A class of integration by parts formulae in stochastic analysis I

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1 Introduction

Consider a Stratonovich stochastic differential equation

$$dx_t = X(x_t) \circ dB_t + A(x_t)dt \tag{1}$$

with C^∞ coefficients on a compact Riemannian manifold M, with associated differential generator $\mathcal{A} = \frac{1}{2}\Delta_M + Z$ and solution flow $\{\xi_t : t \geq 0\}$ of random smooth diffeomorphisms of M. Let $T\xi_t : TM \to TM$ be the induced map on the tangent bundle of M obtained by differentiating ξ_t with respect to the initial point. Using an observation by A. Thalmaier we will extend the basic formula of [EL94] to obtain

$$\mathbb{E}dF\left(T\xi_\cdot\left(h_\cdot\right)\right) = \mathbb{E}F\left(\xi_\cdot(x)\right)\int_0^T \left\langle T\xi_s\left(\dot{h}_s\right), X\left(\xi_s(x)\right)dB_s\right\rangle \tag{2}$$

where $F \in \mathcal{F}C_b^\infty(C_x(M))$, the space of smooth cylindrical functions on the space $C_x(M)$ of continuous paths $\gamma : [0,T] \to M$ with $\gamma(0) = x$, dF is its derivative, and $h_\cdot$ is a suitable adapted process with sample paths in the Cameron-Martin space $L_0^{2,1}([0,T]; T_xM)$. Set $\mathcal{F}_t^x = \sigma\{\xi_s(x) : 0 \leq s \leq t\}$. Taking conditional expectation with respect to $\mathcal{F}_T^x$, formula (2) yields integration by parts formulae on $C_x(M)$ of the form

$$\mathbb{E}dF(\gamma)(\bar{V}^h) = \mathbb{E}F(\gamma)\delta\overline{V}^h(\gamma) \tag{3}$$

where $\bar{V}^h$ is the vector field on $C_x(M)$

$$\bar{V}^h(\gamma)_t = \mathbb{E}\left\{T\xi_t(h_t)\,|\xi_\cdot(x) = \gamma\right\}$$

and $\delta\overline{V}^h : C_x(M) \to \mathbb{R}$ is given by

$$\delta\overline{V}^h(\gamma) = \mathbb{E}\left\{ \int_0^T < T\xi_s(\dot{h}_s), X(\xi_s(x))dB_s > |\xi_\cdot(x) = \gamma \right\}.$$

When $h.$ is adapted to $\mathcal{F}^x_\cdot$ results from [ELJL95] extending [EY93] give explicit expressions for $\bar{V}^h$ and $\delta\bar{V}^h$ in terms of the Ricci curvature of the LeJan-Watanabe connection associated to (1). Equation (3) then reduces to a Driver's integration by parts formula, Theorem 3.3 below, but no hypothesis of torsion skew symmetry of the connection is required: the integration by parts formulae follow for the adjoint of any metric connection. In particular for any such connection there is a Hilbert "tangent space" of "good" directions obtained by parallel translation of the Cameron-Martin space of paths in T_xM. (In fact it is the "Ricci flow" or "Dohrn-Guerra parallel translation" (see Nelson [Nel84]), leading to the "damped gradient" ([FM93]) which occurs more naturally). However, in Remark 2.4, we show that in this case $\bar{V}_h$ is in the class for which integration by parts formulae are known, so that the results of 2.3, 3.3, 3.5 are not claimed to be new in substance.

Although this filtering out of the extraneous noise gives intrinsic results comparable to those of Driver [Dri92], this viewpoint throws away a lot of the structure we have. Moreover integration by parts formulae such as (2) should have some connection with quasi-invariance properties of flows associated to the vector fields. Flows for the $\bar{V}^h$ on $C_x(M)$ do not appear to be easy to analyse in general. However in §3 we show that in the context of Diff M valued processes there are very natural flows associated and (2) has a rather natural geometric interpretation. This leads to another elementary proof of (2) and in Theorem 4.1 we use this method to obtain integration by parts formulae for the free path space.

There are at least 3 proofs of (2). The first given here is via Itô's formula and elementary martingale calculus (it requires F to be cylindrical), the second given here is based on the Girsanov-Maruyama theorem (and works for more general F), and a third method would be to deduce it from the standard integration by parts formula on Wiener space applied to the functional $F \circ \xi$. Indeed this work was stimulated by D. Bell and D. Nualart pointing out that this third approach could be used to deduce the basic formula of [EL94]. The point made (and carried out) in [Elw92] and [EL94] that the first approach can be applied directly to 'Ricci flows' instead of derivative flows to give intrinsic formulae without stochastic flows, also needs to be emphasized: see also [SZ].

There are also now many proofs of Driver's results for $C_x(M)$ and for the free path space and their extensions. See [Hsu95], [ES95], [LN] (with a very concise proof), [AM], [Aid], and [CM].

Acknowledgment: This research was supported by SERC grant GR/H67263 and stimulated and helped by our contacts with A. Thalmaier.

2 The integration by parts formula from finite dimensional manifolds to path spaces

In this section we deduce by induction an integration by parts formula on the path space from a formula on the base manifold M. The key is to obtain formula (10) for M.

Let $h:\Omega\times[0,T]\to T_xM$ be an adapted process with $h(\omega):[0,T]\to T_xM$ in $L^{2,1}$ for almost all ω.

