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A CLASS OF MULTISAMPLE DISTRIBUTION-FREE TESTS[1]

BY JAYANT V. DESHPANDÉ

Case Western Reserve University and the University of Sheffield

1. Summary and introduction. Let $x_{i1}, x_{i2}, \cdots, x_{in_i}$ be a random sample of real observations from the ith population with cumulative distribution function (cdf) $F_i(x)$, $i = 1, 2, \cdots, c$. Let the c samples be independent and the F's continuous. In this paper we shall consider tests for the null hypothesis

$$H_0 : F_1(x) = F_2(x) = \cdots = F_c(x) = F(x), \quad \text{say.}$$

The statistics and tests, proposed in this paper, are based upon c-plets of observations which are formed by selecting one observation from each of the c samples. The total number of distinct c-plets that can be formed in this way is $\prod_{i=1}^{c} n_i$. In each c-plet we compare and rank observations appearing therein. Let v_{ij} be the number of c-plets in which the observation selected from the ith sample is larger than exactly $(j-1)$ observations and smaller than the other $(c-j)$ observations. Since the distributions are assumed to be continuous the probability of the existence of ties is zero. Let us define $u_{ij} = v_{ij}/\prod_{i=1}^{c} n_i$; it is the proportion of c-plets which give rank j to the observation from the ith sample.

Let us have $N = \sum_{i=1}^{c} n_i$, $p_i = n_i/N$, $L_i = \sum_{j=1}^{c} a_j u_{ij}$, where the a's are real constants such that they are not all equal and

$$A = \sum_{j=1}^{c} \sum_{l=1}^{c} a_j a_l \left\{ \frac{\binom{c-1}{j-1}\binom{c-1}{l-1}}{(2c-1)\binom{2c-2}{j+l-2}} - \frac{1}{c^2} \right\}. \tag{1.1}$$

Then we define a class of statistics $\mathscr{L}$ as

$$\mathscr{L} = \frac{N(c-1)^2}{Ac^2} \left[\sum_{i=1}^{c} p_i L_i^2 - \left(\sum_{i=1}^{c} p_i L_i \right)^2 \right]. \tag{1.2}$$

A particular member of the class is found by specifying the real constants a's.

With each member of this class we associate a test of H_0: Reject H_0 at a significance level α if $\mathscr{L}$ exceeds some predetermined constant $\mathscr{L}_\alpha$. We, later in this paper, show that under H_0, $\mathscr{L}$ is distributed as a χ^2 variate with $c-1$ degrees of freedom, in the limit as $N \to \infty$. Hence for sufficiently large N, $\mathscr{L}_\alpha$ may be approximated by the corresponding significance point of the χ^2 distribution with requisite degrees of freedom.

Tests proposed by Bhapkar [2], [3], Sugiura [13], and the author [5], [6] may be seen to belong to this class. In this paper it is attempted to provide a unified treatment of statistics and tests based on c-plets—particularly those based on linear combinations of the u's. The detailed properties of statistics belonging to this class

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are discussed under the null hypothesis and the following two alternative hypotheses. (I) the alternative of different locations or shift, the distributions being equal in all other respects and, (II) the alternative of different scales, the distributions again being equal in all other respects. Haller [7] has discussed the use and the properties of some statistics belonging to this class for testing H_0 against an alternative of stochastically ordered variables and for selection and ranking procedures.

In the fourth section we give a condition on the distributions under which these tests are consistent against specified alternatives. In the fifth section $\mathscr{L}$ is shown to have a limiting noncentral χ^2 distribution with $c-1$ degrees of freedom under the pertinent alternative hypotheses. The noncentrality parameter is seen to be a quadratic form in the constants a's, involving F.

The earlier test statistics, mentioned above, were constructed taking into account the relative magnitudes of the u's under the null and under the alternative hypotheses. The idea was to emphasize the difference between the two magnitudes. This "difference" is, in some sense, maximized if we are able to obtain the statistics, from the class, which has the largest noncentrality parameter under the alternative hypothesis of interest. This statistic would then be recommended to test H_0 whenever the particular alternative is suspected as likely. Also, for this particular alternative hypothesis, this test shall have maximum asymptotic relative efficiency (in the Pitman sense) among the class of statistics proposed.

In the sixth section we show how to obtain the statistics with the above property and do so for certain specified alternatives. In the same section we compute the ARE of these tests with respect to certain of their competitors.

