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A CLASS OF INVERSE THEOREMS ON RECURSIVE PROGRAMMING WITH MONOTONICITY

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Abstract. The author gives a broad class of inverse theorems on mathematical programming problems, where the objective function is either a recursive function with strict increasingness or a recursive function with strict decreasingness, and so is the constraint function. It is also shown that the optimal-value functions of main and inverse problems can be expressed by the successive use of some nonlinear operators defined in this paper. Each expression is based upon either Bellman's Principle of Optimality or its modified principle. Further each inverse theorem accompanies an example.

1. Introduction and summary

Recently the author has established INVERSE THEOREMS I, II AND III IN DYNAMIC PROGRAMMING [3,4,5,6]. In those theorems both objective and constraint functions satisfy dynamic programming structure, that is, they are recursive functions with strict "increasingness".

This paper studies a braod class of inverse theorems on mathematical programming problems, where the objective function is either a recursive function with strict increasingness or a recursive function with strict "decreasingness", and so is the constraint function.

In Section 2 we define both recursive function with strict increasingness and recursive function with strict decreasingness, Considering 5 pairs of main and inverse problems having these functions as the objective and constraint functions, 5 inverse theorems associated with the corresponding pairs are established, respectively, like as in [5].

Section 3 is devoted to proofs of inverse theorems.

In Section 4, given 2-variable functions f and g we define important operators T(f;g), S(f;g), P(f;g) and Q(f;g). Each operator maps one class of continuous and strictly monotone functions into another. According to the monotonicity in 2-nd variable of f and g, pairs (T(f;g), S(g;f)), (P(f;g), P(g;f)) and (Q(f;g), Q(g;f)) preserve an inverse relation in Iwamoto sense [3,4,5,6], respectively. Moreover, it is shown that the optimal-value functions of main and inverse problems can be expressed by the successive use of these operators, provided the objective function is either a recursive function with strict increasingness or a recursive one with strict decreasingness and so is the constraint function. These expressions are immediate consequences from Bellman's Principle of Optimality [1,2,3,11,12] and its modified principle [9, 10, 12].

The last section illustrates an example of each inverse theorem.

2. Inverse theorems

This section deals with N-variable ($N \geq 2$) problems except a pair of main and inverse problems studied extensively by the author [3,4,5]. In this paper the omitted pair is denoted by Main Problem I and Inverse Problem I or simply by MP I and IP I.

Throughout the paper we shall use the following notations [3,4,5,6,8] : For $d < e$, $\langle d,e \rangle$ denotes an arbitrary interval in the real line R^1. Let E be the Cartesian product of intervals $\langle d_k, e_k \rangle$ $1 \leq k \leq N$, namely,

$$E = \langle d_1,e_1 \rangle \times \langle d_2,e_2 \rangle \times \cdots \times \langle d_N,e_N \rangle.$$

A continuous function $f : E \longrightarrow R^1$ is called the recursive function* on E if it is expressed as follows :

$$f(x_1,x_2,\cdots,x_N) = f_1(x_1;f_2(x_2;\cdots f_{N-1}(x_{N-1};f_N(x_N))\cdots)),$$

* Our definition of recursive function is slightly different from definition of recursive function in mathematical logic. See also [10].

where $f_N : \langle d_N, e_N \rangle \longrightarrow R^1$, $f_k : \langle d_k, e_k \rangle \times \text{range}(f_{k+1}) \longrightarrow R^1$ $(1 \leqq k \leqq N-1)$ are continuous. Here note that $\text{range}(f_k) = \{z \; ; \; z = f_k(x;y), \; (x,y) \in \langle d_k, e_k \rangle \times \text{range}(f_{k+1})\}$ $1 \leqq k \leqq N-1$, and $\text{range}(f_N) = \{y \; ; \; y = f_N(x), \; x \in \langle d_N, e_N \rangle\}$. A recursive function f on E is called the recursive function with strict increasingness (resp. decreasingness) on E if each $f_k(x;\cdot)$ $(1 \leqq k \leqq N-1, \; x \in \langle d_k, e_k \rangle)$ is strictly increasing (resp. decreasing) and f_N is strictly increasing.

In this paper "monotone" denotes "either increasing or decreasing". For example, a strictly monotone function denotes either a strictly increasing function or a strictly decreasing function.

If $h : \langle a,b \rangle \longrightarrow \langle c,d \rangle$ is an onto continuous and strictly monotone function, then so is the inverse function $h^{-1} : \langle c,d \rangle \longrightarrow \langle a,b \rangle$. This is an elementary result in mathematical calculus. If $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are functions, we denotes by $g \circ f : X \longrightarrow Z$ the composition

$$(g \circ f)(x) = g(f(x)).$$

First, we consider a pair of main and inverse problems as follows :

MP I' Maximize $f(u_1(x_1), u_2(x_2), \cdots, u_N(x_N))$

subject to (i) $g(x_1,x_2,\cdots,x_N) \leqq c$

(ii) $(x_1,x_2,\cdots,x_N) \in E$

IP I' Minimize $g(v_1(y_1), v_2(y_2), \cdots, v_N(y_N))$

subject to (i)' $f(y_1,y_2,\cdots,y_N) \geqq c$

(ii)' $(y_1,y_2,\cdots,y_N) \in E$.

Here $f : E \longrightarrow \langle \alpha, \beta \rangle$ and $g : E \longrightarrow \langle a,b \rangle$ are onto recursive functions with strict decreasingness on E, $u_k : \langle d_k, e_k \rangle \longrightarrow \langle d_k, e_k \rangle$ is an onto continuous and strictly monotone function for $1 \leqq k \leqq N-1$, $u_N : \langle d_N, e_N \rangle \longrightarrow \langle d_N, e_N \rangle$ is an onto continuous and strictly increasing function, and v_k is the inverse function to u_k for $1 \leqq k \leqq N$.

