A class of Fredholm equations and systems of equations related to the Kontorovich-Lebedev and the Fourier integral transforms

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Abstract: In this article, we solve in closed form a class of Fredholm integral equations and systems of Fredholm integral equations with nondegenerate kernels by using techniques of convolutions and generalized convolutions related to the Kontorovich-Lebedev, Fourier sine, and Fourier cosine integral transforms.

1. Introduction

Fredholm integral equation is of the form (see [4])

$$\alpha(x)f(x) + \lambda \int_{\Gamma} k(x, \tau)f(\tau)d\tau = g(x), \tag{1.1}$$

where $\alpha(x), g(x)$ are given functions, f is unknown function, $\lambda \in \mathbb{C}$, $k(x, \tau)$ is a kernel, and the integral is on the curve Γ, or in general case $\Gamma = \Gamma_0 + \Gamma_1 + ... + \Gamma_n, \Gamma_i$ is collection of curve.

The Fredholm integral equation was studied first by I. Fredholm, and further developed by Riesz [3]. In the last two decades, the theory of abstract Volterra and Fredholm integral equation has undergone rapid development. To a large extent this was due to the applications of this theory to problems in mathematical physics, such as viscoelasticity, heat conduction in materials with memory, electrodynamics with memory, and to the need of tools to tackle the problems arising in these fields. Many interesting phenomena are not found with differential equations but observed in specific examples of integral equations (see [4, 5]).

However, the equation (1.1) is only solved by a method for approximating solutions. The equation (1.1) was only solved in closed form for some classes of degenerate kernels. The solution in closed form of the equation (1.1) in general case is still open.

In recent years, the equation (1.1) with $\Gamma = [0, T]$ and the kernel is a periodic function that has been studied by many authors (see [1] and references therein). In [8–11, 13–15], the authors studied the equation (1.1) in case $\Gamma = (0, +\infty)$ and kernel is of the Toeplitz plus Hankel type.

In this paper, we attempt to solve in closed form a class of Fredholm integral equations and systems of Fredholm integral equations with nondegenerate kernels and $\Gamma = (0, +\infty)$ by using techniques of convolutions

and generalized convolutions related to Kontorovich–Lebedev and Fourier sine, Fourier cosine integral transform. These convolutions and inverse formulas were studied in [6, 12, 15–17].

The main results of the article are presented in Sections 3 and 4. In Section 3, we obtain the closed form of solutions of the equation (1.1) in the case of $\Gamma = (0, +\infty)$, $\alpha(x) = 0$, $\lambda = 1$, and the kernel $k(x, \tau) = k_1(x, \tau) + k_2(x, \tau)$ is certain nondegenerate kernel (3.1). In Section 4, we solve in closed form a class systems of Fredholm integral equations (4.1). The results obtained in Theorems 3.1, 3.2, 4.1, and 4.2 give explicit formulas of solutions which contain of integral transformations such as Kontorovich–Lebedev, Fourier sine, Fourier cosine. Each equation of system (4.1) is of the form (1.1), with $\Gamma = (0, +\infty), \alpha(x) = 1, \lambda = 1$, and the kernels k_3, k_4 are nondegenerated.

The key tool in proofs of results in papers [8–10, 14, 15] is Wiener–Levy theorem. However, we prove Theorems 3.1, 3.2, 4.1 and 4.2 by a method without using Wiener–Levy theorem. Tuan et al. [13] studied the equation (1.1) with a kernel of the generalized convolution. It is harder to make function spaces of solution for the equation in Theorem 3.1, 3.2 if compared with results in [13].

2. Related integral transforms and function spaces

In this section, we recall several integral transform and function spaces which are usefull throughout this article.

The Fourier cosine transform (F_c) and its inverse formula (F_c^{-1}) in $L_1(\mathbb{R}_+)$ are defined by (see [2])

$$(F_c f)(y) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} f(x) \cos(xy) dx, \ y > 0, \tag{2.1}$$

and

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} (F_c f)(y) \cos(xy) dy, \ x > 0. \tag{2.2}$$

The Fourier sine transform (F_s) and its inverse formula (F_s^{-1}) in $L_1(\mathbb{R}_+)$ are defined by (see [2])

$$(F_s f)(y) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} f(x) \sin(xy) dx, \ y > 0, \tag{2.3}$$

and

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} (F_s f)(y) \sin(xy) dy, \ x > 0. \tag{2.4}$$

The Kontorovich–Lebedev integral transform (K) in $L_1(\mathbb{R}_+)$ is of the form (see [12])

$$(K f)(y) = \int_0^{+\infty} K_{iy}(x) f(x) dx, \ y > 0, \tag{2.5}$$

here, $K_{iy}(x)$ is the Macdonald function (see [12])

$$K_{iy}(x) = \int_0^{+\infty} e^{-x \cosh u} \cos(yu) dy, \ y \geq 0, \ x > 0.$$

The inverse Kontorovich–Lebedev transform (K^{-1}) is of the form

$$(K^{-1} f)(x) = f(x) = \frac{2}{\pi^2 x} \int_0^{+\infty} y \sinh(\pi y) K_{iy}(x)(K f)(y) dx, \ y > 0. \tag{2.6}$$

In the space $L_2(\mathbb{R}_+)$, the integral transforms $(F_c), (F_s)$ are of the form

$$\left(F_{\{\begin{smallmatrix}c\\s\end{smallmatrix}\}}f\right)(y) = \lim_{N\to\infty}\int_0^N f(x)\left\{\begin{smallmatrix}\cos xy\\\sin xy\end{smallmatrix}\right\}dx, y > 0. \tag{2.7}$$

The inverse transforms of the (F_c) và (F_s) are defined by formulas (2.2) and (2.4) respectively.

