A Class of Finsler Metrics with Isotropic S-curvature

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Abstract

In this paper, we study a class of Finsler metrics defined by a Riemannian metric and a 1-form. We characterize these metrics with isotropic S-curvature.

1 Introduction

The S-curvature is one of the most important non-Riemannian quantities in Finsler geometry which was first introduced by the second author when he studied volume comparison in Riemann-Finsler geometry [10]. The second author proved that the Bishop-Gromov volume comparison holds for Finsler manifolds with vanishing S-curvature. He also proved that the S-curvature and the Ricci curvature determine the local behavior of the Busemann-Hausdorff measure of small metric balls around a point [16]. Recent study shows that the S-curvature plays a very important role in Finsler geometry (cf. [8][12][15]). It is known that, for a Finsler metric F of scalar flag curvature, if the S-curvature is almost isotropic, i.e.,

$$\mathbf{S} = (n+1)cF + \eta, \tag{1}$$

where $c = c(x)$ is a scalar function and η is a closed 1-form, then the flag curvature must be in the following form

$$\mathbf{K} = \frac{3\tilde{c}_{x^m} y^m}{F} + \sigma, \tag{2}$$

where $\sigma = \sigma(x)$ and $\tilde{c} = \tilde{c}(x)$ are scalar functions with $c - \tilde{c} = constant$ [4]. Therefore it is an important problem to study and characterize Finsler metrics of (almost) isotropic S-curvature.

In Finsler geometry, there is an important class of Finsler metrics—Randers metrics which were introduced and studied by G. Randers. A Randers metric

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is a Finsler metric expressed in the form $F = \alpha + \beta$, where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form with $\|\beta_x\|_\alpha < 1$. Let

$$r_{ij} = \frac{1}{2}(b_{i|j} + b_{j|i}), \qquad s_{ij} = \frac{1}{2}(b_{i|j} - b_{j|i}),$$

$$r_j := b^i r_{ij}, \qquad s_j := b^i s_{ij},$$

where $b_{i|j}$ denote the covariant derivatives of β with respect to α. In [5], we prove that the Randers metric $F = \alpha + \beta$ has isotropic S-curvature, $\mathbf{S} = (n+1)c(x)F$, if and only if

$$r_{ij} + b_i s_j + b_j s_i = 2c(a_{ij} - b_i b_j). \tag{3}$$

See [1] [17] for related work. In this paper, we generalize the above result as follows.

Theorem 1.1 *Let*

$$F = k_1\sqrt{\alpha^2 + k_2\beta^2} + k_3\beta$$

be a Finsler metric of Randers type where $k_1 > 0$ and $k_3 \neq 0$. F is of isotropic S-curvature, $F = (n+1)cF$ if and only if β satisfies

$$r_{ij} + \tau(s_i b_j + s_j b_i) = \frac{2c(1 + k_2 b^2)k_1^2}{k_3}\Big(a_{ij} - \tau b_i b_j\Big), \tag{4}$$

where

$$\tau := \frac{k_3^2}{k_1^2} - k_2.$$

If a Randers metric is of scalar flag curvature, then (1) and (2) are actually equivalent ([7], [18]). In particular, if a Randers metric is of constant flag curvature, then it must be of constant S-curvature ([1], [2]). We have classified Randers metrics of scalar flag curvature and isotropic S-curvature ([4], [7]). Further, we have characterized the locally projectively flat Finsler metrics with isotropic S-curvature ([6]).

It is natural to consider general Finsler metrics defined by a Riemannian metric $\alpha = \sqrt{a_{ij}y^iy^j}$ and a 1-form $\beta = b_i y^i$ with $\|\beta_x\|_\alpha < b_o$. They are expressed in the form $F = \alpha\phi(s)$, $s = \beta/\alpha$, where $\phi(s)$ is a C^∞ positive function on $(-b_o, b_o)$. It is known that $F = \alpha\phi(\beta/\alpha)$ is a (positive definite) Finsler metric for any α and β with $\|\beta_x\|_\alpha < b_o$ if and only if ϕ satisfies the following condition (cf. [13][14]):

$$\phi(s) - s\phi'(s) + (\rho^2 - s^2)\phi''(s) > 0, \qquad (|s| \le \rho < b_o). \tag{5}$$

Such a metric is called an (α, β)-*metric*. Clearly, Finsler metrics of Randers type are special (α, β)-metrics.

For a positive C^∞ function $\phi = \phi(s)$ on $(-b_o, b_o)$ and a number $b \in [0, b_o)$, let

$$\Phi := -(Q - sQ')\Big\{n\Delta + 1 + sQ\Big\} - (b^2 - s^2)(1 + sQ)Q'',$$

where $\Delta := 1 + sQ + (b^2 - s^2)Q'$ and $Q := \phi'/(\phi - s\phi')$. In this paper, we prove the following

Theorem 1.2 *Let $F = \alpha\phi(s), s = \beta/\alpha$, be an (α, β)-metric on a manifold and $b := \|\beta_x\|_\alpha$. Suppose that $\phi \neq k_1\sqrt{1 + k_2 s^2} + k_3 s$ for any constants $k_1 > 0, k_2$ and k_3. Then F is of isotropic S-curvature, $\mathbf{S} = (n + 1)cF$, if and only if one of the following holds*

(i) β satisfies

$$r_j + s_j = 0 \tag{6}$$

and $\phi = \phi(s)$ satisfies

$$\Phi = 0. \tag{7}$$

In this case, $\mathbf{S} = 0$.

(ii) β satisfies

$$r_{ij} = \epsilon\Big\{b^2 a_{ij} - b_i b_j\Big\}, \qquad s_j = 0, \tag{8}$$

where $\epsilon = \epsilon(x)$ is a scalar function, and $\phi = \phi(s)$ satisfies

$$\Phi = -2(n + 1)k\frac{\phi\Delta^2}{b^2 - s^2}, \tag{9}$$

where k is a constant. In this case, $\mathbf{S} = (n + 1)cF$ with $c = k\epsilon$.

(iii) β satisfies

$$r_{ij} = 0, \qquad s_j = 0. \tag{10}$$

In this case, $\mathbf{S} = 0$, regardless of the choice of a particular ϕ.

It is easy to see that (10) implies (8), while (8) implies (6). The condition (6) is equivalent to that $b := \|\beta_x\|_\alpha = constant$. See Lemma 3.2 below. Thus (7) and (9) are independent of $x \in M$.

The mean Landsberg curvature $\mathbf{J}$ is another important non-Riemannian quantity. It has been proved that for an (α, β)-metric $F = \alpha\phi(\beta/\alpha)$, if β has constant length and ϕ satisfies (7), then F is a weakly Landsberg metric, i.e., $\mathbf{J} = 0$. See [9].

