A Class of Integrable Geodesic Flows on the Symplectic Group and the Symmetric Matrices

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Abstract

This paper shows that the left-invariant geodesic flow on the symplectic group relative to the Frobenius metric is an integrable system that is not contained in the Mishchenko–Fomenko class of rigid body metrics. This system may be expressed as a flow on symmetric matrices and is bi-Hamiltonian. This analysis is extended to cover flows on symmetric matrices when an isomorphism

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with the symplectic Lie algebra does not hold. The two Poisson structures associated with this system, inclusive of their Casimirs, are completely analyzed. Since the system integrals are not generated by its Casimirs, it is shown that the nature of integrability is fundamentally different from that exhibited in the Mishchenko–Fomenko setting.

1 Introduction

This paper continues the analysis, begun in Bloch and Iserles [2006], of the set of ordinary differential equations

$$\dot{X} = [X^2, N], \tag{1.1}$$

where $X \in \mathrm{Sym}(n)$, the linear space of $n \times n$ symmetric matrices, $\dot{X}$ denotes the time derivative, $N \in \mathfrak{so}(n)$, the space of skew symmetric $n \times n$ matrices, is given, and where initial conditions $X(0) = X_0 \in \mathrm{Sym}(n)$ are also given.

It is easy to check that $[X^2, N] \in \mathrm{Sym}(n)$, so that if the initial condition is in $\mathrm{Sym}(n)$ then $X(t) \in \mathrm{Sym}(n)$ for all t. Also, because of the straightforward identity $\left[X^2, N\right] = [X, XN + NX]$, this equation may be rewritten in the Lax form

$$\dot{X} = [X, XN + NX], \tag{1.2}$$

again with initial conditions $X(0) = X_0 \in \mathrm{Sym}(n)$.

We show below that this system may be viewed as a Lie–Poisson system on the dual of the symplectic Lie algebra if N invertible, in which case it is geodesic, and on the dual of a more general Lie algebra on symmetric matrices for arbitrary N. The system is bi-Hamiltonian and is not in the Mishchenko–Fomenko class of integrable (geodesic) rigid body systems (Mishchenko and Fomenko [1976]). Despite this, we prove that it is integrable on the generic symplectic leaf of the corresponding phase space if N is invertible or of nullity one. We use the Lax pair with parameter found in Bloch and Iserles [2006] to find a class of integrals that, as we show using the bi-Hamiltonian structure and the technique of Morosi and Pizzocchero [1996], are in involution . Independence is proved directly, since the method in Mishchenko and Fomenko [1976] does not apply to this system, even though the integrals are obtained from the Casimirs with a shifted argument. Indeed, this system appears to be fundamentally different from completely integrable systems either of rigid body or Toda type (on symmetric matrices).

If N is not invertible, there is no isomorphism of the Lie algebra induced by N with the symplectic Lie algebra. We extend our analysis of the system to this case and study the Poisson geometry of the dual of this Lie algebra determining the generic leaves and the Casimir functions of both Poisson structures relative to which the system (1.1) is bi-Hamiltonian.

We want to emphasize that the system (1.1) (or (1.2)) for N invertible is thus *a new integrable geodesic flow* of a left invariant metric on the Lie group $\mathrm{Sp}(n, \mathbb{R})$. So far the only known left invariant metrics whose geodesic flows are integrable on the Lie group $\mathrm{Sp}(n, \mathbb{R})$ are the rigid body metrics of Mishchenko and Fomenko [1976]. Finding integrable geodesic flows on Lie groups for left invariant metrics that are not of rigid body type is a daunting task. System (1.1) is the only one known to us on any semisimple Lie algebra with the exception of $\mathfrak{so}(4)$, where we review the stituation below.

Even for the case of SO(4) there is only one known geodesic flow that is not of rigid body type (see Mishchenko [1970]; Manakov [1976]; Mishchenko and Fomenko [1976]; Ratiu [1980] for the definition of such metrics). There are three integrable cases of left invariant metrics for geodesic flow on SO(4): the metric used in Manakov [1976] (which reduces to the Clebsch case by contraction to the Euclidean group) and two other cases that correspond to left-invariant metrics that are not diagonal in the standard basis of $\mathfrak{so}(4)$. The first one is obtained from deformation of the classical Lyapunov–Steklov integrable case on SE(3) by deforming the Lie algebra $\mathfrak{se}(3)$ to $\mathfrak{so}(4)$; the integrability of the corresponding system is due to Borisov, Mamaev, and Sokolov [2001]. The last case has a fourth quartic constant of the motion and is the genuinely new integrable geodesic case found by Adler and van Moerbeke [1986]; a $\mathfrak{g}_2$ Lax pair for this system was given in Reyman and Semenov-Tian-Shansky [1986]. Sokolov [2001] showed that these two cases are not linearly equivalent. The rigid body metric used by Manakov [1976] is the only algebraic completely integrable case for a left invariant metric that is diagonal in the standard basis of $\mathfrak{so}(4)$ (Adler and van Moerbeke [1982]; Haine [1984]). The state of the art regarding these systems is contained in Theorem 8.3, page 270, of Adler, van Moerbeke, and Vanhaecke [2004]: in a certain large class of metrics (non-degenerate half-diagonal metrics with some weight homogeneity conditions) these three cases are the only algebraically completely integrable geodesic flows. Whether these three cases are the only algebraically completely integrable geodesic flows in the class of *all* left invariant metrics is still an open question. See Sokolov [2002] for a review and references of what is known about a related system, the Kirchhoff case of the motion of a rigid body in an ideal fluid.

The structure of the paper is as follows. In Section 2 we consider the Lie algebra structure on symmetric matrices induced by N and the special case of the isomorphism to $\mathfrak{sp}(\mathbb{R}^n)$. In Section 3 we analyze the bi-Hamiltonian structure as well as the symplectic leaves and Casmirs of both structures. In Section 4 we compare our system with the sectional operator systems of Mishchenko and Fomenko and conclude that (1.1) is not in this family, thereby showing that it is a new geodesic flow that is not of rigid body type on the Lie group Sp($n\mathbb{R}$). In Section 5 we analyze the Lax pair with parameter and find a family with the right number of integrals of motion that is a candidate for Liouville integrability. In Section 6 we prove involution of the integrals using the bi-Hamiltonian structure. In Section 7 we analyze indepedence and finally we discuss some future work in Section 8.

2 The Lie Algebra and the Euler–Poincaré Form

We can regard N as a Poisson tensor on $\mathbb{R}^n$ by defining the bracket of two functions f, g as

$$\{f, g\}_N = (\nabla f)^T N \nabla g. \tag{2.1}$$

The Hamiltonian vector field associated with a function h (with the convention that $\dot{f}(z) = X_h(z) \cdot \nabla f(z) = \{f, h\}(z)$) is given by

$$X_h(z) = N \nabla h(z), \tag{2.2}$$

as is easily checked.

For each $X \in \mathrm{Sym}(n)$ define the quadratic Hamiltonian Q_X by

$$Q_X(z) := \frac{1}{2} z^T X z, \quad z \in \mathbb{R}^n.$$

Let $\mathcal{Q} := \{Q_X \mid X \in \mathrm{Sym}(n)\}$ be the vector space of all such functions. Note that the map $Q : X \in \mathrm{Sym}(n) \mapsto Q_X \in \mathcal{Q}$ is an isomorphism.

Using (2.2) it follows that the Hamiltonian vector field of Q_X has the form

$$X_{Q_X}(z) = NXz. \tag{2.3}$$

Next, we compute the Poisson bracket of two such quadratic functions.

Lemma 2.1. *For $X, Y \in \mathrm{Sym}(n)$, we have*

$$\{Q_X, Q_Y\}_N = Q_{[X,Y]_N}, \tag{2.4}$$

where $[X, Y]_N = XNY - YNX \in \mathrm{Sym}(n)$. In addition, $\mathrm{Sym}(n)$ is a Lie algebra relative to the Lie bracket $[\cdot, \cdot]_N$. Therefore, $Q : X \in (\mathrm{Sym}(n), [\cdot, \cdot]_N) \mapsto Q_X \in (\mathcal{Q}, \{\cdot, \cdot\}_N)$ is a Lie algebra isomorphism.

Proof. Using (2.1) we have

$$\begin{aligned} \{Q_X, Q_Y\}_N(z) &= (\nabla Q_X)(z)^T N (\nabla Q_Y)(z) = (Xz)^T NYz = z^T XNYz \\ &= \frac{1}{2} z^T (XNY - YNX) z = Q_{[X,Y]_N}(z). \end{aligned}$$

Recall that the notation Q_V is reserved only for symmetric matrices V. Since $X, Y \in \mathrm{Sym}(n)$ implies that $[X, Y]_N = XNY - YNX \in \mathrm{Sym}(n)$ we can write $Q_{[X,Y]_N}$ in the preceding equation.