In the space $L_2(\mathbb{R}_+, x^\alpha), \alpha \in \mathbb{R}$, the Kontorovich–Lebedev integral transform (K) is of the form

$$(Kf)(y) = \lim_{N\to\infty}\int_{\frac{1}{N}}^N K_{iy}(x)f(x)dx, y > 0. \tag{2.8}$$

The inverse transform is defined by formula (2.6).

The function space $L_p(\mathbb{R}_+, \gamma)$ with the weight function γ is defined as follow (see [2])

$$L_p(\mathbb{R}_+, \gamma) = \left\{f : \left(\int_0^{+\infty}\gamma(x)|f(x)|^p\ dx\right)^{\frac{1}{p}} < \infty\right\}, 1 \le p < \infty.$$

Note that

$$L_p(\mathbb{R}_+) \subset L_p(\mathbb{R}_+, \gamma).$$

The function space $L_p^{\alpha,\beta}$ is introduced in a research of Yakubovich and Britvina [16]

$$L_p^{\alpha,\beta} \equiv L_p(\mathbb{R}_+, K_0(\beta x)x^\alpha),\ \alpha \in \mathbb{R},\ 0 < \beta \le 1.$$

This function spaces is equived the norm

$$\|f\|_{L_p^{\alpha,\beta}} = \left(\int_0^{+\infty}|f(x)|^p K_0(\beta x)x^\alpha dx\right)^{\frac{1}{p}} < +\infty.$$

Note that $L_p(\mathbb{R}_+) \subset L_p^{\alpha,\beta},\ L_1^{\alpha,1} \equiv L_1(\mathbb{R}_+, K_0(x)x^\alpha)$, and

$$K_0(x) = \int_0^{+\infty} e^{-x\cosh u}du,\ x > 0.$$

3. A class of Fredholm's type integral equations

In this section, we solve in close form a class of integral equations of Fredholm's type with nondegenerate kernel $k(x,\tau) = k_1(x,\tau) + k_2(x,\tau)$ with the help of convolutions and generalized convolutions techniques. Namely, we use the convolutions and generalized convolutions related to the Kontorovich–Lebedev (K), Fourier sine (F_s), Fourier cosine (F_c) transforms (see [12, 16, 17]) and their inverses.

Consider the following problem:

$$\int_0^{+\infty}[k_1(x,\tau) + k_2(x,\tau)]f(\tau)d\tau = g(x),\ x > 0, \tag{3.1}$$

where

$$\begin{cases} k_1(x,\tau) = & \dfrac{1}{2\pi} \displaystyle\int_0^{+\infty} \dfrac{1}{x}[e^{-x\cosh(\tau+\theta)} - e^{-x\cosh(\tau-\theta)}]h(\theta)d\theta, \ x,\tau > 0. \\ k_2(x,\tau) = & \dfrac{1}{2\pi^2} \displaystyle\int_0^{+\infty} [\sinh(\tau+\theta)e^{-x\cosh(\tau+\theta)} + \sinh(\tau-\theta)e^{-x\cosh(\tau-\theta)}]\varphi(\theta)d\theta, x,\tau > 0, \end{cases} \tag{3.2}$$

where h, φ are given functions in $L_2(\mathbb{R}_+)$, $g \in L_2(\mathbb{R}_+, x^\alpha)$ $(\alpha \in \mathbb{R})$ is given function, and f is unknown function.

Theorem 3.1 *Suppose that* $\dfrac{\pi}{2}(F_s h)(y) + y(F_c \varphi) \neq 0$, $\forall y > 0$ *such that*

$$\dfrac{y\sinh(\pi y)(Kg)(y)}{\dfrac{\pi}{2}(F_s h)(y) + y(F_c\varphi)(y)} \in L_2(\mathbb{R}_+),$$

then the equation (3.1) *has the solution in* $L_2(\mathbb{R}_+)$, *which is of the form*

$$f(x) = \sqrt{\dfrac{2}{\pi}} \int_0^{+\infty} \dfrac{y\sinh(\pi y)(Kg)(y)}{\dfrac{\pi}{2}(F_s h)(y) + y(F_c\varphi)(y)} \sin(xy)dy, \ x > 0.$$

Proof In order to prove this theorem, we use the generalized convolution for the Kontorovich–Lebedev, Fourier sine transforms which is defined by (see [17]):

$$(f \overset{\gamma_1}{\underset{1}{*}} h)(x) := \dfrac{1}{2\pi} \int_{\mathbb{R}_2^+} \dfrac{1}{x}[e^{-x\cosh(\tau+\theta)} - e^{-x\cosh(\tau-\theta)}]f(\tau)h(\theta)d\tau d\theta, \ x > 0, \tag{3.3}$$

where $\gamma_1(y) = \dfrac{1}{y\sinh(\pi y)}$.