Lemma 2.1 *If $h:\Omega\times[0,T]\to T_xM$ is adapted, $L^{2,1}$ for a.s. ω and $\left(\int_0^T|\dot h_s|^2ds\right)^{1/2}\in L^{1+\epsilon}$ for some $\epsilon>0$. Then for $t<T$,*

$$\begin{aligned}&\mathbb{E}\left\{\int_0^t <T\xi_s(\dot h_s), X(\xi_s(x))dB_s> |\xi_T(x)\right\}\\ &=\mathbb{E}\left\{\int_t^T <T\xi_s(-), X(\xi_s(x))dB_s> \tfrac{h_t-h_0}{T-t}\,|\xi_T(x)\right\}.\end{aligned}\tag{4}$$

If furthermore $h.$ is non-random then for $t\le T$,

$$\begin{aligned}&\mathbb{E}\left\{\int_0^t <T\xi_s(\dot h_s), X(\xi_s(x))dB_s> |\xi_T(x)\right\}\\ &=\mathbb{E}\left\{\int_0^t <T\xi_s(-), X(\xi_s(x))dB_s> \left(\tfrac{h_t-h_0}{t}\right)|\xi_T(x)\right\}.\end{aligned}\tag{5}$$

Proof. First by the Burkholder-Davis-Gundy inequality, for some constant c_1,

$$\mathbb{E}\left|\int_0^T <T\xi_s(\dot h_s), X(\xi_s(x))dB_s>\right|\le c_1\mathbb{E}\left(\int_0^T|T\xi_s(\dot h_s)|^2ds\right)^{\frac12}$$
$$\le c_1\left(\mathbb{E}\sup_{0\le s\le T}|T_x\xi_s|^{\frac{1+\epsilon}{\epsilon}}\right)^{\frac{\epsilon}{1+\epsilon}}\left[\mathbb{E}\left(\int_0^T|\dot h_s|^2ds\right)^{\frac{1+\epsilon}{2}}\right]^{\frac{1}{1+\epsilon}}.$$

This is finite since $\sup_{0\le s\le t}|T_x\xi_s|\in L^q$ for all $1\le q<\infty$, e.g. see [Li94]. Moreover, since the adapted processes in $L^\infty(\Omega,\mathcal{F},\mathbb{P};C^1([0,T];T_xM))$ are dense in the subspace of adapted processes in $L^{1+\epsilon}(\Omega, \mathcal{F}, \mathbb{P}; L^{2,1}([0,T]; T_xM))$, this estimate allows us to assume that h belongs to the former space.

Set $M_t = \int_0^t < T_x\xi_s(-), X(\xi_s(x))dB_s >$. Then $\{M_\cdot\}$ is a T_x^*M valued local martingale. If $0 = t_0 < t_1 < \ldots < t_l = t$ is a partition of $[0,t]$, $\Delta_j t = t_{j+1} - t_j$, and $\Delta_j M = M_{t_{j+1}} - M_{t_j}$, then

$$\sum_{j=1}^{l-1} \Delta_j M(\dot{h}_{t_j}) \to \int_0^t \dot{h}_s dM_s = \int_0^t < T\xi_s(\dot{h}_s), X(\xi_s(x))dB_s > \tag{6}$$

and the convergence is in L^1.

On the other hand if $v_0 \in T_xM$ and P_t is the probabilistic semigroup associated to the S.D.E. and f a bounded measurable function then

$$d(P_t f)(v_0) = \frac{1}{T}\mathbb{E} f(\xi_T(x)) \int_0^T \langle T\xi_s(v_0), X(\xi_s(x))dB_s \rangle . \tag{7}$$

See [EL94]. However by an observation of Thalmaier: the same proof shows that for any $r, h \in [0,T]$ with $h > 0$ and $r + h \le T$

$$d(P_t f)(v_0) = \frac{1}{h}\mathbb{E} f(\xi_T(x)) \int_r^{r+h} \langle T\xi_s(v_0), X(\xi_s(x))dB_s \rangle$$

c.f. [SZ]. From these two formulae we obtain:

$$\begin{aligned} &\mathbb{E}\left\{ \tfrac{1}{T} \textstyle\int_0^T < T\xi_s(v_0), X(\xi_s(x))dB_s > |\xi_T(x) \right\} \\ &= \mathbb{E}\left\{ \tfrac{1}{h} \textstyle\int_r^{r+h} < T\xi_s(v_0), X(\xi_s(x))dB_s > |\xi_T(x) \right\}. \end{aligned} \tag{8}$$

For any $0 \le r \le T$, let $\{\xi_s^r(x) : r \le s \le T, x \in M\}$ be the solution flow to (1) starting from x at time r. The flow $\xi_\cdot^r$ can be taken to be adapted to a filtration $\{\mathcal{F}_s^r : r \le s \le T\}$ independent of $\mathcal{F}_r$, and then we have $\xi_s^r \xi_r = \xi_s$, almost surely, $r \le s \le T$. From this, time homogeneity, and (8),

$$\begin{aligned} &\mathbb{E}\left\{ \sum_{j=1}^{l-1} \Delta_j M(\dot{h}_{t_j})\, |\xi_T(x) \right\} \\ &= \mathbb{E}\left\{ \sum_{j=1}^{l-1} \Delta_j t \frac{1}{\Delta_j t} \int_{t_j}^{t_{j+1}} \left\langle T\xi_s^{t_j}\left(T\xi_{t_j}\left(\dot{h}_{t_j}\right)\right), X\left(\xi_s^{t_j}\left(\xi_{t_j}(x)\right)\right) dB_s \right\rangle \Big| \xi_T^{t_j}(\xi_{t_j}(x)) \right\} \\ &= \mathbb{E}\left\{ \sum_{j=1}^{l-1} \Delta_j t \frac{1}{T-t} \int_t^T \left\langle T\xi_s^{t_j}\left(T\xi_{t_j}(\dot{h}_{t_j})\right), X\left(\xi_s^{t_j}\left(\xi_{t_j}(x)\right)\right) dB_s \right\rangle \Big| \xi_T^{t_j}(\xi_{t_j}(x)) \right\} \end{aligned}$$

$$= \mathbb{E}\left\{\sum_{j=1}^{l-1}\Delta_j t\frac{1}{T-t}\int_t^T < T\xi_s(\dot{h}_{t_j}), X(\xi_s(x))dB_s > |\xi_T(x)\right\}$$
$$\to \mathbb{E}\left\{\int_t^T < T\xi_s(-), X(\xi_s(x))dB_s > \frac{h_t - h_0}{T-t}\,|\xi_T(x)\right\}.$$