The bracket $[\cdot, \cdot]_N$ on $\mathrm{Sym}(n)$ is clearly bilinear and antisymmetric. The Jacobi identity follows by a straightforward direct verification. ■

It is a general fact that Hamiltonian vector fields and Poisson brackets are related by

$$[X_f, X_g] = -X_{\{f,g\}}, \tag{2.5}$$

where the bracket on the left hand side is the Jacobi–Lie bracket. Thus, it is natural to look at the corresponding algebra of Hamiltonian vector fields on the Poisson manifold $(\mathbb{R}^n, \{\cdot, \cdot\}_N)$ associated to quadratic Hamiltonians. If we take $f = Q_X$ and $g = Q_Y$, with $X_f = NX$ and $X_g = NY$, and recall that the Jacobi–Lie bracket of *linear* vector fields is the negative of the commutator of the associated matrices, then we have the following result.

Proposition 2.2. *Equations (2.4) and (2.5) imply*

$$N[X,Y]_N = [NX, NY]\,. \tag{2.6}$$

This can, of course, be easily verified by hand.

Letting $\mathcal{LH}$ denote the Lie algebra of linear Hamiltonian vector fields on $\mathbb{R}^n$ relative to the commutator bracket of matrices, (2.6) states that the map

$$X \in (\mathrm{Sym}(n), [\cdot,\cdot]_N) \mapsto NX \in (\mathcal{LH}, [\cdot,\cdot])$$

is a homomorphism of Lie algebras[1].

Invertible Case. If N is invertible, then this homomorphism is an isomorphism. In addition, the non-degeneracy of N implies that n is even and that $\mathbb{R}^n$ is a symplectic vector space relative to the symplectic form defined by N^{-1}. Therefore, the Lie algebra $(\mathcal{LH}, [\cdot,\cdot])$ is isomorphic to the Lie algebra $\mathfrak{sp}(\mathbb{R}^n, N^{-1})$ of linear symplectic maps of $\mathbb{R}^n$ relative to the symplectic form N^{-1}, that is, to the classical Lie algebra $\mathfrak{sp}(n,\mathbb{R})$. Note that this means that $(NX)^T N^{-1} + N^{-1}(NX) = 0$.

We summarize these considerations in the following statement.

Proposition 2.3. *Let* $N \in \mathfrak{so}(n)$. *The map* $Q : X \in (\mathrm{Sym}(n), [\cdot,\cdot]_N) \mapsto Q_X \in (\mathcal{Q}, \{\cdot,\cdot\}_N)$ *is a Lie algebra isomorphism. The map* $X \in (\mathrm{Sym}(n), [\cdot,\cdot]_N) \mapsto NX \in (\mathcal{LH}, [\cdot,\cdot])$ *is a Lie algebra homomorphism and if* N *is invertible it induces an isomorphism of* $(\mathrm{Sym}(n), [\cdot,\cdot]_N)$ *with* $\mathfrak{sp}(n,\mathbb{R})$.

The Euler–Poincaré Form The Euler–Poincaré form for the equations can be derived as follows. Identify $\mathrm{Sym}(n)$ with its dual using the the positive definite inner product

$$\langle\langle X, Y\rangle\rangle := \mathrm{trace}\,(XY)\,, \quad \text{for} \quad X, Y \in \mathrm{Sym}(n). \tag{2.7}$$

Remark. The inner product $\langle\langle X, Y\rangle\rangle$ is not ad-invariant relative to the N-bracket, but another one, namely $\kappa_N(X,Y) := \mathrm{trace}(NXNY)$ is invariant, as is easy to check.

Define the Lagrangian $l : \mathrm{Sym}(n) \to \mathbb{R}$ on the Lie algebra $(\mathrm{Sym}(n), [\cdot,\cdot]_N)$ by

$$l(X) = \frac{1}{2}\,\mathrm{trace}\left(X^2\right) = \frac{1}{2}\,\mathrm{trace}\left(XX^T\right) =: \frac{1}{2}\,\langle\langle X, X\rangle\rangle\,. \tag{2.8}$$

Proposition 2.4. *The equations*

$$\dot{X} = [X^2, N] \tag{2.9}$$

are the Euler–Poincaré equations[2] *corresponding to the Lagrangian* (2.8) *on the Lie algebra* $(\mathrm{Sym}(n), [\cdot,\cdot]_N)$.

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[1] We thank Gopal Prasad for suggesting isomorphisms of this type; they are closely related to well-known properties of linear Hamiltonian vector fields, as in Marsden and Ratiu [1994], Proposition 2.7.8.

[2] For a general discussion of the Euler–Poincaré equations, see, for instance, Marsden and Ratiu [1994].

Proof. Recall that the general (left) Euler–Poincaré equations on a Lie algebra $\mathfrak{g}$ associated with a Lagrangian $l : \mathfrak{g} \to \mathbb{R}$ are given by

$$\frac{d}{dt} Dl(\xi) = \operatorname{ad}^*_\xi Dl(\xi),$$

where $Dl(\xi) \in \mathfrak{g}^*$ is the Fréchet derivative of l at ξ. Equivalently, for each fixed $\eta \in \mathfrak{g}$, we have

$$\frac{d}{dt} Dl(\xi) \cdot \eta = Dl(\xi) \cdot [\xi, \eta]. \tag{2.10}$$

In our case, letting $\xi = X$ and $\eta = Y$ arbitrary, time-independent, equations (2.10) become

$$\begin{aligned} \frac{d}{dt} \langle\!\langle X, Y \rangle\!\rangle &= \langle\!\langle X, [X, Y]_N \rangle\!\rangle \\ &= \langle\!\langle X, XNY - YNX \rangle\!\rangle \,; \end{aligned}$$

that is,

$$\begin{aligned} \operatorname{trace}\left(\dot{X} Y\right) &= \operatorname{trace}\left(X(XNY - YNX)\right) \\ &= \operatorname{trace}\left((X^2 N - N X^2) Y\right), \end{aligned}$$

which gives the result. ■

General Case–Noninvertible N. We next determine the structure of the Lie algebra $(\operatorname{Sym}(n), [\cdot,\cdot]_N)$ for a general skew-symmetric matrix N. The point of departure is the fact that if N is nondegenerate, then $X \in (\operatorname{Sym}(n), [\cdot,\cdot]_N) \mapsto NX \in (\mathcal{LH}, [\cdot,\cdot]) = (\mathfrak{sp}(\mathbb{R}^n, N^{-1}), [\cdot,\cdot])$ is a Lie algebra isomorphism. Recall that if $\mathbb{R}^n$ has an inner product, which we shall take in what follows to be the usual dot product associated to the basis in which the skew-symmetrix matrix N is given, and $L : \mathbb{R}^n \to \mathbb{R}^n$ is a linear map, then $\mathbb{R}^n$ decomposes orthogonally as $\mathbb{R}^n = \operatorname{im} L^T \oplus \ker L$. Taking $L = N$ in this statement and recalling that $N^T = -N$, we get the orthogonal decomposition $\mathbb{R}^n = \operatorname{im} N \oplus \ker N$. Let $2p = \operatorname{rank} N$ and $d := n - 2p$. Then $\bar{N} := N|_{\operatorname{im} N} : \operatorname{im} N \to \operatorname{im} N$ defines a nondegenerate skew symmetric bilinear form and, by the previous proposition, $(\operatorname{Sym}(2p), [\cdot,\cdot]_{\bar{N}})$ is isomorphic as a Lie algebra to $(\mathfrak{sp}(\mathbb{R}^{2p}, \bar{N}^{-1}), [\cdot,\cdot])$. In this direct sum decomposition of $\mathbb{R}^n$, the skew- symmetric matrix N takes the form

$$N = \begin{bmatrix} \bar{N} & 0 \\ 0 & 0 \end{bmatrix},$$

where $\bar{N}$ is a $(2p) \times (2p)$ skew-symmetric nondegenerate matrix.

The Lie algebra $(\operatorname{Sym}(2p), [\cdot,\cdot]_{\bar{N}})$ acts on the vector space $\mathcal{M}_{(2p)\times d}$ of $(2p) \times d$ matrices (which we can think of as linear maps of $\ker N$ to $\operatorname{im} N$) by $S \cdot A := S\bar{N}A$, where $S \in (\operatorname{Sym}(2p), [\cdot\,\cdot]_{\bar{N}})$ and $A \in \mathcal{M}_{(2p)\times d}$. Indeed, if $S, S' \in \operatorname{Sym}(2p)$ and $A \in \mathcal{M}_{(2p)\times d}$, then

$$\begin{aligned} [S, S']_{\bar{N}} \cdot A &= (S\bar{N}S' - S'\bar{N}S)\bar{N}A = S\bar{N}S'\bar{N}A - S'\bar{N}S\bar{N}A \\ &= S \cdot (S' \cdot A) - S' \cdot (S \cdot A). \end{aligned} \tag{2.11}$$

Now form the semidirect product $\mathrm{Sym}(2p) \circledS \mathcal{M}_{(2p)\times d}$. Its bracket is defined by

$$\begin{aligned}[(S,A),(S',A')] &= ([S,S']_{\bar{N}}, S\cdot A' - S'\cdot A) \\ &= (S\bar{N}S' - S'\bar{N}S, S\bar{N}A' - S'\bar{N}A) \end{aligned} \tag{2.12}$$

for any $S, S' \in \mathrm{Sym}(2p)$ and $A, A' \in \mathcal{M}_{(2p)\times d}$.

