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A class of non-holomorphic modular forms I

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Abstract

This introductory paper studies a class of real analytic functions on the upper half plane satisfying a certain modular transformation property. They are not eigenfunctions of the Laplacian and are quite distinct from Maass forms. These functions are modular equivariant versions of real and imaginary parts of iterated integrals of holomorphic modular forms and are modular analogues of single-valued polylogarithms. The coefficients of these functions in a suitable power series expansion are periods. They are related both to mixed motives (iterated extensions of pure motives of classical modular forms) and to the modular graph functions arising in genus one string perturbation theory.

This paper studies examples of real analytic functions on the upper half plane satisfying a modular transformation property of the form

$$f\Big(\frac{az+b}{cz+d}\Big) = (cz+d)^r (c\bar{z}+d)^s f(z) \tag{0.1}$$

for integers r, s. They *do not* satisfy a simple condition involving the Laplacian. The *raison d'être* for this class of functions is two-fold:

(1) Holomorphic modular forms f with rational Fourier coefficients correspond to certain pure motives M_f over $\mathbb{Q}$. Using iterated integrals, we can construct non-holomorphic modular forms which are associated with iterated extensions of the pure motives M_f. Their coefficients are periods.
(2) In genus one closed string perturbation theory, one assigns a lattice sum to a graph [16], which defines a real analytic function on the upper half plane invariant under $\mathrm{SL}_2(\mathbb{Z})$. It is an open problem to give a complete description of this class of functions and prove their conjectured properties.

In this introductory paper, we describe elementary properties of a class $\mathcal{M}$ of modular forms. Within this class are modular iterated integrals, which are analogues of single-valued polylogarithms, and are obtained by solving a differential equation in $\mathcal{M}$. The basic prototype are real analytic Eisenstein series, defined by

$$\mathcal{E}_{r,s}(z) = \frac{w!}{(2\pi i)^{w+1}} \frac{1}{2} \sum_{(m,n)\neq(0,0)} \frac{i\,\mathrm{Im}(z)}{(mz+n)^{r+1}(m\bar{z}+n)^{s+1}}$$

for all $r, s \geq 0$ such that $w = r + s > 0$ is even. It is known that the functions $\mathrm{Im}(z)^r\, \mathcal{E}_{r,r}(z)$ all occur as modular graph functions (2). Their relation with motives (1) comes about by expressing the $\mathcal{E}_{r,s}$ as integrals. Indeed, they are equivariant or 'modified single-valued versions' of regularised Eichler integrals of holomorphic Eisenstein series, and their Fourier expansion involves the Riemann zeta values $\zeta(w+1)$, which are periods of simple extensions of Tate motives. We shall say very little about motives in this paper and instead refer to [1,18] for geometric motivation.

This paper is based on a talk at a conference in honour of Don Zagier's birthday, and connects with his work in several ways: through his work on modular graph functions [10], on single-valued polylogarithms [33], on period polynomials [22], on periods [21], on multiple zeta values [15], on double Eisenstein series [19], and doubtless many others. It is a great pleasure to dedicate it to him on his 65th birthday.

1 Modular graph functions

For motivation, we briefly recall the definition of modular graph functions.

Definition 1.1 Let G be a connected graph with no self-edges. It is permitted to have a number of half-edges. Denote its set of vertices by V_G and number its edges (including the half-edges) $1, \ldots, r$. Choose an orientation of G. The associated modular graph function is defined, when it converges, by the sum [10] (3.12):

$$I_G(z) = \pi^{-r} \sum_{m_1,n_1}{}' \cdots \sum_{m_r,n_r}{}' \frac{\mathrm{Im}(z)}{|m_1 z + n_1|^2} \cdots \frac{\mathrm{Im}(z)}{|m_r z + n_r|^2} \prod_{v \in V_G} \delta(m_v)\delta(n_v),$$

where z is a variable in the upper half plane $\mathfrak{H}$, the prime over a summation symbol denotes a sum over $(m,n) \in \mathbb{Z}^2 \backslash (0,0)$, and for every vertex $v \in V_G$,

$$m_v = \sum_{i=1}^{r} \varepsilon_{v,i} m_i \quad \text{and} \quad n_v = \sum_{i=1}^{r} \varepsilon_{v,i} n_i,$$

where $\varepsilon_{v,i}$ is 0 if the edge i is not incident to the vertex v, $+1$ if i is oriented towards the vertex v, and -1 if it is oriented away from v.