Comparing with (6) this gives the first required identity. When $h.$ is non-random the second follows immediately from (8). ∎

Remark:

As in [SZ] a further modification is possible replacing (8) by:

$$\frac{1}{T}\mathbb{E}\left\{\int_0^T < T\xi_s(v_0), X(\xi_s(x))dB_s > |\xi_T(x)\right\}$$
$$= \frac{1}{\int_0^T \Psi(r)dr}\mathbb{E}\left\{\int_0^T \Psi(s) < T\xi_s(v_0), X(\xi_s(x))dB_s > |\xi_T(x)\right\}$$

for $\Psi : [0,T] \to \mathbb{R}$ integrable with $\int_0^T \Psi(r)dr \neq 0$. The argument leads to, for non-random h,

$$\begin{aligned}&\mathbb{E}\left\{\int_0^t < T\xi_s(\dot{h}_s), X(\xi_s(x))dB_s > |\xi_T(x)\right\}\\ &= \mathbb{E}\left\{\int_0^T \Psi(s) < T\xi_s(-), X(\xi_s(x))dB_s > \left(\frac{h_t - h_0}{\int_0^T \Psi(r)dr}\right)|\xi_T(x)\right\}.\end{aligned} \tag{9}$$

Corollary 2.2 *Under the conditions of the lemma, for any C^1 function $f : M \to \mathbb{R}$,*

$$\mathbb{E}f\left(\xi_T(x)\right)\int_0^T < T\xi_s(\dot{h}_s), X(\xi_s(x))dB_s >= \mathbb{E}df\left(T\xi_T\left(h_T - h_0\right)\right). \tag{10}$$

Proof. First by the composition property of solution flows,

$$\mathbb{E}\left\{\int_t^T < T\xi_s(-), X(\xi_s(x))dB_s > \frac{h_t - h_0}{T-t}\,|\xi_T(x)\right\}$$
$$= \mathbb{E}\left\{\int_t^T < T\xi_s^t(-), X(\xi_s^t\left(\xi_t(x)\right))dB_s > \frac{T\xi_t(h_t - h_0)}{T-t}\,|\xi_T^t\left(\xi_t(x)\right)\right\}.$$

As in the proof of the lemma, (4) yields

$$\begin{aligned}
&\mathbb{E} f(\xi_T(x)) \int_0^t < T\xi_s(\dot h_s), X(\xi_s(x)) dB_s > \\
&= \mathbb{E} f(\xi_T^t(\xi_t(x)) \int_t^T \left\langle T\xi_s^t(-), X(\xi_s^t(\xi_t(x)) dB_s \right\rangle \frac{T\xi_t(h_t - h_0)}{T-t} \\
&= \mathbb{E}\left\{ dP_{T-t}(f)\left(T\xi_t(h_t - h_0)\right)\right\}
\end{aligned}$$

by [EL94], since $\mathcal{F}^t_\cdot$ is independent of $\mathcal{F}_t$. Now let t increase to T and the required result follows. ∎

Next consider a cylindrical function F on $C_x(M)$, the space of continuous paths with base point x. Write

$$F(\gamma_\cdot) = f(\gamma_{t_1}, \dots, \gamma_{t_k}),$$

for $(t_1, \dots, t_k) \in [0,T]^k$, $\gamma \in C_x(M)$ and f a smooth function on M^k. Suppose $h_0 = 0$ and consider the tangent vector field $V^h(\xi_\cdot(x))$ along $\{\xi_t(x) : 0 \le t \le T\}$ on $C_x(M)$ given by

$$V^h(\xi_\cdot)_t = T_x\xi_t(h_t).$$

Then

$$dF(V^h(\xi_\cdot)) = \sum_{j=1}^k d^j f_{\xi_{\underline{t}}}\left(V^h(\xi_\cdot)_{t_j}\right). \tag{11}$$

Here $\xi_{\underline{t}} = (\xi_{t_1}, \dots, \xi_{t_k})$ and $d^j f$ is the partial derivative of f in the jth direction.

Let

$$\delta V^h(\xi_\cdot) = \int_0^T < T_x\xi_s(\dot h_s), X(\xi_s(x)) dB_s > .$$

Theorem 2.3 *Let $h : [0,T] \times \Omega \to T_xM$ be an adapted stochastic process with almost surely all $h(\omega) \in L_0^{2,1}$ and $\mathbb{E}\left(\int_0^T |\dot h_s|^2 ds\right)^{\frac{1+\epsilon}{2}} < \infty$ for some $\epsilon > 0$. Then*

$$\mathbb{E} dF(V^h(\xi_\cdot)) = \mathbb{E} F(\xi_\cdot(x)) \delta V^h(\xi_\cdot). \tag{12}$$

Proof. We prove by induction on k. When $k = 1$, this is just (10), the formula for functions. Let $\Omega = C_0([0,T]; \mathbb{R}^n)$ be the canonical probability space. We set $\Omega_1 = C_0([0,t_1]; \mathbb{R}^n)$ and $\Omega_2 = C_0([t_1,T]; \mathbb{R}^n)$. There is then the standard decomposition of filtered spaces

$$\begin{aligned}
& \{\Omega, \mathcal{F}, \mathcal{F}_t, 0 \le t \le T, \mathbb{P}\} \\
= \quad & \{\Omega_1, \mathcal{F}, \mathcal{F}_t, 0 \le t \le t_1, \mathbb{P}_{\not\vDash}\} \times \{\Omega_2, \mathcal{F}, \mathcal{F}_t^{t_1}, t_1 \le t \le T, \mathbb{P}_{\not\vDash}\}
\end{aligned}$$

in the sense that $\mathcal{F}_t = \mathcal{F}_t * \Omega_2$ if $t \le t_1$, and $\mathcal{F}_t = \mathcal{F}_{t_1} * \mathcal{F}_t^{t_1}$ if $t \ge t_1$. As before let $\xi_t^{t_1}(y_0), t_1 \le t \le T, y_0 \in M$ be the solution flow to (1) starting at time t_1, i.e. $\xi_{t_1}^{t_1}(y_0) = y_0$. We will consider it as a function of $\omega_2 \in \Omega_2$, adapted to $\mathcal{F}_\cdot^{t_1}$, while $\{\xi_t : 0 \le t \le t_1\}$ will be considered on Ω_1, and $\{\xi_t : t_1 \le t \le T\}$ on $\Omega_1 \times \Omega_2 = \Omega$. The composition property for flows gives

$$\xi_t^{t_1}\left(\xi_{t_1}(x,\omega_1),\omega_2\right) = \xi_t\left(x,(\omega_1,\omega_2)\right), \quad \text{each } t_1 \le t \le T, a.s.$$