Next, define the $\mathrm{Sym}(d)$-valued Lie algebra two cocycle

$$C : \mathrm{Sym}(2p) \circledS \mathcal{M}_{(2p)\times d} \times \mathrm{Sym}(2p) \circledS \mathcal{M}_{(2p)\times d} \to \mathrm{Sym}(d)$$

by

$$C((S,A),(S',A')) := A^T\bar{N}A' - (A')^T\bar{N}A \tag{2.13}$$

for any $S, S' \in \mathrm{Sym}(2p)$ and $A, A' \in \mathcal{M}_{(2p)\times d}$. The cocycle identity

$$\begin{aligned} &C([(S,A),(S',A')],(S'',A'')) + C([(S',A'),(S'',A'')],(S,A)) \\ &\quad + C([(S'',A''),(S,A)],(S',A')) = 0 \end{aligned}$$

for any $S, S', S'' \in \mathrm{Sym}(2p)$ and $A, A', A'' \in \mathcal{M}_{(2p)\times d}$ is a straightforward verification. Now extend $\mathrm{Sym}(2p) \circledS \mathcal{M}_{(2p)\times d}$ by this cocycle. That is, form the vector space $(\mathrm{Sym}(2p) \circledS \mathcal{M}_{(2p)\times d}) \oplus \mathrm{Sym}(d)$ and endow it with the bracket

$$[(S,A,B),(S',A',B')]^C := (S\bar{N}S' - S'\bar{N}S, S\bar{N}A' - S'\bar{N}A, A^T\bar{N}A' - (A')^T\bar{N}A) \tag{2.14}$$

for any $S, S' \in \mathrm{Sym}(2p)$, $A, A' \in \mathcal{M}_{(2p)\times d}$, and $B, B' \in \mathrm{Sym}(d)$.

Proposition 2.5. *The map*

$$\Psi : ((\mathrm{Sym}(2p) \circledS \mathcal{M}_{(2p)\times d}) \oplus \mathrm{Sym}(d), [\cdot,\cdot]^C) \to (\mathrm{Sym}(n), [\cdot,\cdot]_N)$$

given by

$$\Psi(S,A,B) := \begin{bmatrix} S & A \\ A^T & B \end{bmatrix} \tag{2.15}$$

is a Lie-algebra isomorphism.

Proof. It is obvious that Ψ is a vector space isomorphism, therefore only the Lie-algebra homomorphism condition needs to be verified. So, let $(S,A,B), (S',A',B') \in (\mathrm{Sym}(2p) \circledS \mathcal{M}_{(2p)\times d}) \oplus \mathrm{Sym}(d)$ and compute

$$\begin{aligned} \Psi([(S,A,B),(S',A',B')]) &= \Psi(S\bar{N}S' - S'\bar{N}S, S\bar{N}A' - S'\bar{N}A, A^T\bar{N}A' - (A')^T\bar{N}A) \\ &= \begin{bmatrix} S\bar{N}S' - S'\bar{N}S & S\bar{N}A' - S'\bar{N}A \\ (S\bar{N}A' - S'\bar{N}A)^T & A^T\bar{N}A' - (A')^T\bar{N}A \end{bmatrix} \\ &= \begin{bmatrix} S & A \\ A^T & B \end{bmatrix} \begin{bmatrix} \bar{N} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} S' & A' \\ (A')^T & B' \end{bmatrix} - \begin{bmatrix} S' & A' \\ (A')^T & B' \end{bmatrix} \begin{bmatrix} \bar{N} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} S & A \\ A^T & B \end{bmatrix} \\ &= [\Psi(S,A,B), \Psi(S',A',B')]_N \end{aligned}$$

as required. ■

3 Poisson Structures

Identifying $\mathrm{Sym}(n)$ with its dual using the inner product (2.7) endows $\mathrm{Sym}(n)$ with the the (left, or minus) Lie–Poisson bracket

$$\{f, g\}_N (X) = -\operatorname{trace}\Big[X\Big(\nabla f(X) N \nabla g(X) - \nabla g(X) N \nabla f(X)\Big)\Big], \tag{3.1}$$

where ∇f is the gradient of f relative to the inner product $\langle\langle\cdot,\cdot\rangle\rangle$ on $\mathrm{Sym}(n)$. It is easy to check that the equations $\dot{X} = \left[X^2, N\right]$ are Hamiltonian relative to the function l defined in (2.8) and the Lie–Poisson bracket (3.1).

Later on we shall also need the *frozen* Poisson bracket

$$\{f, g\}_{FN} (X) = -\operatorname{trace}\Big(\nabla f(X) N \nabla g(X) - \nabla g(X) N \nabla f(X)\Big). \tag{3.2}$$

It is a general fact that the Poisson structures (3.1) and (3.2) are *compatible* in the sense that their sum is a Poisson structure (see e.g. Exercise 10.1-5 in Marsden and Ratiu [1994]).

For what follows it is important to compute the Poisson tensors corresponding to the above Poisson brackets. Recall that the Poisson tensor can be viewed as a vector bundle morphism $B : T^*(\mathrm{Sym}(n)) \to T(\mathrm{Sym}(n))$ covering the identity. It is defined by $B(\mathbf{d}h) = \{\cdot, h\}_N$ for any locally defined smooth function h on $\mathrm{Sym}(n)$. Since $\mathrm{Sym}(n)$ is a vector space, these bundles are trivial and hence the value B_X at $X \in \mathrm{Sym}(n)$ of the Poisson tensor B is a linear map $B_X : \mathrm{Sym}(n) \to \mathrm{Sym}(n)$ by identifying $\mathrm{Sym}(n)$ with its dual using the inner product $\langle\langle\cdot,\cdot\rangle\rangle$.

Proposition 3.1. *Denote the value at $X \in \mathrm{Sym}(n)$ of the Poisson tensors corresponding to the Lie–Poisson (3.1) and frozen (3.2) brackets by B_X and C_X, respectively. Then for any $Y \in \mathrm{Sym}(n)$ we have*

$$B_X(Y) = XYN - NYX \tag{3.3}$$
$$C_X(Y) = YN - NY. \tag{3.4}$$

Proof. Let f and g be locally defined smooth functions on $\mathrm{Sym}(n)$. The definition of B_X gives

$$\begin{aligned}
\langle\langle \nabla f(X), B_X(\nabla g(X)\rangle\rangle &= \{f, g\}_N(X) \\
&= -\operatorname{trace}\Big[X\Big(\nabla f(X) N \nabla g(X) - \nabla g(X) N \nabla f(X)\Big)\Big] \\
&= \operatorname{trace}\Big[\nabla f(X)\Big(X \nabla g(X) N - N \nabla g(X) X\Big)\Big] \\
&= \langle\langle \nabla f(X), X \nabla g(X) N - N \nabla g(X) X\rangle\rangle,
\end{aligned}$$

which implies (3.3) since any $Y \in \mathrm{Sym}(n)$ is of the form $\nabla g(X)$, where $g(X) =$

$\langle\!\langle X, Y\rangle\!\rangle$. Similarly, the definition of C_X gives

$$\begin{aligned}\langle\!\langle \nabla f(X), C_X(\nabla g(X))\rangle\!\rangle &= \{f,g\}_{FN}(X)\\ &= -\operatorname{trace}\Big(\nabla f(X)N\nabla g(X) - \nabla g(X)N\nabla f(X)\Big)\\ &= \operatorname{trace}\Big[\nabla f(X)\Big(\nabla g(X)N - N\nabla g(X)\Big)\Big]\\ &= \langle\!\langle \nabla f(X), \nabla g(X)N - N\nabla g(X)\rangle\!\rangle,\end{aligned}$$

which proves (3.4). ∎

Proposition 3.2. *Let $n = 2p + d$, where $2p = \operatorname{rank} N$. The generic leaves of the Lie–Poisson bracket $\{\cdot,\cdot\}_N$ are $2p(p+d)$-dimensional.*

Proof. As in the proof of Proposition 2.5, we orthogonally decompose $\mathbb{R}^n = \operatorname{im} N \oplus \ker N$ so that $\bar N = N|\operatorname{im} N : \operatorname{im} N \to \operatorname{im} N$ is an isomorphism. In this decomposition the matrix N takes the form

$$N = \begin{bmatrix}\bar N & 0\\ 0 & 0\end{bmatrix}$$

and, according to the isomorphism Ψ in Proposition 2.5, the matrix X can be written as

$$X = \begin{bmatrix} S & A\\ A^T & B\end{bmatrix},$$

where $S \in \operatorname{Sym}(2p)$, $B \in \operatorname{Sym}(d)$, and $A \in \mathcal{M}_{(2p)\times d}$. Therefore, if

$$Y = \begin{bmatrix} U & C\\ C^T & D\end{bmatrix} \in \operatorname{Sym}(n)$$

with $U \in \operatorname{Sym}(2p)$, $D \in \operatorname{Sym}(d)$, $C \in \mathcal{M}_{(2p)\times d}$, the Poisson tensor of the Lie–Poisson bracket $\{\cdot,\cdot\}_N$ takes the form (see Proposition 3.1)

$$\begin{aligned}B_X(Y) &= XYN - NYX\\ &= \begin{bmatrix} S & A\\ A^T & B\end{bmatrix}\begin{bmatrix} U & C\\ C^T & D\end{bmatrix}\begin{bmatrix}\bar N & 0\\ 0 & 0\end{bmatrix} - \begin{bmatrix}\bar N & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix} U & C\\ C^T & D\end{bmatrix}\begin{bmatrix} S & A\\ A^T & B\end{bmatrix}\\ &= \begin{bmatrix} SU\bar N - \bar N US + AC^T\bar N - \bar N CA^T & -\bar N UA - \bar N CB\\ A^TU\bar N + BC^T\bar N & 0\end{bmatrix}.\end{aligned}$$