2. Distribution of u_{ij} under H_0. In this section we prove

THEOREM 2.1. *Let X_{ij} be independent random variables with continuous* cdf $F_i(x)$, $i = 1, 2, \cdots, c; j = 1, 2, \cdots, n_i$. *Then if* $F_1(x) = \cdots = F_c(x) = F(x)$, *the distribution of* $w_{ij} = N^{\frac{1}{2}}(u_{ij} - 1/c)$, *in the limit as* $N \to \infty$ *in such a way as* p_i $(i = 1, 2, \cdots, c)$ *remain fixed, is normal with mean zero and the elements of the covariance matrix* $\sum$ *given by* $\sigma_{ij,kl} = \operatorname{Cov}(w_{ij}, w_{kl})$ *where*

$$\sigma_{ij,il} = \frac{1}{(c-1)^2}\left[\frac{\binom{c-1}{j-1}\binom{c-1}{l-1}}{\binom{2c-2}{j+l-2}(2c-1)} - \frac{1}{c^2}\right]\left[\frac{(c-1)^2}{p_i} + \sum_{r \neq i} \frac{1}{p_r}\right] \tag{2.1}$$

and

$$\sigma_{\substack{ij,kl \\ i \neq k}} = \frac{1}{(c-1)^2}\left[\frac{\binom{c-1}{j-1}\binom{c-1}{l-1}}{\binom{2c-2}{j+l-2}(2c-1)} - \frac{1}{c^2}\right]\left[\sum_{r=1}^{c} \frac{1}{p_r} - \frac{c}{p_i} - \frac{c}{p_k}\right].$$

u_{ij}, N and p_i being the same as defined in Section 1.

PROOF. Let us define

$$\varphi_{ij}(x_{1t_1}, x_{2t_2}, \cdots, x_{ct_c}) = 1 \quad \text{if } x_{it_i} \text{ is larger than exactly } (j-1) \text{ other } x\text{'s},$$
$$= 0 \quad \text{otherwise.}$$

Then it is seen that

$$u_{ij} = (\prod n_i)^{-1} \sum_{t_1=1}^{n_1} \sum_{t_2=1}^{n_2} \cdots \sum_{t_c=1}^{n_c} \varphi_{ij}(x_{1t_1}, x_{2t_2}, \cdots, x_{ct_c}). \tag{2.2}$$

Obviously u_{ij} are U-statistics generalized to the c-sample case (see [9], [14]). They, therefore, under H_0 have asymptotically, as $N \to \infty$ in such a way that p_i remain constant, a multivariate normal distribution with

$$\mathscr{E}(u_{ij}) = \mathscr{E}(\varphi_{ij}) = \eta_{ij} \quad \text{say, and}$$

$$\lim_{N\to\infty} N \operatorname{Cov}(u_{ij}, u_{kl}) = \sum_{r=1}^{c} p_r^{-1} \xi_{ij,kl}^{(r)} \quad \text{where} \tag{2.3}$$

$$\xi_{ij,kl}^{(r)} = \mathscr{E}[\varphi_{ij}(X_1, X_2, \cdots, X_r, \cdots, X_c) \times \varphi_{kl}(X_1', X_2', \cdots, X_r, \cdots, X_c')] - \eta_{ij}\eta_{kl}.$$

We evaluate these quantities.

$$\begin{aligned}\eta_{ij} &= \mathscr{E}[\varphi_{ij}(X_{1t_1}, X_{2t_2}, \cdots, X_{ct_c})]\\ &= \Pr[X_{it_i} \text{ is larger than } (j-1)\ X\text{'s and smaller than } (c-j)\ X\text{'s}]\\ &= c^{-1}.\end{aligned}$$

$$\begin{aligned}\xi_{ij,il}^{(i)} &= \mathscr{E}[\varphi_{ij}(X_1, \cdots, X_i, \cdots, X_c) \times \varphi_{il}(X_1', \cdots, X_i, \cdots, X_c')] - \eta_{ij}\eta_{il}\\ &= \Pr[X_i \text{ is larger than } (j-1)\ X\text{'s and } (l-1)\ X''\text{s and smaller than } (c-j)\ X\text{'s and } (c-l)\ X''\text{s}] - c^{-2}\\ &= \frac{\binom{c-1}{j-1}\binom{c-1}{l-1}}{\binom{2c-2}{j+l-2}(2c-1)} - \frac{1}{c^2}.\end{aligned} \tag{2.4}$$

$$\begin{aligned}\xi_{ij,kl,r\neq i,k}^{(r)} &= \mathscr{E}[\varphi_{ij}(X_1, \cdots, X_r, \cdots, X_c) \times \varphi_{kl}(X_1', \cdots, X_r, \cdots, X_c')] - c^{-2}\\ &= \frac{1}{(c-1)^2}\left[\frac{\binom{c-1}{j-1}\binom{c-1}{l-1}}{\binom{2c-2}{j+l-2}(2c-1)} - \frac{2}{c} + 1\right] - \frac{1}{c^2}.\end{aligned} \tag{2.5}$$

And lastly

$$\begin{aligned}\xi_{ij,kl,i\neq k}^{(i)} &= \mathscr{E}[\varphi_{ij}(X_1, \cdots, X_i, \cdots, X_c) \times \varphi_{kl}(X_1', \cdots, X_i, \cdots, X_c')] - c^{-2}\\ &= \frac{1}{(c-1)}\left[\frac{1}{c} - \frac{\binom{c-1}{j-1}\binom{c-1}{l-1}}{\binom{2c-2}{j+l-2}(2c-1)}\right] - \frac{1}{c^2}.\end{aligned} \tag{2.6}$$

It is seen that $\xi_{ij,kl}^{(i)} = \xi_{ij,kl}^{(k)}$.