Second, let us consider four pairs of main and inverse problems as follows :

MP II (resp. II') Maximize $f(u_1(x_1), u_2(x_2), \cdots, u_N(x_N))$

subject to (i) $g(x_1,x_2,\cdots,x_N) \geqq c$

(ii) $(x_1,x_2,\cdots,x_N) \in E$

IP II (resp. II') Maximize $g(v_1(y_1), v_2(y_2), \cdots, v_N(y_N))$

subject to (i)' $f(y_1,y_2,\cdots,y_N) \geqq c$

(ii)' $(y_1,y_2,\cdots,y_N) \in E$,

and

MP III (resp. III') Minimize $f(u_1(x_1), u_2(x_2), \cdots, u_N(x_N))$

subject to (i) $g(x_1,x_2,\cdots,x_N) \leqq c$

(ii) $(x_1,x_2,\cdots,x_N) \in E$

IP III (resp. III') Minimize $g(v_1(y_1), v_2(y_2), \cdots, v_N(y_N))$

subject to (i)' $f(y_1,y_2,\cdots,y_N) \leqq c$

(ii)' $(y_1,y_2,\cdots,y_N) \in E$,

where $f : E \longrightarrow \langle\alpha,\beta\rangle$ and $g : E \longrightarrow \langle a,b\rangle$ are onto recursive functions with strict increasingness (resp. decreasingness) on E, $u_k : \langle d_k,e_k\rangle \longrightarrow \langle d_k,e_k\rangle$ is an onto continuous and strictly monotone function for $1 \leqq k \leqq N-1$, $u_N : \langle d_N,e_N\rangle \longrightarrow \langle d_N,e_N\rangle$ is an onto continuous and strictly decreasing function, and v_k is the inverse function to u_k for $1 \leqq k \leqq N$.

Then we have Theorem X which establishes an inverse relation between MP X and IP X, where X = I', II, II', III, III'.

THEOREM X. (i) MP X has an onto continuous and strictly monotone optimum-value function $U : \langle a,b\rangle \longrightarrow \langle\alpha,\beta\rangle$ and an optimum-point function $(x_1^*,x_2^*,\cdots,x_N^*) : \langle a,b\rangle \longrightarrow E$ if and only if IP X has an onto continuous and strictly monotone optimum-value function $U^{-1} : \langle\alpha\ \beta\rangle \longrightarrow \langle a,b\rangle$ and an optimum-point function $(u_1 \circ x_1^* \circ U^{-1}, u_2 \circ x_2^* \circ U^{-1}, \cdots, u_N \circ x_N^* \circ U^{-1}) : \langle\alpha,\beta\rangle \longrightarrow E$.

(ii) IP X has an onto continuous and strictly monotone optimum-value function $V : \langle\alpha,\beta\rangle \longrightarrow \langle a,b\rangle$ and an optimum-point function $(\hat{y}_1,\hat{y}_2,\cdots \hat{y}_N) : \langle\alpha,\beta\rangle \longrightarrow E$ if and only if MP X has an onto continuous and strictly monotone optimum-value function $V^{-1} : \langle a,b\rangle \longrightarrow \langle\alpha,\beta\rangle$ and an optimum-point function $(v_1 \circ \hat{y}_1 \circ V^{-1}, v_2 \circ \hat{y}_2 \circ V^{-1}, \cdots, v_N \circ \hat{y}_N \circ V^{-1}) : \langle a,b\rangle \longrightarrow E$.

Note that according to X "monotone" becomes either "increasing" or "decreasing" and "optimum" does either "maximum" or "minimum". These correspondences are immediate from the structure of MP X and IP X.

3. Proofs

This section gives proofs of theorems stated in Section 2. Note that in each theorem (i) is equivalent to (ii), and that in (i), (ii) "if" part can be proved by the same method as "only if" part. Therefore we shall prove only " only if" part of (i) in each theorem.

3.1. Proof of Theorem I'

Let U and $(x_1^*, x_2^*, \cdots, x_N^*)$ be a continuous and strictly increasing maximum-value function and a maximum-point function. Then we have for any fixed $c \in \langle a,b \rangle$

$$U(c) = f(u_1(x_1^*(c)),\ u_2(x_2^*(c)),\ \cdots,\ u_N(x_N^*(c))) \tag{3.1}$$

$$g(x_1^*(c),\ x_2^*(c),\ \cdots,\ x_N^*(c)) \leqq c. \tag{3.2}$$

Now assume that

$$g(x_1^*(c),\ x_2^*(c),\ \cdots,\ x_N^*(c)) = c' < c$$

in (3.2). Then the strict increasingness of U implies

$$U(c') < U(c). \tag{3.3}$$

On the other hand, since $(x_1^*(c),\ x_2^*(c),\ \cdots,\ x_N^*(c))$ is a feasible solution of MP I' with c' :

$$\begin{aligned} &\text{Maximize} \quad f(u_1(x_1),\ u_2(x_2),\ \cdots,\ u_N(x_N)) \\ &\text{subject to (i)} \quad g(x_1, x_2, \cdots, x_N) \leqq c' \\ &\qquad\qquad\quad \text{(ii)} \quad (x_1, x_2, \cdots, x_N) \in E, \end{aligned}$$

