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A Class of Linear Programming Problems Requiring a Large Number of Iterations

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ABSTRACT

A coordinate-free description of the simplex algorithm (for nondegenerate linear programming problems) is supplied, and is used to show that the number of iterations can be larger than was previously known. For $0 < m < n$, there is constructed a non-degenerate linear programming problem whose bounded $(n - m)$-dimensional feasible region is defined by means of m linear equality constraints in n nonnegative variables, and in which, after starting from the worst choice of an initial feasible vertex, $m(n - m - 1) + 1$ simplex iterations are required in order to reach the optimal vertex. It is conjectured that this is the maximum possible number of iterations (for arbitrary $0 < m < n$), but the conjecture is proved only for $n < m + 4$.

INTRODUCTION

Before attacking a linear programming problem with the simplex algorithm, it can be very comforting to have a good estimate of the number of iterations which may be required in order to reach a solution. For problems of a given size, the number of iterations is nearly proportional to the actual computation time, and it may also give some indication of the extent to which round-off errors will become troublesome (depending on the computer program by means of which the algorithm is implemented). Of primary interest is the <u>expected number</u> of iterations. This can be defined mathematically in various ways, but the resulting systems appear to be too complex for mathematical analysis and in any case are of doubtful relevance to the practical situation. It seems that the only reliable guide to the expected number of iterations is computational experience such as that reported by Kuhn & Quandt [11], Wolfe & Cutler [14] or Dantzig [2] (p. 160).

In addition to the primary interest in the expected number of iterations, there is a strong secondary interest in the <u>maximum number</u> of iterations for problems of a given size. Here the analytical approach seems more relevant as well as more feasible, and the still-unsolved problem of determining the maximum number has been of central interest since the inception of the simplex algorithm. The present note contributes to this problem by supplying a coordinate-free description of the simplex algorithm which helps to clarify the underlying geometry, and by using the description to show that the number of iterations can be larger than was previously known. (Here the iterations are those in the most commonly used version of the simplex algorithm [1,2], where each pivot operation maximizes the gradient in the space of nonbasic variables; that is, the pivot column is chosen so that introduction of the associated variable into the basis permits the greatest possible improvement in the objective function per unit level of the variable.) For $0 < m < n$ there is constructed a nondegenerate linear programming program whose bounded $(n - m)$-dimensional feasible region is defined by means of m linear equality constraints in n nonnegative variables, and in which, after starting from the "worst" choice of an initial feasible vertex, $m(n - m - 1) + 1$ simplex iterations are required in order to reach the optimal vertex. It is conjectured that this is the maximum possible number of iterations, for arbitrary $0 < m < n$, but the conjecture is proved only for $n < m + 4$.

I. A COORDINATE-FREE DESCRIPTION OF THE SIMPLEX ALGORITHM

The linear programming problems which are described in the next section are not given in an explicit numerical form. Instead, their existence is established by means of a geometrical construction based on mathematical induction. For this purpose, it is convenient to have a coordinate-free description of the simplex algorithm. In the interest of simplicity, attention will be confined to nondegenerate problems.

Let X be an affine set (or flat) in a real vector space; that is, $\lambda x + (1 - \lambda)x' \in X$ for all $x \in X$, $x' \in X$ and $\lambda \in \underset{\rightarrow}{R}$ (the real number field). An affine form on X is a real-valued function ξ such that $\xi(\lambda x + (1 - \lambda)x') = \lambda\xi(x) + (1 - \lambda)\xi(x')$ whenever $x \in X$, $x' \in X$ and $\lambda \in \underset{\rightarrow}{R}$; if ξ is not constant on X, then the sets $H(\xi) = \{x \in X : \xi(x) = 0\}$ and $J(\xi) = \{x \in X : \xi(x) \geq 0\}$ are respectively the hyperplane and the halfspace associated with ξ.

A d-polyhedron in X is a d-dimensional set P which is the intersection of a finite number of halfspaces. The faces of P are intersections of P with its various supporting hyperplanes and thus are sets of the form $P \cap H(\xi)$ where $P \subset J(\xi)$. The 0-faces, bounded 1-faces, and $(d - 1)$-faces of P are called respectively its vertices, edges, and facets, and two vertices are adjacent provided they are joined by an edge. A d-polyhedron P is called proper provided it has at least one vertex, and a proper d-polyhedron is simple provided each vertex is on exactly d 1-faces or, equivalently, on exactly d facets.