We have the following two interesting examples.

Example 1.1 Let $F = \alpha + \beta$ be the family of Randers metrics on S^3 constructed in [3] (see also [16]). It is shown that $r_{ij} = 0$ and $s_j = 0$. Thus for any C^∞ positive function $\phi = \phi(s)$ satisfying (5), the (α, β)-metric $F = \alpha\phi(\beta/\alpha)$ has vanishing S-curvature.

Example 1.2 Let $F = \alpha\phi(\beta/\alpha)$ be an (α, β)-metric defined on an open subset in R^3. At a point $\mathbf{x} = (x, y, z) \in R^3$ and in the direction $\mathbf{y} = (u, v, w) \in T_\mathbf{x}R^3$, $\alpha = \alpha(\mathbf{x}, \mathbf{y})$ and $\beta = \beta(\mathbf{x}, \mathbf{y})$ are given by

$$\begin{aligned} \alpha &:= \sqrt{u^2 + e^{2x}(v^2 + w^2)}, \\ \beta &:= u. \end{aligned}$$

Then β satisfies (8) with $\epsilon = 1$, $b = 1$. Thus if $\phi = \phi(s)$ satisfies (9) for some constant k, then $F = \alpha\phi(\beta/\alpha)$ is of constant S-curvature $\mathbf{S} = (n + 1)cF$.

2 Volume forms

The S-curvature is associated with a volume form. There are two important volume forms in Finsler geometry. One is the Busemann-Hausdorff volume form and the other is the Holmes-Thompson volume form.

The Busemann-Hausdorff volume form $dV_{BH} = \sigma_{BH}(x)dx$ is given by

$$\sigma_{BH}(x) = \frac{\omega_n}{\mathrm{Vol}\big\{(y^i) \in R^n | F\big(x, y^i \frac{\partial}{\partial x^i}\big) < 1\big\}}$$

and the Holmes-Thompson volume form $dV_{HT} = \sigma_{HT}(x)dx$ is given by

$$\sigma_{HT}(x) = \frac{1}{\omega_n} \int_{\big\{(y^i)\in R^n | F\big(x,y^i \frac{\partial}{\partial x^i}\big)<1\big\}} \det(g_{ij})dy.$$

Here Vol denotes the Euclidean volume and

$$\omega_n := \mathrm{Vol}(\mathrm{B}^n(1)) = \frac{1}{n}\mathrm{Vol}(\mathrm{S}^{n-1}) = \frac{1}{n}\mathrm{Vol}(\mathrm{S}^{n-2}) \int_0^\pi \sin^{n-2}(t)dt$$

denotes the Euclidean volume of the unit ball in R^n. When $F = \sqrt{g_{ij}(x)y^iy^j}$ is a Riemannian metric, both volume forms are reduced to the same Riemannian volume form

$$dV_{BH} = dV_{HT} = \sqrt{\det(g_{ij}(x))}dx.$$

For an (α, β)-metric, we have the following formulas for the volume forms dV_{BH} and dV_{HT}.

Proposition 2.1 *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an (α, β)-metric on an n-dimensional manifold M. Let $dV = dV_{BH}$ or dV_{HT}. Let*

$$f(b) := \begin{cases} \dfrac{\int_0^\pi \sin^{n-2} t dt}{\int_0^\pi \frac{\sin^{n-2} t}{\phi(b\cos t)^n} dt} & \text{if } dV = dV_{BH} \\ \dfrac{\int_0^\pi (\sin^{n-2} t) T(b\cos t) dt}{\int_0^\pi \sin^{n-2} t dt} & \text{if } dV = dV_{TH}, \end{cases}$$

where $T(s) := \phi(\phi - s\phi')^{n-2}[(\phi - s\phi') + (b^2 - s^2)\phi'']$. Then the volume form dV is given by

$$dV = f(b)dV_\alpha,$$

where $dV_\alpha = \sqrt{\det(a_{ij})}dx$ denotes the Riemannian volume form of α.

Proof: In a coordinate system, the determinant of $g_{ij} := \frac{1}{2}[F^2]_{y^iy^j}$ is given by (cf. [14])

$$\det(g_{ij}) = \phi^{n+1}(\phi - s\phi')^{n-2}[(\phi - s\phi') + (b^2 - s^2)\phi''] \det(a_{ij}).$$

First we take an orthonormal basis at x with respect to α so that

$$\alpha = \sqrt{\sum (y^i)^2}, \qquad \beta = by^1,$$

where $b = \|\beta_x\|_\alpha$. Then the volume form $dV_\alpha = \sigma_\alpha dx$ at x is given by

$$\sigma_\alpha = \sqrt{\det(a_{ij})} = 1.$$

In order to evaluate the integrals

$$\mathrm{Vol}\{(y^i) \in R^n \big| F\big(x, y^i \frac{\partial}{\partial x^i}\big) < 1\} = \int_{F(x,y)<1} dy = \int_{\alpha\phi(\beta/\alpha)<1} dy$$

and

$$\int_{F(x,y)<1} \det(g_{ij}) dy = \int_{\alpha\phi(\beta/\alpha)<1} \det(g_{ij}) dy,$$

we take the following coordinate transformation, $\psi : (s, u^a) \to (y^i)$:

$$y^1 = \frac{s}{\sqrt{b^2 - s^2}} \bar{\alpha}, \qquad y^a = u^a, \tag{11}$$

where $\bar{\alpha} = \sqrt{\sum_{a=2}^n (u^a)^2}$. Then

$$\alpha = \frac{b}{\sqrt{b^2 - s^2}} \bar{\alpha}, \qquad \beta = \frac{bs}{\sqrt{b^2 - s^2}} \bar{\alpha}.$$

Thus

$$F = \alpha\phi(\beta/\alpha) = \frac{b\phi(s)}{\sqrt{b^2 - s^2}} \bar{\alpha}$$

and the Jacobian of the transformation $\psi : (s, u^a) \to (y^i)$ is given by

$$\frac{b^2}{(b^2 - s^2)^{3/2}} \bar{\alpha}.$$

Then

$$\begin{aligned}
\mathrm{Vol}\{(y^i) \in R^n | F(x, y) < 1\} &= \int_{\frac{b\phi(s)}{\sqrt{b^2-s^2}}\bar{\alpha}<1} \frac{b^2}{(b^2 - s^2)^{3/2}} \bar{\alpha} ds du \\
&= \int_{-b}^{b} \frac{b^2}{(b^2 - s^2)^{3/2}} \Big[\int_{\bar{\alpha}<\frac{\sqrt{b^2-s^2}}{b\phi(s)}} \bar{\alpha} du \Big] ds \\
&= \frac{1}{n} \mathrm{Vol}(\mathrm{S}^{n-2}) \int_{-b}^{b} \frac{b^2}{(b^2 - s^2)^{3/2}} \Big(\frac{\sqrt{b^2 - s^2}}{b\phi(s)} \Big)^n ds \\
&= \frac{1}{n} \mathrm{Vol}(\mathrm{S}^{n-2}) \int_{-b}^{b} \frac{(b^2 - s^2)^{(n-3)/2}}{b^{n-2}\phi(s)^n} ds \\
&= \frac{1}{n} \mathrm{Vol}(\mathrm{S}^{n-2}) \int_0^{\pi} \frac{\sin^{n-2} t}{\phi(b\cos t)^n} dt \quad (s = b\cos t).
\end{aligned}$$