Assume the required result holds for cylindrical functions depending on $k-1$ times, some $k \in \{2, 3\ldots\}$. Take $y_0 \in M$ and define $f_1^{y_0} : M^{k-1} \to \mathbb{R}$ and $F_1^{y_0} : \Omega_2 \to \mathbb{R}$ by:

$$f_1^{y_0}(x_1,\ldots,x_{k-1}) = f(y_0,x_1,\ldots,x_{k-1})$$

and

$$F_1^{y_0}(\omega_2) = f(y_0, \xi_{t_2}^{t_1}(y_0,\omega_2),\ldots,\xi_{t_k}^{t_1}(y_0,\omega_2)).$$

Take $h_\cdot^1 : \Omega_2 \to L_0^{2,1}\left([t_1,T]; T_{y_0}M\right)$, adapted to $\mathcal{F}_\cdot^{t_1}$, and with $\mathbb{E}\left(\int_{t_1}^T |\dot{h}_s^1|^2 ds\right)^{\frac{1+\epsilon}{2}}$ finite. By time homogeneity our inductive hypothesis gives

$$\begin{aligned}
&\textstyle\sum_{j=2}^k \int_{\Omega_2} d^j f\left(y_0, \xi_{t_2}^{t_1}(y_0,\omega_2),\ldots,\xi_{t_k}^{t_1}(y_0,\omega_2)\right)\left(T\xi_{t_j}^{t_1}(h_{t_j}^1(\omega_2),\omega_2)\right) d\mathbb{P}_{\not{2}}(\omega_2)\\
&= \textstyle\int_{\Omega_2} f\left(y_0, \xi_{t_2}^{t_1}(y_0,\omega_2),\ldots,\xi_{t_k}^{t_1}(y_0,\omega_2)\right) \times\\
&\quad \textstyle\int_{t_1}^T \left\langle T\xi_r^{t_1}(\dot{h}_r^1(\omega_2),\omega_2), X(\xi_r^{t_1}(y_0,\omega_2)) dB_r(\omega_2)\right\rangle d\mathbb{P}_{\not{2}}(\omega_2).
\end{aligned} \tag{13}$$

Now for $\omega_1 \in \Omega_1$ (outside of a certain measure zero set) we can take $y_0 = \xi_{t_1}(x_0,\omega_1)$ and

$$h_t^1(\omega_2) = T\xi_{t_1}\left(h_t(\omega_1,\omega_2) - h_{t_1}(\omega_1),\omega_1\right).$$

Then, for almost all $\omega_1 \in \Omega_1$, we have $h_\cdot^1$ adapted to $\mathcal{F}_\cdot^{t_1}$. Substitute this in (13). Using the composition property, and then integrating over Ω_1 yields

$$\begin{aligned}
&\textstyle\sum_{j=2}^k \mathbb{E} d^j f(\xi_{\underline{t}})\left(T\xi_{t_j}(h_{t_j} - h_{t_1})\right)\\
&= \mathbb{E} f(\xi_{\underline{t}}(x)) \textstyle\int_{t_1}^T \left\langle T\xi_r(\dot{h}_r), X(\xi_r(x)) dB_r\right\rangle.
\end{aligned} \tag{14}$$

On the other hand we can define $g : M \to \mathbb{R}^1$ by

$$g(x) = \int_{\Omega_2} f\left(x, \xi_{t_2}^{t_1}(x,\omega_2),\ldots,\xi_{t_k}^{t_1}(x,\omega_2)\right)$$

and apply formula (10) to g to obtain:

$$\int_{\Omega_1} dg(T\xi_{t_1}(h_{t_1}))d\mathbb{P}_{\not k}(\omega_1) = \int_{\Omega_1} g(\xi_{t_1}(x)) \int_0^{t_1} \left\langle T\xi_r(\dot{h}_r)), X(\xi_r(x_0))dB_r \right\rangle d\mathbb{P}_{\not k}(\omega_1).$$

But note that

$$\int_{\Omega_1} dg(T\xi_{t_1}(h_{t_1}))d\mathbb{P}_{\not k}(\omega_1) = \sum_{j=1}^{k} \mathbb{E}d^k f_{\xi_{\underline{t}}}(T\xi_{t_j}(h_{t_1}))d\mathbb{P}_{\not k}(\omega_1),$$

and therefore

$$\sum_{j=1}^{k} \mathbb{E}d^j f_{\xi_{\underline{t}}}(T\xi_{t_j}(h_{t_1})) = \mathbb{E}f(\xi_{\underline{t}}) \int_0^{t_1} \left\langle T\xi_r(\dot{h}_r), X(\xi_r(x))dB_r, \right\rangle \tag{15}$$

Adding (14) we arrive at (12):

$$\sum_{j=1}^{k} \mathbb{E}d^j f_{\xi_{\underline{t}}}(T\xi_{t_j}(h_{t_j})) \;=\; \mathbb{E}f(\xi_{\underline{t}}(x)) \int_0^{T} \left\langle T\xi_r(\dot{h}_r), X(\xi_r(x))dB_r \right\rangle.$$