we have

$$U(c') \geqq f(u_1(x_1^*(c)),\ u_2(x_2^*(c)),\ \cdots,\ u_N(x_N^*(c))).$$

Hence (3.1) yields that

$$U(c') \geqq U(c).$$

This contradicts (3.3). Therefore

(3.4) $$g(x_1^*(c), x_2^*(c), \cdots, x_N^*(c)) = c.$$

Let $\hat{y}_n = u_n \circ x_n^*$. Then $x_n^* = v_n \circ \hat{y}_n$. Hence (3.4) and (3.1) yield

$$g(v_1(\hat{y}_1(c)), v_2(\hat{y}_2(c)), \cdots, v_N(\hat{y}_N(c))) = c$$

$$f(\hat{y}_1(c), \hat{y}_2(c), \cdots, \hat{y}_N(c)) = U(c).$$

Let $V(c)$ be the infimum-value of IP I'. Since $(\hat{y}_1(c), \hat{y}_2(c), \cdots, \hat{y}_N(c))$ is a feasible solution of the minimizing problem :

(3.5) $$\text{Minimize } g(v_1(y_1), v_2(y_2), \cdots, v_N(y_N))$$

(3.6) $$\text{subject to (i)' } f(y_1,y_2,\cdots,y_N) \geq U(c)$$

$$\text{(ii)' } (y_1,y_2,\cdots,y_N) \in E,$$

we have

$$V(d) \leq c,$$

where $d = U(c) \in \langle\alpha,\beta\rangle$. If $V(d) < c$, then we may without loss of generality choose $(\hat{\hat{y}}_1(d), \hat{\hat{y}}_2(d), \cdots, \hat{\hat{y}}(d))$ in E such that

(3.8) $$g(\hat{\hat{v}}_1(y_1(d)), v_2(\hat{\hat{y}}_2(d)), \cdots, v_N(\hat{\hat{y}}_N(d))) = V(d) < c$$

(3.9) $$f(\hat{\hat{y}}_1(d), \hat{\hat{y}}_2(d), \cdots, \hat{\hat{y}}_N(d)) = d.$$

By replacing $x_n^{**}(c) = v_n(\hat{\hat{y}}_n(d))$ for $1 \leq n \leq N$, (3.9) and (3.8) reduce to

$$f(u_1(x_1^{**}(c)), u_2(x_2^{**}(c)), \cdots, u_N(x_N^{**}(c))) = d$$

$$g(x_1^{**}(c), x_2^{**}(c), \cdots, x_N^{**}(c)) < c.$$

Let

$$g(x_1^{**}(c), x_2^{**}(c), \cdots, x_N^{**}(c)) = c'' < c.$$

Then the strict increasingness of U implies

(3.10) $$U(c'') < U(c).$$

On the other hand, since $(x_1^{**}(c), x_2^{**}(c), \cdots, x_N^{**}(c))$ is a feasible solution of MP I' with c" :

$$\text{Maximize} \quad f(u_1(x_1), u_2(x_2), \cdots, u_N(x_N))$$

$$\text{subject to (i)} \quad g(x_1,x_2,\cdots,x_N) \leq c"$$

$$\text{(ii)} \quad (x_1,x_2,\cdots,x_N) \in E,$$

we have

$$U(c") \geq f(u_1(x_1^{**}(c)), u_2(x_2^{**}(c)), \cdots, u_N(x_N^{**}(c))).$$

That is,

$$U(c") \geq d = U(c).$$

This contradicts (3.10). Therefore we have for $c \in \langle a,b \rangle$

$$V(U(c)) = c$$

and $(\hat{y}_1(U(c)), \hat{y}_2(U(c)), \cdots, \hat{y}_N(U(c)))$ is a minimum-point of the problem (3.5), (3.6), (3.7). This implies that $V = U^{-1}$ is a continuous strictly increasing minimum-value function of IP I' and that $(u_1 \circ x_1^* \circ U^{-1}, u_2 \circ x_2^* \circ U^{-1}, \cdots, u_N \circ x_N^* \circ U^{-1})$ is a minimum-point function of IP I'. This completes the proof.

3.2. Proofs of other theorems

The proofs are similar, mutatis mutadis, with the proof of Theorem I'.

4. Operator expressions of optimal-value functions

In this section we define fundamental operators $T(f;g)$, $S(f;g)$, $P(f;g)$ and $Q(f;g)$ for given recursive functions f,g with monotonicity on $F = \langle d_1,e_1\rangle \times \langle d_2,e_2\rangle$. By decomposing MP X (resp. IP X) into subproblems MP X(N-n) (resp. IP X(N-n)) $1 \leq n \leq N$, we will find that a successive use of above operators yields a continuous and strictly monotone optimal-value function of MP X (resp. IP X), where = II, III, I', II', III'.

DEFINITION. Let $f : F \longrightarrow \langle \alpha,\beta \rangle$, $g : F \longrightarrow \langle a,b \rangle$ be onto recursive functions with monotonicity on F. Let $u : \langle d_2, e_2\rangle \longrightarrow \langle d_2, e_2\rangle$ be an onto continuous and strictly monotone function. Then functions $T(f;g)u$, $S(f;g)u$, $P(f;g)u$ and $Q(f;g)u : \langle a,b\rangle \longrightarrow \langle \alpha,\beta\rangle$ are defined (if they exist) by

$$T(f;g)u(c) = \underset{\substack{g(x,y)\leqq c \\ (x,y)\in F}}{\mathrm{Max}} f(x, u(y)), \tag{4.1}$$

$$S(f;g)u(c) = \underset{\substack{g(x,y)\geqq c \\ (x,y)\in F}}{\mathrm{Min}} f(x, u(y)), \tag{4.2}$$

$$P(f;g)u(c) = \underset{\substack{g(x,y)\geqq c \\ (x,y)\in F}}{\mathrm{Max}} f(x, u(y)), \tag{4.3}$$

and

$$Q(f;g)u(c) = \underset{\substack{g(x,y)\leqq c \\ (x,y)\in F}}{\mathrm{Min}} f(x, u(y)), \tag{4.4}$$

respectively.