Therefore

$$\sigma_{BH} = \frac{\int_0^\pi \sin^{n-2} t dt}{\int_0^\pi \frac{\sin^{n-2} t}{\phi(b\cos t)^n} dt}. \tag{12}$$

Let

$$T(s) := \phi(\phi - s\phi')^{n-2}[(\phi - s\phi') + (b^2 - s^2)\phi'']. \tag{13}$$

Then

$$\det(g_{ij}) = \phi(s)^n T(s) \det(a_{ij}).$$

By a similar argument, we get

$$\begin{aligned}
\sigma_{HT} &= \frac{1}{\omega_n} \int_{F(x,y)<1} \phi(s)^n T(s) dy^1 \cdots dy^n \\
&= \frac{1}{n\omega_n} \mathrm{Vol}(\mathrm{S}^{n-2}) \int_{-b}^{b} \frac{b^2}{(b^2 - s^2)^{3/2}} \Big(\frac{\sqrt{b^2 - s^2}}{b} \Big)^n T(s) ds \\
&= \frac{\int_0^\pi (\sin^{n-2} t) T(b\cos t) dt}{\int_0^\pi \sin^{n-2} t dt}.
\end{aligned}$$

Thus

$$\sigma_{HT} = \frac{\int_0^\pi (\sin^{n-2} t) T(b\cos t) dt}{\int_0^\pi \sin^{n-2} t dt}. \tag{14}$$

The above formulas for σ_{BH} and σ_{HT} are given in a special coordinate system at x and $\sigma_\alpha = 1$. Thus $dV = f(b) dV_\alpha$. This proves the proposition. Q.E.D.

Note that if $b = constant$, then $f(b) = constant$. In this case, both dV_{BH} and dV_{HT} are constant multiples of dV_α.

It is surprised to see that $dV_{TH} = dV_\alpha$ for certain functions ϕ.

Corollary 2.2 *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an (α, β)-metric on an n-dimensional manifold M. Let $T = T(s)$ be defined in (13). Suppose that $T(s) - 1$ is an odd function of s. Then $dV_{TH} = dV_\alpha$.*

Proof: Let $h(s) = T(s) - 1$. By assumption $h(-s) = -h(s)$. It is easy to see that

$$\int_0^\pi (\sin^{n-2} t) h(b\cos t) dt = 0.$$

Thus

$$\int_0^\pi (\sin^{n-2} t) T(b\cos t) dt = \int_0^\pi \sin^{n-2} t dt.$$

This implies that $\sigma_{HT} = 1$ in the above special coordinate system at x. Then in a general coordinate system $\sigma_{HT} = \sigma_\alpha$. Q.E.D.

If $\phi = 1 + s$, then $T = 1 + s$ and $T(s) - 1$ is an odd function of s. Then for a Randers metric, $dV_{HT} = dV_\alpha$. This fact is known to Y. B. Shen.

3 The S-Curvature

In this section, we are going to find a formula for the S-curvature of an (α,β)-metric on an n-dimensional manifold M.

Let $F = F(x,y)$ be a Finsler metric on an n-dimensional manifold M. Let $G = y^i\frac{\partial}{\partial x^i} - 2G^i\frac{\partial}{\partial y^i}$ denote the spray of F and $dV = \sigma dx$ be a volume form on M. The spray coefficients G^i are defined by

$$G^i = \frac{1}{4}g^{il}\Big\{[F^2]_{x^jy^l}y^j - [F^2]_{x^l}\Big\}.$$

Then the S-curvature (with respect to dV) is defined by

$$\mathbf{S} = \frac{\partial G^m}{\partial y^m} - y^m\frac{\partial}{\partial x^m}\Big(\ln\sigma\Big).$$

By the definition, S-curvature $\mathbf{S}(y)$ measures the average rate of changes of (T_xM, F_x) in the direction $y \in T_xM$. An important property is that $\mathbf{S} = 0$ for Berwald spaces with respect to the Busemann-Hausdorff volume form dV_{BH} [10][11].

Definition 3.1 Let F be a Finsler metric on an n-dimensional manifold M. Let $\mathbf{S}$ denote the S-curvature of F with respect to dV_{BH}. F is of isotropic S-curvature if

$$\mathbf{S} = (n+1)cF,$$

where $c = c(x)$ is a scalar function. F is of constant S-curvature if $c = constant$.

We now compute the S-curvature of an (α,β)-metric on a manifold. Let

$$F = \alpha\phi(s), \qquad s = \beta/\alpha.$$

We have the following formula for the spray coefficients G^i of F:

$$G^i = \bar{G}^i + \alpha Q{s^i}_0 + \Theta\Big\{-2Q\alpha s_0 + r_{00}\Big\}\frac{y^i}{\alpha} + \Psi\Big\{-2Q\alpha s_0 + r_{00}\Big\}b^i,$$

where $\bar{G}^i$ denote the spray coefficients of α and

$$Q := \frac{\phi'}{\phi - s\phi'}, \qquad \Theta = \frac{Q - sQ'}{2\Delta}, \qquad \Psi = \frac{Q'}{2\Delta}, \tag{15}$$

where $\Delta := 1 + sQ + (b^2 - s^2)Q'$.

It is easy to see that if $\phi = \phi(s)$ satisfies

$$b^2Q + s = 0,$$

then

$$\phi = a_0\sqrt{b^2 - s^2},$$

where a_0 is a number independent of s.

If $\phi = \phi(s)$ satisfies

$$\Psi = constant,$$

then

$$\phi = k_1\sqrt{1 + k_2 s^2} + k_3 s,$$

where k_1, k_2 and k_3 are numbers independent of s.