Using the above results the expressions in (2.1) are easily obtained. Hence we conclude that w_{ij} is, in the limit, distributed normally with zero means and the elements of covariance matrix given by (2.1).

3. The statistic $\mathscr{L}$ and the class of tests. Let us consider linear forms L_i of u_{ij} for $i = 1, 2, \cdots, c$. We define

$$L_i = a_1 u_{i1} + a_2 u_{i2} + \cdots + a_c u_{ic}. \tag{3.1}$$

It is assumed that a_j are not all equal and are real constants. Then we have

$$\mathscr{E}(L_i) = \sum_{j=1}^{c} a_j \mathscr{E}(u_{ij}) = c^{-1} \sum_{j=1}^{c} a_j = \bar{a} \quad \text{say}. \tag{3.2}$$

Let $\lambda_{ik} = \lim_{N\to\infty} N \operatorname{Cov}(L_i, L_k)$ for $i, k = 1, 2, \cdots, c$. Then

(3.3) $$\lambda_{ii} = \sum_{j=1}^{c} \sum_{l=1}^{c} a_j a_l \sigma_{ij,il}.$$
$$= \frac{1}{(c-1)^2}\left[\frac{(c-1)^2}{p_i} + \sum_{r\neq i} \frac{1}{p_r}\right]\left[\sum_{j=1}^{c}\sum_{l=1}^{c} a_j a_l \left\{\frac{\binom{c-1}{j-1}\binom{c-1}{l-1}}{\binom{2c-2}{j+l-2}(2c-1)} - \frac{1}{c^2}\right\}\right].$$

And

(3.4) $$\lambda_{ik, i\neq k} = \sum_{j=1}^{c}\sum_{l=1}^{c} a_j a_l \sigma_{ij,kl}$$
$$= \frac{1}{(c-1)^2}\left[\sum_{r=1}^{c} \frac{1}{p_r} - \frac{c}{p_i} - \frac{c}{p_k}\right]\left[\sum_{j=1}^{c}\sum_{l=1}^{c} a_j a_l \left\{\frac{\binom{c-1}{j-1}\binom{c-1}{l-1}}{\binom{2c-2}{j+l-2}(2c-1)} - \frac{1}{c^2}\right\}\right].$$

(3.3) and (3.4) can be rewritten using the definition of A given in (1.1) as

(3.5) $$\lambda_{ii} = \frac{A}{(c-1)^2}\left[\frac{(c-1)^2}{p_i} + \sum_{r\neq i}\frac{1}{p_r}\right]\text{ and}$$
$$\lambda_{ik} = \frac{A}{(c-1)^2}\left[\sum_{r=1}^{c}\frac{1}{p_r} - \frac{c}{p_i} - \frac{c}{p_k}\right].$$

Hence we conclude that $N^{\frac{1}{2}}(\mathbf{L}-\bar{a}\mathbf{J})$ has, in the limit as $N\to\infty$, a multivariate normal distribution with mean vector $\mathbf{0}$ and covariance matrix $\mathbf{\Lambda}$. Here $\mathbf{L}' = (L_1, L_2, \cdots, L_c)_{1\times c}$, $\mathbf{J}' = (1, 1, \cdots, 1)_{1\times c}$, $\mathbf{0}' = (0, 0, \cdots, 0)_{1\times c}$ and $\mathbf{\Lambda} = (\lambda_{ik})_{c\times c}$ $i, k = 1, 2, \cdots, c$.

The multivariate distribution is singular since $\sum L_i = \sum a_i = K$. In fact it may, trivially, be observed that $\mathbf{\Lambda J} = \mathbf{0}$. We consider the distribution of $N^{\frac{1}{2}}(\mathbf{L}_0 - \bar{a}\mathbf{J}_0)$. It is nonsingular with $\mathbf{0}_0$ mean and $\mathbf{\Lambda}_0$ as covariance matrix. Here $\mathbf{L}_0' = (L_1, \cdots, L_{c-1})_{1\times c-1}$, $\mathbf{J}_0' = (1, 1, \cdots, 1)_{1\times c-1}$, $\mathbf{0}_0' = (0, \cdots, 0)_{1\times c-1}$ and $\mathbf{\Lambda}_0 = (\lambda_{ik})_{c-1\times c-1}$; $i, k = 1, 2, \cdots, c-1$.