As for the properties of $T(f;g)$, $S(f;g)$, see [3,4,5]. The reader will find that according to the strict monotonicity in the 2-nd variable of f, g each operator maps one class of continuous and strictly monotone functions into another.. Note that $S(g;f)v$, $P(g;f)v$ and $Q(g;f)v$ are also defined by (4.2), (4.3) and (4.4), respectively, where v is the inverse function to u. Moreover, under some appropriate conditions, pairs $(P(f;g)u, P(g;f)v)$, $(Q(f;g)u, Q(g;f)v)$ become inverse functions each other like as a pair $(T(f;g)u, S(g;f)v)$, respectively. The detailed analysis of the former pairs is omitted, since it is similar to one of the latter pair [3,4,5].

Throughout the remainder of this section we shall use the following notations : Given a recursive function

$$h(z_1,z_2,\cdots,z_N) = h_1(z_1;h_2(z_2;\cdots h_{N-1}(z_{N-1};h_N(z_N))\cdots))$$

on $E = \langle d_1,e_1\rangle \times \langle d_2,e_2\rangle \times \cdots \times \langle d_N,e_N\rangle$, we define h_n $(1\leqq n\leqq N)$ by

$$h_n(z_n,z_{n+1},\cdots,z_N) = h_n(z_n;h_{n+1}(z_{n+1};\cdots h_{N-1}(z_{N-1};h_N(z_N))\cdots)).$$

Clearly $h_n(z_n,z_{n+1},\cdots,z_N)$ is also a recurisve function on $\langle d_n,e_n\rangle \times \langle d_{n+1},e_{n+1}\rangle \times \cdots \times \langle d_N,e_N\rangle$ for $1\leqq n\leqq N-1$ and

$$h_1(z_1,z_2,\cdots,z_N) = h(z_1,z_2,\cdots,z_N).$$

It will be clear from the context whether a function h_n is considered $h_n(z;w)$ or $h_n(z_n,z_{n+1},\cdots,z_N)$. Further, given a 2-variable function $h = h(x;y):\langle a,b\rangle\times\langle c,d\rangle \longrightarrow R^1$, we define for $x\in\langle a,b\rangle$ 1-variable function $h^x : \langle c,d\rangle \longrightarrow R^1$ by

$$h^x(y) = h(x;y).$$

Let us now consider subproblems of problems discussed in Section 2. First we define (N-n)-subproblems of MP II, IP II, MP III and IP III as follows :
For $1\leqq n\leqq N$

MP II(N-n) $\quad$ Max $f_n(u_n(x_n),\ u_{n+1}(x_{n+1}),\ \cdots,\ u_N(x_N))$

s.t. (i) $g_n(x_n,x_{n+1},\cdots,x_N) \geqq c$

(ii) $x_k\in\langle d_k,e_k\rangle \quad n\leqq k\leqq N$,

IP II(N-n) $\quad$ Max $g_n(v_n(y_n),\ v_{n+1}(y_{n+1}),\ \cdots,\ v_N(y_N))$

s.t. (i)' $f_n(y_n,y_{n+1},\cdots,y_N) \geqq c$

(ii)' $y_k\in\langle d_k,e_k\rangle \quad n\leqq k\leqq N$,

MP III(N-n) $\quad$ Min $f_n(u_n(x_n),\ u_{n+1}(x_{n+1}),\ \cdots,\ u_N(x_N))$

s.t. (i) $g_n(x_n,x_{n+1},\cdots,x_N) \leqq c$

(ii) $x_k\in\langle d_k,e_k\rangle \quad n\leqq k\leqq N$,

IP III(N-n) $\quad$ Min $g_n(v_n(y_n),\ v_{n+1}(y_{n+1}),\ \cdots,\ v_N(y_N))$

s.t. (i)' $f_n(y_n,y_{n+1},\cdots,y_N) \leqq c$

(ii)' $x_k\in\langle d_k,e_k\rangle \quad n\leqq k\leqq N$.

On the other hand, according to either oddness or evenness we define subproblems of the problems I', II' and III' as follows :
For n odd (resp. even), where $1\leqq n\leqq N$

MP I'(N-n) $\quad$ Max (resp. Min) $f_n(u_n(x_n),\ u_{n+1}(x_{n+1}),\ \cdots,\ u_N(x_N))$

s.t. (i) $g_n(x_n,x_{n+1},\cdots,x_N) \leqq$ (resp. $\geqq$) c

(ii) $x_k\in\langle d_k,e_k\rangle \quad n\leqq k\leqq N$, IP I'(N-n) Min (resp. Max) $g_n(v_n(y_n), v_{n+1}(y_{n+1}), \cdots, v_N(y_N))$