For $f, h \in L_2(\mathbb{R}_+)$, the generalized convolution $(f \overset{\gamma_1}{\underset{1}{*}} h) \in L_2(\mathbb{R}_+, x^\alpha)$ and satisfies the following factorization equality:

$$K(f \overset{\gamma_1}{\underset{1}{*}} h)(y) = \dfrac{\pi}{2}\gamma_1(y)(F_s f)(y)(F_s h)(y), \ \forall y > 0. \tag{3.4}$$

The generalized convolution for the Kontorovich–Lebedev, Fourier transforms is defined by (see [12])

$$(f \overset{\gamma_2}{\underset{2}{*}} h)(x) := \dfrac{1}{2\pi^2} \int_{\mathbb{R}_2^+} [\sinh(\tau+\theta)e^{-x\cosh(\tau+\theta)} + \sinh(\tau-\theta)e^{-x\cosh(\tau-\theta)}]f(\tau)h(\theta)d\tau d\theta, \ x > 0, \tag{3.5}$$

where $\gamma_2(y) = \dfrac{1}{\sinh(\pi y)}$.

For $f, h \in L_2(\mathbb{R}_+)$, the generalized convolution $(f \overset{\gamma_2}{\underset{2}{*}} h) \in L_2(\mathbb{R}_+, x^\alpha), \alpha \in \mathbb{R}$ and satisfies the following factorization equality

$$K(f \overset{\gamma_2}{\underset{2}{*}} h)(y) = \gamma_2(y)(F_c f)(y)(F_s h)(y), \ \forall y > 0. \tag{3.6}$$

When the kernels k_1, k_2 are defined by formula (3.2), the equation (3.1) can be rewritten in the dual convolutions equation as follow:

$$(f \overset{\gamma_1}{\underset{1}{*}} h)(x) + (\varphi \overset{\gamma_2}{\underset{2}{*}} f)(x) = g(x), \ x > 0, \tag{3.7}$$

where f is unknown function, $\gamma_1(y) = \dfrac{1}{y \sinh(\pi y)}, \gamma_2(y) = \dfrac{1}{\sinh(\pi y)}$, h, φ are given functions in $L_2(\mathbb{R}_+)$, g is given function in $L_2(\mathbb{R}_+, x^\alpha), \alpha \in \mathbb{R}$.

Applying the (K) transform on both sides of equation (3.7), with the help of factorization equalities $(3.4), (3.6)$, we have

$$K(f \overset{\gamma_1}{\underset{1}{*}} h)(y) + K(\varphi \overset{\gamma_2}{\underset{2}{*}} f)(y) = (Kg)(y), \ \forall y > 0.$$

This implies that

$$\frac{\pi}{2} \frac{1}{y \sinh(\pi y)}(F_s f)(y)(F_s h)(y) + \frac{1}{\sinh(\pi y)}(F_c \varphi)(y)(F_s f)(y) = (Kg)(y), \ \forall y > 0,$$

or equivalent,

$$(F_s f)(y) = \frac{y \sinh(\pi y)(Kg)(y)}{\dfrac{\pi}{2}(F_s h)(y) + y(F_c \varphi)(y)} \in L_2(\mathbb{R}_+).$$

Thanks to the inverse formular of Fourier sine transform (2.4), we have the solution in $L_2(\mathbb{R}_+)$ as follows

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} \frac{y \sinh(\pi y)(Kg)(y)}{\dfrac{\pi}{2}(F_s h)(y) + y(F_c \varphi)(y)} \sin(xy) dy, \ \ x > 0. \tag{3.8}$$

$\square$

Example 1. We use Theorem 3.1 to solve in closed form in the case

$$\varphi(x) = h(x) = \sqrt{\frac{2}{\pi}} e^{-x} \in L_2(\mathbb{R}_+),$$

$$g(x) = \frac{e^{-x}}{\pi^2 x} \left(\sqrt{\frac{f \pi x}{2}} - \pi x e^{2x} \operatorname{erfe}(\sqrt{2x}) \right) \in L_2(\mathbb{R}_+, x^\alpha), \ \alpha \in \mathbb{R},$$

then,

$$(F_s h)(y) = \frac{2}{\pi} \frac{y}{1 + y^2}, \ (F_c \varphi)(y) = \frac{1}{1 + y^2}.$$

Therefore, $\dfrac{\pi}{2}(F_s h)(y) + y(F_c \varphi)(y) = (\dfrac{2}{\pi} + 1)\dfrac{y}{1 + y^2} \neq 0, \ y > 0.$

Thanks to formular $(2.16.48.8)$ in [7], we have $(Kg)(y) = \dfrac{y}{\sinh(\pi y) \cosh(\pi y)}$. Thus,

$$\frac{y \sinh(\pi y)(Kg)(y)}{\dfrac{\pi}{2}(F_s h)(y) + y(F_c \varphi)(y)} = \frac{y(1 + y^2)}{(\dfrac{2}{\pi} + 1) \cosh(\pi y)} \in L_2(\mathbb{R}_+), \ \forall y > 0.$$

The solution for this case is as follows

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} \frac{y(1+y^2)}{(\frac{2}{\pi}+1)\cosh(\pi y)} \sin(xy) dy, \ x > 0.$$

Therefore,

$$f(x) = \sqrt{\frac{2}{\pi}} \frac{\pi+2}{\pi} \frac{3}{64} \operatorname{sech}^4 \frac{x}{2} \left(9 \sinh \frac{x}{2} + \sinh \frac{3x}{2} \right), \ f \in L_2(\mathbb{R}_+).$$