Therefore $\mathscr{L} = N(\mathbf{L}_0 - \bar{a}\mathbf{J}_0)'\mathbf{\Lambda}_0^{-1}(\mathbf{L}_0 - \bar{a}\mathbf{J}_0)$ is distributed under the null hypothesis as $N\to\infty$, as a χ^2 variate with $c-1$ degrees of freedom. Following Bhapkar [2], we simplify and obtain

(3.6) $$\mathscr{L} = (c-1)^2 N(c^2 A)^{-1}\left[\sum_{i=1}^{c} p_i L_i^2 - \left\{\sum_{i=1}^{c} p_i L_i\right\}^2\right].$$

We have proved the following theorem:

THEOREM 3.1. *If* $F_1 = F_2 = \cdots = F_c$ *and* $n_i = Np_i$ *where the* p_i *are fixed numbers such that* $\sum_{i=1}^{c} p_i = 1$, *then the statistic* $\mathscr{L}$, *as defined in* (3.6) *above, for any real* a_j *such that they are not all equal, has a limiting* χ^2 *distribution with* $c-1$ *degrees of freedom.*

4. Consistency of tests based on $\mathscr{L}$. In this section we give a condition for the consistency of tests based on $\mathscr{L}$. Using Lemma 4.2 of Bhapkar [2], it may be concluded that tests of the type which reject $H: F_1 = F_2 = \cdots = F_c$ if $\mathscr{L} > \mathscr{L}_\alpha$ are consistent for all F_i, $i = 1, 2, \cdots, c$, such that $\eta^{(i)}$ is different from $\bar{a}$ for at least one i, where

$$\eta^{(i)} = \sum_{j=1}^{c} a_j \left\{\int_{-\infty}^{\infty} \sum{}^{*} \prod_{(j-1)\text{ terms}} F_r(x) \prod_{(c-j)\text{ terms}} [1 - F_s(x)]\, dF_i(x)\right\}.$$

Here $\sum^*$ indicates summation over all possible choices of $(j-1)$ F's out of $(c-1)$ F's (all except F_i). It may be noted that $\mathscr{L}$ is a nonnegative function of L_i and equal to zero only when $L_i = \bar{a}$ for each i.

5. Distribution of u_{ij} and $\mathscr{L}$ under alternative hypotheses. In this section we derive the limiting distribution of u_{ij} and $\mathscr{L}$ under the following two sequences of alternative hypotheses.

$$H_{L_n}: F_i(x) = F(x-n^{-\frac{1}{2}}\theta_i) \quad \text{and} \tag{5.1}$$

$$H_{S_n}: F_i(x) = F(x(1+n^{-\frac{1}{2}}\delta_i)). \tag{5.2}$$

Here n is given by the relation $n_i = ns_i$ where s_i are fixed integers, all θ are not equal, $\delta_i > 0$ for each i and all δ_i are not equal

THEOREM 5.1. (a) w_{ij}, *as defined in Theorem* 2.1, *have jointly in the limit as* $n \to \infty$, *under* H_{L_n} *multivariate normal distribution, with means*

$$\begin{aligned} \eta_{ij}^{Ln} = (\textstyle\sum_{i=1}^c s_i)^{\frac{1}{2}} \sum_{r=1}^c (\theta_r-\theta_i) \{ \binom{c-2}{j-1} \int_{-\infty}^{\infty} f^2(x)[F(x)]^{j-1}[1-F(x)]^{c-j-1}\,dx \\ -\binom{c-2}{j-2} \int_{-\infty}^{\infty} f^2(x)[F(x)]^{j-2}[1-F(x)]^{c-j}\,dx \} \end{aligned} \tag{5.3}$$

and elements of the covariance matrix given by (2.1) *under the following two conditions.*

(i) F *is absolutely continuous with derivative* f *and*

(ii) *There exists a function* g *such that*

$$|[f(y+h)-f(y)]/h| \leqq g(y) \quad \text{for small} \quad h \quad \text{and} \quad \int_{-\infty}^{\infty} g(y)f(y)\,dy < \infty.$$

(b) *Under* H_{L_n} *and the above two conditions* $\mathscr{L}$ *has, in the limit as* $n \to \infty$, *noncentral* χ^2 *distribution with* $c-1$ *degrees of freedom and noncentrality parameter given by*

$$\mu_{L_n} = \frac{(\mathbf{a}'\mathbf{b})^2}{A} \sum_{i=1}^{c} s_i(\theta_i-\bar{\theta})^2 \quad \text{where} \tag{5.4}$$

$$\mathbf{a}' = (a_1, a_2, \cdots, a_c), \qquad \mathbf{b}' = (b_1, b_2, \cdots, b_c),$$

$$\begin{aligned} b_j = \binom{c-1}{j-1} \{ (c-j) \int_{-\infty}^{\infty} f^2(x)[F(x)]^{j-1}[1-F(x)]^{c-j-1}\,dx \\ -(j-1) \int_{-\infty}^{\infty} f^2(x)[F(x)]^{j-2}[1-F(x)]^{c-j}\,dx \}; \end{aligned}$$

A as defined in (1.1) *and* $\bar{\theta} = \sum_{i=1}^c s_i\theta_i / \sum_{i=1}^c s_i$.