In order that our terminology may suggest the appropriate correspondents in the usual treatment of linear programming, an affine form ξ on X will be called a variable provided it is not constant on P.

Now suppose that P is a simple d-polyhedron in X and φ_o is a variable. We are concerned with the problem of minimizing φ_o over P. Let $\underset{\sim}{F}$ denote the set of all facets of P, and for each $F \in \underset{\sim}{F}$ let φ_F be a variable such that $F \subset H(\varphi_F)$ and $P \subset J(\varphi_F)$. Let $\Phi = \{\varphi_F : F \in \underset{\sim}{F}\}$, whence of course P is the intersection of the set $\bigcap_{\varphi \in \Phi} J(\varphi)$ with the affine hull aff P of P. Henceforth, our discussion will be relative to the system (P,Φ,φ_o). For an arbitrary choice of numbers $\alpha_F > 0$, the systems $(P,\{\alpha_F\varphi_F : F \in \underset{\sim}{F}\},\varphi_o)$ and (P,Φ,φ_o) correspond to the same linear programming problem, but the simplex algorithm is materially affected by (nonuniform) rescaling of its variables and would not in general require the same number of iterations for the two systems.

If v is a vertex of the simple d-polyhedron P, a variable $\varphi \in \Phi$ will be called <u>basic</u> or <u>nonbasic</u> for v according as $\varphi(v) > 0$ or $\varphi(v) = 0$. The set of all nonbasic variables will be denoted by Φ_v, so that $\varphi \in \Phi_v$ if and only if $v \in H(\varphi)$, or equivalently, $\varphi = \varphi_F$ for some facet $F \ni v$. Since P is simple, each set Φ_v is of cardinality d, and two vertices v and v' are adjacent if and only if there is exactly one variable $\varphi^{v,v'}$ in $\Phi_v \sim \Phi_{v'}$. The <u>nonbasic</u> (v,v') <u>gradient</u> of φ_o is then defined as the quotient

$$\frac{\varphi_o(v') - \varphi_o(v)}{|\varphi^{v,v'}(v') - \varphi^{v,v'}(v)|} .$$

let $C(h,i,j,k)$ denote the assertion that $\zeta(h,i,j,k) < \alpha_j$. For $2 \leq m \leq d$, let A_m denote the assertion that $\gamma_1 > \alpha_1\gamma_1 > \alpha_2\gamma_2 > \cdots > \alpha_m\gamma_m$, B_m the assertion that $\alpha_2 < \cdots < \alpha_m$, and C_m the assertion that $C(h,i,j,k)$ holds whenever $1 \leq h < i < j \leq k \leq d$ and $j \leq m$. We want to choose the α_i's so that A_m, B_m and C_m are true for all m, and this will be accomplished by induction on m.

Choose α_2 so that $\alpha_1\gamma_1 > \alpha_2\gamma_2 > 0$. Then A_2 holds, and B_2 and C_2 are vacuously satisfied. In order to satisfy A_3, B_3 and C_3 the number α_3 must be chosen subject to the following restrictions:

$$\alpha_2 < \alpha_3 < \frac{\alpha_2\gamma_2}{\gamma_3}$$

$$\frac{\alpha_1\alpha_2\gamma_1\gamma_k}{\gamma_3(\alpha_1\gamma_1 - \alpha_2\gamma_2 + \alpha_2\gamma_k)} < \alpha_3 < \frac{\alpha_2\gamma_2}{\gamma_3} \quad (3 \leq k \leq d).$$

But $\alpha_2 < \alpha_2\gamma_2/\gamma_3$ because $\gamma_2 > \gamma_3$, so the desired choice of α_3 can be made if and only if

$$\frac{\alpha_1\gamma_1\gamma_k}{\alpha_1\gamma_1 - \alpha_2\gamma_2 + \alpha_2\gamma_k} < \gamma_2 \qquad (3 \leq k \leq d).$$

This is evidently equivalent to the condition that

$$\alpha_2\gamma_2(\gamma_2 - \gamma_k) < \alpha_1\gamma_1(\gamma_2 - \gamma_k),$$

an inequality which follows from A_2 in conjunction with the fact that $\gamma_2 > \gamma_k$.

