A class of group divisible designs

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Let Q be the incidence matrix of a cyclic projective plane of order q, which exists whenever q is a prime power, then

$$QQ^T = qI + J, \quad QJ = (q+1)J.$$

Let $Q = \sum_{d \in D} T^d$ where T is the matrix which is 1 in the $(i,i+1)$ position (reduced mod q^2+q+1) for $i = 1,\ldots,q^2+q+1$. Then

$$Q^2 = \sum_{d \in D} T^{2d} + 2 \sum_{\substack{e,f \in D \\ e \neq f}} T^{e+f}.$$

It is shown in Geramita and Seberry [3,Theorem 4.124] that Q^2 has entries 0,1,2. Let $C = \sum_{d \in D} T^{2d}$. Then D is a difference set. C satisfies

$$CC^T = qI + J, \; CJ = (q+1)J.$$

Let $W = Q^2-J$ which has entries $(0,1,-1)$. Let A be the matrix which is 1 where W is 1 and zero elsewhere. Let B be the matrix which is 1 where W is -1 and zero elsewhere. Then

$$W = A - B \quad \text{and} \quad A + B = J - C.$$

Now W and $A+B$ satisfy

$$WW^T = AA^T + BB^T - AB^T - BA^T = q^2I,$$

$$(A+B)(A+B)^T = AA^T + BB^T + AB^T + BA^T = qI + (q^2-q)J.$$

Hence

$$AA^T + BB^T = \tfrac{1}{2}(q^2+q)I + \tfrac{1}{2}(q^2-q)J,$$

$$AB^T + BA^T = \tfrac{1}{2}(q-q^2)I + \tfrac{1}{2}(q^2-q)J.$$

So

$$\begin{pmatrix} A & B \\ B & A \end{pmatrix}$$

ARS COMBINATORIA, Vol. 6 (1978), pp. 151-152.

is a (0,1) matrix with q^2 elements in each row and column with inner product between any two rows 0 (if they are in different submatrices of the partition) or $\frac{1}{2}(q^2-q)$ (otherwise).

We refer the reader to [1,2] for the definition and some important properties of symmetric regular group divisible designs. Now we can say: Suppose q is a prime power; then there is a symmetric regular group divisible design with parameters

$$(v,b,r,k,\lambda_1,\lambda_2,m,n) = (2(q^2+q+1),\ 2(q^2+q+1), q^2, q^2, 0, \tfrac{1}{2}(q^2-q), q^2+q+1, 2).$$

REFERENCES

[1] R.C. Bose and W.S. Connors, *Combinatorial properties of group divisible incomplete block designs*, Annals of Mathematical Statistics, 23 (1952), 367-383.

[2] R.C. Bose, S.S. Shrikhande and K.N. Bhattacharya, *On the construction of group divisible incomplete block designs*, Annals of Mathematical Statistics, 24(1953), 167-195.

[3] Anthony V. Geramita and Jennifer Seberry, *Orthogonal designs: quadratic forms and Hadamard matrices*, Marcel Dekker, New York 1978.

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