PROOF.

(a) $$\begin{aligned} \mathscr{E}(u_{ij} \mid H_{L_n}) = \textstyle\sum' \int_{-\infty}^{\infty} \prod_{(j-1)\,\text{terms}} F(x-n^{-\frac{1}{2}}\theta_r) \prod_{(c-j)\,\text{terms}} [1-F(x-n^{-\frac{1}{2}}\theta_k)] \\ \cdot dF(x-n^{-\frac{1}{2}}\theta_i). \end{aligned}$$

[In the above expression $\sum'$ indicates summation over all possible choices of $(j-1)$ F's out of $(c-1)$ F's (all except F_i).] Under conditions of the theorem we have

$$\begin{aligned} \mathscr{E}(u_{ij} \mid H_{L_n}) = c^{-1} + n^{-\frac{1}{2}} \textstyle\sum_{r=1}^c (\theta_r-\theta_i) \{ \binom{c-2}{j-1} \int_{-\infty}^{\infty} f^2(x)[F(x)]^{j-1}[1-F(x)]^{c-j-1}\,dx \\ -\binom{c-2}{j-2} \int_{-\infty}^{\infty} f^2(x)[F(x)]^{j-2}[1-F(x)]^{c-j}\,dx \} + O(n^{-1}). \end{aligned}$$

Proceeding on exactly similar lines and using Conditions (i) and (ii) of the theorem we see that

$$\lim_{N\to\infty} N \operatorname{Cov}(u_{ij}, u_{kl} \mid H_{Ln}) = \lim_{N\to\infty} N \operatorname{Cov}(u_{ij}, u_{kl} \mid H_0).$$

Hence part (a) of the theorem follows.

(b) It follows that $\mathscr{L}$, in the limit as $n \to \infty$, is distributed as noncentral χ^2 with $c-1$ degrees of freedom and the noncentrality parameter

$$A^{-1}(c-1)^2 \textstyle\sum_{i=1}^{c} s_i(\theta_i-\bar{\theta})^2 [\sum_{j=1}^{c} a_j\{\binom{c-2}{j-1}\int_{-\infty}^{\infty} f^2(x)[F(x)]^{j-1}[1-F(x)]^{c-j-1}\,dx$$
$$-\binom{c-2}{j-2}\int_{-\infty}^{\infty} f^2(x)[F(x)]^{j-2}[1-F(x)]^{c-j}\,dx\}].$$

which further simplifies to (5.4).

THEOREM 5.2. (a) *w_{ij} as defined in Theorem* 2.1 *have jointly in the limit as $n \to \infty$, under H_{S_n}, multivariate normal distribution, with means*

$$\eta_{ij}^{S_n} = (\textstyle\sum_{i=1}^{c} s_i)^{\frac{1}{2}} \sum_{r=1}^{c} (\delta_r-\delta_i)\{\binom{c-2}{j-2}\int_{-\infty}^{\infty} xf^2(x)[F(x)]^{j-2}[1-F(x)]^{c-j}\,dx$$
$$-\binom{c-2}{j-1}\int_{-\infty}^{\infty} xf^2(x)[F(x)]^{j-1}[1-F(x)]^{c-j-1}\,dx\} \qquad (5.5)$$

and the elements of the covariance matrix given by (2.1) *under the following conditions*:

(i) *F is absolutely continuous with derivative f.*

(ii) *There exists a function g such that*

$$|[f(x)-f(x+hx)]/h| \leqq g(x) \quad \textit{for small} \quad h \quad \textit{and}$$
$$\int_{-\infty}^{\infty} [xg(x)]^i f(x)\,dx < \infty \quad \textit{for} \quad i = 1, 2, \cdots, 2c-1.$$

(iii) *There exists $A < \infty$ such that $P_F[|Xf(X)| < A] = 1$.*

(b) *Under H_{S_n} and the above three conditions, $\mathscr{L}$ has, in the limit as $n \to \infty$, a non-central χ^2 distribution with $c-1$ degrees of freedom and noncentrality parameter given by*

$$\mu_{S_n} = \frac{(\mathbf{a}'\mathbf{d})^2}{A}\sum_{i=1}^{c} s_i(\delta_i-\bar{\delta})^2 \qquad \textit{where}$$

$$\mathbf{d}' = (d_1, \cdots, d_c) \qquad \textit{with} \qquad (5.6)$$

$$d_j = \binom{c-1}{j-1}\{(j-1)\int_{-\infty}^{\infty} xf^2(x)[F(x)]^{j-2}[1-F(x)]^{c-j}\,dx$$
$$-(c-j)\int_{-\infty}^{\infty} xf^2(x)[F(x)]^{j-1}[1-F(x)]^{c-j-1}\,dx\}$$

and $\bar{\delta} = \sum_{i=1}^{c} s_i\delta_i/\sum_{i=1}^{c} s_i$. **a** *and A are the same as in Theorem* 5.1.