For the general inductive step, we suppose that $4 \leq m \leq d$ and that the numbers $\alpha_1, \cdots, \alpha_{m-1}$ have been chosen so that conditions A_{m-1}, B_{m-1} and C_{m-1} are all satisfied. We want to choose α_m so that A_m, B_m and C_m are all satisfied, and as in the case of A_3 the choice is possible if and only if

$$(2) \qquad \zeta(h,i,m,k) < \frac{\alpha_{m-1}\gamma_{m-1}}{\gamma_m} \quad \text{for} \quad 1 \leq h < i < m \leq k \leq d.$$

If $m = i + 1$, the inequality (2) reduces to the assertion that

$$\alpha_i\gamma_i(\gamma_i - \gamma_k) < \alpha_h\gamma_h(\gamma_i - \gamma_k),$$

which is a consequence of A_i and the fact that $\gamma_i > \gamma_k$. If $m > i + 1$, (2) is equivalent to the inequality

$$\frac{\alpha_h\alpha_i\gamma_h\gamma_k}{\gamma_{m-1}(\alpha_h\gamma_h - \alpha_i\gamma_i + \alpha_i\gamma_k)} < \alpha_{m-1},$$

which is merely $C(h,i,m-1,k)$ and hence is a consequence of C_{m-1}.

This completes the proof of the lemma. (Actually, condition (B) of the lemma will not be used in the sequel.)

A bounded polyhedron will be called a <u>polytope</u>.

THEOREM 1 <u>For</u> $n > d > 1$, <u>there exists a simple</u> d-<u>polytope</u> P <u>having exactly</u> n <u>facets, a variable</u> φ_0, <u>and a set</u> Φ <u>of</u>

variables corresponding to the facets of P *such that the system* (P,Φ,φ_o) *admits an unambiguous simplex* φ_o*-path of length* $(n - d)(d - 1) + 1$ *running through all the vertices of* P.

Proof. For an arbitrary fixed d, the proof is by induction on n. For the case in which $n = d + 1$, let the numbers γ_i be such that $\gamma_1 > \gamma_2 > \cdots > \gamma_{d+1} > 0$ and let the α_i's be as in the lemma. Let P be the d-simplex in $\underset{\to}{R}^{d+1}$ whose vertices are $\alpha_1\delta_1, \alpha_2\delta_2,\cdots,\alpha_{d+1}\delta_{d+1}$, where the points δ_i are the Kronecker deltas, and for each point $x = (x^1,x^2,\cdots,x^{d+1}) \in \underset{\to}{R}^{d+1}$ let $\varphi_o(x) = \Sigma_{i=1}^{d+1} \gamma_i x^i$. By condition (A) of the lemma, the sequence $(\alpha_1\delta_1,\alpha_2\delta_2,\cdots,\alpha_{d+1}\delta_{d+1})$ is a φ_o-path in P. If F_i is the facet of P determined by all of P's vertices other than $\alpha_i\delta_i$, let the corresponding variable $\varphi_i (= \varphi_{F_i}) \in \Phi$ be the i^{th} coordinate function. Then the nonbasic $(\alpha_h\delta_h,\alpha_j\delta_j)$ gradient of φ_o is equal to $(\alpha_h\gamma_h - \alpha_j\gamma_j)/\alpha_j$, and the path in question is an unambiguous simplex φ_o-path provided

$$\frac{\alpha_h\gamma_h - \alpha_{h+1}\gamma_{h+1}}{\alpha_{h+1}} > \frac{\alpha_h\gamma_h - \alpha_j\gamma_j}{\alpha_j} \quad \text{for } 1 \leq h < h + 1 < j \leq d + 1.$$

But this inequality is equivalent to

$$\frac{\alpha_h\alpha_{h+1}\gamma_h}{\alpha_h\gamma_h - \alpha_{h+1}\gamma_{h+1} + \alpha_{h+1}\gamma_j} < \alpha_j,$$

which in turn is equivalent to the inequality $C(h,h + 1, j, j)$ of the lemma. (Note that $d + 1$ here corresponds to d in the lemma.) This completes the proof for the case in which $n = d + 1$.