s.t. (i)' $f_n(y_n, y_{n+1}, \cdots, y_N) \geqq$ (resp. $\leqq$) c

(ii)' $y_k \in \langle d_k, e_k \rangle \quad n \leqq k \leqq N$,

MP II'(N-n) Max (resp. Min) $f_n(u_n(x_n), u_{n+1}(x_{n+1}), \cdots, u_N(x_N))$

s.t. (i) $g_n(x_n, x_{n+1}, \cdots, x_N) \geqq$ (resp. $\leqq$) c

(ii) $x_k \in \langle d_k, e_k \rangle \quad n \leqq k \leqq N$,

IP II'(N-n) Max (resp. Min) $g_n(v_n(y_n), v_{n+1}(y_{n+1}), \cdots, v_N(y_N))$

s.t. (i)' $f_n(y_n, y_{n+1}, \cdots, y_N) \geqq$ (resp. $\leqq$) c

(ii)' $y_k \in \langle d_k, e_k \rangle \quad n \leqq k \leqq N$,

MP III'(N-n) Min (resp. Max) $f_n(u_n(x_n), u_{n+1}(x_{n+1}), \cdots, u_N(x_N))$

s.t. (i) $g_n(x_n, x_{n+1}, \cdots, x_N) \leqq$ (resp. $\geqq$) c

(ii) $x_k \in \langle d_k, e_k \rangle \quad n \leqq k \leqq N$,

IP III'(N-n) Min (resp. Max) $g_n(v_n(y_n), v_{n+1}(y_{n+1}), \cdots, v_N(y_N))$

s.t. (i)' $f_n(y_n, y_{n+1}, \cdots, y_N) \leqq$ (resp. $\geqq$) c

(ii)' $y_k \in \langle d_k, e_k \rangle \quad n \leqq k \leqq N$.

Note that (N-1)-subproblem is identical with the original problem and that the parameter c of MP X(N-n) (resp. IP X(N-n)) ranges over the range(g_n) (resp. range(f_n)), where X = II, III, I', II' and III'. Of course, range(f_n) and range (g_n) are intervals of R^1.

Let us define u^{N-n} : range(g_n) $\longrightarrow$ range(f_n) and v^{N-n} : range(f_n) $\longrightarrow$ range (g_n)by

$$u^{N-n}(c) = \text{the optimum-value of MP X(N-n)}$$

$$v^{N-n}(c) = \text{the optimum-value of IP X(N-n)}$$

, if they exist, respectively, where X = II, III, I', II', III'. Here note that according to X and n "optimum" means either "maximum" or "minimum". For example, $u^{N-n}(c)$ denotes the maximum-value of MP II(N-n) for $n=1,2,\cdots,N$. On the other hand, $v^{N-n}(c)$ denotes the minimum-(resp. maximum-) value of IP I'(N-n) for n odd (resp. even).

The usual dynamic programming analysis [1,2,3,9,10,11] combines both (N-n)- and (N-n-1)-subproblems as follows :

THEOREM IV (BELLMAN'S PRINCIPLE OF OPTIMALITY)

MP II (resp. III)
$$u^{N-n}(c) = \mathop{\mathrm{Max}\ (\mathrm{resp.\ Min})}_{\substack{x_n \in \langle d_n, e_n \rangle \\ (g_n^{x_n})^{-1}(c) \in \mathrm{range}(g_{n+1})}} f_n(u_n(x_n); u^{N-n-1}((g_n^{x_n})^{-1}(c))) \qquad 1 \leq n \leq N-1$$

IP II (resp. III)
$$v^{N-n}(c) = \mathop{\mathrm{Max}\ (\mathrm{resp.\ Min})}_{\substack{y_n \in \langle d_n, e_n \rangle \\ (f_n^{y_n})^{-1}(c) \in \mathrm{range}(f_{n+1})}} g_n(v_n(y_n); v^{N-n-1}((f_n^{y_n})^{-1}(c))) \qquad 1 \leq n \leq N-1$$

MP II, III $\qquad u^0(c) = f_N(u_N(g_N^{-1}(c)))$

IP II, III $\qquad v^0(c) = g_N(v_N(f_N^{-1}(c)))$

PROOF. Follow the same line as in Proposition 3.2 of [3].

On the other hand, the recursive programming analysis [9,10 p.378, 12 p.44] yields the following version of Bellman's principle for the problems I', II' and III' :

THEOREM IV'

MP I', II'
$$u^{N-n}(c) = \mathop{\mathrm{Max}\ (\mathrm{resp.\ Min})}_{\substack{x_n \in \langle d_n, e_n \rangle \\ (g_n^{x_n})^{-1}(c) \in \mathrm{range}(g_{n+1})}} f_n(u_n(x_n); u^{N-n-1}((g_n^{x_n})^{-1}(c))) \qquad n = \text{odd (resp. even)} \quad 1 \leq n \leq N-1$$

IP I', III'
$$v^{N-n}(c) = \mathop{\mathrm{Min}\ (\mathrm{resp.\ Max})}_{\substack{y_n \in \langle d_n, e_n \rangle \\ (f_n^{y_n})^{-1}(c) \in \mathrm{range}(f_{n+1})}} g_n(v_n(y_n); v^{N-n-1}((f_n^{y_n})^{-1}(c))) \qquad n = \text{odd (resp. even)} \quad 1 \leq n \leq N-1$$

IP II' $\quad v^{N-n}(c) = \underset{y_n \in <d_n, e_n>}{\text{Max (resp. Min)}}\ g_n(v_n(y_n); v^{N-n-1}((f_n^{y_n})^{-1}(c)))$

$\qquad (f_n^{y_n})^{-1}(c) \in \text{range}(f_{n+1}) \qquad n = \text{odd (resp. even)}\ 1 \leq n \leq N-1$

MP III' $\quad u^{N-n}(c) = \underset{x_n \in <d_n, e_n>}{\text{Min (resp. Max)}}\ f_n(u_n(x_n); u^{N-n-1}((g_n^{x_n})^{-1}(c)))$

$\qquad (g_n^{x_n})^{-1}(c) \in \text{range}(g_{n+1}) \qquad n = \text{odd (resp. even)}\ 1 \leq n \leq N-1$

MP I', II', III' $\qquad u^0(c) = f_N(u_N(g_N^{-1}(c)))$

IP I', II', III' $\qquad v^0(c) = g_N(v_N(f_N^{-1}(c)))$

PROOF. A more generalized form of this theorem has been stated and proved by the author [9].