∎

B. Let $\tilde{\nabla}$ be a metric connection for the manifold M with torsion T, and $\tilde{\nabla}'$ its adjoint connection defined by

$$\tilde{\nabla}'_{V_1}V_2 = \tilde{\nabla}_{V_1}V_2 - T(V_1, V_2).$$

Here V_1, V_2 are vector fields. Let $\tilde{R}$ be the curvature tensor of $\tilde{\nabla}$ and define $\tilde{Ric}^{\#} : TM \to TM$ by $\tilde{Ric}^{\#}(v) = \text{trace } \tilde{R}(v, -)-$. If $\{x_s\}$ is a diffusion on M with generator $\frac{1}{2}\text{trace}\tilde{\nabla}\text{grad} + L_Z$ denote by $\tilde{//}_s$ the parallel transport along $\{x_s\}$, and $\{\tilde{B}_s : 0 \le s \le t\}$ the martingale part of the anti-development of $\{x_s : 0 \le s \le t\}$ using $\tilde{//}_s$, a Brownian motion on $T_{x_0}M$. Let $v_s = \tilde{W}^Z_s(v_0)$ be the solution to

$$\frac{\tilde{D}'}{\partial s}v_s = -\frac{1}{2}\tilde{\text{Ric}}^{\#}(v_s) + \tilde{\nabla}Z(v_s)$$

starting from $v_0 \in T_{x_0}M$. Here $\tilde{D}'$ denotes the covariant differentiation along the paths of $\{x_t\}$ using the adjoint connection. We will show that (12) implies Driver's integration by parts formula. However we do not need to assume $\tilde{\nabla}'$ (or equivalently $\tilde{\nabla}$) is torsion skew symmetric.

Corollary 2.4 *Let F be a cylindrical function on $C_{x_0}(M)$. Suppose $h : [0,T] \times \Omega \to T_{x_0}M$ is adapted to the filtration of $\{x_s : 0 \le s < \infty\}$ and such* *that $h(\omega)$ is in $L_0^{2,1}$ for almost all ω and $h \in L^{1+\epsilon}\left(\Omega, \mathcal{F}, \mathbb{P}; L_0^{2,1}([0,T]; T_{x_0}M)\right)$ for some $\epsilon > 0$. Then*

$$\mathbb{E}dF(\tilde{W}_\cdot^Z(h_\cdot)) = \mathbb{E}F(\xi_\cdot(x_0)) \int_0^T < \tilde{W}_s^Z(\dot{h}_s), \tilde{//}_s d\tilde{B}_s > . \qquad (16)$$

When $\tilde{\nabla}'$ is metric for some Riemannian metric on M, it suffices to have $h \in L^1\left(\Omega, \mathcal{F}, \mathbb{P}; L_0^{2,1}([0,T])\right)$.

Proof. By a result of [ELJL95] we can choose X such that $\tilde{\nabla}$ equals the Le Jan-Watanabe connection induced from the stochastic differential equation

$$dx_t = X(x_t) \circ dB_t + Z(x_t)dt$$

and the solution flow $\{\xi_\cdot(x)\}$ has generator $\frac{1}{2}\mathrm{trace}\tilde{\nabla}\mathrm{grad} + L_Z$ (c.f. Corollary 3.4 of [ELJL95]). Moreover the conditioned process of the derivative flow $T\xi_t(v_0)$ with respect to the natural filtration of $\{\xi_\cdot(x_0)\}$ is given by $\{\tilde{W}_\cdot^Z(v_0)\}$:

$$\mathbb{E}\{T\xi_t(v_0) \mid \mathcal{F}_T^{x_0}\} = \tilde{W}_t^Z(v_0),$$

by Theorem 3.2 of [ELJL95] extending [EY93]. The result follows since $\tilde{B}_t$ equals $\int_0^t \tilde{//}_s^{-1} X(\xi_s(x_0))dB_s$.

If $\tilde{\nabla}'$ is metric for some Riemannian metric then $\sup_{0\le s\le t} |\tilde{W}_s^Z|$ is in $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ and so the Burkholder-Davis-Gundy inequality used as in the proof of Lemma 2.1 allows us to take $\epsilon = 0$. ∎

Remarks 2.5. (i). Let $S : TM \times TM \to TM$ be a tensor fields of type (1,2), and let ∇ refer to the Levi-Civita connection of M. Then, by [KN69] p.146, a connection $\tilde{\nabla}$ can be defined by

$$\tilde{\nabla}_{V_1}(V_2) = \nabla_{V_1}(V_2) + S(V_1, V_2)$$

for vector fields V_1, V_2. and all linear connections on M can be obtained this way. It is easy to see that $\tilde{\nabla}$ is metric if and only if

$$< S(W,U), V >= - < U, S(W,V) >$$

for all vector fields U, V, W, i.e. if and only if $S(W,-)$ is skew symmetric. On the other hand the adjoint connection is given by

$$\tilde{\nabla}'_{V_1}(V_2) = \nabla_{V_1}(V_2) + S(V_2, V_1)$$

so that it is torsion skew symmetric if also $S(-,W)$ is skew symmetric. In terms of the Levi-Civita connection our vector fields $\bar{V}^h$ for which the integration by parts formula hold therefore satisfy an equation of the form

$$D\bar{V}_t^h = -S(\bar{V}_t^h, \circ dx_t) + \Lambda_t(\bar{V}_t^h)dt + W_t^h(\dot{h}_t)dt + \nabla A(\bar{v}_t^h)dt$$

where Λ_t is linear (also depending on S). In particular they are "tangent processes" in the sense proposed by Driver, for which integration by parts formulae are known: see [Dri95b], [CM], [AM], and [Aid], [Dri95a].

(ii) For cylinder functions depending on one time only such integration by parts formulae go back to Bismut [Bis84].