Suppose the theorem is known for $n = r \geq d + 1$, and let (P, Φ, φ_o) be the corresponding system. Let $(v_o, v_1, \cdots, v_\ell)$ be an unambiguous simplex φ_o-path running through all the vertices of P, where $\ell = (r - d)(d - 1) + 1$, The P-minimum of φ_o is attained only at the vertex v_ℓ. We may assume without loss of generality that the d-polytope P lies in $\underset{\rightarrow}{R}^d$, with the vertex v_ℓ at the origin. Since P is simple, there are exactly d variables in Φ which are nonbasic for v_ℓ, and there is a nonsingular linear transformation t of $\underset{\rightarrow}{R}^d$ onto $\underset{\rightarrow}{R}^d$ such that the variables φt (for $\varphi \in \Phi$) are exactly the d coordinate functions on $\underset{\rightarrow}{R}^d$. The system $(tP, \{\varphi t : \varphi \in \Phi\}, \varphi_o t)$ will then have the properties required of the system (P, Φ, φ_o). Thus we may assume without loss of generality that t is the identity transformation, and the d vertices of P which are adjacent to v_ℓ must then lie along the positive coordinate axes. We may assume that the P-minimum of φ_o is 0, whence φ_o is a linear form on $\underset{\rightarrow}{R}^d$ and there are positive numbers γ_i such that $\varphi_o(x) = \Sigma_{i=1}^d \gamma_i x_i$ for all $x \in \underset{\rightarrow}{R}^d$. The relevant aspects of the situation are clearly unchanged by a sufficiently small perturbation of φ_o, so we may assume that the d numbers γ_i are all distinct. By a uniform contraction or dilation together with a suitable permutation of the coordinates, we may assume that $v_{\ell-1} = \delta_1$ and that $\gamma_2 > \gamma_3 > \cdots > \gamma_d$.

Now let the positive numbers $\alpha_1,\cdots,\alpha_d$ be as in the lemma, and note that for each $\lambda \in]0,1[$ the conditions of the lemma are also satisfied by the sequence $\lambda\alpha_1,\cdots,\lambda\alpha_d$. Let $v_j(\lambda) = v_j$ for $0 \leq j < \ell$ and let $v_j(\lambda) = \lambda\alpha_{j-\ell+1}$ for $\ell \leq j \leq \ell + d - 1$. Let P_λ denote the convex hull of the set $\{v_j(\lambda) : 0 \leq j \leq \ell + d - 1\}$. Then it can be verified that P_λ is a simple d-polytope having the points $v_j(\lambda)$ as its vertices. Each facet of P is contained in a facet of P_λ, with the sole exception of the facet $F_\lambda =$ $\text{con}\{\lambda\alpha_i\delta_i : 1 \leq i \leq d\}$ of P_λ. Thus P_λ has exactly $r + 1$ facets. Let Φ_λ be obtained from Φ by the addition of a variable φ_{F_λ} corresponding to F_λ. Then the sequence $(v_0(\lambda),v_1(\lambda),\cdots,v_{\ell+d-1}(\lambda))$ is a path of length $(r + 1 - d)(d - 1) + 1$ which runs through all the vertices of P_λ. Indeed, it is a φ_0-path, for $(v_0(\lambda),v_1(\lambda),\cdots,v_{\ell-1}(\lambda))$ is identical with the φ_0-path $(v_0,v_1,\cdots,v_{\ell-1})$ and $(v_{\ell-1}(\lambda),\cdots,v_{\ell+d-1}(\lambda))$ is a φ_0-path by condition (A) of the lemma, because $\varphi_0(v_{\ell-1}(\lambda)) = \gamma_1$ and $\varphi_0(v_j(\lambda)) = \lambda\alpha_j\gamma_j$ for $\ell \leq j \leq \ell + d - 1$. To complete the proof of the theorem, it suffices to show that for λ sufficiently small (in $]0,1[$) the path $(v_0(\lambda),v_1(\lambda),\cdots,v_{\ell+d-1}(\lambda))$ is an unambiguous simplex φ_0-path relative to the system $(P_\lambda,\Phi_\lambda,\varphi_0)$.