The function I_G depends neither on the edge numbering, nor on the choice of orientation of G. It defines a function I_G on the upper half plane which is real analytic and invariant under the action of $\mathrm{SL}_2(\mathbb{Z})$ (Fig. 1).

Examples 1.2 Consider the graph with 3 half-edges depicted on the left:

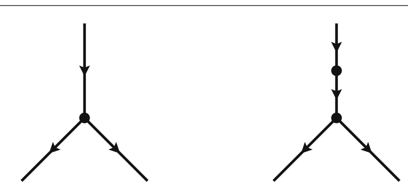


Fig. 1 Two graphs

The associated modular graph function is called

$$C_{1,1,1}(z) = \pi^{-3} \sum_{m_1,n_1,m_2,n_2}' \frac{\mathrm{Im}(z)^3}{|m_1 z + n_1|^2 |m_2 z + n_2|^2 |(m_1+m_2)z + n_1 + n_2|^2},$$

where the sum is over $(m_1, n_1) \in \mathbb{Z}^2$, $(m_2, n_2) \in \mathbb{Z}^2$ such that

$$(m_1, n_1) \neq (0,0), \quad (m_2, n_2) \neq (0,0), \quad (m_1 + m_2, n_1 + n_2) \neq (0,0).$$

Zagier showed, in one of the first calculations of a modular graph function, that

$$C_{1,1,1}(z) = \frac{2}{3} \mathbb{L}^2 \, \mathcal{E}_{2,2} + \zeta(3),$$

where throughout this paper we use the non-standard notation $\mathbb{L} = -2\pi \,\mathrm{Im}(z)$ to stand for 'Lefschetz' (as in the Lefschetz motive). It has the following advantages: it takes care of the powers of π, reflects an underlying integral structure, and carries a weight grading. See [10] Appendix B for another derivation of Zagier's result.

1.1 Properties

The literature on modular graph functions is too extensive to review in detail here. Instead, we give an incomplete list of the expected and conjectural properties of these functions and refer to [10,11,13,16,36] for further details.

(1) Zerbini [36] has shown that in all known examples, the 'zeroth modes' of modular graph functions involve a certain class of multiple zeta values

$$\zeta(n_1, \ldots, n_r) = \sum_{1 \leq k_1 < \cdots < k_r} \frac{1}{n_1^{k_1} \ldots n_r^{k_r}},$$

where $n_1, \ldots, n_r \in \mathbb{N}$ and $n_r \geq 2$, which are called 'single-valued' multiple zeta values. The quantity r is called the depth. The 'single-valued' subclass is generated in depth one by odd zeta values $\zeta(2n+1)$ for $n \geq 1$, in depth two by products $\zeta(2m+1)\zeta(2n+1)$, but starting from depth three includes the following combination of triple zeta values

$$\zeta_{\mathrm{sv}}(3,5,3) := 2\zeta(3,5,3) - 2\zeta(3)\zeta(3,5) - 10\zeta(3)^2\zeta(5)\,.$$

(2) The I_G satisfy some mysterious inhomogeneous Laplace eigenvalue equations. A simple example of this is the Eq. [13] (1.4)

$$(\Delta + 2)\, C_{2,1,1}(z) = 16\, \mathbb{L}^2\, \mathcal{E}_{1,1}^2 - \tfrac{2}{5}\, \mathbb{L}^3\, \mathcal{E}_{3,3}, \tag{1.1}$$

where Δ is the Laplace–Beltrami operator. The function $C_{2,1,1}$ corresponds to the modular graph function of the graph with four edges and two vertices depicted above on the right. As an illustration of our methods, we shall solve this Laplace eigenvalue equation in Sect. 9.3 using a new family of functions constructed here and determine its kernel. Note that the operator Δ in the physics literature has the opposite sign from the usual convention (2.21).