Since $\bar N$ is invertible, the kernel of $B_X : \operatorname{Sym}(n) \to \operatorname{Sym}(n)$ is therefore given by all $U \in \operatorname{Sym}(2p)$, $C \in \operatorname{Sym}(d)$, and $C \in \mathcal{M}_{(2p)\times d}$ such that

$$SU\bar N - \bar N US + AC^T\bar N - \bar N CA^T = 0 \quad \text{and} \quad UA + CB = 0.$$

To compute the dimension of the maximal symplectic leaves, we assume that the matrix X is generic. So, supposing that B is invertible, we have $C = -UAB^{-1}$ and

$$\left(S - AB^{-1}A^T\right)U\bar N - \bar N U\left(S - AB^{-1}A^T\right) = 0.$$

Since $S - AB^{-1}A^T \in \mathrm{Sym}(2p)$ is given, this condition is identical to the vanishing of the Poisson tensor on the dual of the Lie algebra $(\mathrm{Sym}(2p), [\cdot\,,\cdot]_{\bar{N}})$ evaluated at $S - AB^{-1}A^T$. But $\bar{N}$ is invertible so, according to Proposition 2.3, this Lie algebra is isomorphic to $\mathfrak{sp}(2p, \mathbb{R})$ whose rank is p. Therefore, the kernel of the map

$$U \in \mathrm{Sym}(2p) \mapsto \left(S - AB^{-1}A^T\right) U\bar{N} - \bar{N}U \left(S - AB^{-1}A^T\right) \in \mathrm{Sym}(2p)$$

for generic $S - AB^{-1}A^T$ has dimension p.

Since $C = -UAB^{-1}$ is uniquely determined and $D \in \mathrm{Sym}(d)$ is arbitrary, we see that the dimension of the kernel of B_X for generic X has dimension $p + d(d+1)/2$.

Thus, the dimension of the generic leaf of the Lie–Poisson bracket $\{\cdot, \cdot\}_N$ is

$$\frac{1}{2}(2p + d)(2p + d + 1) - p - \frac{1}{2}d(d + 1) = 2p(p + d)$$

as claimed in the statement of the proposition. ■

Proposition 3.3. *All the leaves of the frozen Poisson bracket $\{\cdot, \cdot\}_{FN}$ are*

(i) *$2p(p + d)$-dimensional if N is generic, that is, all its non-zero eigenvalues are distinct, and*

(ii) *$p(p + 1 + 2d)$-dimensional if all non-zero eigenvalue pairs of N are equal.*

Proof. Proceeding as in the proof of the previous proposition and using the same notation for N, X, and Y, the Poisson tensor of the frozen bracket takes the form

$$\begin{aligned} C_X(Y) = YN - NY &= \begin{bmatrix} U & C \\ C^T & D \end{bmatrix} \begin{bmatrix} \bar{N} & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} \bar{N} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U & C \\ C^T & D \end{bmatrix} \\ &= \begin{bmatrix} U\bar{N} - \bar{N}U & \bar{N}C \\ C^T\bar{N} & 0 \end{bmatrix}. \end{aligned}$$

Thus, since $\bar{N}$ is invertible, the kernel of C_X is given by all $U \in \mathrm{Sym}(2p)$, $D \in \mathrm{Sym}(d)$, $C \in \mathcal{M}_{(2p)\times d}$ such that $C = 0$ and $U\bar{N} - \bar{N}U = 0$.

Since $\bar{N}$ is non-degenerate, there exists an orthogonal matrix Q such that

$$\bar{N} = Q^T \begin{bmatrix} 0 & V \\ -V & 0 \end{bmatrix} Q,$$

where $V = \mathrm{diag}(v_1, \ldots, v_p)$ and $v_i \in \mathbb{R}$, $v_i \neq 0$ for all $i = 1, \ldots, p$. Therefore,

$$\begin{aligned} 0 = U\bar{N} - \bar{N}U &= UQ^T \begin{bmatrix} 0 & V \\ -V & 0 \end{bmatrix} Q - Q^T \begin{bmatrix} 0 & V \\ -V & 0 \end{bmatrix} QU \\ &= Q^T \left(QUQ^T \begin{bmatrix} 0 & V \\ -V & 0 \end{bmatrix} - \begin{bmatrix} 0 & V \\ -V & 0 \end{bmatrix} QUQ^T \right) Q \end{aligned}$$

is equivalent to

$$\tilde{U} \begin{bmatrix} 0 & V \\ -V & 0 \end{bmatrix} - \begin{bmatrix} 0 & V \\ -V & 0 \end{bmatrix} \tilde{U} = 0 \tag{3.5}$$

where $\tilde{U} := QUQ^T \in \mathrm{Sym}(2p)$. Write

$$\tilde{U} = \begin{bmatrix} U_{11} & U_{12} \\ U_{12}^T & U_{22} \end{bmatrix}$$

with U_{11} and U_{22} symmetric $p \times p$ matrices and U_{12} an arbitrary $p \times p$ matrix. Then (3.5) is equivalent to

$$U_{22} = VU_{11}V^{-1} = V^{-1}U_{11}V \quad \text{and} \quad U_{12}^T = -V^{-1}U_{12}V = -VU_{12}V^{-1}. \tag{3.6}$$

(i) Assume now that $v_i \neq v_j$ if $i \neq j$. Since $VU_{11}V^{-1} = V^{-1}U_{11}V$ is equivalent to $V^2U_{11}V^{-2} = U_{11}$, it follows that

$$\frac{v_i^2}{v_j^2} u_{11,ij} = u_{11,ij} \quad \text{for all} \quad i, j = 1, \ldots, p,$$

where $u_{11,ij}$ are the entries of the symmetric matrix U_{11}. Since the fraction on the left hand side is never equal to one for $i \neq j$, this relation implies that $u_{11,ij} = 0$ for all $i \neq j$. Thus U_{11} is diagonal and $U_{22} = U_{11}$. A similar argument shows that U_{12} is diagonal. However, then it follows that $U_{12} = -U_{12}^T$ which implies that $U_{12} = 0$. Therefore, the kernel of the map $U \mapsto U\bar{N} - \bar{N}U$ is p-dimensional.

Concluding, the dimension of every leaf of the frozen Poisson structure equals $\frac{1}{2}(2p+d)(2p+d+1) - p - \frac{1}{2}d(d+1) = 2p(p+d)$.

(ii) The other extreme case is when $v_i = v_j =: v$ for all $i, j = 1, \ldots, p$. Then $V = vI$, where I is the identity matrix, and (3.6) becomes $U_{22} = U_{11}$, $U_{12}^T = -U_{12}$. Therefore, the kernel of the map $U \mapsto U\bar{N} - \bar{N}U$ has dimension equal to $\frac{1}{2}p(p+1) + \frac{1}{2}p(p-1) = p^2$.

Concluding, the dimension of every leaf of the frozen Poisson structure equals $\frac{1}{2}(2p+d)(2p+d+1) - p^2 - \frac{1}{2}d(d+1) = p(p+1+2d)$. ■

Proposition 3.4 (Casimir functions)**.** *Let the skew symmetric matrix N have rank $2p$ and size $n := 2p+d$. Choose an orthonormal basis of $\mathbb{R}^{2p+d}$ in which N is written as*

$$N = \begin{bmatrix} 0 & V & 0 \\ -V & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where V is a real diagonal matrix whose entries are $v_1, \ldots, v_p$.

(i) *If $v_i \neq v_j$ for all $i \neq j$, the $p + d(d+1)/2$ Casimir functions for the frozen Poisson structure (3.2) are given by*

$$C_F^i(X) = \mathrm{trace}(E_iX), \quad i = 1, \ldots, p + \frac{1}{2}d(d+1),$$

where E_i is any of the matrices

$$\begin{bmatrix} S_{kk} & 0 & 0 \\ 0 & S_{kk} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & S_{ab} \end{bmatrix}.$$

Here S_{kk} is the $p \times p$ matrix all of whose entries are zero except the diagonal (k,k) entry which is one and S_{ab} is the $d \times d$ symmetric matrix having all entries equal to zero except for the (a,b) and (b,a) entries that are equal to one.

(ii) *If $v_i = v_j$ for all $i, j = 1, \ldots, p$, the $p^2 + d(d+1)/2$ Casimir functions for the frozen Poisson structure (3.2) are given by*

$$C_F^i(X) = \operatorname{trace}(E_i X), \quad i = 1, \ldots, p^2 + \frac{1}{2}d(d+1),$$

where E_i is any of the matrices

$$\begin{bmatrix} S_{kl} & 0 & 0 \\ 0 & S_{kl} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & A_{kl} & 0 \\ -A_{kl} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & S_{ab} \end{bmatrix}.$$

Here S_{kl} is the $p \times p$ symmetric matrix having all entries equal to zero except for the (k,l) and (l,k) entries that are equal to one and A_{kl} is the $p \times p$ skew symmteric matrix with all entries equal to zero except for the (k,l) entry which is 1 and the (l,k) entry which is -1.