PROOF.

(a) $$\mathscr{E}(u_{ij} \mid H_{S_n}) = \textstyle\sum' \int_{-\infty}^{\infty} \prod_{(j-1)\,\text{terms}} [F_r(x)] \prod_{(c-j)\,\text{terms}} [1-F_k(x)]\,dF_i(x),$$

($\sum'$ indicates summation as in Theorem 5.1), thus

$$\mathscr{E}(u_{ij} \mid H_{S_n}) = c^{-1} + n^{-\frac{1}{2}} \textstyle\sum_{r=1}^{c} (\delta_r-\delta_i)\{\binom{c-2}{j-2}\int_{-\infty}^{\infty} xf^2(x)[F(x)]^{j-2}[1-F(x)]^{c-j}\,dx$$
$$-\binom{c-2}{j-1}\int_{-\infty}^{\infty} xf^2(x)[F(x)]^{j-1}[1-F(x)]^{c-j-1}\,dx\} + O(n^{-1}).$$

After lengthy derivation on similar lines we obtain that

$$\lim_{N\to\infty} N\,\mathrm{Cov}(u_{ij}, u_{kl} \mid H_{S_n}) = \lim_{N\to\infty} N\,\mathrm{Cov}(u_{ij}, u_{kl} \mid H_0).$$

Hence part (a) of the theorem is proved.

(b) It follows easily that $\mathscr{L}$, in the limit as $n \to \infty$, is distributed as noncentral χ^2 with $c-1$ degrees of freedom and the noncentrality parameter which simplifies to

$$A^{-1}(c-1)^2 \sum_{i=1}^{c} s_i(\delta_i - \bar{\delta})^2 \{\sum_{j=1}^{c} a_j\{\tbinom{c-2}{j-2} \int_{-\infty}^{\infty} xf^2(x)[F(x)]^{j-2}[1-F(x)]^{c-j}\,dx$$
$$-\tbinom{c-2}{j-1} \int_{-\infty}^{\infty} xf^2(x)[F(x)]^{j-1}[1-F(x)]^{c-j-1}\,dx\}$$

which can be seen to be equal to μ_{S_n}.

6. Asymptotic relative efficiency. We know from Hannan's [8] and Andrews' [1] work that the asymptotic relative efficiency (ARE), in the Pitman sense, of one test with respect to another, is equal to the ratio of the noncentrality parameters of the two test statistics, provided that they are asymptotically distributed as noncentral χ^2 variates with the same degrees of freedom under the given sequence of alternative hypotheses (e.g. H_{L_n} or H_{S_n}). Hence, to obtain a test statistic, from the class of statistics $\mathscr{L}$, which will have maximum ARE, we need to maximize the noncentrality parameter over all real a, for a given sequence of hypotheses.

We know that under H_{L_n} and H_{S_n}, $\mathscr{L}$ is in the limit, distributed as noncentral χ^2 with $c-1$ degrees of freedom and noncentrality parameters given by μ_{L_n} and μ_{S_n} respectively. Let us take μ_{L_n} first.

In it we need to maximize only $A^{-1}(\mathbf{a}'\mathbf{b})^2$ since the other factor does not involve a.

Let us define $\mathbf{D} = (d_{ij})_{c\times c}$ $i,j = 1, 2, \cdots, c$ where

$$d_{ij} = \left\{\frac{\binom{c-1}{i-1}\binom{c-1}{j-1}}{\binom{2c-2}{i+j-2}(2c-1)} - \frac{1}{c^2}\right\}$$

and $\mathbf{D}_0 = (d_{ij})_{c-1\times c-1}$, $i,j = 1, 2, \cdots, c-1$.

It is then obvious that $A = \mathbf{a}'\mathbf{D}\mathbf{a}$. However, we see that $\mathbf{D}$ is singular and of rank $c-1$. But $\mathbf{D}_0$ is nonsingular and positive definite. We note that $\sum_{i=1}^{c} d_{ij} = \sum_{j=1}^{c} d_{ij} = \sum_{j=1}^{c} b_j = 0$. In view of these we may assume, without loss of generality, that $\sum_{i=1}^{c} a_i = 0$; the value of $\mathbf{a}'\mathbf{b}$ or $\mathbf{a}'\mathbf{D}\mathbf{a}$ remains unchanged even if $\mathbf{a}$ is replaced by $\mathbf{a} - \bar{a}\mathbf{J}$.