Further we may restate Theorems IV and IV' in terms of operators defined by (4.1) — (4.4).

COROLLARY. The optimum-value functions of MP X, IP X can be represented by the successive use of the modified operators as follows :

MP II $\qquad P(f_1^*; g_1)P(f_2^*; g_2) \cdots P(f_{N-1}^*; g_{N-1})u^0$

IP II $\qquad P(\hat{g}_1; f_1)P(\hat{g}_2; f_2) \cdots P(\hat{g}_{N-1}; f_{N-1})v^0$

MP III $\qquad Q(f_1^*; g_1)Q(f_2^*; g_2) \cdots Q(f_{N-1}^*; g_{N-1})u^0$

IP III $\qquad Q(\hat{g}_1; f_1)Q(\hat{g}_2; f_2) \cdots Q(\hat{g}_{N-1}; f_{N-1})v^0$

MP I' $\qquad T(f_1^*; g_1)S(f_2^*; g_2)T(f_3^*; g_3)S(f_4^*; g_4) \cdots (f_{N-1}^*; g_{N-1})u^0$

IP I' $\qquad S(\hat{g}_1; f_1)T(\hat{g}_2; f_2)S(\hat{g}_3; f_3)T(\hat{g}_4; f_4) \cdots (\hat{g}_{N-1}; f_{N-1})v^0$

MP II' $\qquad P(f_1^*; g_1)Q(f_2^*; g_2)P(f_3^*; g_3)Q(f_4^*; g_4) \cdots (f_{N-1}^*; g_{N-1})u^0$

IP II' $\qquad P(\hat{g}_1; f_1)Q(\hat{g}_2; f_2)P(\hat{g}_3; f_3)Q(\hat{g}_4; f_4) \cdots (\hat{g}_{N-1}; f_{N-1})v^0$

MP III' $\qquad Q(f_1^*; g_1)P(f_2^*; g_2)Q(f_3^*; g_3)P(f_4^*; g_4) \cdots (f_{N-1}^*; g_{N-1})u^0$

IP III' $\qquad Q(\hat{g}_1; f_1)P(\hat{g}_2; f_2)Q(\hat{g}_3; f_3)P(\hat{g}_4; f_4) \cdots (\hat{g}_{N-1}; f_{N-1})v^0$

, where $f_n^*(x;y) = f_n(u_n(x);y)$, $\hat{g}_n(x;y) = g_n(v_n(x);y)$ $1 \leqq n \leqq N-1$, $u^0 = f_N \circ u_N \circ g_N^{-1}$ and $v^0 = g_N \circ v_N \circ f_N^{-1}$.

PROOF. These are immediate consequences from Theorems IV and IV'.

Note that the corollary tells us that for N even (resp. odd) the maximum-value function of MP I' is expressed by

$$T(f_1^*;g_1)S(f_2^*;\hat{g}_2)T(f_3^*;g_3)S(f_4^*;g_4) \cdots S(f_{N-1}^*;g_{N-1})u^0$$

$$(\text{resp. } T(f_1^*;g_1)S(f_2^*;g_2)T(f_3^*;g_3)S(f_4^*;g_4) \cdots T(f_{N-1}^*;g_{N-1})u^0).$$

The similar interpretation is valid for IP I', MP II', IP II', MP III' and IP III'. Moreover, by Theorems IV and IV', we have an algorithm to obtain both optimum-value function and optimum-point function of MP X and IP X, where X = II, III, I', II' and III'. The algorithm for the cases X = I, II and III is the usual dynamic programming algorithm. On the other hand, the algorithm for the cases X = I', II' and III' is the recursive programming one. Nevertheless, inverse theorems are free from this algorithm, since they claim that the solution of one (main) problem is transformed into the solution of the other (inverse) problem in an inverse sense.

5. Examples

In this section Example X denotes an example of Theorem X, where X = I', II, III, II', III'. We sketch an outline of each example. The author has given a full detail of it [7].

EXAMPLE I' Let us consider a pair of N-variable problems as follows :

MP I' Maximize

$$\cfrac{x_1}{1+2\cfrac{x_2}{1+2\cfrac{x_3}{1+2 \cdots \cfrac{}{1+2\cfrac{x_{N-1}}{1+2x_N}}}}}$$

$$\text{subject to (i)} \quad \cfrac{x_1}{1+\cfrac{x_2}{1+\cfrac{x_3}{\ddots\; 1+\cfrac{x_{N-1}}{1+x_N}}}} \leqq c \; (\geqq 0)$$

$$\text{(ii)} \quad x_n \geqq 0 \qquad 1 \leqq n \leqq N,$$

IP I'

$$\text{Minimize} \quad \cfrac{y_1}{1+\cfrac{y_2}{1+\cfrac{y_3}{\ddots\; 1+\cfrac{y_{N-1}}{1+y_N}}}}$$

$$\text{subject to (i)'} \quad \cfrac{y_1}{1+2\cfrac{y_2}{1+2\cfrac{y_3}{\ddots\; 1+2\cfrac{y_{N-1}}{1+2y_N}}}} \geqq c \; (\geqq 0)$$