(iii) *Denote*

$$\bar{N} = \begin{bmatrix} 0 & V \\ -V & 0 \end{bmatrix} \quad \textit{and} \quad \hat{N} = \begin{bmatrix} \bar{N}^{-1} & 0 \\ 0 & 0 \end{bmatrix}.$$

The $p + d(d+1)/2$ Casimir functions for the Lie–Poisson bracket $\{\cdot,\cdot\}_N$ are given by

$$C^k(X) = \frac{1}{2k} \operatorname{trace}\left[\left(X\hat{N}\right)^{2k}\right], \quad \textit{for} \quad k = 1, \ldots, p$$

and

$$C^k(X) = \operatorname{trace}(XE_k), \quad \textit{for} \quad k = p+1, \ldots, p + \frac{1}{2}d(d+1),$$

where E_k is any matrix of the form

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & S_{ab} \end{bmatrix}.$$

In the special case when N is full rank the Casimirs are just

$$C^k(X) = \frac{1}{2k} \operatorname{trace}\left[\left(XN^{-1}\right)^{2k}\right], \quad \textit{for} \quad k = 1, \ldots, p,$$

Proof. To prove (i), recall from Proposition 3.3(i) that the kernel of the Poisson tensor C_X has dimension $p + \frac{1}{2}d(d+1)$. Moreover, if E belongs to this kernel, then the linear function given by $X \mapsto \operatorname{trace}(EX)$ has gradient E, which is annihilated by the Poisson tensor C_X. Thus all C_F^i are Casimir functions. Since the gradients of all these functions are the $p + \frac{1}{2}d(d+1)$ matrices in the statement which are obviously linearly independent, it follows that the functions C_F^i form a functionally independent set of Casimir functions for the frozen bracket $\{\cdot,\cdot\}_{FN}$.

Part (ii) has an identical proof.

In the proof of (iii) we do not need the detailed 3×3 block decomposition of N and X and shall use exclusively the 2×2 block decomposition, where the $(1,1)$ block has size $(2p)\times(2p)$. Consider first the functions $C^k(X)$ for $k=1,\dots,p$. Note that $\nabla C^k(X)=\hat N X\hat N\cdots\hat N X\hat N$ (with $(2k-1)$ factors of X) and hence (3.3) gives

$$B_X(\nabla C^k(X)) = X\left(\hat N X\hat N\cdots\hat N X\hat N\right)N - N\left(\hat N X\hat N\cdots\hat N X\hat N\right)X. \tag{3.7}$$

Note firstly that in the case when N is invertible this is just $X\hat N\cdots\hat N X - X\hat N\cdots\hat N X$ which is clearly 0.

Now consider the general case. We first observe that

$$\hat N N = N\hat N = \begin{bmatrix} I & 0\\ 0 & 0\end{bmatrix}. \tag{3.8}$$

The product of the last four factors in the first term of equation (3.7) is thus

$$\hat N X\hat N N = \begin{bmatrix} \bar N^{-1}S & 0\\ 0 & 0\end{bmatrix}.$$

Similarly, the product of the first four factors of the second term of (3.7) is

$$N\hat N X\hat N = \begin{bmatrix} S\bar N^{-1} & 0\\ 0 & 0\end{bmatrix}.$$

Continuing the multiplication in both terms in this fashion (always taking a group of three consecutive factors from the right and left, respectively) we see that both terms have only nonzero $(1,1)$ blocks which are identical and equal to $S\bar N^{-1}S\bar N^{-1}....\bar N^{-1}S$. Thus, again, (3.7) is identically zero.

However, $\mathfrak{sp}(2p,\mathbb{R})$ is identified with the subalgebra consisting of the $(1,1)$ blocks of elements of $\mathrm{Sym}(n)$ (see Proposition 2.5). The isomorphism $S\in\mathrm{Sym}(2p)\mapsto \bar N S\in\mathfrak{sp}(2p,\mathbb{R})$ given in Proposition 2.3 identifies the basis of p Casimirs in the dual of $\mathfrak{sp}(2p,\mathbb{R})$ (given by the even traces of the powers of a matrix) with the functions $S\mapsto \operatorname{trace}\left[(S\bar N^{-1})^{2k}\right]/2k$. Therefore the functions C^k for $k=1,\dots,p$ given in the statement of the proposition are functionally independent Casimirs for the Lie–Poisson bracket of $\mathrm{Sym}(n)$.

To see that the remaining functions $C^k(X)=\operatorname{trace}(XE_k)$ are Casimirs observe that in this case

$$\nabla C^k(X) = \begin{bmatrix} 0 & 0\\ 0 & S_{ab}\end{bmatrix}$$

and

$$B_X(\nabla C^k(X)) = \begin{bmatrix} S & A\\ A^T & B\end{bmatrix}\begin{bmatrix} 0 & 0\\ 0 & S_{ab}\end{bmatrix}\begin{bmatrix} \bar N & 0\\ 0 & 0\end{bmatrix} - \begin{bmatrix} \bar N & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix} 0 & 0\\ 0 & S_{ab}\end{bmatrix}\begin{bmatrix} S & A\\ A^T & B\end{bmatrix} = 0.$$

Since the matrices S_{ab} span symmetric $k \times k$ matrices, these Casimirs are functionally independent. The two sets of Casimirs are also independent taken together, since each set depends only on a subset of independent variables and these two sets of variables are disjoint. We have thus obtained $p + d(d+1)/2$ Casimirs, which is the codimension of the generic leaf, thus proving that they generate the space of all Casimir functions of the Lie–Poisson bracket. ■

The equations in the degenerate case. If N is degenerate, representing it and the matrix $X \in \mathrm{Sym}(n)$ as in Proposition 2.5, the equations $\dot{X} = [X^2, N]$ are equivalent to the system

$$\begin{cases} \dot{S} = [S^2 + A^T A, \bar{N}] \\ \dot{A} = -\bar{N}(SA + AB) \\ \dot{B} = 0 \end{cases}$$

4 The Sectional Operator Equations

This section shows that the flow (1.1) is not of the sectional operator type discussed in Mishchenko and Fomenko [1976]; in fact, this is the case already for 2×2 matrices with the canonical choice of N.

Let

$$N = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{4.1}$$

and denote elements of Sym(2) by

$$X = \begin{bmatrix} a & b \\ b & d \end{bmatrix}, \quad a, b, c \in \mathbb{R}. \tag{4.2}$$

One can readily check that a maximal Abelian subalgebra of Sym(2), that is, a Cartan subalgebra, consists of purely off-diagonal matrices

$$A = \begin{bmatrix} 0 & \alpha \\ \alpha & 0 \end{bmatrix}, \quad \alpha \in \mathbb{R}. \tag{4.3}$$

A complementary subspace is $\mathrm{Sym}_d(2)$, the space of diagonal 2×2 matrices. Notice that for any $X \in \mathrm{Sym}(2)$ we have

$$[A, X]_N = \begin{bmatrix} -2\alpha a & 0 \\ 0 & 2\alpha d \end{bmatrix} \tag{4.4}$$

and hence, also in accordance with general theory, if $\alpha \neq 0$, then $\mathrm{ad}_A : \mathrm{Sym}_d(2) \to \mathrm{Sym}_d(2)$ is an isomorphism. Thus the inverse $\mathrm{ad}_A^{-1} : \mathrm{Sym}_d(2) \to \mathrm{Sym}_d(2)$ is defined and hence

$$\mathrm{ad}_A^{-1}(\mathrm{ad}_B X) = \frac{\beta}{\alpha} \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \quad \text{for} \quad A = \begin{bmatrix} 0 & \alpha \\ \alpha & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & \beta \\ \beta & 0 \end{bmatrix}, \quad \alpha \neq 0. \tag{4.5}$$

An operator of this form is called a *sectional operator* in the sense of Mishchenko and Fomenko [1976]. The equations defined by a sectional operator are

$$\dot{X} = \left[X, \operatorname{ad}_A^{-1}\left(\operatorname{ad}_B X\right)\right]_N = \frac{\beta}{\alpha}\begin{bmatrix} -2ab & 0 \\ 0 & 2bd \end{bmatrix}. \tag{4.6}$$

We shall now prove that (1.1) *is not in this family.* Indeed, since

$$XN + NX = \begin{bmatrix} 0 & a+d \\ -a-d & 0 \end{bmatrix} = (a+d)N \tag{4.7}$$

equation (1.2) becomes

$$\dot{X} = [X, XN + NX] = (a+d)\begin{bmatrix} -2b & a-d \\ a-d & 2b \end{bmatrix}. \tag{4.8}$$

The only way equations (4.6) and (4.8) can be identical is if one requires that $a = d$, which is not allowed since X is arbitrary in $\mathrm{Sym}(2)$. Therefore the system (1.1) *is not in the list of equations of generalized rigid body type on* $\mathfrak{sp}(2,\mathbb{R})$ *described by a sectional operator in* Mishchenko and Fomenko [1976].