(3) Modular graph functions satisfy many relations [11], which suggests that they should lie in a finite-dimensional space of modular-invariant functions.

(4) The zeroth modes of modular graph functions are homogeneous [10], Sect. 6.1, for a grading called the weight, in which rational numbers have weight 0, and multiple zeta values have weight $n_1 + \cdots + n_r$. The weight of $\mathrm{Im}(z)$ is zero.

In the continuation of this paper, we construct a class of functions $\mathcal{MI}^E \subset \mathcal{M}$ satisfying (1)–(5) (see Sect. 10). They are associated with universal mixed elliptic motives [18], which are in turn related to mixed Tate motives over the integers. We presently explain why we strongly expect that the modular graph functions are contained in our class, which would imply all the conjectures above, and a number of other consequences.

1.2 Landscape

A heuristic explanation for the connection between string theory, and our modular iterated integrals can be summarised in the following picture:

	Open string	Closed string
Genus 0	Multiple polylogs	Single-valued polylogs
Genus 1	Multiple elliptic polylogs	Equivariant iterated Eisenstein integrals

The open genus zero amplitudes are integrals on the moduli spaces of curves of genus 0 with n marked points $\mathfrak{M}_{0,n}$. They involve multiple polylogarithms, whose values are multiple zeta values. The genus one string amplitudes are integrals on the moduli space $\mathfrak{M}_{1,n}$ and are expressible [5] in terms of multiple elliptic polylogarithms [3]. Viewed as a function of the modular parameter, the latter are given by certain products of iterated integrals of Eisenstein series. The passage from the open to the closed string involves a 'single-valued' construction [31]. The closed superstring amplitudes in genus one are thus linear combinations of products of iterated of Eisenstein series and their complex conjugates which are *modular*. This is the definition of the space $\mathcal{MI}^E$. A rigorous proof of the relation between closed superstring amplitudes and our class $\mathcal{MI}^E$ might go along the broad lines of the author's thesis, generalised to genus one using [3].

2 A class of functions $\mathcal{M}$

Throughout this paper, z will denote a variable in the upper half plane

$$\mathfrak{H} = \{z : \mathrm{Im}\, z > 0\}$$

equipped with the standard action of $\mathrm{SL}_2(\mathbb{Z})$:

$$\gamma(z) = \frac{az+b}{cz+d} \quad \text{where} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}). \tag{2.1}$$

We shall write $z = x + iy$, and $q = \exp(2i\pi z)$. Let

$$\mathbb{L} = \log|q| = \frac{1}{2}\log q\overline{q} = i\pi(z - \overline{z}) = -2\pi y. \tag{2.2}$$

The quantity $2\,\mathbb{L}$ is the single-valued period of a family of Kummer motives over the punctured q-disc. The latter normalisation (i.e. $2\mathbb{L}$ rather than $\mathbb{L}$), simplifies some formulae and may be preferred.

2.1 First definitions

Definition 2.1 Call a real analytic function $f : \mathfrak{H} \to \mathbb{C}$ *modular of weights* (r, s), where $r, s \in \mathbb{Z}$, if for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ of the form (2.1) it satisfies

$$f(\gamma(z)) = (cz+d)^r (c\overline{z}+d)^s f(z). \tag{2.3}$$

If $r+s$ is odd, then f vanishes [put $\gamma = -\mathrm{id}$ in (2.3)]. Let $\mathcal{M}_{r,s}$ denote the space of real analytic functions of modular weights (r,s) which admit an expansion of the form

$$f(q) \quad \in \quad \mathbb{C}[[q,\overline{q}]][\mathbb{L}^{\pm}]. \tag{2.4}$$