Similarly, one can solve the equation (3.1) with the following kernels

$$\begin{cases} k_1(x,\tau) & = \frac{1}{2\pi} \int_0^{+\infty} \frac{1}{x} [e^{-x\cosh(\tau+\theta)} + e^{-x\cosh(\tau-\theta)}] h(\theta) d\theta, \ \forall x, \tau > 0 \\ k_2(x,\tau) & = \frac{1}{\pi} \int_0^{+\infty} \frac{1}{x} K_0(\sqrt{x^2+\theta^2+2x\theta\cosh\tau}) \psi(\theta) d\theta, \ \forall x, \tau > 0, \end{cases} \tag{3.9}$$

where f is unknown function, and $\psi \in L_1(\mathbb{R}_+), h \in L_2(\mathbb{R}_+), g \in L_2(\mathbb{R}_+, x^\alpha)$ are given functions.

Theorem 3.2 *Suppose that* $\sqrt{\frac{\pi}{2}}(F_s h)(y) + (K\psi)(y) \neq 0, \ \forall y > 0$ *such that*

$$\frac{y\sinh(\pi y)(Kg)(y)}{\sqrt{\frac{\pi}{2}}(F_s h)(y) + (K\psi)(y)} \in L_2(\mathbb{R}_+),$$

then the equation (3.1) *has the solution in* $f \in L_2(\mathbb{R}_+)$*, which can be presented in the form*

$$f(x) = \int_0^{+\infty} \frac{y\sinh(\pi y)(Kg)(y)}{\sqrt{\frac{\pi}{2}}(F_s h)(y) + (K\psi)(y)} \cos(xy) dy, \ x > 0.$$

Proof First, we recall the generalized convolution for the Kontorovich–Lebedev, Fourier cosine (see [17]), defined by

$$(f \overset{\gamma_1}{\underset{3}{*}} h)(x) := \frac{1}{2\pi} \int_{\mathbb{R}_2^+} \frac{1}{x} [e^{-x\cosh(\tau+\theta)} + e^{-x\cosh(\tau-\theta)}] f(\tau) h(\theta) d\tau d\theta, \ x > 0, \tag{3.10}$$

where $\gamma_1(y) = \dfrac{1}{y\sinh(\pi y)}$.

For $f, h \in L_2(\mathbb{R}_+)$, the generalized convolution $(f \overset{\gamma_1}{\underset{3}{*}} h) \in L_2(\mathbb{R}_+, x^\alpha)$ satisfies the following factorization equality

$$K(f \overset{\gamma_1}{\underset{3}{*}} h)(y) = \frac{\pi}{2} \gamma_1(y)(F_c f)(y)(F_c h)(y), \ \forall y > 0. \tag{3.11}$$

The generalized convolution for the Kontorovich–Lebedev, Fourier cosine transforms (see [16]) is defined by

$$(f \overset{\gamma_1}{\underset{4}{*}} h)(x) := \frac{1}{\pi} \int_{\mathbb{R}_2^+} \frac{1}{x} K_0(\sqrt{x^2+\theta^2+2x\theta\cosh\tau}) f(\tau) h(\theta) d\tau d\theta, \ \forall x > 0, \tag{3.12}$$

where $\gamma_1(y) = \dfrac{1}{y \sinh(\pi y)}$.

For $f \in L_1(\mathbb{R}_+), h \in L_2^{0,1}$, the generalized convolution $(f \overset{\gamma_1}{\underset{4}{*}} h) \in L_2(\mathbb{R}_+, x^\alpha), \alpha \in \mathbb{R}$ and satisfies the following factorization equality

$$K(f \overset{\gamma_1}{\underset{4}{*}} h)(y) = \sqrt{\frac{\pi}{2}} \gamma_1(y)(F_c f)(y)(Kh)(y), \ \forall y > 0. \tag{3.13}$$

With the kernel k_1, k_2 defined by formula (3.9), the equation (3.1) can be rewritten in the following dual convolutions form

$$(f \overset{\gamma_1}{\underset{3}{*}} h)(x) + (\psi \overset{\gamma_1}{\underset{4}{*}} f)(x) = g(x), \ x > 0, \tag{3.14}$$

where f is unknown function, h, ψ, g are given functions, and $h \in L_2(\mathbb{R}_+), \psi \in L_1(\mathbb{R}_+), g \in L_2(\mathbb{R}_+, x^\alpha)$.

Applying Kontorovich–Lebedev transform on both sides of equation (3.14), we have

$$K(f \overset{\gamma_1}{\underset{3}{*}} h)(y) + K(\psi \overset{\gamma_1}{\underset{4}{*}} f)(y) = (Kg)(y), \ \forall y > 0,$$

By the factorization equalities (3.11), (3.13), we obtain

$$\frac{\pi}{2} \frac{1}{y \sinh(\pi y)}(F_c f)(y)(F_c h)(y) + \sqrt{\frac{\pi}{2}} \frac{1}{y \sinh(\pi y)}(F_c f)(y)(K\psi)(y) = (Kg)(y), \ \forall y > 0.$$

Moreover, by conditions of the theorem, we have

$$(F_c f)(y) = \frac{y \sinh(\pi y)(Kg)(y)}{\frac{\pi}{2}(F_s h)(y) + \sqrt{\frac{\pi}{2}}(K\psi)(y)} \in L_2(\mathbb{R}_+).$$