To compute the S-curvature, we need the following identities:

$$\frac{\partial s}{\partial y^m} = \frac{1}{\alpha}\Big\{ b_m - s\frac{y_m}{\alpha}\Big\},$$

$$\frac{\partial \alpha}{\partial y^m} = \frac{y_m}{\alpha},$$

$$\frac{\partial \bar{G}^m}{\partial y^m} = y^m \frac{\partial}{\partial x^m}\Big(\ln \sigma_\alpha\Big).$$

Using the above identities, we obtain

$$\frac{\partial G^m}{\partial y^m} = y^m \frac{\partial}{\partial x^m}\Big(\ln \sigma_\alpha\Big) + 2\Psi(r_0 + s_0) - \alpha^{-1}\frac{\Phi}{2\Delta^2}(r_{00} - 2\alpha Q s_0),$$

where

$$\Phi := -(Q - sQ')\{n\Delta + 1 + sQ\} - (b^2 - s^2)(1 + sQ)Q''. \tag{16}$$

By Proposition 2.1, $dV = \sigma dx = f(b)\sigma_\alpha dx$. Thus

$$y^m \frac{\partial}{\partial x^m}\Big(\ln \sigma\Big) = \frac{f'(b)}{f(b)} y^m \frac{\partial b}{\partial x^m} + y^m \frac{\partial}{\partial x^m}\Big(\ln \sigma_\alpha\Big).$$

$$y^m \frac{\partial b}{\partial x^m} = \frac{b^i b_{i|m} y^m}{b} = \frac{r_0 + s_0}{b}. \tag{17}$$

Then the S-curvature is given by

$$\mathbf{S} = \Big\{ 2\Psi - \frac{f'(b)}{bf(b)}\Big\}(r_0 + s_0) - \alpha^{-1}\frac{\Phi}{2\Delta^2}(r_{00} - 2\alpha Q s_0). \tag{18}$$

Lemma 3.2 *Let β be a 1-form on a Riemannian manifold (M, α). Then $b(x) := \|\beta_x\|_\alpha = constnt$ if and only if β satisfies the following equation:*

$$r_j + s_j = 0. \tag{19}$$

Proof: This follows immediately from (17). Q.E.D.

In the case when $b = constant$, the S-curvature is given by

$$\mathbf{S} = -\alpha^{-1}\frac{\Phi}{2\Delta^2}(r_{00} - 2\alpha Q s_0). \tag{20}$$

We can prove the following

Proposition 3.3 *Let $F = \alpha\phi(\beta/\alpha)$ be an (α,β)-metric on an n-manifold. If β and ϕ satisfy conditions in Theorem 1.2 (i) or (ii) or (iii), then F has isotropic S-curvature.*

Proof: If β satisfies (6) and ϕ satisfies (7), then it follows from (18) that $\mathbf{S} = 0$.
If β satisfies (8), then

$$r_{00} = \epsilon(b^2 - s^2)\alpha^2, \quad r_0 = 0, \quad s_0 = 0.$$

By (9) and the above equations, we get from (18) that

$$\mathbf{S} = -\alpha\epsilon(b^2 - s^2)\frac{\Phi}{2\Delta^2} = (n+1)k\epsilon\alpha\phi = (n+1)k\epsilon F.$$

If β satisfies (10), then

$$r_{00} = 0, \quad r_0 = 0, \quad s_0 = 0.$$

It follows from (18) that $\mathbf{S} = 0$. Q.E.D.

To prove the necessary conditions in Theorem 1.2, we consider an (α,β)-metric $F = \alpha\phi(\beta/\alpha)$ with isotropic S-curvature, $\mathbf{S} = (n+1)cF$. By (18), the equation $\mathbf{S} = (n+1)cF$ is equivalent to the following equation:

$$\alpha^{-1}\frac{\Phi}{2\Delta^2}(r_{00} - 2\alpha Q s_0) - 2\Psi(r_0 + s_0) = -(n+1)cF + \theta, \tag{21}$$

where

$$\theta := -\frac{f'(b)}{bf(b)}(r_0 + s_0). \tag{22}$$

To simplify the equation (21), we choose special coordinates. Fix an arbitrary point x. Take a local coordinate system at x as in (11). We have

$$r_1 = br_{11}, \quad r_\alpha = br_{1\alpha},$$

$$s_1 = 0, \quad s_\alpha = bs_{1\alpha}.$$

Let

$$\bar{r}_{10} := \sum_{\alpha=2}^{n} r_{1\alpha}y^\alpha, \quad \bar{s}_{10} := \sum_{\alpha=2}^{n} s_{1\alpha}y^\alpha \quad \bar{r}_{00} := \sum_{\alpha,\beta=2}^{n} r_{\alpha\beta}y^\alpha y^\beta,$$

$$\bar{r}_0 := \sum_{\alpha=2}^{n} r_\alpha y^\alpha \quad \bar{s}_0 := \sum_{\alpha=2}^{n} s_\alpha y^\alpha.$$

We have

$$\bar{r}_0 = b\bar{r}_{10}, \quad \bar{s}_0 = b\bar{s}_{10}.$$

Let $\theta = t_i y^i$. Then t_i are given by

$$t_1 = -\frac{f'(b)}{f(b)}r_{11}, \quad t_\alpha = -\frac{f'(b)}{f(b)}(r_{1\alpha} + s_{1\alpha}). \tag{23}$$

(21) is equivalent to the following two equations:

$$\frac{\Phi}{2\Delta^2}(b^2-s^2)\bar{r}_{00} = -\Big\{s\Big(\frac{s\Phi}{2\Delta^2}-2\Psi b^2\Big)r_{11}+(n+1)cb^2\phi - sbt_1\Big\}\bar{\alpha}^2, \tag{24}$$

$$\Big(\frac{s\Phi}{\Delta^2}-2\Psi b^2\Big)(r_{1\alpha}+s_{1\alpha})-(b^2Q+s)\frac{\Phi}{\Delta^2}s_{1\alpha}-bt_\alpha = 0. \tag{25}$$

Let

$$\Upsilon := \Big[\frac{s\Phi}{\Delta^2}-2\Psi b^2\Big]'.$$

We see that $\Upsilon = 0$ if and only if

$$\frac{s\Phi}{\Delta^2}-2\Psi b^2 = b^2\mu,$$

where μ is a number independent of s. We shall divide the problem into three cases: (i) $\Phi = 0$, (ii) $\Phi \neq 0$, $\Upsilon = 0$ and (iii) $\Phi \neq 0$, $\Upsilon \neq 0$.

4 $\Phi = 0$

In this section, we study the simplest case when $\Phi = 0$.

Proposition 4.1 *Let $F = \alpha\phi(\beta/\alpha)$ be an (α,β)-metric. Assume that $\Phi = 0$ but $\phi \neq k_1\sqrt{1+k_2s^2}$ for any constants $k_1 > 0$ and k_2. If F has isotropic S-curvature, then*

$$r_0 + s_0 = 0.$$

In this case, $\mathbf{S} = 0$.