3 Geometric intepretation and a shorter proof

A. The processes $T_x\xi_t(h_t)$ cannot strictly speaking be considered as tangent vectors or vector fields on $C_x(M)$. In some sense they form tangent vectors at $\xi_\cdot(x,-)$ to the space of processes (or semi-martingales)

$$[0,T] \times \Omega \to M$$

since $T_x\xi_t(h_t(\omega),\omega) \in T_{\xi_t(x,\omega)}M$ for $(t,\omega) \in [0,T] \times \Omega$ or equivalently as 'tangent vectors' to the space of random variables

$$\Omega \to C_x(M)$$

at $\omega \mapsto \xi_\cdot(x,\omega)$. However c.f. [Dri92] there is still no natural associated flow. In fact the most natural interpretation takes into account the variable x and replaces $C_x(M)$ by $P_{id}\mathrm{Diff}M$ the space of paths on the diffeomorphism group of M, as we now describe.

Let Diff$M$ be the space of C^∞ diffeomorphisms of M. We can consider it with a rather formal differential structure or if the reader prefers it can be replaced by a suitable Sobolev space of diffeomorphisms, to give a Hilbert manifold (as in [Elw82] following [EM70]). In any case the tangent space $T_\alpha(\mathrm{Diff}M)$ will be identified with all vector fields on M over α i.e. smooth $v : M \to TM$ such that $v(x) \in T_{\alpha(x)}M$ for all $x \in M$. If $P\mathrm{Diff}M$ refers to continuous paths $\phi : [0,T] \to \mathrm{Diff}M$ with $\phi(0) = id_M$ then $T_\phi P\mathrm{Diff}M$ will be identified with continuous $v : [0,T] \to T\mathrm{Diff}M$ vanishing at $t = 0$, such that $v(t) \in T_{\phi(t)}\mathrm{Diff}M$, or equivalently $v : [0,T] \times M \to TM$ with $v(t)(x) \in T_{\phi(t)(x)}M$.

B. Given our S.D.E. (1) now take $h \in L_0^{2,1}([0,T];\mathbb{R}^n)$. There is $X^{h_\cdot}$, the time dependent vector field $X(\cdot)(h_t)$ on M. From this we obtain a field U^h on $P\mathrm{Diff}M$ by

$$U^h(\phi)_t(x) = T_x\phi_t(X(x)h_t). \tag{17}$$

This is just the left invariant vector field on $P\mathrm{Diff}M$ corresponding to $X^{h_\cdot} \in T_eP\mathrm{Diff}M$ for $e(t) = id_M$, $0 \leq t \leq T$.

For each $0 \leq t \leq T$ let $H_t^\tau : M \to M$, $\tau \in \mathbb{R}$ be the solution flow to the vector field $X(\cdot)(h_t)$ so

$$\begin{cases} \frac{\partial}{\partial\tau} H_t^\tau(x) & = & X(H_t^\tau(x))h_t \\ H_t^0(x) & = & x. \end{cases} \tag{18}$$

Lemma 3.1 *The vector field U^h on PDiffM has solution flow $\Phi_\tau : PDiffM \to PDiffM$, $\tau \in \mathbb{R}$ given by $\Phi_\tau(\phi)_t(x) = \phi_t(H_t^\tau(x))$.*

Proof. By left invariance we can suppose $\phi = e$. We then need only to observe that

$$\frac{\partial}{\partial\tau} H_t^\tau(x) = TH_t^\tau\left(X(x)h_t\right)$$

for each $0 \leq t \leq T$: a standard property of ordinary, time-independent dynamical systems which is seen by differentiating the identity

$$H_t^{\tau+\sigma} = H_t^\tau \circ H_t^\sigma(x)$$

with respect to σ at $\sigma = 0$. ∎

C. In the case where h is random, with $h : \Omega \to L_0^{2,1}([0,T]; \mathbb{R}^d)$ adapted, we can use the same notation to obtain a variation of our stochastic flow $\{\xi_t : 0 \leq t \leq T\}$ on M generated by the vector field V^h, and given explicitly by

$$\xi_\cdot^\tau = \Phi_\tau(\xi_\cdot),$$

i.e.

$$\xi_t^\tau(x) = \xi_t(H_t^\tau(x)). \tag{19}$$

In particular

$$\frac{\partial}{\partial\tau}\xi_t^\tau(x)\,|_{\tau=0} = T\xi_t\left(X(x)h_t\right). \tag{20}$$

Using the structure of $C_x(M)$ as a C^∞ Banach manifold let $BC^1(C_x(M))$ be the space of C^1 maps $F : C_x(M) \to \mathbb{R}$ such that there is a constant $|dF|_\infty$ with

$$|dF(v)| \leq |dF|_\infty \sup_{0\leq t\leq T} |v_t| \tag{21}$$

for all tangent vectors $v : [0,T] \to TM$ to $C_x(M)$. Set $V_t^{X(h)}(x) = T\xi_t\left(X(x)(h_t)\right)$, which gives rise to a vector field along $\{\xi_\cdot(x)\}$ on $C_x(M)$.

Proposition 3.2 *Suppose $h : [0,T] \times \Omega \to T_xM$ is adapted, belongs to $L_0^{2,1}$ a.s. and such that $\mathbb{E}\left(\int_0^T |\dot{h}_s|^2 ds\right)^{\frac{1+\epsilon}{2}} < \infty$ for some $\epsilon > 0$. Then for each $x \in M$ the processes $\xi^\tau_\cdot(x)$, $\tau \in \mathbb{R}$ have mutually equivalently laws $\mathbb{P}^x_\tau$, $\tau \in \mathbb{R}$ on $C_x(M)$ with*

$$\frac{d\mathbb{P}^x_\tau}{d\mathbb{P}^x_0} = exp\left\{\int_0^T < X(\xi^\tau_s(x))^* T\xi_s\left(\frac{\partial}{\partial s}H^\tau_s(x)\right), dB_s > -\frac{1}{2}\int_0^T |T\xi_s\left(\frac{\partial}{\partial s}H^\tau_s(x)\right)|^2 ds\right\}.$$