$$\text{(ii)'} \quad y_n \geqq 0 \qquad 1 \leqq n \leqq N.$$

Then the objective function $f = f(x_1,x_2,\cdots,x_N)$ of MP I' is a recursive function with strict decreasingness on R^N_+, because we may choose $f_n(x;y) = \frac{x}{1+2y}$ $1 \leqq n \leqq N-1$, $f_N(y) = y$. Similarly, letting $g_n(x;y) = \frac{x}{1+y}$ $1 \leqq n \leqq N-1$, $g_N(y) = y$, the constraint function $g = g(y_1,y_2,\cdots,y_N)$ of MP I' is also a recursive function with strict decreasingness on R^N_+. A successive use of inequality

$$\frac{1}{2}\left(\frac{x}{1+y}\right) \leqq \frac{x}{1+2y} \leqq \frac{x}{1+y} \qquad \text{on } R^2_+$$

yields the inequality

$$(5.1) \quad \cfrac{x_1}{1+2\cfrac{x_2}{1+2\cfrac{x_3}{\ddots\; 1+\cfrac{x_{N-1}}{1+2x_N}}}} \leqq \cfrac{x_1}{1+\cfrac{x_2}{1+\cfrac{x_3}{\ddots\; 1+\cfrac{x_{N-1}}{1+x_N}}}} \qquad \text{on } R^N_+$$

The sign of equality holds if and only if $x_1 = 0$ or $x_2 = 0$. The inequality (5.1) shows us the following solutions of main and inverse problems : MP I' has a continuous and strictly increasing maximum-value function $U(c) = c$ and a maximum-point function $(x_1^*(c), x_2^*(c), \cdots, x_N^*(c))$, where $x_1^*(c) = c$, $x_2^*(c) = 0$ and $x_n^*(c)$ is arbitrary for $3 \leqq n \leqq N$. IP I' has a continuous and strictly increasing minimum-value function $V(c) = c$ and a minimum-point function $(\hat{y}_1(c), \hat{y}_2(c), \cdots, \hat{y}_N(c))$, where $\hat{y}_1(c) = c$, $\hat{y}_2(c) = 0$ and $\hat{y}_n(c)$ is arbitrary for $3 \leqq n \leqq N$. It is obvious that U^{-1} is a continuous and strictly increasing minimum-value function of IP I' and $(x_1^* \circ U^{-1}, x_2^* \circ U^{-1}, \cdots, x_N^* \circ U^{-1})$ is a minimum-pcint function of IP I', and that V^{-1} is a continuous and strictly maximum-value function of MP I' and $(\hat{y}_1 \circ V^{-1}, \hat{y}_2 \circ V^{-1}, \cdots, \hat{y}_N \circ V^{-1})$ is a maximum-point function of MP I'. This fact is also a direct application of Theorem I'.

EXAMPLE II Let us consider the following pair :

MP II Maximize $\dfrac{x_1x_2x_3x_4}{x_5}$

subject to (i) $\dfrac{x_5}{x_1x_5+x_2x_5+x_3x_5+x_4x_5+1} \geqq c \quad (\in (0,\infty))$

(ii) $x_n > 0 \quad 1 \leqq n \leqq 5$,

IP II Maximize $\dfrac{1}{y_1+y_2+y_3+y_4+y_5}$

subject to (i)' $y_1y_2y_3y_4y_5 \geqq c \quad (\in (0,\infty))$

(ii)' $y_n > 0 \quad 1 \leqq n \leqq 5$.

Note that

$$\frac{x_5}{x_1x_5+x_2x_5+x_3x_5+x_4x_5+1} = \frac{1}{x_1+x_2+x_3+x_4+\dfrac{1}{x_5}}$$

$$= \cfrac{1}{x_1 + \cfrac{1}{\cfrac{1}{x_2 + \cfrac{1}{\cfrac{1}{x_3 + \cfrac{1}{\cfrac{1}{x_4 + \frac{1}{x_5}}}}}}}}$$

is a recursive function with strict increasingness on $(0,\infty)^5$, where $(0,\infty)^n = (0,\infty) \times (0,\infty) \times \cdots \times (0,\infty)$ (n factors). Then by dynamic programming analysis [1,2,3,12] MP II has a continuous and strictly decreasing maximum-value function $U(c) = (\frac{1}{5c})^5$ and a maximum-point function $(x_1^*(c), x_2^*(c), x_3^*(c), x_4^*(c), x_5^*(c)) = (\frac{1}{5c}, \frac{1}{5c}, \frac{1}{5c}, \frac{1}{5c}, 5c)$. IP II has a continuous and strictly decreasing maximum-value function $V(c) = \frac{1}{5c^{1/5}}$ and a maximum-point function $(\hat{y}_1(c), \hat{y}_2(c), \hat{y}_3(c), \hat{y}_4(c), \hat{y}_5(c)) = (c^{1/5}, c^{1/5}, c^{1/5}, c^{1/5}, c^{1/5})$. Of course, Theorem II holds true in this problem.