By the inverse Fourier cosine transform, we obtain

$$f(x) = \int_0^{+\infty} \frac{y \sinh(\pi y)(Kg)(y)}{\sqrt{\frac{\pi}{2}}(F_s h)(y) + (K\psi)(y)} \cos(xy)dy, \ x > 0. \tag{3.15}$$

It is clear that $f \in L_2(\mathbb{R}_+)$. $\qquad\qquad\square$

4. A class of systems of two integral equations of Fredholm's type

In this section, we consider a class of systems of two integral equation of Fredholm's type with nondegenerate kernels $k_3(x, \tau), k_4(x, \tau)$. We will obtain the solution in close form of these systems by using the techniques of convolution, generalized convolution related to the Kontorovick–Lebedev, Fourier sine and Fourier cosine transforms. These convolutions, generalized convolution were studied in [6, 15, 17].

Consider the following system of Fredholm's integral equations

$$\begin{cases} f(x) + \displaystyle\int_0^{+\infty} k_3(x, \tau)g(\tau)d\tau = q_1(x) \\ \displaystyle\int_0^{+\infty} k_4(x, \tau)f(\tau)d\tau + g(x) = q_2(x), \end{cases} \tag{4.1}$$

where k_3, k_4 are nondegenerate kernel as follows

$$\begin{cases} k_3(x, \tau) & = \displaystyle\int_0^{+\infty} \frac{1}{2\pi x} \left[e^{-x\cosh(\tau+\theta)} - e^{-x\cosh(\tau-\theta)} \right] \varphi(\theta) d\theta, \ \forall x, \tau > 0, \\ k_4(x, \tau) & = \displaystyle\int_0^{+\infty} \frac{1}{\tau} \left[e^{-\tau\cosh(x-\theta)} - e^{-\tau\cosh(x+\theta)} \right] \psi(\theta) d\theta, \ \forall x, \tau > 0, \end{cases} \tag{4.2}$$

where $q_2, \varphi \in L_2(\mathbb{R}_+), \psi \in L_2^{0,\beta} \ (0 < \beta \leq 1), q_1 \in L_2(\mathbb{R}_+, x^\alpha), (\alpha \in \mathbb{R})$ are given functions, and f, g are unknown functions.

Theorem 4.1 *Suppose that the following conditions hold true*

$(C_1): \ 1 - \dfrac{\pi}{2y\sinh \pi y} \dfrac{1}{\sin y} F_s \left(\varphi \overset{\gamma_2}{\underset{F_s}{*}} \psi \right) (y) \neq 0, \forall y > 0,$

$(C_2): \ \left(\dfrac{2y\sinh(\pi y) \sin y (Kq_1)(y) - \pi F_s \left(\varphi \overset{\gamma_2}{\underset{F_s}{*}} q_2 \right) (y)}{2y\sinh(\pi y) \sin y - \pi F_s \left(\varphi \overset{\gamma_2}{\underset{F_s}{*}} \psi \right) (y)} \right) \in L_1(\mathbb{R}_+) \cap L_2(\mathbb{R}_+),$

$(C_3): \ \left(\dfrac{2y\sinh(\pi y) \sin y [(F_s q_2)(y) - F_s \left(q_1 \underset{5}{*} \psi \right) (y)]}{2y\sinh(\pi y) \sin y - \pi F_s \left(\varphi \overset{\gamma_2}{\underset{F_s}{*}} \psi \right) (y)} \right) \in L_1(\mathbb{R}_+) \cap L_2(\mathbb{R}_+).$

Then the problem (4.1) *has the solution in the form*

$$f(x) = K^{-1} \left(\dfrac{2y\sinh(\pi y) \sin y (Kq_1)(y) - \pi F_s \left(\varphi \overset{\gamma_2}{\underset{F_s}{*}} q_2 \right) (y)}{2y\sinh(\pi y) \sin y - \pi F_s \left(\varphi \overset{\gamma_2}{\underset{F_s}{*}} \psi \right) (y)} \right) (x), \ \forall x > 0,$$

$$g(x) = F_s^{-1} \left(\dfrac{2y\sinh(\pi y) \sin y [(F_s q_2)(y) - F_s \left(q_1 \underset{5}{*} \psi \right) (y)]}{2y\sinh(\pi y) \sin y - \pi F_s \left(\varphi \overset{\gamma_2}{\underset{F_s}{*}} \psi \right) (y)} \right) (x), \ \forall x > 0.$$

Morover, $f, g \in L_1(\mathbb{R}_+) \cap L_2(\mathbb{R}_+).$

Proof

In order to prove the theorem (4.1), we will use the generalized convolution with the weight function $\gamma_1(y)$ for the Kontorovich–Lebedev and the Fourier sine transform $\left(f \overset{\gamma_1}{\underset{1}{*}} h \right)$ (see [17]) defined by (3.3) and satisfy the equality (3.4). We also use the generalized convolution for the Fourier sine, Kontorovich–Lebedev transforms (see [15]) which are defined by

$$\left(f \underset{5}{*} h \right) (x) := \int_{\mathbb{R}_+^2} \frac{1}{\tau} \left[e^{-\tau\cosh(x-\theta)} - e^{-\tau\cosh(x+\theta)} \right] f(\tau) h(\theta) d\tau d\theta, \ \forall x > 0. \tag{4.3}$$

For $f \in L_1 \left(\mathbb{R}_+, \dfrac{1}{\sqrt{x^3}} \right), h \in L_1(\mathbb{R}_+)$, we get that $\left(f \underset{5}{*} h \right) \in L_1(\mathbb{R}_+)$, and the following factorization equality holds

$$F_s \left(f \underset{5}{*} h \right) (y) = (Kf)(y)(F_s h)(y), \ \forall y > 0. \tag{4.4}$$

As proof presented in Tuan et al. [15], we can show that, if $f \in L_2(\mathbb{R}_+)$, and $h \in L_2^{0,\beta}$, with $0 < \beta \le 1$, then $\left(f \underset{5}{*} h \right) \in L_2(\mathbb{R}_+)$.