Proof: Take a special coordinate system at x as in (11). (24) and (25) are reduced to

$$s\Big\{\frac{f'(b)}{bf(b)}-2\Psi\Big\}b^2r_{11}+(n+1)cb^2\phi = 0 \tag{26}$$

$$\Big\{\frac{f'(b)}{bf(b)}-2\Psi\Big\}b^2(r_{1\alpha}+s_{1\alpha}) = 0. \tag{27}$$

Letting $s = 0$ in (26) yields

$$c = 0$$

and

$$\Big\{\frac{f'(b)}{bf(b)}-2\Psi\Big\}r_{11} = 0. \tag{28}$$

If

$$\frac{f'(b)}{bf(b)}-2\Psi = 0,$$

then

$$\phi = k_1\sqrt{1+k_2s^2}+k_3s,$$

where $k_1 > 0$, k_2 and k_3 are numbers independent of s. Plugging it into the equation $\Phi = 0$ yields that $k_3 = 0$ and

$$\phi = k_1\sqrt{1 + k_2 s^2}.$$

But this is impossible by assumption. Thus

$$\frac{f'(b)}{bf(b)} - 2\Psi \neq 0.$$

From (26) and (27), we conclude that

$$r_{11} = 0, \qquad r_{1\alpha} + s_{1\alpha} = 0.$$

Q.E.D.

5 $\Phi \neq 0$, $\Upsilon = 0$

First, note that $\Upsilon = 0$ implies that

$$\frac{s\Phi}{\Delta^2} - 2\Psi b^2 = b^2\mu, \tag{29}$$

where μ is a number independent of s. First, we prove the following

Lemma 5.1 *Let $F = \alpha\phi(\beta/\alpha)$ be an (α, β)-metric. Assume that $\Phi \neq 0$ and $\Upsilon = 0$. If F has isotropic S-curvature, $\mathbf{S} = (n+1)cF$, then β satisfies*

$$r_{ij} = ka_{ij} - \epsilon b_i b_j + \frac{1}{b^2}(r_i b_j + r_j b_i), \tag{30}$$

where $k = k(x)$, $\epsilon = \epsilon(x)$, and $\phi = \phi(s)$ satisfies the following ODE:

$$(k - \epsilon s^2)\frac{\Phi}{2\Delta^2} = \Big\{\nu + (k - \epsilon b^2)\mu\Big\}s - (n+1)c\phi, \tag{31}$$

where $\nu = \nu(x)$. If $s_0 \neq 0$, then ϕ satisfies the following additional ODE:

$$\frac{\Phi}{\Delta^2}(Qb^2 + s) = b^2(\mu + \lambda), \tag{32}$$

where $\lambda = \lambda(x)$.

Proof: Since $\Phi \neq 0$, it follows from (24) and (25) that in a special coordinate system (s, y^a) at a point x,

$$r_{ab} = k\delta_{ab}, \tag{33}$$

$$s\Big(\frac{s\Phi}{2\Delta^2} - 2\Psi b^2\Big)r_{11} + (n+1)cb^2\phi + k\frac{\Phi}{2\Delta^2}(b^2 - s^2) = bst_1, \tag{34}$$

$$\Big(\frac{s\Phi}{\Delta^2} - 2\Psi b^2\Big)(r_{1\alpha} + s_{1\alpha}) - (b^2 Q + s)\frac{\Phi}{\Delta^2}s_{1\alpha} - bt_\alpha = 0. \tag{35}$$

Let

$$r_{11} = -(k - \epsilon b^2).$$

Then (30) holds. By (29), we have

$$\frac{s\Phi}{2\Delta^2} - 2\Psi b^2 = b^2\mu - \frac{s\Phi}{2\Delta^2}.$$

Then (34) and (35) become

$$b(k - \epsilon s^2)\frac{\Phi}{2\Delta^2} = st_1 + sb\mu(k - b^2\epsilon) - (n+1)cb\phi. \tag{36}$$

$$b^2\mu(r_{1\alpha} + s_{1\alpha}) - \frac{\Phi}{\Delta^2}(Qb^2 + s)s_{1\alpha} - bt_\alpha = 0. \tag{37}$$

Letting $t_1 = b\nu$ in (36) we get (31).

Suppose that $s_0 \neq 0$. Rewrite (37) as

$$\Big\{b^2\mu - \frac{\Phi}{\Delta^2}(Qb^2 + s)\Big\}s_{1\alpha} = bt_a - b^2\mu r_{1\alpha}.$$

We can see that there is a number λ such that

$$\mu b^2 - \frac{\Phi}{\Delta^2}(Qb^2 + s) = -b^2\lambda.$$

This gives (32). Q.E.D.

Lemma 5.2 *Let $F = \alpha\phi(\beta/\alpha)$ be an (α, β)-metric. Assume that $\Upsilon = 0$. Then $b = constant$.*

Proof: Suppose that $b \neq constant$. Then b can be viewed as a variable over the manifold. By assumption,

$$\frac{s\Phi}{\Delta^2} - 2\Psi b^2 = b^2\mu,$$

where $\mu = \mu(x)$. Note that $\Delta^2\Big(\frac{s\Phi}{\Delta^2} - 2\Psi b^2 - b^2\mu\Big)$ is a polynomial of degree six in b by (15). More precisely, we have

$$-\mu Q'^2 b^6 - \Big\{Q'^2 - 2\mu Q'(1 + sQ - s^2Q')\Big\}b^4 + (\cdots)b^2 + (\cdots) = 0. \tag{38}$$

Thus

$$\mu Q'^2 = 0, \qquad Q'^2 - 2\mu Q'(1 + sQ - s^2Q') = 0.$$

Then $Q' = 0$, which implies that $\phi = 1 + cs$. This is impossible. Q.E.D.

Proposition 5.3 *Let $F = \alpha\phi(\beta/\alpha)$ be an (α, β)-metric. Suppose that $b^2Q+s \neq 0$, $\Phi \neq 0$ and $\Upsilon = 0$. If F has isotropic S-curvature, $\mathbf{S} = (n+1)cF$, then*

$$r_{ij} = \epsilon(b^2 a_{ij} - b_i b_j), \qquad s_j = 0, \tag{39}$$

where $\epsilon = \epsilon(x)$ is a scalar function, and $\phi = \phi(s)$ satisfies

$$\epsilon(b^2 - s^2)\Phi = -2(n+1)c\phi\Delta^2. \tag{40}$$

If $\epsilon \neq 0$, then $c/\epsilon = constant$.