It may then be seen that $\mathbf{a}'\mathbf{b} = \mathbf{a}_0'\mathbf{b}_0$ and $\mathbf{a}'\mathbf{D}\mathbf{a} = \mathbf{a}_0'\mathbf{E}\mathbf{a}_0$ where $\mathbf{a}_0' = (a_1, a_2, \cdots, a_{c-1})$, $\mathbf{b}_0' = (b_1 - b_c, b_2 - b_c, \cdots, b_{c-1} - b_c)$ and $\mathbf{E} = (e_{ij})$ $i,j = 1, 2, \cdots, c-1$ with $e_{ij} = (d_{ij} - d_{ic} - d_{cj} + d_{cc})$.

Using the facts that $\sum_i d_{ij} = \sum_j d_{ij} = 0$, it is seen that $\mathbf{E} = \mathbf{T}\mathbf{D}_0\mathbf{T}$ where $T = (t_{ij})$, $i,j = 1, 2, \cdots, c-1$ and $t_{ii} = 2$ and $t_{ij} = 1$ if $i \neq j$. Therefore, $\mathbf{E}$ is positive definite.

Using Cauchy's inequality it may be seen that $(\mathbf{a}'\mathbf{b})^2/(\mathbf{a}'\mathbf{D}\mathbf{a}) = (\mathbf{a}_0'\mathbf{b}_0)^2/(\mathbf{a}_0'\mathbf{E}\mathbf{a}_0) \leqq \mathbf{b}_0'\mathbf{E}^{-1}\mathbf{b}_0$ for all real a and the equality is attained whenever $\mathbf{a}_0 \propto \mathbf{E}^{-1}\mathbf{b}_0$. On similar lines it can be proved that $(\mathbf{a}'\mathbf{d})^2/(\mathbf{a}'\mathbf{D}\mathbf{a}) \leqq \mathbf{d}_0'\mathbf{E}^{-1}\mathbf{d}_0$ where $\mathbf{d}_0' = (d_1 - d_c, \cdots, d_{c-1} - d_c)$. Hence, we have proved the following theorem.

THEOREM 6.1. (a) *The maximum of* μ_{L_n} *for all real a is* $\sum s_i(\theta_i-\bar{\theta})^2(\mathbf{b}_0'\mathbf{E}^{-1}\mathbf{b}_0)$ *and is obtained when* $\mathbf{a}_0 \propto \mathbf{E}^{-1}\mathbf{b}_0$ *and* $a_c = \sum_{i=1}^{c-1} a_i$.

(b) *The maximum of* μ_{S_n} *for all real a is* $\sum s_i(\delta_i-\bar{\delta})^2(\mathbf{d}_0'\mathbf{E}^{-1}\mathbf{d}_0)$ *and is obtained when* $\mathbf{a}_0 \propto \mathbf{E}^{-1}\mathbf{d}_0$ *and* $a_c = \sum_{i=1}^{c-1} a_i$.

It has not been possible to compute E^{-1} in any simple or manageable form for general c. However, the following results are available. E and E^{-1} are computed for $c = 2, 3, \cdots, 12$. Maximum values of the noncentrality parameters and the a's (or any multiples thereof) that lead to these maximum values are computed for the normal distribution and the sequences of alternative hypotheses mentioned earlier. These results are tabulated below along with the asymptotic relative efficiencies of the tests based on these statistics with respect to their parametric counterparts for the normal distribution. The test based on the statistic that has the largest noncentrality parameter under the sequence of alternatives H_{L_n} (we call it the $\mathscr{L}_L$-test) is compared with the classical F test for the equality of means of several normal populations. The test based on the statistic that has the largest noncentrality parameter under the sequence H_{S_n} (we call it the $\mathscr{L}_S$ test) of alternative hypotheses is compared with a test proposed by Lehmann ([11] pages 273–275) called the L-test. The values are quoted up to four decimal places.