The generalized convolution with the weight function $\gamma_2(y)$ for the Fourier sine transform is of the form (see [6])

$$\left(f \underset{F_s}{\overset{\gamma_2}{*}} h \right)(x) := \frac{1}{2\sqrt{2\pi}} \int_0^{+\infty} f(y) \left[\text{sign}(x+y-1)h(|x+y-1|) - h(x+y+1) \right.$$
$$+\text{sign}(x-y+1)h(|x-y+1|)$$
$$\left. -\text{sign}(x-y-1)h(|x-y-1|) \right] dy, \ \forall x > 0, \tag{4.5}$$

where $\gamma_2(y) = \sin y$.

For $f, h \in L_1(\mathbb{R}_+)$, the convolution $\left(f \underset{F_s}{\overset{\gamma_2}{*}} h \right) \in L_1(\mathbb{R}_+)$ and satisfies the following factorization equality

$$F_s \left(f \underset{F_s}{\overset{\gamma_2}{*}} h \right)(y) = \sin y (F_s f)(y)(F_s h)(y), \ \forall y > 0. \tag{4.6}$$

In case the kernels k_3, k_4 are defined by formula (4.2), system (4.1) becomes

$$\begin{cases} f(x) + \left(g \underset{1}{\overset{\gamma_1}{*}} \varphi \right)(x) = q_1(x), \ \forall x > 0 \\ \left(f \underset{5}{*} \psi \right)(x) + g(x) = q_2(x), \ \forall x > 0, \end{cases} \tag{4.7}$$

where $\varphi, q_2 \in L_2(\mathbb{R}_+), \psi \in L_2^{0,\beta} \ (0 < \beta \le 1), q_1 \in L_2(\mathbb{R}_+, x^\alpha) \ (\alpha \in \mathbb{R})$ are given functions, and f, g are unknown functions.

Applying the Kontorovich–Lebedev transform, Fourier sine transform respectively on both sides of the first and the second in the system of equations (4.7)

$$\begin{cases} (Kf)(y) + K \left(g \underset{1}{\overset{\gamma_1}{*}} \varphi \right)(y) &= (Kq_1)(y), \ \forall y > 0, \\ F_s \left(f \underset{5}{*} \psi \right)(y) + (F_s g)(y) &= (F_s q_2)(y), \ \forall y > 0. \end{cases}$$

Combining the factorization equalities (3.4), (4.4),(4.6) we obtain

$$\begin{cases} (Kf)(y) + \dfrac{\pi}{2y \sinh(\pi g)}(F_s g)(y)(F_s \varphi)(y) &= (Kq_1)(y), \ \forall y > 0, \\ (Kf)(y)(F_s \psi)(y) + (F_s g)(y) &= (F_s q_2)(y), \ \forall y > 0. \end{cases}$$

We have

$$\Delta = \begin{vmatrix} 1 & \dfrac{\pi}{2y\sinh(\pi y)}(F_s\varphi)(y) \\ (F_s\psi)(y) & 1 \end{vmatrix} = 1 - \dfrac{\pi}{2y\sinh(\pi y)}F_s\varphi(y)(F_s\psi)(y)$$

$$= 1 - \dfrac{\pi}{2y\sinh(\pi y)}\dfrac{1}{\sin y}F_s\left(\varphi \overset{\gamma_2}{\underset{F_s}{*}} \psi\right)(y), \ \forall y > 0,$$

$$\Delta_1 = \begin{vmatrix} (Kq_1)(y) & \dfrac{\pi}{2y\sinh(\pi y)}(F_s\varphi)(y) \\ (F_sq_2)(y) & 1 \end{vmatrix}$$

$$= (Kq_1)(y) - \dfrac{\pi}{2y\sinh(\pi y)}\dfrac{1}{\sin y}F_s\left(\varphi \overset{\gamma_2}{\underset{F_s}{*}} q_2\right)(y), \ \forall y > 0,$$

$$\Delta_2 = \begin{vmatrix} 1 & (Kq_1)(y) \\ (F_s\psi)(y) & (F_sq_2)(y) \end{vmatrix} = (F_sq_2)(y) - F_s\left(q_1 \underset{5}{*} \psi\right)(y), \ \forall y > 0.$$