A more general class of functions was considered in [27]. A function in $\mathcal{M}_{r,s}$ can be written explicitly, for some $N \in \mathbb{N}$, in the form

$$f = \sum_{k=-N}^{N} \sum_{m,n\geq 0} a_{m,n}^{(k)} \mathbb{L}^k q^m \overline{q}^n, \tag{2.5}$$

where $a_{m,n}^{(k)} \in \mathbb{C}$. For any ring $R \subset \mathbb{C}$, let $\mathcal{M}(R)$ be the bigraded subspace of modular forms whose coefficients $a_{m,n}^{(k)}$ lie in R. Define a bigraded vector space

$$\mathcal{M} = \bigoplus_{r,s} \mathcal{M}_{r,s},$$

which is a bigraded algebra since $\mathcal{M}_{r,s}\mathcal{M}_{k,l} \subset \mathcal{M}_{r+k,s+l}$. Complex conjugation induces an involution

$$f(z) \mapsto \overline{f(z)} : \mathcal{M}_{r,s} \xrightarrow{\sim} \mathcal{M}_{s,r}$$

which fixes $\mathbb{L} \in \mathcal{M}_{-1,-1}$. Of special importance are the quantities

$$w = r+s \quad \text{and} \quad h = r-s. \tag{2.6}$$

We call w the total weight, and let w, h be even.

2.2 q-Expansions and pole filtration

Lemma 2.2 *Suppose that* $f : \mathfrak{H} \to \mathbb{C}$ *satisfies Eq.* (2.3) *and admits an expansion in the ring* $\mathbb{C}[[q,\overline{q}]][\log q, \log \overline{q}]$*. Then,* $f \in \mathbb{C}[[q,\overline{q}]][\mathbb{L}]$.

Proof Setting $\gamma = T = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ in Eq. (2.3) gives $f(z+1) = f(z)$. Since q and $\overline{q}$ are invariant under translations $z \mapsto z+1$, it suffices to show that

$$\mathbb{C}[\log q, \log \overline{q}]^T = \mathbb{C}[\log |q|],$$

where T denotes analytic continuation of q around a loop around 0 in the punctured q-disc. We have $T \log q = \log q + 2i\pi$ and $T \log \overline{q} = \log \overline{q} - 2i\pi$. It is a simple exercise in invariant theory to show that every T-invariant polynomial in $\log q$ and $\log \overline{q}$ is a polynomial in $2\log |q| = \log q + \log \overline{q}$. □

Every element $f \in \mathcal{M}$ admits a q-expansion of the form (2.5) for some N. This expansion is unique. Define the *constant part* of f to be

$$f^0 = \sum_k a_{0,0}^{(k)} \mathbb{L}^k \in \mathbb{C}[\mathbb{L}^{\pm}].$$

The reason for calling this 'constant', although it is not constant as a function on $\mathfrak{H}$, is that it is constant with respect to differential operators to be defined below. Note that the word 'constant' has a different meaning in the context of quasi-modular forms [20].

In the physics literature, the constant parts of modular graph functions are called 'zeroth Fourier modes'. The space $\mathcal{M}$ is filtered by the order of poles in $\mathbb{L}$. Set

$$P^p\mathcal{M} = \left\{ f \in \mathcal{M} : a^{(k)}_{m,n}(f) = 0 \text{ if } k < p \right\}. \tag{2.7}$$

It is a decreasing filtration. It satisfies $P^a\mathcal{M} \times P^b\mathcal{M} \subset P^{a+b}\mathcal{M}$, and $P^0\mathcal{M}$ is the subalgebra of functions admitting expansions in $\mathbb{C}[[q,\overline{q}]][\mathbb{L}]$ with no poles in $\mathbb{L}$. Multiplication by $\mathbb{L}$ is an isomorphism $\mathbb{L} : P^a\mathcal{M}_{r,s} \overset{\sim}{\rightarrow} P^{a+1}\mathcal{M}_{r-1,s-1}$.