If $0 \leq h < \ell \leq j \leq \ell + d - 1$ and the vertices $v_h(\lambda)$ and $v_j(\lambda)$ are adjacent in P_λ, then v_j and $v_\ell(= 0)$ are adjacent in P. Further, the variable in Φ_λ which is nonbasic for $v_h(\lambda)$ but basic for $v_j(\lambda)$ is identical with the variable in Φ which is nonbasic for v_h but basic for v_ℓ. Thus for λ very small, the nonbasic $(v_h(\lambda), v_j(\lambda))$ gradient of φ_o is very close to the nonbasic (v_h, v_j) gradient of φ_o. The desired conclusion then follows from this fact in conjunction with the facts that $(v_o, v_1, \cdots, v_\ell)$ is an unambiguous simplex φ_o-path in P, that $v_\ell(\lambda)$ is the only vertex of P_λ which is adjacent to $v_{\ell-1}$ and gives a smaller value to φ_o, and that $(v_\ell(\lambda), \cdots, v_{\ell+d-1}(\lambda))$ is (by the first paragraph of the proof of the theorem) an unambiguous simplex φ_o-path relative to the system $(P_\lambda, \Phi_\lambda, \varphi_o)$.

III. REPRESENTATION IN STANDARD FORM

Our main result will be stated in a somewhat redundant form in order to exhibit explicitly its various features of possible interest.

THEOREM 2 *For* $0 < m < n$, *there exists a linear programming problem in standard form such that the following conditions are all satisfied:*

(a) *the feasible region is defined by means of* m *linear equality constraints in* n *nonnegative variables;*

(b) *the feasible region is a simple* $(n - m)$-*polytope which has* n *facets and* $m(n - m - 1) + 2$ *vertices;*

(c) *the objective function does not assume equal values at any two distinct vertices of the feasible region;*

(d) *starting from the feasible vertex which maximizes the*

<u>objective function,</u> $m(n - m - 1) + 1$ <u>simplex iterations are required in order to reach the minimizing vertex;</u>

(e) <u>for a random choice of the initial feasible vertex,</u> $\frac{1}{2}(m(n - m - 1) + 1)$ <u>is the expected number of simplex iterations required to reach the minimizing vertex.</u>

Proof. Let $d = n - m$ and let the system (P,Φ,φ_0) be as in the preceding theorem. Let $\underset{\to}{O}^n$ be the positive orthant in $\underset{\to}{R}^n$ and let G be the affine hull of P. By a theorem of Davis [3] (also 4.2 of [6]), there exists a nonsingular affine transformation t of G into $\underset{\to}{R}^n$ such that $tP = (tG) \cap \underset{\to}{O}^n$. The d-flat tG does not pass through the origin, because P is bounded and consists of more than one point. Since tP has exactly n facets, its facets are precisely the sets of the form $(tP) \cap H_j$, where H_j is the set of all points $x = (x^1,x^2,\ldots,x^n) \in \underset{\to}{R}^n$ such that $x^j = 0$. If φ_j is the variable in Φ corresponding to the facet $t^{-1}((tP) \cap H_j)$, then the variable $\varphi_j t^{-1}$ on tG must be a positive multiple of the j^{th} coordinate function—say $\varphi_j(t^{-1}(x)) = \mu_j x_j$ for all $x \in tG$. Let the transformation s of $\underset{\to}{R}^n$ onto $\underset{\to}{R}^n$ be defined by the condition that $x(x^1,\ldots,x^n) = (x^1/\mu_1,\ldots,x^n/\mu_n)$. Finally, let $Q = stP$, let Ψ be the set of all coordinate functions on $\underset{\to}{R}^n$, and let z be a linear form on $\underset{\to}{R}^n$ whose restriction to stG (the smallest flat containing Q) is equal to the function $\varphi_0 t^{-1} s^{-1}$. Then the system (Q,Ψ,z) has all of the relevant properties of the system (P,Φ,φ_0). Since the flat aff stP is of dimension $d = n - m$ in $\underset{\rightarrow}{R}^n$, it can be determined by means of m linear equality constraints--equations of the form $\eta_i(x) = \beta_i$, where η_i is a nonconstant linear form on $\underset{\rightarrow}{R}^n$ and β_i is a real number $\neq 0$ $(1 \leq i \leq m)$.