Under the conditions (C_1) and (C_2) we get

$$1 - \dfrac{\pi}{2y\sinh(\pi y)}\dfrac{1}{\sin y}F_s\left(\varphi \overset{\gamma_2}{\underset{F_s}{*}} \psi\right)(y) \neq 0, \ \forall y > 0,$$

or,

$$(Kf)(y) = \dfrac{2y\sinh(\pi y)\sin y(Kq_1)(y) - \pi F_s\left(\varphi \overset{\gamma_2}{\underset{F_s}{*}} q_2\right)(y)}{2y\sinh(\pi y)\sin y - \pi F_s\left(\varphi \overset{\gamma_2}{\underset{F_s}{*}} \psi\right)(y)}, \ \forall y > 0,$$

and $(Kf)(y) \in L_1(\mathbb{R}_+) \cap L_2(\mathbb{R}_+)$. Using the inverse Kontorovich–Lebedev transform K^{-1} we have

$$f(x) = K^{-1}\left(\dfrac{2y\sinh(\pi y)\sin y(Kq_1)(y) - \pi F_s\left(\varphi \overset{\gamma_2}{\underset{F_s}{*}} q_2\right)(y)}{2y\sinh(\pi y)\sin y - \pi F_s\left(\varphi \overset{\gamma_2}{\underset{F_s}{*}} \psi\right)(y)}\right)(x), \ \forall x > 0, \tag{4.8}$$

and $f \in L_1(\mathbb{R}_+) \cap L_2(\mathbb{R}_+)$.

Under the conditions (C_1), (C_3) we have

$$(F_sg)(y) = \dfrac{2y\sinh(\pi y)\sin y\left[(F_sq_2)(y) - F_s\left(q_1 \underset{5}{*} \psi\right)(y)\right]}{2y\sinh(\pi y)\sin y - \pi F_s\left(\varphi \overset{\gamma_2}{\underset{F_s}{*}} \psi\right)(y)}, \ \forall y > 0.$$

From the inverse Fourier sine transform, we have

$$g(x) = F_s^{-1}\left(\dfrac{2y\sinh(\pi y)\sin y[(F_sq_2)(y) - F_s\left(q_1 \underset{5}{*} \psi\right)(y)]}{2y\sinh(\pi y)\sin y - \pi F_s\left(\varphi \overset{\gamma_2}{\underset{F_s}{*}} \psi\right)(y)}\right)(x), \ \forall x > 0. \tag{4.9}$$

and $g \in L_1(\mathbb{R}_+) \cap L_2(\mathbb{R}_+.)$ □

Simillarly, one can solve system (4.1) in close form for the following case of kernel

$$
\begin{cases}
k_3(x,\tau) & = \displaystyle\int_0^{+\infty} \frac{1}{\pi x} K_0\left(\sqrt{x^2 + \theta^2 + 2x\theta\cosh\tau}\right)\xi(\theta)d\theta, \ \forall x,\tau > 0, \\
k_4(x,\tau) & = \displaystyle\int_0^{+\infty} \frac{1}{2x}\exp\left[-\frac{1}{2}\left(\frac{\tau\theta}{x} + \frac{\tau x}{\theta} + \frac{\theta x}{\tau}\right)\right]\eta(\theta)d\theta, \ \forall x,\tau > 0,
\end{cases}
\tag{4.10}
$$

where $\xi \in L_2^{0,1}, \eta \in L_1(\mathbb{R}_+, K_0(x)), \ q_1, q_2 \in L_2(\mathbb{R}_+, x^\alpha), \alpha \in \mathbb{R}$ are given functions, and f, g are unknown functions.

Theorem 4.2 *Suppose that the following conditions hold true*

$(C_4):\ 1 - K\left(\eta \overset{\gamma_1}{\underset{4}{*}} \xi\right)(y) \neq 0, \ \forall y > 0,$

$(C_5): \left(\dfrac{K\left(q_1 - \left(q_2 \overset{\gamma_1}{\underset{4}{*}} \xi\right)\right)(y)}{1 - K\left(\eta \overset{\gamma_1}{\underset{4}{*}}\right)(y)}\right) \in L_2(\mathbb{R}_+, x^\alpha),$

$(C_6): \left(\dfrac{K\left(q_2 - \left(\eta \underset{6}{*} q_1\right)\right)(y)}{1 - K\left(\eta \overset{\gamma_1}{\underset{4}{*}} \xi\right)(y)}\right) \in L_2(\mathbb{R}_+, x^\alpha).$

Then the system (4.1) *has the solution which can be written in the following form*

$$
f(x) = K^{-1}\left(\frac{K\left(q_1 - \left(q_2 \overset{\gamma_1}{\underset{4}{*}} \xi\right)\right)(y)}{1 - K\left(\eta \overset{\gamma_1}{\underset{4}{*}}\right)(y)}\right)(x),
$$

$$
g(x) = K^{-1}\left(\frac{K\left(q_2 - \left(\eta \underset{6}{*} q_1\right)\right)(y)}{1 - K\left(\eta \overset{\gamma_1}{\underset{4}{*}} \xi\right)(y)}\right)(x).
$$

Moreover, $f, g \in L_2(\mathbb{R}_+, x^\alpha)\ (\alpha \in \mathbb{R}).$

Proof In order to prove this theorem, we will use the generalized convolution with the weight function $\gamma_1(y)$ for the Kontorovich–Lebedev, Fourier cosine transforms $\left(f \overset{\gamma_1}{\underset{4}{*}} h\right)$ which can be defined by (3.12) and the respectively factorization equality (3.13). We also use the Kontorovich–Lebedev convolution which is of the form (see [6]):

$$
\left(f \underset{6}{*} h\right)(x) := \int_{\mathbb{R}_2^+} \frac{1}{2x}\exp\left[-\frac{1}{2}\left(\frac{\tau\theta}{x} + \frac{\tau x}{\theta} + \frac{\theta x}{\tau}\right)\right]f(\tau)h(\theta)d\tau d\theta, \forall x > 0.
\tag{4.11}
$$