TABLE 6.1

c	$a_1, a_2, \cdots, a_c$ which maximize μ_{L_n}	$\sup_{\mathbf{a}} \frac{(\mathbf{a}'\mathbf{b})^2}{A}$	$e_{\mathscr{L}_L,F}$
2	any $a_1 \neq a_2$	0.9549	0.9549
3	$1, 0, -1$	0.9549	0.9549
4	$2.1768, -0.8884, 0.8884, -2.1768$	0.9884	0.9884
5	$2.1768, -0.1221, 0, 0.1221, -2.1768$	0.9884	0.9884
6	$2.4477, -1.0172, 2.6608, -2.6608, 1.0172, -2.4477$	0.9951	0.9951
7	$2.4477, -0.4397, 1.4348, 0, -1.4348, 0.4397, -2.4477$	0.9951	0.9951
8	$2.6326, -1.3212, 4.7812, -5.8550, 5.8550, -4.7812, 1.3212, -2.6326$	0.9974	0.9974
9	$2.6326, -0.8269, 3.2556, -1.8664, 0, 1.8664, -3.2556, 0.8269, -2.6326$	0.9974	0.9974
10	$2.7715, -1.6932, 7.3510, -11.8320, 16.6798, -16.6798, 11.8320, -7{,}3510, 1.6932, -2.7715$	0.9984	0.9984
11	$2.7715, -1.2467, 5.5422, -6.0771, 5.2751, 0, -5.2751, 6.0771, -5.5422, 1.2467, -2.7715$	0.9984	0.9984
12	$2.8822, -2.0989, 10.3953, -21.1706, 37.6720, -48.7371, 48.7371, -37.6720, 21.1706, -10.3953, 2.0989, -2.8822$	0.9989	0.9989

TABLE 6.2

c	$a_1, a_2, \cdots, a_c$ which maximize μ_{S_n}	$\underset{\mathbf{a}}{\text{Sup}} \frac{(\mathbf{a}'\mathbf{d})^2}{A}$	$e_{\mathscr{L}_S, L}$
3	1, −2, 1	1.5198	0.7599
4	1, −1, −1, 1	1.5198	0.7599
5	4.3200, −7.6318, 6.6236, −7.6318, 4.3200	1.7914	0.8957
6	4.3200, −5.2414, 0.9214, 0.9214, −5.2414, 4.3200	1.7914	0.8957
7	5.4180, −10.2355, 15.5336, −21.1037, 15.5336, −10.2355, 5.4180	1.8841	0.9420
8	5.4180, −7.9993, 8.0300, −5.4487, −5.4487, 8.0300, −7.9993, 5.4180	1.8841	0.9420
9	6.2656, −13.1035, 27.5726, −47.8633, 53.8877, −47.8633, 27.5726, −13.1035, 6.2656	1.9264	0.9632
10	6.2656, −10.9513, 18.6771, −22.6565, 8.6651, 8.6651, −22.6565, 18.6771, −10.9513, 6.2656	1.9264	0.9632
11	6.9565, −16.1388, 43.8426, −93.1663, 141.2287, −165.4457, 141.2287, −93.1663, 43.8426, −16.1388, 6.9565	1.9491	0.9745
12	6.9565, −14.0392, 32.9369, −55.8002, 55.9943, −26.0483, −26.0483, 55.9943, −55.8002, 32.9369, −14.0392, 6.9565	1.9491	0.9745

7. Remarks. Both tabular displays in the last section reinforce the conjecture that the efficiency of the "best" tests in this class will monotonically increase to one with c when compared with their "best" parametric counterparts for the normal distribution. Unfortunately the author has not been able to get an analytic proof of it and must leave it as an open problem. It may be noticed that the lowest efficiency, that for $c = 2, 3$, is the same as that of the Wilcoxon test [12], [15] or the Kruskal test [10] for shift alternatives.

It is possible to take a different approach to construct tests. Bhapkar [3] has constructed a test for the c sample problem based on pairs (X_i, X_j) of observations where X_i and X_j are from different samples. Chatterjee [4] has proposed a test based on triplets (X_i, X_j, X_k) of observations for the same problem. Let us consider t-plets formed by $t(\leqq c)$ observations such that each of them represents a distinct sample. Let us define a function

$$\varphi^t_{\alpha_i j}(x_{\alpha_1 s_{\alpha_1}}, \cdots, x_{\alpha_t s_{\alpha_t}}) = m_j \quad \text{whenever} \quad x_{\alpha_i s_{\alpha_i}} \quad \text{is larger} \tag{7.1}$$

than exactly $(j-1)$ x's and smaller than the rest. Here $(\alpha_1, \cdots, \alpha_t)$ are t members of $(1, 2, \cdots, c)$. To assure symmetry between all the c samples we will have to consider a function based collectively on all the t-plets in which an observation from the α_ith sample occurs.

It may easily be seen that we shall develop identical tests if we base them on t-plets using the function defined in (7.1) or if we base them on c-plets using the function (7.2) defined below.

$$\varphi_{ik}(x_{1t_1}, x_{2t_2}, \cdots, x_{ct_c}) = \sum_{j=1}^{t} \binom{k-1}{j-1}\binom{c-k}{t-j} m_j \quad \text{whenever } x_{it_i} \text{ is larger than exactly } k-1\ x\text{'s} \tag{7.2}$$

$$= 0 \quad \text{otherwise.}$$

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