The desired linear programming problem is that of minimizing the linear form $z(x)$ subject to the m linear equality constraints $\eta_i(x) = \beta_i$ and the n nonnegativity constraints $x^j \geq 0$. Assertions (a), (b) and (c) are obviously true, and (e) follows from (d) in conjunction with the fact that all of the vertices of the feasible region lie on the simplex path described in (e). In order to justify (d), it suffices to compare the previous section's description of simplex φ_0-paths with any of the usual descriptions of the simplex algorithm. The descriptions in Dantzig's Chapter 7 [2] and in Kuhn & Quandt [11] are especially good for this purpose, as they emphasize the geometry of the situation.

The standard form of a linear programming problem, involving linear equality constraints, is the one to which the simplex algorithm is applied. However, the form most directly related to the underlying practical situation usually involves linear <u>inequality</u> constraints which are then replaced by equality constraints (with the introduction of a slack variable for each inequality) in order to convert the problem to standard form. In this connection, we note the following.

COROLLARY <u>For</u> $k > 1 \leq m$, <u>there exists a linear programming</u>

problem in inequality form such that the following conditions are satisfied:

(i) *the feasible region is defined by means of* m *linear inequality constraints in* k *nonnegative variables;*

(ii) *if the problem is converted to standard form by adding a slack variable for each of the* m *inequality constraints, then with* $n = m + k$ *the resulting problem in standard form satisfies all the conditions* (a) - (e) *of the preceding theorem; in particular, for the worst choice of an initial feasible vertex,* $m(k - 1) + 1$ *simplex iterations will be required to reach the optimal vertex.*

Proof. Let $n = m + k$ and consider the problem in standard form described in the preceding theorem. Let v_o be the feasible vertex which maximizes the objective function. Then exactly k of the coordinate functions are nonbasic for v_o, and we may assume that they are the variables x^j for $m + 1 \leq j \leq n$. For $1 \leq j \leq m$, let β_j denote the j^{th} coordinate of v_o; that is, $\beta_j = v_o^j > 0$. The problem can be transformed into feasible canonical form ([2], p. 94) with respect to v_o, and in this form it will ask for values of $x^1 \geq 0$, $x^2 \geq 0, \cdots, x^n \geq 0$, and minimum z such that

$$
\begin{array}{lllll}
x^1 & & + \alpha_{1,m+1}x^{m+1} + \cdots + \alpha_{1,n}x^n & = & \beta_1 \\
& x^2 & + \alpha_{2,m+1}x^{m+1} + \cdots + \alpha_{2,n}x^n & = & \beta_2 \\
& \ddots & \vdots \qquad\qquad\qquad \vdots & & \\
& x^m & + \alpha_{m,m+1}x^{m+1} + \cdots + \alpha_{m,n}x^n & = & \beta_n \\
& (-z) & + \gamma_{m+1}x^{m+1} + \cdots + \gamma_n x^n & = & -z_o,
\end{array}
$$

where the numbers z_o, $\alpha_{i,j}$ and γ_j are constants.

Now consider the problem of finding $x^{m+1} \geq 0, \ldots, x^n \geq 0$ and minimum z such that

$$
\begin{aligned}
\alpha_{1,m+1}x^{m+1} + \cdots + \alpha_{1,n}x^n &\leq \beta_1 \\
\vdots \qquad\qquad\qquad & \quad \vdots \\
\alpha_{m,m+1}x^{m+1} + \cdots + \alpha_{m,n}x^n &\leq \beta_n \\
\gamma_{m+1}x^{m+1} + \cdots + \gamma_n x^n &= z - z_o.
\end{aligned}
$$

The two problems are clearly equivalent, and the former results from the latter under the usual conversion to standard form by the addition of slack variables. The desired conclusion follows.