If $f \in L_2(\mathbb{R}_+, x), h \in L_1(\mathbb{R}_+, K_0(x))$, then $\left(f \underset{6}{*} h\right) \in L_2(\mathbb{R}_+, x)$ and the following factorization equality holds

$$
K\left(f \underset{6}{*} h\right)(y) = (Kf)(y)(Kh)(y), \ \forall y > 0.
\tag{4.12}
$$

For the kernels k_3, k_4 defined by (4.10), the system (4.1) becomes

$$\begin{cases} f(x) + \left(\xi \overset{\gamma_1}{\underset{4}{*}} g \right)(x) = q_1(x), \ \forall x > 0, \\ \left(\eta \underset{6}{*} f \right)(x) + g(x) = q_2(x), \ \forall x > 0, \end{cases} \tag{4.13}$$

where $\xi \in L_2^{0,1}, \eta \in L_1(\mathbb{R}_+, K_0(x)), q_1, q_2 \in L_2(\mathbb{R}_+, x^\alpha), \alpha \in \mathbb{R}$ are given functions.

Applying the Kontorovich–Lebedev transform on both sides of (4.13)

$$\begin{cases} (Kf)(y) + K\left(\xi \overset{\gamma_1}{\underset{4}{*}} g \right)(y) = (Kq_1)(y), \ \forall y > 0, \\ K\left(\eta \underset{6}{*} f \right)(y) + (Kg)(y) = (Kq_2)(y), \ \forall y > 0, \end{cases}$$

combining the formula (3.13), (4.12) we have

$$\begin{cases} (Kf)(y) + \sqrt{\dfrac{\pi}{2}} \gamma_1(y)(F_c\xi)(y)(Kg)(y) = (Kq_1)(y), \ \forall y > 0, \\ (K\eta)(y)(Kf) + (Kg)(y) = (Kq_2)(y), \ \forall y > 0. \end{cases}$$

We have

$$\Delta = \begin{vmatrix} 1 & \sqrt{\dfrac{\pi}{2}} \dfrac{1}{y\sinh(\pi y)}(F_c\xi)(y) \\ (K\eta)(y) & 1 \end{vmatrix} = 1 - K\left(\eta \overset{\gamma_1}{\underset{4}{*}} \xi \right)(y), \ \forall y > 0,$$

$$\Delta_1 = \begin{vmatrix} (Kq_1)(y) & \sqrt{\dfrac{\pi}{2}} \dfrac{1}{y\sinh \pi y}(F_c\xi)(y) \\ (Kq_2)(y) & 1 \end{vmatrix}$$

$$= (Kq_1)(y) - K\left(q_2 \overset{\gamma_1}{\underset{4}{*}} \xi \right)(y) = K\left(q_1 - \left(q_2 \overset{\gamma_1}{\underset{4}{*}} \xi \right) \right)(y), \ \forall y > 0,$$

$$\Delta_2 = \begin{vmatrix} 1 & (Kq_1)(y) \\ (K\eta)(y) & (Kq_2)(y) \end{vmatrix} = (Kq_2)(y) - K\left(\eta \underset{6}{*} q_1 \right)(y), \ \forall y > 0,$$

$$= K\left(q_2 - \left(\eta \underset{6}{*} q_1 \right) \right)(y), \ \forall y > 0.$$

Under conditions (C_4) and (C_5) we get $1 - K\left(\eta \overset{\gamma_1}{\underset{4}{*}} \xi \right)(y) \neq 0, \ \forall y > 0$, then

$$(Kf)(y) = \dfrac{K\left(q_1 - \left(q_2 \overset{\gamma_1}{\underset{4}{*}} \xi \right) \right)(y)}{1 - K\left(\eta \overset{\gamma_1}{\underset{4}{*}} \right)(y)} \in L_2(\mathbb{R}_+, x^\alpha), \alpha \in \mathbb{R},$$

or,

$$f(x) = K^{-1}\left(\dfrac{K\left(q_1 - \left(q_2 \overset{\gamma_1}{\underset{4}{*}} \xi \right) \right)(y)}{1 - K\left(\eta \overset{\gamma_1}{\underset{4}{*}} \right)(y)} \right)(x) \in L_2(\mathbb{R}_+, x^\alpha). \tag{4.14}$$

Under conditions (C_4) and (C_6) we have

$$(Kg)(y) = \dfrac{K\left(q_2 - \left(\eta \underset{6}{*} q_1 \right) \right)(y)}{1 - K\left(\eta \overset{\gamma_1}{\underset{4}{*}} \xi \right)(y)} \in L_2(\mathbb{R}_+, x^\alpha), \alpha \in \mathbb{R}.$$

This implies that

$$g(x) = K^{-1} \left(\frac{K \left(q_2 - \left(\eta \underset{6}{*} q_1 \right) \right)(y)}{1 - K \left(\eta \overset{\gamma_1}{\underset{4}{*}} \xi \right)(y)} \right)(x) \in L_2(\mathbb{R}_+, x^\alpha).$$

□

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