Proof: First by Lemma 5.2 and Lemma 3.2, we have

$$r_0 + s_0 = 0.$$

Then by (18), we get

$$\mathbf{S} = -\alpha^{-1}\frac{\Phi}{2\Delta^2}\Big\{r_{00} - 2\alpha Q s_0\Big\}.$$

By Lemma 5.1,

$$r_{00} = (k - \epsilon s^2)\alpha^2 + \frac{2s}{b^2}r_0\alpha.$$

Then

$$\mathbf{S} = -(k - \epsilon s^2)\frac{\Phi}{2\Delta^2}\alpha + \frac{\Phi}{b^2\Delta^2}(b^2Q + s)s_0.$$

By (31), we get

$$\mathbf{S} = -s\Big\{\nu + (k - \epsilon b^2)\mu\Big\}\alpha + \frac{\Phi}{b^2\Delta^2}(b^2Q + s)s_0 + (n+1)c\phi\alpha. \tag{41}$$

By our assumption, $\mathbf{S} = (n+1)cF$, we get from (41) that

$$-s\Big\{\nu + (k - \epsilon b^2)\mu\Big\}\alpha + \frac{\Phi}{b^2\Delta^2}(b^2Q + s)s_0 = 0. \tag{42}$$

Letting $y^i = \delta b^i$ for a sufficiently small $\delta > 0$ yields

$$-\delta\Big\{\nu + (k - \epsilon b^2)\mu\Big\}b^2 = 0.$$

We conclude that

$$\nu + (k - \epsilon b^2)\mu = 0. \tag{43}$$

Then (42) is reduced to

$$\frac{\Phi}{b^2\Delta^2}(b^2Q + s)s_0 = 0.$$

Since $\Phi \neq 0$ and $b^2Q + s \neq 0$, we conclude that

$$s_0 = 0.$$

Then

$$r_0 = -s_0 = 0.$$

It follows from (30) that

$$r_{ij} = ka_{ij} - \epsilon b_i b_j. \tag{44}$$

Contracting (44) with b^i gives

$$r_j = (k - \epsilon b^2) b_j = 0.$$

Since $\beta \neq 0$, we get

$$k = \epsilon b^2 \tag{45}$$

and (44) becomes

$$r_{ij} = \epsilon(b^2 a_{ij} - b_i b_j).$$

Finally, (40) follows from (31), (43) and (45).

If $\epsilon \neq 0$, then letting $s = 0$ in (40) yields that $c/\epsilon = constant$ since $b = constant$. Q.E.D.

6 $\Phi \neq 0$ and $\Upsilon \neq 0$

In this section, we shall consider the case when $\phi = \phi(s)$ satisfies

$$\Phi \neq 0, \qquad \Upsilon \neq 0. \tag{46}$$

First we need the following

Lemma 6.1 *Let $F = \alpha\phi(s), s = \beta/\alpha$, be an (α, β)-metric on an n-dimensional manifold. Assume that $\phi = \phi(s)$ satisfies (46). Suppose that F has isotropic S-curvature, $\mathbf{S} = (n+1)cF$. Then*

$$r_{ij} = ka_{ij} - \epsilon b_i b_j - \lambda(s_i b_j + s_j b_i), \tag{47}$$

where $\lambda = \lambda(x), k = k(x)$ and $\epsilon = \epsilon(x)$ are scalar functions of x and

$$-2s(k - \epsilon b^2)\Psi + (k - \epsilon s^2)\frac{\Phi}{2\Delta^2} + (n+1)c\phi - s\nu = 0, \tag{48}$$

where

$$\nu := -\frac{f'(b)}{bf(b)}(k - \epsilon b^2). \tag{49}$$

If in addition $s_0 \neq 0$, then

$$-2\Psi - \frac{Q\Phi}{\Delta^2} - \lambda\Big(\frac{s\Phi}{\Delta^2} - 2\Psi b^2\Big) = \delta, \tag{50}$$

where

$$\delta := -\frac{f'(b)}{bf(b)}(1 - \lambda b^2). \tag{51}$$

Proof: By assumption, $\Phi \not\equiv 0$. It follows from (24) that there is a number k at x, independent of s, such that

$$\bar{r}_{00} = k\bar{\alpha}^2, \tag{52}$$

$$s\Big(\frac{s\Phi}{2\Delta^2} - 2\Psi b^2\Big)r_{11} + (n+1)cb^2\phi + k\frac{\Phi}{2\Delta^2}(b^2 - s^2) = sbt_1. \tag{53}$$

Let

$$r_{11} = k - \epsilon b^2,$$

where ϵ is a number independent of s. By (23), $t_1 = b\nu$, where ν is given by (49). Plugging them into (53) yields (48).

Suppose that $s_0 = 0$. Then

$$bs_{1\alpha} = s_\alpha = 0.$$

Then (25) is reduced to

$$\Big(\frac{s\Phi}{\Delta^2} - 2\Psi b^2\Big)r_{1\alpha} - bt_\alpha = 0. \tag{54}$$

By assumption, $\Upsilon \neq 0$, we know that $\frac{s\Phi}{\Delta^2} - 2\Psi b^2 \neq constant$. It follows from (54) that

$$r_{1\alpha} = 0, \qquad t_\alpha = 0.$$

The above identities together with $r_{11} = k - \epsilon b^2$ and $t_1 = b\nu$ imply the following identities

$$r_{ij} = ka_{ij} - \epsilon b_i b_j. \tag{55}$$

Suppose that $s_0 \neq 0$. Then $s_{\alpha_o} = bs_{1\alpha_o} \neq 0$ for some α_o.
Differentiating (25) with respect to s yields

$$\Upsilon\, r_{1\alpha} - \Big[\frac{Q\Phi}{\Delta^2} + 2\Psi\Big]' b^2 s_{1\alpha} = 0. \tag{56}$$

Let

$$\lambda := -\frac{r_{1\alpha_o}}{b^2 s_{1\alpha_o}}.$$

Plugging it into (56) yields

$$-\lambda\Upsilon - \Big[\frac{Q\Phi}{\Delta^2} + 2\Psi\Big]' = 0. \tag{57}$$

It follows from (57) that

$$\delta := -\frac{Q\Phi}{\Delta^2} - 2\Psi - \lambda\Big[\frac{s\Phi}{\Delta^2} - 2\Psi b^2\Big]$$

is a number independent of s. By assumption that $\Upsilon \neq 0$, we obtain from (56) and (57) that

$$r_{1\alpha} + \lambda b^2 s_{1\alpha} = 0. \tag{58}$$

(52) and (58) together with $r_{11} = k - \epsilon b^2$ implies that

$$r_{ij} + \lambda(b_i s_j + b_j s_i) = k a_{ij} - \epsilon b_i b_j. \tag{59}$$

By (23) and (58),

$$t_\alpha = \frac{f'(b)}{f(b)}(b^2\lambda - 1)s_{1\alpha}.$$

On the other hand, by (25) and (58), we obtain

$$b t_\alpha = \delta b^2 s_{1\alpha}.$$

Combining the above identities, we get (51). Q.E.D.

