial a_{p+1})^{m_p}$ can be expanded in the form

$$\sum_{s=1}^{m_p} b_s a_{p+1}^s \frac{\partial^s}{\partial a_{p+1}^s}$$

where $b_{m_p} = 1$.

Since $a_{p+1} - a_p$ appears m_p times in the product term that we are going to differentiate and since we are going to set $a_{p+1} = a_p$, the only nonzero term of

$$\left(a_{p+1}\frac{\partial}{\partial a_{p+1}}\right)^{m_p} \prod_{j<p+1}(a_{p+1} - a_j)^{m_j}\Big|_{a_{p+1}\to a_p}$$

will be

$$[\prod_{j<p}(a_{p+1} - a_j)^{m_j}] a_{p+1}^{m_p} \frac{\partial^{m_p}}{\partial a_{p+1}^{m_p}}(a_{p+1} - a_p)^{m_p}\,|_{a_{p+1}\to a_p}$$

which is

$$a_p^{m_p} m_p! \prod_{j<p}(a_p - a_j)^{m_j}.$$

Thus

Corollary 3

$$D_n\begin{bmatrix} a_1 & a_2 & \cdots & a_p \\ m_1 & m_2 & \cdots & m_p\end{bmatrix}$$

is singular if, and only if, $a_i = a_j$ for some i and j or $a_i = 0$ for some i such that $m_i > 1$.

Proof: Follows immediately from Theorem 3. Q.E.D.

It then follows in the proof of Lemma 1 that the $\mathbf{g}_{ij}$'s are linearly independent, since the $a_i = e^{\alpha_i}$ are distinct and nonzero. It will also be noted that the condition $|\text{Im}\ \alpha_i| < \pi$ can be relaxed to the condition $e^{\alpha_i} \neq e^{\alpha_j}$ unless $i = j$.

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The Nonlinear Theory of a Class of Transistor Oscillators

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***Abstract*—The nonlinear theory of a class of transistor oscillators is developed, using the Ebers–Moll large-signal model for the transistors. Simplified versions of tuned-collector, tuned-base, and Hartley transistor oscillators are shown to be characterized by a nonlinear differential equation of the form**

$$\ddot{x} - \mu[e^{a\dot{x}} - \kappa e^{(a+b)\dot{x}}] + \gamma\dot{x} + x = 0,$$

where μ, κ, γ, a, and b are positive constants and $\kappa \ll 1$. Approximate solutions of the above equation, which are derived in a very simple manner using the phase plane approach, are compared favorably with experimental results. A push–pull version of the tuned-collector oscillator characterized by the above equation with the exponential terms replaced by hyperbolic sines is discussed.

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