IV. COMMENTS

Saaty [13] conjectured that in applying the simplex algorithm to nondegenerate linear programming problems involving m linear inequality constraints in k nonnegative variables, the number of iterations required to reach an optimal vertex is at most $\frac{2}{m}\binom{m+k}{2}$. This is contradicted by the corollary, with its number $m(k - 1) + 1$. In particular, when $m = k$ Saaty's number and that of the corollary are respectively $4k - 2$ and $k^2 - k + 1$. When $k = 3$, they are $m^2 - 3m - 6$ and $2m^2 + 2m$. Thus there is a nondegenerate problem whose feasible region is defined by means of 5 linear inequalities in 3 nonnegative variables, in which Saaty's conjectured bound is exceeded. (Quandt & Kuhn [12] mention some 25×25 problems in which the bound is exceeded when a very inefficient pivot rule (not the standard one) is employed.)

* * * * * * * * * * *

A result similar to Theorem 1 appears in [7], but there the paths maximize the gradient in the containing normed space and thus are not based on the usual pivot rule. (See also Goldman & Kleinman [4].)

* * * * * * * * * * *

Now let us consider the number $(n - d)(d - 1) + 1$ of Theorem 1. I conjecture that this is the maximum length of simplex φ_o-paths for a system (P,Φ,φ_o) in which P is a simple d-polytope having n facets. More than that, I conjecture that it is the maximum length of φ_o-paths in d-polytopes having n facets. However, the conjecture has been proved only for $d \leq 3$ [7] and hence the corresponding conjecture for Theorem 2 is established only for $0 < m < n < m + 4$. If the conjecture is correct, then <u>at its worst, the usual simplex algorithm behaves at least as badly as any of its variants</u>. That is, the maximum number of iterations which may be required for problems of a given size is at least as much for the usual pivot rule as it is for any algorithm which improves the value of the objective function at each stage.

* * * * * * * * * * *

In closing, we repeat from [8] a comparison which is of interest for linear programming and in connection with the above conjecture. For $1 \leq i \leq 5$, let $L_i(d,n)$ denote the smallest number ℓ such that the statement (i) below is true whenever P is a d-polytope having n facets and φ_o is a nonconstant affine form on P.

(1) any two vertices of P can be joined by a path of length $\leq \ell$;

(2) any vertex of P can be joined to a φ_o-minimizing vertex of P by a φ_o-path of length $\leq \ell$;

(3) if a φ_o-path $(v_o, \cdots, v_k)$ in P maximizes the improvement of φ_o at each stage (that is, if $\varphi_o(v_{i-1}) \leq \varphi_o(w)$ for each vertex w adjacent to v_{i-1}), then its length is $\leq \ell$;

(4) every φ_0-path in P is of length $\leq \ell$;

(5) every simple path in P is of length $\leq \ell$.

(A simple path is one in which no vertex is repeated.)

It is conjectured that the values of the numbers $L_i(d,n)$ are as follows, where $[r]$ denotes the greatest integer $\leq r$.

(1) $\left[\frac{d-1}{d} n\right] - d + 2$;

(2) $n - d$;

(3) $n - 2$ if $d = 2$; $\left[\frac{3n-1}{2}\right] - 4$ if $d = 3$; $2(n-d) - 1$ if $d \geq 4$;

(4) $(d - 1)(n - d) + 1$;

(5) $\binom{n - \left[\frac{d+1}{2}\right]}{n - d} + \binom{n - \left[\frac{d+1}{2}\right]}{n - d} - 1$.

Each of these conjectures has been established for $d \leq 3$; in addition, (1) is known when $n \leq d + 4$ and (5) when $d \leq 6$ or $n \leq d + 3$ or $n \geq (d/2)^2 - 1$. In each case, the number given is known to be a lower bound for $L_i(d,n)$ and to be attained for a simple d-polytope having n facets. (This statement requires a slight modification for (1).) The evidence supporting these conjectures is of varying strength; it seems rather weak in the case of (3) with $d \geq 4$, although this is of great interest for linear programming. For details and additional references, see [5] for (1), [10] for (2), [7] for (3) and (4), and [9] for (5).

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