Despite the fact that our system is not in the class of integrable systems studied in Mishchenko and Fomenko [1976], we shall see in the next sections, using the techniques of Manakov [1976] and Magri [1978] (the method of recursion operators), that the system is nonetheless integrable.

5 Lax Pairs with Parameter

To prove that system (1.1) is integrable for any choice of N, we will compute its flow invariants. Bear it in mind that, by virtue of the isospectral representation (1.2), we already know that the eigenvalues of X, or alternatively, the quantities trace X^k for $k = 1, 2, \ldots, n-1$, are invariants.

One way to compute additional invariants is to rewrite the system as a Lax pair with a parameter. One can do this in a fashion similar to that for the generalized rigid body equations (see Manakov [1976]).

Theorem 5.1. *Let λ be a real parameter. The system* (1.2) *is equivalent to the following Lax pair system*

$$\frac{d}{dt}(X + \lambda N) = \left[X + \lambda N, NX + XN + \lambda N^2\right] \tag{5.1}$$

Proof. The proof is a computation. The only nontrivial power of λ to check is the first. In fact, the coefficient of λ on the right hand side of equation (5.1) is

$$\begin{aligned} &[N, NX + XN] + [X, N^2] \\ &\qquad = N^2X + NXN - NXN - XN^2 + XN^2 - N^2X = 0, \end{aligned}$$

which proves (5.1). ∎

We recall from Manakov [1976] and Ratiu [1980] that the left-invariant generalized rigid body equations on $SO(n)$ may be written as

$$\dot{M} = [M, \Omega], \qquad M(0) = M_0 \in \mathfrak{so}(n), \tag{5.2}$$

where $\Omega = Q^{-1}\dot{Q} \in \mathfrak{so}(n)$ is the body angular velocity, $Q \in SO(n)$ denotes the configuration space variable (the attitude of the body), and

$$M = J(\Omega) := \Lambda\Omega + \Omega\Lambda \in \mathfrak{so}(n)$$

is the body angular momentum. Here $J : \mathfrak{so}(n) \to \mathfrak{so}(n)$ is the symmetric, positive definite (and hence invertible) operator defined by

$$J(\Omega) := \Lambda\Omega + \Omega\Lambda,$$

where Λ is a diagonal matrix satisfying $\Lambda_i + \Lambda_j > 0$ for all $i \neq j$. For $n = 3$ the elements of Λ_i are related to the standard diagonal moment of inertia tensor I by $I_1 = \Lambda_2 + \Lambda_3$, $I_2 = \Lambda_3 + \Lambda_1$, $I_3 = \Lambda_1 + \Lambda_2$.

Manakov [1976] has noticed that the generalized rigid body equations (5.2) can be written as a Lax equation with a parameter in the form

$$\frac{d}{dt}(M + \lambda\Lambda^2) = [M + \lambda\Lambda^2, \Omega + \lambda\Lambda]. \tag{5.3}$$

Note the following contrast with our setting: in the Manakov case the system matrix M is in $\mathfrak{so}(n)$ and the parameter Λ is a symmetric matrix while in our case X is symmetric and the parameter $N \in \mathfrak{so}(n)$.

For the generalized rigid body the nontrivial coefficients of $\lambda^i, 0 < i < k$ in the traces of the powers of $M + \lambda\Lambda^2$ then yield the right number of independent integrals in involution to prove integrability of the flow on a generic adjoint orbit of $SO(n)$ (identified with the corresponding coadjoint orbit). The case $i = 0$ needs to be eliminated, because these are Casimir functions.

Similarly, in our case, the nontrivial coefficients of $\lambda^i, 0 \leq i \leq k$, in

$$h_k^\lambda(X) := \frac{1}{k}\operatorname{trace}(X + \lambda N)^k, \qquad k = 1, 2, \ldots, n-1 \tag{5.4}$$

yield the conserved quantities. The coefficient of $\lambda^r, 0 \leq r \leq k$, in (5.4) is

$$\operatorname{trace} \sum_{|i|=k-r} \sum_{|j|=r} X^{i_1} N^{j_1} X^{i_2} \cdots X^{i_s} N^{j_s}, \qquad r = 0, \ldots, k, \quad k = 1, \ldots, n-1,$$

where $i = (i_1, i_2, \ldots i_s)$, $j = (j_1, j_2, \ldots j_s)$ are multi-indices, $i_q, j_q = 0, 1, \ldots, k$, and $|i| = \sum_{q=1}^s i_q$, $|j| = \sum_{q=1}^s j_q$. The coefficient of λ^k is the constant N^k so it should not be counted. Thus we have $r < k$. In addition, since the trace of a matrix equals the trace of its transpose, $X \in \operatorname{Sym}(n)$, and $N \in \mathfrak{so}(n)$, it follows that

$$\operatorname{trace} X^{i_1} N^{j_1} X^{i_2} \cdots X^{i_s} N^{j_s} = (-1)^{|j|} \operatorname{trace} N^{j_s} X^{j_s} \cdots X^{i_2} N^{j_1} X^{i_1}.$$

Therefore, if r is odd, then necessarily

$$\operatorname{trace} \sum_{|i|=k-r} \sum_{|j|=r} X^{i_1} N^{j_1} X^{i_2} \cdots X^{i_s} N^{j_s} = 0$$

and only for even r we get an invariant. Thus, we are left with the invariants

$$h_{k,2r}(X) := \operatorname{trace} \sum_{|i|=k-2r} \sum_{|j|=2r} X^{i_1} N^{j_1} X^{i_2} \cdots X^{i_s} N^{j_s} \tag{5.5}$$

for $k = 1, \ldots, n-1$, $i_q = 1, \ldots, k$, $j_q = 0, \ldots, k-1$, $r = 0, \ldots, \left[\frac{k-1}{2}\right]$, where $[\ell]$ denotes the integer part of $\ell \in \mathbb{R}$.

The integrals (5.5) are thus the coefficients of λ^{2r}, $0 < 2r < k$, in the expansion of $\frac{1}{k}\operatorname{trace}(X + \lambda N)^k$. For example, if $k = 1$ or $k = 2$ then we have one integral, the coefficient of λ^0. If $k = 3$ or $k = 4$, only the coefficients of λ^2 and λ^0 yield non-trivial integrals. If $k = 5$ or $k = 6$ it is the coefficients of λ^4, λ^2, and λ^0 that give non-trivial integrals. In general, for power k we have $\left[\frac{k+1}{2}\right]$ integrals. Recall that $k = 1, \ldots, n-1$. If $n - 1 = 2\ell$, we have hence

$$\begin{aligned} 1 + 1 + 2 + 2 + \cdots + \left[\frac{n-1+1}{2}\right] + \left[\frac{n-1+1}{2}\right] &= 1 + 1 + 2 + 2 + \cdots + \ell + \ell \\ = \ell(\ell+1) = \frac{n-1}{2}\left(\frac{n-1}{2} + 1\right) &= \frac{n-1}{2}\frac{n+1}{2} \end{aligned}$$

integrals. If $n - 1 = 2\ell + 1$ then we have

$$\begin{aligned} 1 + 1 + 2 + 2 + \cdots &+ \left[\frac{n-2+1}{2}\right] + \left[\frac{n-2+1}{2}\right] + \left[\frac{n-1+1}{2}\right] \\ &= 1 + 1 + 2 + 2 + \cdots + \ell + \ell + (\ell + 1) \\ &= \ell(\ell+1) + (\ell+1) = (\ell+1)^2 = \left(\frac{n}{2}\right)^2 \end{aligned}$$

integrals. However,

$$\left[\frac{n}{2}\right]\left[\frac{n+1}{2}\right] = \begin{cases} \dfrac{n-1}{2}\dfrac{n+1}{2}, & \text{if} \quad n \quad \text{is odd} \\ \left(\dfrac{n}{2}\right)^2, & \text{if} \quad n \quad \text{is even} \end{cases}$$

Concluding, we have

$$\left[\frac{n}{2}\right]\left[\frac{n+1}{2}\right]$$

invariants which are the coefficients of λ^{2r}, $0 < 2r < k$, in the expansion of $\frac{1}{k}\operatorname{trace}(X + \lambda N)^k$ for $k = 1, \ldots, n-1$.

Are these integrals the right candidates to prove complete integrability of the system $\dot{X} = [X^2, N]$?

- If N is invertible, then $n = 2p$ and hence

$$\left[\frac{n}{2}\right]\left[\frac{n+1}{2}\right] = \left[\frac{2p}{2}\right]\left[\frac{2p+1}{2}\right] = p^2 = \frac{1}{2}\left(2p^2 + p - p\right)$$
$$= \frac{1}{2}\left(\dim \mathfrak{sp}(2p, \mathbb{R}) - \operatorname{rank} \mathfrak{sp}(2p, \mathbb{R})\right)$$

which is half the dimension of the generic adjoint orbit in $\mathfrak{sp}(2p, \mathbb{R})$. Therefore, these conserved quantities are the right candidates to prove that this system is integrable on the generic coadjoint orbit of $\mathrm{Sym}(n)$. This will be proved in the next sections.