Moreover, for any $F \in BC^1(C_x(M))$,

$$\mathbb{E}dF(V_\cdot^{X(h)}) = \mathbb{E}F(\xi_\cdot)\int_0^T \left\langle X(\xi_s(x))dB_s, V_s^{X(\dot{h})}(x))\right\rangle.$$

Proof. For the equivalent part note that $\{\xi^\tau_t : 0 \le t \le T\}$ satisfies the equation:

$$d\xi^\tau_t(x) = X\left(\xi^\tau_t(x)\right) \circ dB_t + A(\xi^\tau_t(x))dt + T\xi_t\left(\frac{\partial}{\partial t}H^\tau_t(x)\right)dt.$$

A straightforward argument shows that

$$\int_0^T \left|X(\xi^\tau_s(x))^* T\xi_s\left(\frac{\partial}{\partial s}H^\tau_s(x)\right)\right|^2 < \infty, \ \ a.s.$$

Therefore if we set

$$M^\tau_t = \int_0^t \left\langle X(\xi^\tau_s(x))^* T\xi_s\left(\frac{\partial}{\partial s}H^\tau_s(x)\right), dB_s\right\rangle,$$

then by the Girsanov-Maruyama theorem, P^x_τ is equivalent to P^x_0 and

$$\frac{d\mathbb{P}^x_\tau}{d\mathbb{P}^x_0} = \mathrm{e}^{M^\tau_T - \frac{1}{2}<M>^\tau_T}. \tag{22}$$

Consequently,

$$\mathbb{E}F(\xi^\tau_\cdot(x)) = \mathbb{E}F(\xi_\cdot(x))\frac{d\mathbb{P}^x_\tau}{d\mathbb{P}^x_0}.$$

Now suppose $h_\cdot$ and $\int_0^\cdot |\dot{h}_s|^2 ds$ are bounded on $[0,T]\times\Omega$. Differentiating with respect to τ at $\tau = 0$ and using (18) gives

$$\mathbb{E}dF(T\xi_\cdot(X(x)h_\cdot)) = \mathbb{E}F(\xi_\cdot(x))\frac{\partial}{\partial\tau}\left(\frac{d\mathbb{P}^x_\tau}{d\mathbb{P}^x_0}\right)_{\tau=0},$$

since $|dF|$ is bounded and $\sup_{0\le s\le T}|T\xi_s| \in \cap_{1\le p\le\infty} L^p$.

The second statement follows from differentiation of (22), using the fact that $\left(\frac{d\mathbb{P}^x_\tau}{d\mathbb{P}^x_{\curvearrowright}}\right)_{\tau=0} = 1$ and $\frac{\partial}{\partial t}H^\tau_t(x)\,|_{\tau=0} = 0$:

$$\begin{aligned}
\frac{\partial}{\partial\tau}\left(\frac{d\mathbb{P}^x_\tau}{d\mathbb{P}^x_{\curvearrowright}}\right)_{\tau=0} &= \left(\frac{d\mathbb{P}^x_\tau}{d\mathbb{P}^x_{\curvearrowright}}\right)_{\tau=0}\cdot\left[\left(\frac{\partial}{\partial\tau}M^\tau_T\right)_{\tau=0} - \frac{1}{2}\left(\frac{\partial}{\partial\tau}\langle M^\tau_T\rangle^2\right)_{\tau=0}\right]\\
&= \int_0^T\left\langle X(\xi^\tau_s(x)dB_s, \frac{D}{\partial\tau}\left[T\xi_s\left(\frac{\partial}{\partial s}H^\tau_s(x)\right)\right]\right\rangle_{\tau=0}\\
&= \int_0^T\left\langle X(\xi_s(x))dB_s, T\xi_s(\frac{D}{\partial s}X(H^\tau_s(x))h_s)\Big|_{\tau=0}\right\rangle\\
&= \int_0^T\left\langle X(\xi_s(x))dB_s, T\xi_s(X(x)\dot{h}_s)\right\rangle.
\end{aligned}$$

For general h take a sequence of bounded h_n which converges to h in $L^{\frac{1+\epsilon}{2}}(\Omega, L^{2,1}_0([0,T]))$ to finish the proof. See the proof of theorem 4.1. ∎

The following is an analogue of Corollary 2.4: here $\tilde{\nabla}$ is any metric connection and $\tilde{W}^Z_\cdot$ is as in Corollary 2.4,

Theorem 3.3 *Let $F \in BC^1(C_x(M))$ and $h(\omega) \in L^{2,1}_0([0,T];\mathbb{R}^n)$ a.s.. Suppose $h_\cdot$ is adapted to the filtration of $\{\mathcal{F}^x_\cdot\}$ and such that $\mathbb{E}\left(\int_0^T|\dot{h}_s|^2ds\right)^{\frac{1+\epsilon}{2}} < \infty$ for some $\epsilon > 0$. Then*

$$\mathbb{E}dF(\tilde{W}^Z_\cdot(h_\cdot)) = \mathbb{E}F(\xi_\cdot(x))\int_0^T < \tilde{W}^Z_s(\dot{h}_s), \tilde{//}_s d\tilde{B}_s > . \tag{23}$$

If $\tilde{\nabla}'$ is metric for some Riemannian metric, we can take $\epsilon = 0$.