EXAMPLE III Let us consider the following pair of N-variable problems :

MP III $\quad \text{Min} \sum_{n=1}^{N} \frac{q_n}{x_n^{t_n}}$

s.t. (i) $\prod_{n=1}^{N} x_n^{p_n} \leqq c \ (\in (0,\infty))$

(ii) $x_n > 0 \quad 1 \leqq n \leqq N$

IP III $\quad \text{Min} \prod_{n=1}^{N} y_n^{p_n}$

s.t. (i)' $\sum_{n=1}^{N} \frac{q_n}{y_n^{t_n}} \leqq c \ (\in (0,\infty))$

(ii)' $y_n > 0 \quad 1 \leqq n \leqq N$

where p_n, q_n and t_n are positive for $1 \leqq n \leqq N$. Then usual dynamic programming analysis [1,2,3,11] yields the following solutions : MP III has a continuous and strictly decreasing minimum-value function U and a minimum-point function $(x_1^*, x_2^*, \cdots, x_N^*)$, and IP III has a continuous and strictly decreasing minimum-value function V and a minimum-point function $(\hat{y}_1, \hat{y}_2, \cdots, \hat{y}_N)$, where

$$U(c) = \prod_{k=1}^{N} \left(\frac{q_k t_k}{p_k}\right)^{\left(\frac{p_k/t_k}{\sum_1^N p_n/t_n}\right)} \cdot \left(\sum_1^N \frac{p_k}{t_k}\right) \cdot c^{-\left(\frac{1}{\sum_1^N p_k/t_k}\right)},$$

$$x_k^*(c) = \left(\frac{q_k t_k}{p_k}\right)^{1/t_k} \cdot \left\{ \prod_{k=1}^{N} \left(\frac{q_k t_k}{p_k}\right)^{\left(\frac{p_k/t_k}{\sum_1^N p_k/t_k}\right)} \right\}^{(-1/t_k)} \cdot c^{\left(\frac{1/t_k}{\sum_1^N p_n/t_n}\right)} \qquad 1 \leqq k \leqq N,$$

$$V(c) = \prod_{k=1}^{N} \left(\frac{q_k t_k}{p_k}\right)^{p_k/t_k} \cdot \left(\sum_1^N \frac{p_k}{t_k}\right)^{\left(\sum_1^N p_k/t_k\right)} \cdot c^{-\left(\sum_1^N p_k/t_k\right)},$$

$$\hat{y}_k(c) = \left(\frac{q_k t_k}{p_k}\right)^{1/t_k} \cdot \left(\sum_1^N \frac{p_k}{t_k}\right)^{1/t_k} \cdot c^{-1/t_k} \qquad 1 \leqq k \leqq N.$$

Note that

$$U \circ V = V \circ U = I, \text{ where } I(x) = x \text{ on } (0,\infty),$$

$$x_n^* \circ V = \hat{y}_n,\ \hat{y}_n \circ U = x_n^* \qquad 1 \leqq n \leqq N.$$

This completes Example III.

EXAMPLE II' Consider the following pair of main and inverse problems :

MP II' $\quad \text{Max } \frac{y}{x}$

$\quad$ s.t. (i) $\frac{1}{\frac{1}{x} + y} \geqq c \quad (\in (0,\infty))$

$\quad$ (ii) $x > 0,\ y > 0$

IP II' $\quad \text{Max } \frac{1}{\frac{1}{x} + \frac{1}{y}}$

$\quad$ s.t. (i)' $\frac{1}{xy} \geqq c \quad (\in (0,\infty))$

$\quad$ (ii)' $x > 0,\ y > 0.$

This is the case when

$$N=2,\ f_1(x;y) = \frac{1}{xy},\ f_2(y) = y,\ u_1(x) = x,\ u_2(y) = \frac{1}{y},\ g_1(x;y) = \frac{1}{\frac{1}{x} + y},$$

(5.2) $\quad g_2(y) = y,\ v_1(x) = x,\ v_2(y) = \frac{1}{y},\ \langle d_1,e_1\rangle \times \langle d_2,e_2\rangle = (0,\infty) \times (0,\infty),$

and $<\alpha,\beta> = <a,b> = (0,\infty)$.

It is easily verified that MP II' has a continuous and strictly decreasing maximum-value function $U(c) = \frac{1}{4c^2}$ and a maximum-point function $(x^*(c), y^*(c)) = (2c, \frac{1}{2c})$, and that IP II' has a continuous and strictly decreasing maximum-value function $V(c) = \frac{1}{2\sqrt{c}}$ and a maximum-point function $(\hat{x}(c), \hat{y}(c)) = (\frac{1}{\sqrt{c}}, \frac{1}{\sqrt{c}})$. Note that

(5.3) $U \circ V = V \circ U = I,\ \hat{x} = u_1 \circ x^* \circ V,\ \hat{y} = u_2 \circ y^* \circ V,\ x^* = v_1 \circ \hat{x} \circ U,\ y^* = v_2 \circ \hat{y} \circ U.$

Therefore, Theorem II' holds true.

EXAMPLE III' Let us consider the following pair :

MP III' $\text{Min } \frac{1}{x} + y$
s.t. (i) $\frac{x}{y} \leqq c \ (\in (0,\infty))$
(ii) $x > 0,\ y > 0$

IP III' $\text{Min } xy$
s.t. (i)' $\frac{1}{x} + \frac{1}{y} \leqq c \ (\in (0,\infty))$
(ii)' $x > 0,\ y > 0$

This is the case when the other elements except for $f_1(x;y) = \frac{1}{x} + \frac{1}{y}$, $g_1(x;y) = \frac{x}{y}$ are the same ones as in (5.2). MP III' has a continuous and strictly decreasing minimum-value function $U(c) = \frac{2}{\sqrt{c}}$ and a minimum-point function $(x^*(c), y^*(c)) = (\sqrt{c}, \frac{1}{\sqrt{c}})$. IP III' has a continuous and strictly decreasing minimum-value function $V(c) = \frac{4}{c^2}$ and a minimum-point function $(\hat{x}(c), \hat{y}(c)) = (\frac{2}{c}, \frac{2}{c})$. Since (5.3) holds true in this problem, Theorem III' holds true.

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