Lemma 6.2 *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an (α, β)-metric. Suppose that $\phi = \phi(s)$ satisfies (46) and $\phi \neq k_1\sqrt{1 + k_2 s^2} + k_3 s$ for any constants $k_1 > 0$, k_2 and k_3. If F has isotropic S-curvature, then*

$$r_j + s_j = 0.$$

Proof: Suppose that $r_j + s_j \neq 0$, then $b := \|\beta_x\|_\alpha \neq$ *constant* in a neighborhood. We view b as a variable in (48) and (50). Since $\phi = \phi(s)$ is a function independent of x, (48) and (50) actually give rise infinitely many ODEs on ϕ.

First, we consider (48). Let

$$eq := \Delta^2\Big\{ - 2s(k - \epsilon b^2)\Psi + (k - \epsilon s^2)\frac{\Phi}{2\Delta^2} + (n+1)c\phi - s\nu\Big\}.$$

We have

$$eq = \Xi_0 + \Xi_2 b^2 + \Xi_4 b^4,$$

where

$$\Xi_4 := \Big\{(\epsilon - \nu)s + (n+1)c\phi\Big\}\frac{\phi^2}{(\phi - s\phi')^4}(\phi'')^2.$$

It follows from (48) that $eq = 0$. Thus

$$\Xi_0 = 0, \qquad \Xi_2 = 0, \qquad \Xi_4 = 0.$$

Since $\phi'' \neq 0$, the equation $\Xi_4 = 0$ is equivalent to the following Ode:

$$(\epsilon - \nu)s + (n+1)c\phi = 0.$$

we conclude that

$$\epsilon = \nu, \qquad c = 0.$$

Then by a direct computation we get

$$\Xi_0 + \Xi_2 s^2 = -\frac{1}{2}(1+sQ)\Big\{(n-1)(k-\epsilon s^2)(Q-sQ') + 2kQ + 2\epsilon s\Big\}.$$

Then $\Xi_0 = 0$ and $\Xi_2 = 0$ imply that

$$(n-1)(k-\epsilon s^2)(Q-sQ') + 2kQ + 2\epsilon s = 0, \tag{60}$$

Suppose that $(k,\epsilon) \neq 0$. We claim that $k \neq 0$. If this is not true, i.e., $k = 0$, then $\epsilon \neq 0$ and (60) is reduced to

$$-(n-1)s(Q-sQ') + 2 = 0.$$

Letting $s = 0$, we get a contradiction.

Now we have that $k \neq 0$. It is easy to see that $Q(0) = 0$. Let

$$\tilde{Q} := Q(s) - sQ'(0).$$

Plugging it into (60) yields

$$(n-1)(k-\epsilon s^2)(\tilde{Q} - s\tilde{Q}') + 2k\tilde{Q} + 2(kQ'(0) + \epsilon)s = 0.$$

Since $\tilde{Q} = q_m s^m + o(s^m)$ where $m > 1$ is an integer, we see that $kQ'(0) + \epsilon = 0$. The above equation is reduced to

$$(n-1)(k-\epsilon s^2)(\tilde{Q} - s\tilde{Q}') + 2k\tilde{Q} = 0.$$

We obtain

$$\tilde{Q} = c_1 \frac{s^{\frac{n+1}{n-1}}}{(k-\epsilon s^2)^{\frac{1}{n-1}}}.$$

We must have $c_1 = 0$, that is, $\tilde{Q} = 0$. We get

$$Q(s) - sQ'(s) = 0.$$

Then it follows that

$$Q(s) = Q'(0)s.$$

In this case, $\phi = c_1\sqrt{1+c_2 s^2}$ where $c_1 > 0$ and c_2 are numbers independent of s. This case is excluded in the assumption. Therefore $k = 0$ and $\epsilon = 0$. Then (47) is reduced to

$$r_{ij} = -\lambda(s_j b_i + s_i b_j).$$

Then

$$r_j + s_j = (1-\lambda b^2)s_j.$$

By the assumption at the beginning of the proof, $r_j + s_j \neq 0$, we conclude that $1 - \lambda b^2 \neq 0$ and $s_j \neq 0$. By Lemma 6.1, $\phi = \phi(s)$ satisfies (50). Let

$$EQ := \Delta^2\Big\{-2\Psi - \frac{Q\Phi}{\Delta^2} - \lambda\Big(\frac{s\Phi}{\Delta^2} - 2\Psi b^2\Big) - \delta\Big\}.$$

We have

$$EQ = \Omega_0 + \Omega_2 b^2 + \Omega_4 b^4,$$

where

$$\Omega_4 = (Q')^2(\lambda - \delta).$$

By (50), $EQ = 0$. Thus

$$\Omega_0 = 0, \quad \Omega_2 = 0, \quad \Omega_4 = 0.$$

Since $Q' \neq 0$, $\Omega_4 = 0$ implies that

$$\delta = \lambda.$$

By a direct computation, we get

$$\Omega_0 + \Omega_2 s^2 = (1 + sQ)\Big\{(n+1)Q(Q - sQ') - Q' + \lambda\Big[ns(Q - sQ') - 1\Big]\Big\}.$$

The equations $\Omega_0 = 0$ and $\Omega_2 = 0$ imply that $\Omega_0 + \Omega_2 s^2 = 0$, that is,

$$(n+1)Q(Q - sQ') - Q' + \lambda\Big[ns(Q - sQ') - 1\Big] = 0.$$

We obtain

$$Q = -\frac{[k_0 n(n+1) - 1]\lambda s \pm \sqrt{\lambda k_0(k_0(1+n)^2 - 1 + \lambda s^2)}}{k_0(n+1)^2 - 1}.$$

Plugging it into $\Omega_2 = 0$ yields

$$k_0 \lambda = 0.$$

Then

$$Q = \frac{\lambda s}{k_0(n+1)^2 - 1}.$$

This implies that $\phi = k_1\sqrt{1 + k_2 s^2}$ where $k_1 > 0$ and k_2 are numbers independent of s. This case is excluded in the assumption of the lemma. Therefore, $r_j + s_j = 0$. Q.E.D.