4 Integration by parts for the free path space

It is easy to modify the proof of Proposition 3.2 to the case where $h(0)\neq 0$ and so obtain an integration by parts formula for the free path space $PM = \cup_{x\in M}P_xM$ with uniform topology and measure given by the Riemannian measure of M together with the laws of $\{\xi_\cdot(x) : x \in M\}$. In fact it is straightforward to generalize to the case of an x-dependent $h_\cdot$. For this let $C^1(TM)$ be the space of C^1 vector fields on M with its usual topology:

Theorem 4.1 *Let $h : [0,T]\times\Omega \to C^1(TM)$ be a cadlag adapted process such that the T_xM valued process $h_\cdot(x)$ has sample paths in $L^{2,1}([0,T];T_xM)$*

for each $x \in M$ *with* $|h_0(\cdot)| + \sqrt{\int_0^t |\dot{h}_s(\cdot)|^2 ds}$ *in* $L^{1+\epsilon}(\Omega \times M; \mathbb{R})$ *for some* $\epsilon > 0$*. Let* F *be in* $BC^1(PM;\mathbb{R})$*. Then*

$$\begin{aligned} & \mathbb{E}\int_M dF\left(T_x\xi_\cdot(h_\cdot(\omega)(x))\right)dx \\ = \; & \mathbb{E}\int_M F(\xi_\cdot(x))\left\{-div h_0(x) + \int_0^T \left\langle T\xi_s(\dot{h}_s(x)), X(\xi_s(x))dB_s\right\rangle\right\}dx. \end{aligned} \tag{24}$$

Proof. Proceed as for Proposition 3.2 but with $X(x)h_t$ replaced by $h_t(x)$. In particular the definition (6) of H_t^τ becomes

$$\begin{aligned} \frac{\partial}{\partial\tau}H_t^\tau(x) &= h_t\left(H_t^\tau(x)\right) \\ H_t^0(x) &= x. \end{aligned}$$

while ξ_t^τ is defined by (19). However now $\xi_0^\tau(x) = \xi_0\left(H_0^\tau(x)\right)$: the starting point is transported by the flow of $h_0(x)$.

We first assume $h_\cdot$ and $\int_0^\cdot |\dot{h}_s|^2 ds$ are bounded on $\Omega \times M$. Then the Girsanov-Maruyama theorem gives us equivalence between the measures P_τ^x and $P_0^{H_0^\tau(x)}$ with

$$\int_M \mathbb{E}F\left(\xi_\cdot^\tau(x)\right)dx = \int_M \mathbb{E}F\left(\xi_\cdot(H_0^\tau(x))\right)\frac{d\mathbb{P}_\tau^x}{d\mathbb{P}_0^{H_0^\tau(x)}}dx.$$

On differentiating this there is the extra term

$$\begin{aligned} & \int_M dF\left(T\xi_\cdot\left(\frac{\partial}{\partial\tau}H_0^\tau(x)\Big|_{\tau=0}\right)\right) \\ = \; & \int_M dF\left(T_x\xi_\cdot\left(h_0(x)\right)\right)dx \\ = \; & \int_M d_x\left(F\circ\xi_\cdot\right)\left(h_0(x)\right)dx \end{aligned}$$

where $d_x\left(F\circ\xi_\cdot\right)$ refers to the derivative in M of $F\circ\xi_\cdot : M\times\Omega \to \mathbb{R}$. Now apply the classical Stokes theorem on M to get:

$$\begin{aligned} & \mathbb{E}\int_M dF(T_x\xi_\cdot(h_\cdot(\omega)(x)))dx \\ = \; & \mathbb{E}\int_M F(\xi_\cdot(x))\left\{-div h_0(x) + \int_0^T < T_x\xi_s(\dot{h}_s(x)), X(\xi_s(x))dB_s >\right\}dx. \end{aligned}$$

For general h let τ_R be the first exit time of $||h.||_{C^1} + \int_0^\cdot |h_s(x)|^2 ds$ from $[0, R)$. Set $h_t^R(x) = h_{t\wedge\tau_R}(x)\chi_{\{||h_0||_{C^1}<R\}}$. We have:

$$\begin{aligned} & \mathbb{E}\int_M dF(T_x\xi_\cdot(h_\cdot^R(\omega)(x)))dx \\ = \quad & \mathbb{E}\chi_{\{||h_0||_{C^1}<R\}}\int_M F(\xi_\cdot(x))\left\{-divh_0(x) + \int_0^{T\wedge\tau_R} < T_x\xi_s(\dot{h}_s(x)), X(\xi_s(x))dB_s >\right\}dx. \end{aligned}$$

Now let $R \to \infty$. The left hand side converges to $\mathbb{E}\int_M dF(T\xi_\cdot(h_\cdot(\omega)(x)))dx$ since

$$|dF(T\xi_\cdot(h_\cdot^R(\omega)(-)))| \le \tilde{c}\sup_t |T\xi_t(\omega)| \sup_t |h_t(-,\omega)|$$

and $\sup_x \mathbb{E}\left(\sup_t |T\xi_t| \int_M \sup_t |h_t(x,\omega)|dx\right) < \infty$ from

$$\begin{aligned} \sup_t |h_t(x)| &\le |h_0(\omega)| + \int_0^T |\dot{h}_s(\omega)|ds \\ &\le |h_0(\omega)| + \sqrt{T}\sqrt{\int_0^T |\dot{h}_s(\omega)|^2 ds} \in L^{1+\epsilon}(\Omega \times M) \end{aligned}$$

Using Burkholder-Davies-Gundy inequality to justify the integration on the right hand side we see that it converges to the right hand side of (24). ∎

Just as before the intrinsic formulae can be deduced using [ELJL95]:

Theorem 4.2 *Let F be in $BC^1(PM;\mathbb{R})$ and h be as in Theorem 4.1 but with $h_\cdot(x)$ adapted to the filtration of $\{\mathcal{F}_\cdot^x\}$, and $divh_0 \in L^1(\Omega \times M, \mathbb{R})$. Then for any metric connection $\tilde{\nabla}$ on M,*

$$\begin{aligned} & \mathbb{E}\textstyle\int_M dF\left(\tilde{W}_\cdot^Z(h_\cdot(\omega)(x))\right)dx \\ = \quad & \mathbb{E}\textstyle\int_M F(\xi_\cdot(x))\left\{-divh_0(x) + \int_0^T \left\langle \tilde{W}_s^Z(\dot{h}_s(x)), \tilde{//}_s d\tilde{B}_s\right\rangle\right\}dx. \end{aligned} \tag{25}$$

If furthermore $\tilde{\nabla}'$ is metric with respect to a Riemannian metric, we can take $\epsilon = 0$.

Proof. The proof is just as that of Theorem 3.3. ∎

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