- If N is non-invertible (which is equivalent to $d \neq 0$), then $n = 2p + d$ and hence

$$\begin{aligned}\left[\frac{n}{2}\right]\left[\frac{n+1}{2}\right] &= \left[\frac{2p+d}{2}\right]\left[\frac{2p+d+1}{2}\right] \\ &= \left(p + \left[\frac{d}{2}\right]\right)\left(p + \left[\frac{d+1}{2}\right]\right) \\ &= p^2 + p\left(\left[\frac{d}{2}\right] + \left[\frac{d+1}{2}\right]\right) + \left[\frac{d}{2}\right]\left[\frac{d+1}{2}\right] \\ &= p^2 + pd + \left[\frac{d}{2}\right]\left[\frac{d+1}{2}\right].\end{aligned}$$

The right number of integrals is $p(p+d)$ according to Proposition 3.2, so this calculation seems to indicate that there are additional integrals. The situation is not so simple since there are redundancies due to the degeneracy of N. Note, however, that if $d = 1$, then we do get the right number of integrals. We shall return to the study of the degenerate case in §7.

6 Involution

In this section we prove involution of the integrals found in the previous section for arbitrary $N \in \mathfrak{so}(n)$.

Bi-Hamiltonian structure. We begin with the following observation.

Proposition 6.1. *The system $\dot{X} = X^2N - NX^2$ is Hamiltonian with respect to the bracket $\{f, g\}_N$ defined in (3.1) using the Hamiltonian $h_2(X) := \frac{1}{2}\operatorname{trace}(X^2)$ and is also Hamiltonian with respect to the compatible bracket $\{f, g\}_{FN}$ defined in (3.2) using the Hamiltonian $h_3(X) := \frac{1}{3}\operatorname{trace}(X^3)$.*

Proof. We have already implicitly checked the first statement using Euler–Poincaré theory, but here is a direct verification. We want to show that the condition $\dot{f} = \{f, h_2\}_N$ for any f determines the equations $\dot{X} = X^2N - NX^2$. First note that

$\dot{f} = \operatorname{trace}(\nabla f(X)\dot{X})$. Second, since $\nabla h_2(X) = X$, the right hand side $\{f, h_2\}_N$ becomes by (3.1)

$$\begin{aligned}\{f, h_2\}_N(X) &= -\operatorname{trace}\left[X\Big(\nabla f(X)NX - XN\nabla f(X)\Big)\right] \\ &= -\operatorname{trace}\Big(\nabla f(X)NX^2 - \nabla f(X)X^2N\Big).\end{aligned}$$

Thus, $\dot{X} = X^2N - NX^2$ as required.

To show that the same system is Hamiltonian in the frozen structure, we proceed in a similar way. Noting that $\nabla h_3(X) = X^2$, we get from (3.2)

$$\begin{aligned}\{f, h_3\}_{FN}(X) &= -\operatorname{trace}\Big(\nabla f N X^2 - X^2 N \nabla f\Big) \\ &= -\operatorname{trace}\Big(\nabla f N X^2 - \nabla f X^2 N\Big),\end{aligned}$$

and hence $\dot{X} = X^2N - NX^2$, as before. ∎

Involution. Next we begin the proof that the $\left[\frac{n}{2}\right]\left[\frac{n+1}{2}\right]$ integrals given in (5.5), namely

$$h_{k,2r}(X) := \operatorname{trace} \sum_{|i|=k-2r} \sum_{|j|=2r} X^{i_1}N^{j_1}X^{i_2}\cdots X^{i_s}N^{j_s},$$

where $k = 1, \ldots, n-1$, $i_q = 1, \ldots, k$, $j_q = 0, \ldots, k-1$, $r = 0, \ldots, \left[\frac{k-1}{2}\right]$, are in involution. It will be convenient below to write the expansion of h_k^λ starting with the highest power of λ, that is,

$$h_k^\lambda(X) = \frac{1}{k}\operatorname{trace}\,(X + \lambda N)^k = \sum_{r=0}^{k} \lambda^{k-r} h_{k,k-r}(X)\,. \tag{6.1}$$

As explained before, not all of these coefficients should be counted: roughly half of them vanish and the last one, namely, $h_{k,k}$, is the constant N^k. Consistently with our notation for the Hamiltonians, we set $h_k = h_{k,0}$.

Firstly we require the gradients of the functions h_k^λ.

Lemma 6.2. *The gradients ∇h_k^λ are given by*

$$\nabla h_k^\lambda(X) = \frac{1}{2}(X + \lambda N)^{k-1} + \frac{1}{2}(X - \lambda N)^{k-1}. \tag{6.2}$$

Proof. We have for any $Y \in \operatorname{Sym}(n)$,

$$\begin{aligned}\langle\!\langle \nabla h_k^\lambda(X), Y \rangle\!\rangle = \mathbf{d}h_k^\lambda(X)\cdot Y &= \operatorname{trace}\Big((X + \lambda N)^{k-1}Y\Big) \\ &= \frac{1}{2}\operatorname{trace}\Big(\Big((X + \lambda N)^{k-1} + (X - \lambda N)^{k-1}\Big)\,Y\Big)\,.\end{aligned}$$

Since $\langle\!\langle\, , \rangle\!\rangle$ is nondegenerate on $\operatorname{Sym}(n)$, the result follows. ∎

Proposition 6.3.

$$B_X(\nabla h_k^\lambda(X)) = C_X(\nabla h_{k+1}^\lambda(X)) \tag{6.3}$$

Proof. By (3.3) we have

$$\begin{aligned}
B_X(\nabla h_k^\lambda(X)) &= X\nabla h_k^\lambda(X)N - N\nabla h_k^\lambda(X)X \\
&= \frac{1}{2}\Big[X(X+\lambda N)^{k-1}N + X(X-\lambda N)^{k-1}N \\
&\qquad -N(X+\lambda N)^{k-1}X - N(X-\lambda N)^{k-1}X\Big] \\
&= \frac{1}{2}\Big[(X+\lambda N)^k N - \lambda N(X+\lambda N)^{k-1}N + (X-\lambda N)^k N + \lambda N(X-\lambda N)^{k-1}N \\
&\qquad -N(X+\lambda N)^k + \lambda N(X+\lambda N)^{k-1}N - N(X-\lambda N)^k - \lambda N(X-\lambda N)^{k-1}N\Big] \\
&= \frac{1}{2}\Big[(X+\lambda N)^k N + (X-\lambda N)^k N - N(X+\lambda N)^k - N(X-\lambda N)^k\Big] \\
&= \nabla h_{k+1}^\lambda(X)N - N\nabla h_{k+1}^\lambda(X) = C_X(\nabla h_{k+1}^\lambda(X))
\end{aligned}$$

by (3.4), which proves the formula. ■

Proposition 6.4. *The functions $h_{k,k-r}$ satisfy the recursion relation*

$$B_X(\nabla h_{k,k-r}(X)) = C_X(\nabla h_{k+1,k-r}(X)) \tag{6.4}$$

Proof. Substituting (6.1) into (6.3) we obtain

$$\sum_{r=0}^{k} \lambda^{k-r} B_X\left(\nabla h_{k,k-r}(X)\right) = \sum_{r=0}^{k+1} \lambda^{k+1-r} C_X\left(\nabla h_{k+1,k+1-r}(X)\right).$$

Since $\nabla h_{k+1,k+1}(X) = N^{k+1}$, formula (3.4) implies that $C_X\left(\nabla h_{k+1,k+1}(X)\right) = 0$. Thus on the right hand side the sum begins at $r = 1$. Changing the summation index on the right hand side from r to $r-1$ and identifying the coefficients of like powers of λ yields (6.4). ■

Remark. It is worth making a few remarks about Propositions 6.3 and 6.4. Note that unlike the similar recursion for the rigid body Mankov integrals (see e.g. Morosi and Pizzocchero [1996]), our polynomial recursion relation (6.3) does not have a premultiplier λ on the right-hand side and the polynomials on the left and right hand sides appear to be of different order. This cannot be and indeed is not so. Indeed the highest-order order coefficient on the right hand side vanishes by virtue of following result.

Corollary 6.5. *The functions $h_{k,k-1}(X)$ are Casimirs for the frozen Poisson structure, i.e.*

$$C_X\left(\nabla h_{k,k-1}(X)\right) = 0 \tag{6.5}$$

for all k.

Proof. By (6.1), $h_{k,k-1}(X) = \operatorname{trace}\left(N^{k-1}X\right)$, so its gradient equals $\nabla h_{k,k-1}(X) = N^{k-1}$. So (3.4) immediately gives (6.5). ■

The recursion relations (6.4) for $r = 0$ also imply the following relation between the Hamiltonians that can also be easily checked by hand.

Corollary 6.6.

$$B_X\left(\nabla h_k(X)\right) = C_X\left(\nabla h_{k+1}(X)\right) \tag{6.6}$$

Example: An interesting nontrivial example of the recursion relation to check is $B_X(dh_{3,2}(X)) = C_X(dh_{4,2}(X))$ where $h_{3,2}(X) = \operatorname{trace}(N^2X)$ and $h_{4,2}(X) = \operatorname{trace}(N^2X^2)+\frac{1}{2}\operatorname{trace}(NXNX)$. This example illustrates how the recursion relation works despite the apparent inconsistency in order.

Involution follows immediately, using the recursion relations.