Proposition 6.3 *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an (α, β)-metric. Suppose that $\phi = \phi(s)$ satisfies (46) and $\phi \neq k_1\sqrt{1 + k_2 s^2} + k_3 s$ for any constants $k_1 > 0$, k_2 and k_3. If F is of isotropic S-curvature, $\mathbf{S} = (n+1)cF$, then*

$$r_{ij} = \epsilon(b^2 a_{ij} - b_i b_j), \qquad s_j = 0, \tag{61}$$

where $\epsilon = \epsilon(x)$ is a scalar function on M and $\phi = \phi(s)$ satisfies

$$\epsilon(b^2 - s^2)\frac{\Phi}{2\Delta^2} = -(n+1)c\phi. \tag{62}$$

Proof: Contracting (47) with b^i yields

$$r_j + s_j = (k - \epsilon b^2) b_j + (1 - \lambda b^2) s_j. \tag{63}$$

By Lemma 6.2, $r_j + s_j = 0$. It follows from (63) that

$$(1 - \lambda b^2) s_j + (k - \epsilon b^2) b_j = 0. \tag{64}$$

Contracting (64) with b^j yields

$$(k - \epsilon b^2) b^2 = 0.$$

We get

$$k = \epsilon b^2.$$

Then (47) is reduced to

$$r_{ij} = \epsilon (b^2 a_{ij} - b_i b_j) - \lambda (b_i s_j + b_j s_i).$$

By (49),

$$\nu = 0.$$

Then (48) is reduced to (62).

We claim that $s_0 = 0$. Suppose that $s_0 \neq 0$. By (64), we conclude that

$$\lambda = \frac{1}{b^2}.$$

By (51),

$$\delta = 0.$$

It follows from (50) that

$$(b^2 Q + s)\Phi = 0.$$

This is impossible by the assumption $\Phi \neq 0$. Q.E.D.

7 Proof of Theorem 1.1

Notice that in Lemma 6.1, there is no restriction on ϕ other than (46). Let $\phi = k_1\sqrt{1 + k_2 s^2} + k_3 s$ where $k_1 > 0$, k_2 and k_3 are numbers independent of s. It is easy to check that, if $k_3 \neq 0$, then ϕ satisfies (46). Let $F = \alpha\phi(\beta/\alpha)$, where α is a Riemannian metric and β is a 1-form on an n-dimensional manifold. It is easy to see that if F is a Finsler metric, then $1 + k_2 b^2 > 0$, where $b := \|\beta_x\|_\alpha$. By Lemma 6.1, we can easily prove Theorem 1.1.

Proof of Theorem 1.1: Assume that F is of isotropic S-curvature, $\mathbf{S} = (n+1)cF$. By Lemma 6.1, β satisfies (47) and ϕ satisfies (48) and further it satisfies (50) if $s_0 \neq 0$.

First, we plug $\phi = k_1\sqrt{1 + k_2 s^2} + k_3 s$ into

$$eq := -2s(k - \epsilon b^2)\Psi + (k - \epsilon s^2)\frac{\Phi}{2\Delta^2} + (n+1)c\phi - s\nu.$$

By (48), the coefficients of the Taylor expansion of eq in s must be zero. We obtain

$$\begin{aligned} c &= \frac{k_3 k}{2(1+k_2 b^2)k_1^2} \\ \nu &= \Big\{\Big(\frac{n}{1+k_2b^2}+1\Big)\frac{k_3^2}{k_1^2}-k_2\Big\}k \\ \epsilon &= \Big\{\frac{k_3^2}{k_1^2}-k_2\Big\}k. \end{aligned}$$

Assume that $s_0 \neq 0$. We plug $\phi = k_1\sqrt{1+k_2s^2}+k_3 s$ into

$$EQ = -2\Psi - \frac{Q\Phi}{\Delta^2} - \lambda\Big(\frac{s\Phi}{\Delta^2} - 2\Psi b^2\Big) - \delta.$$

By (50), the coefficients of the Taylor expansion of EQ in s must be zero. We obtain

$$\begin{aligned} \lambda &= \frac{k_3^2}{k_1^2} - k_2 \\ \delta &= \Big(\frac{n}{1+k_2b^2}+1\Big)\frac{k_3^2}{k_1^2} - k_2. \end{aligned}$$

This proves the necessary conditions by (47).

Conversely, if β satisfies (4), then F is of isotropic S-curvature by (18). The proof is direct, so is omitted. Q.E.D.

References

[1] D. Bao and C. Robles, *On Randers metrics of constant curvature*, Rep. on Math. Phys., **51**(2003), 9-42.

[2] D. Bao, C. Robles and Z. Shen, *Zermelo navigation on Riemannian manifolds*, J. Differential Geometry, **66**(2004),377-435.

[3] D. Bao and Z. Shen, *Finsler metrics of constant flag curvature on the Lie group S^3*, J. London Math. Soc. **66**(2002), 453-467.

[4] X. Chen, X. Mo and Z. Shen, *On the flag curvature of Finsler metrics of scalar curvature*, J. of London Math. Soc., **68**(2)(2003),762-780.

[5] X. Chen and Z. Shen, *Randers metrics with special curvature properties*, Osaka J. of Math., **40**(2003), 87-101.

[6] X. Cheng and Z. Shen, *Projectively flat Finsler metrics with almost isotropic S-curvature*, Acta Math. Scientia, **26B**(2)(2006), 307-313.

[7] X. Cheng and Z. Shen, *Randers metrics of scalar flag curvature*, preprint, 2005.

[8] C.-W. Kim and J.-W. Yim, *Finsler manifolds with positive constant flag curvature*, Geom. Dedicata (to appear).

[9] B. Li and Z. Shen, On a class of weakly Landsberg metrics, preprint, 2006.

[10] Z. Shen, *Volume compasion and its applications in Riemann-Finsler geometry*, Advances in Math. **128**(1997), 306-328.

[11] Z. Shen, *Differential Geometry of Spray and Finsler Spaces* , Kluwer Academic Publishers, 2001.

[12] Z. Shen, *Finsler metrics with K=0 and S=0*, Canadian J. Math. **55**(2003),112-132.

[13] Z. Shen, *On a class of Landsberg metircs*, preprint, 2006.

[14] Z. Shen, *Landsberg curvature, S-curevature, and Riemann curvature, In " A Sampler of Finsler Geometry " MSRI series*, Cambridge University Press,2004.

[15] Z. Shen, *Nonpositively curved Finsler manifolds with constant S-curvature*, Math. Z. **249**(2005), 625-639

[16] Z. Shen, *Lectures on Finsler Geometry*, World Scientific Publishers, 2001.

[17] Z. Shen and H. Xing, *On Randers metrics with isotropic S-curvature* , Acta Math. Sinica, to appear.

[18] Z. Shen and G. C. Yildirim, *Randers metrics with special flag curvature*, preprint, 2005.

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