Proposition 6.7. *The invariants $h_{k,k-r}$ are in involution with respect to both Poisson brackets $\{f,g\}_N$ and $\{f,g\}_{FN}$.*

Proof. The definition of the Poisson tensors B_X and C_X and the recursion relation (6.4) give

$$\begin{aligned}
\{h_{k,k-r}, h_{l,l-q}\}_N &= \langle\!\langle \nabla h_{k,k-r}(X), B_X(\nabla h_{l,l-q}(X))\rangle\!\rangle \\
&= \langle\!\langle \nabla h_{k,k-r}(X), C_X(\nabla h_{l+1,l-q}(X))\rangle\!\rangle \\
&= \{h_{k,k-r}, h_{l+1,l-q}\}_{FN} = -\{h_{l+1,l-q}, h_{k,k-r}\}_{FN} \\
&= -\langle\!\langle \nabla h_{l+1,l-q}(X), C_X(\nabla h_{k,k-r}(X))\rangle\!\rangle \\
&= -\langle\!\langle \nabla h_{l+1,l-q}(X), B_X(\nabla h_{k-1,k-r}(X))\rangle\!\rangle \\
&= -\{h_{l+1,l-q}, h_{k-1,k-r}\}_N = \{h_{k-1,k-r}, h_{l+1,l-q}\}_N
\end{aligned}$$

for any $k,l = 1,\ldots,n-1$, $r = 1,\ldots,k$ and $q = 0,\ldots,l-1$. Of course, in these relations we assume that $k-r$ and $l-q$ are even, for if at least one of them is odd, the identity above has zeros on both sides. Repeated application of this relation eventually leads to Hamiltonians $h_{k,k-r}$ where either $k-r$ is a power of λ that does not exist for k, in which case the Hamiltonian is zero, or one is led to $h_{0,0}$ which is constant. This shows that $\{h_{k,k-r}, h_{l,l-q}\}_N = 0$ for any pair of indices.

In a similar way one shows that $\{h_{k,k-r}, h_{l,l-q}\}_{FN} = 0$. ■

7 Independence

To complete the proof of integrability we need to show that the integrals $h_{k,2r}$ are independent. We will demonstrate this first in the generic case when N is invertible with distinct eigenvalues.

By (5.5), the gradients of the integrals $h_{k,2r}$ have the form

$$\nabla h_{k,2r}(X) := \sum_{|i|=k-2r-1}\sum_{|j|=2r} X^{i_1}N^{j_1}X^{i_2}\cdots X^{i_s}N^{j_s} \tag{7.1}$$

where $k = 1,\ldots,n-1$, $i_q = 1,\ldots,k$, $j_q = 0,\ldots,k-1$, $r = 0,\ldots,\left[\frac{k-1}{2}\right]$.

The Generic Case. We consider the case N invertible with distinct eigenvalues. Therefore $d = 0$ and $n = 2p$. In this case we show that the integrals $h_{k,2r}$ given in (5.5) are independent, and hence the system (1.1) is integrable.

Theorem 7.1. *For generic N the integrals $h_{k,2r}$ given by equation (5.5) are independent.*

Proof. We are concerned with the linear independence (in a generic sense) of (7.1) where $k = 1, \dots, n-1$, $i_q = 1, \dots, k$, $j_q = 0, \dots, k-1$ and $r = 0, \dots [\frac{1}{2}(k-1)]$. We recall that N is invertible with distinct eigenvalues and, without loss of generality, assume that X is diagonal,

$$X = \operatorname{diag} \mu.$$

This reduces the problem to a problem about the independence of polynomials in single matrix variable.

Now, we aim to prove a stronger statement: the terms

$$v_{i,j} = X^{i_1} N^{j_1} X^{i_2} \cdots X^{i_s} N^{j_s}$$

are independent for all multi-indices i and j in the above range. Note however that each $v_{i,j}$ is a q-degree polynomial in $\mu_1, \mu_2, \dots, \mu_n$, where $q = k - 2r - 1 \in \{0, \dots, n-2\}$. Let

$$\mathcal{H}_q = \{v_{i,j} \,:\, |i| = q,\ |j| \,\text{even}\}.$$

Clearly, in a generic sense, if linear dependence exists, it must exist *within* a set $\mathcal{H}_q$. In other words, if we can prove that there is no linear dependence within each $\mathcal{H}_q$, we are done. (Note that since $k \le n-1$ in the expression (7.1) there is no dependence of powers of X on lower powers through the characteristic polynomial.)

There is nothing to prove for $q = 0$ For $q = 1$ we have

$$\mathcal{H}_1 = \{XN^j \,:\, j \text{ even}\} \cup \{N^j X \,:\, j \text{ even}\}.$$

Suppose that there exists linear dependence in $\mathcal{H}_1$. Then there necessarily exist $\rho_0, \rho_2, \dots, \rho_{n-2}$ and $\kappa_0, \kappa_2, \dots, \kappa_{n-2}$, not all zero, such that

$$X\left(\sum \rho_{2j} N^{2j}\right) + \left(\sum \kappa_{2j} N^{2j}\right) X = 0 = XR(N) + K(N)X = 0.$$

Therefore,

$$\mu_a [R(N)]_{a,b} + [K(N)]_{a,b} \mu_b = 0, \qquad a, b = 1, \dots, n.$$

Generically (i.e., for all μ except for a set of measure zero) this can hold only if $R(N), K(N) = 0$. But $\deg R, \deg K \le n-1$ and, since the eigenvalues of N are distinct, the degree of the minimal polynomial of N is n. Therefore $K, R \equiv 0$, a contradiction. Hence there is no linear dependence.

We continue to $s = 2$. Now

$$\mathcal{H}_2 = \{X^{i_1} N^{j_1} X^{i_2} N^{j_2} X^{i_3} \,:\, i_1 + i_2 + i_3 = 2,\ j_1 + j_2 \text{ even}\}.$$

Assume that there exist $\rho_{i,j}$, not all zero, such that

$$\sum_{i,j} \rho_{i,j} X^{i_1} N^{j_1} X^{i_2} N^{j_2} X^{i_3} = 0.$$

Therefore

$$\sum_{i,j} \rho_{i,j} \sum_{b} \mu_a^{i_1} \mu_b^{i_2} \mu_c^{i_3} (N^{j_1})_{a,b} (N^{j_2})_{b,c} = 0, \qquad a, c = 1, \ldots, n.$$

Note that we want the above to hold for all real μ_k, but this is possible only if

$$0 = \sum_{i,j} \rho_{i,j} \sum_{b} (N^{j_1})_{a,b} (N^{j_2})_{b,c} = \sum_{i,j} \rho_{i,j} (N^{j_1+j_2})_{a,c}, \qquad a, c = 1, \ldots, n,$$

thus

$$\sum_{i.j} \rho_{i,j} N^{j_1+j_2} = 0.$$

We again obtain a polynomial in N^2 of degree $< n/2$, which cannot be zero: a contradiction.

We can continue for higher s in an identical manner. ■

Hence, since we have involution and independence, we have proved the following.

Theorem 7.2. *For N invertible with distinct eigenvalues the system (1.1) is completely integrable.*

Corollary 7.3. *For N odd with distinct eigenvalues and nullity one, the system (1.1) is completely integrable.*

Proof. In this case we have $d = 1$ and $n = 2p + 1$. All eigenvalues are distinct with one of them being zero. The above proof of indepdence still holds, the only change being that the characteristic (and mininal) polynomial of N is of form $Nw(N^2)$, where w is a polynomiail of degree $(n - 1)/2$. ■

8 Conclusion and future work

We have demonstrated integrability of the system (1.1) for appropriate N by showing involution and independence of a sufficient number of integrals. It is also of interest to analyze linearization on the Jacobi variety of the curve

$$\det(zI - \lambda N - X) = 0$$

using the theory discussed in Adler, van Moerbeke, and Vanhaecke [2004] and Griffiths [1985], for example. We shall discuss these algebro-geometric aspects in a future paper. Independently Li and Tomei [2006] have shown the integrablity of the same system in precisely the two cases discussed in this paper employing different techniques; they use the loop group approach suggested by the Lax equation with parameter (5.1) and give the solution in terms of factorization and the Riemann-Hilbert problem.

Another interesting variation of this system that we shall consider in future work is the following.

A generalized system. The flow of (1.1) can be rendered more general by complexification. Generalizing it to evolution in $\mathfrak{su}(n)$ yields an n^2-dimensional flow of generalized rigid body type with two natural Hamiltonian structures. Let $X_0 \in \mathfrak{su}(n)$, $N \in \mathrm{Sym}(n, \mathbb{R})$, and consider

$$\dot{X} = [X^2, N] = [X, XN + NX], \qquad X(0) = X_0. \tag{8.1}$$

Note that $X(t)$ evolves in $\mathfrak{su}(n)$ since one readily checks that $[X, XN+NX] \in \mathfrak{su}(n)$.

Moreover, one can generalize this still further and take $N \in \mathfrak{su}(n)$. We define

$$\begin{aligned} H_1(X) &= \frac{1}{4} \operatorname{trace} X(XN + NX), \\ H_2(X) &= \frac{1}{2} \operatorname{trace} X^2. \end{aligned}$$

Note that both Hamiltonians are real and that H_2 gives us our earlier Hamiltonian in the case that X is symmetric but that H_1 is zero in this case.

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