Example 2.3 Consider the Eisenstein series, defined for all even $k \geq 4$ by

$$\mathbb{G}_k(q) = -\frac{b_k}{2k} + \sum_{n\geq 1} \sigma_{k-1}(n)q^n \in \mathcal{M}_{k,0}(\mathbb{Q}), \tag{2.8}$$

where σ denotes the divisor function. The Eisenstein series of weight two

$$\mathbb{G}_2(q) = \frac{-1}{24} + \sum_{n=1}^{\infty} \sigma_1(n)q^n = -\frac{1}{24} + q + 3q^2 + 4q^3 + 7q^4 + 6q^5 + \cdots$$

is not modular invariant, but can be modified [34] §2.3 by defining

$$\mathbb{G}_2^* = \mathbb{G}_2 - \frac{1}{4\mathbb{L}}, \tag{2.9}$$

which is modular of weight 2 and therefore defines an element in $\mathcal{M}_{2,0}$. Then, for example, the function $\mathbb{L}^2\mathbb{G}_2^*\overline{\mathbb{G}_2^*} \in \mathcal{M}_{0,0}$ is modular invariant, where $\overline{\mathbb{G}_2^*} = \overline{\mathbb{G}_2} - \frac{1}{4\mathbb{L}}$.

Recall that the polynomial ring

$$\widetilde{M} := M[\mathbb{G}_2^*], \tag{2.10}$$

where M is the ring of holomorphic modular forms, is called the ring of almost holomorphic modular forms. By the previous example, it is contained in $\mathcal{M}$.

2.3 Differential operators (Maass)

Definition 2.4 For any integers $r, s \in \mathbb{Z}$, define a pair of operators

$$\partial_r = (z - \overline{z})\frac{\partial}{\partial z} + r, \quad \overline{\partial}_s = (\overline{z} - z)\frac{\partial}{\partial \overline{z}} + s. \tag{2.11}$$

They act on real analytic functions $f : \mathfrak{H} \rightarrow \mathbb{C}$.

These operators satisfy a version of the Leibniz rule:

$$\partial_{r+s}(fg) = \partial_r(f)g + f\partial_s(g) \tag{2.12}$$

for any r, s and $f, g : \mathfrak{H} \rightarrow \mathbb{C}$, and in addition the formula

$$\partial_r\big((z - \overline{z})^k f\big) = (z - \overline{z})^k \partial_{r+k} f \tag{2.13}$$

for any integers r, k. Both formulae (2.12) and (2.13) remain true on replacing ∂ by $\overline{\partial}$ and are verified by straightforward computation. Finally, one checks that

$$\partial_{r-1}\big(\overline{\partial}_s f\big) - \overline{\partial}_{s-1}\big(\partial_r f\big) = (r-s)f. \tag{2.14}$$

The following lemma implies that these operators respect modular transformations.

Lemma 2.5 *For all* $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_{2\times 2}(\mathbb{R})$ *and* $z \in \mathfrak{H}$, *we have*

$$\partial_r\Big((cz+d)^{-r} f(\gamma z)\Big) = (cz+d)^{-r-1}(c\overline{z}+d)\,(\partial_r f)(\gamma z),$$
$$\overline{\partial}_s\Big((c\overline{z}+d)^{-s} f(\gamma z)\Big) = (cz+d)(c\overline{z}+d)^{-s-1}(\partial_s f)(\gamma z).$$

Proof Direct computation. □

See Sect. 7 for another interpretation of $\partial_r, \overline{\partial}_s$ in terms of sections of vector bundles.

Lemma 2.6 *The operators* $\partial_r, \overline{\partial}_s$ *preserve the expansions* (2.5), *the filtration* (2.7), *and are defined over* $\mathbb{Z}$. *Their action is given explicitly for any* k, m, n *by*

$$\begin{aligned} \partial_r(\mathbb{L}^k q^m \overline{q}^n) &= (2m\,\mathbb{L} + r + k)\,\mathbb{L}^k q^m \overline{q}^n, \\ \overline{\partial}_s(\mathbb{L}^k q^m \overline{q}^n) &= (2n\,\mathbb{L} + s + k)\,\mathbb{L}^k q^m \overline{q}^n. \end{aligned} \tag{2.15}$$

Proof The first part follows immediately from the formulae (2.15), which are easily derived from the definitions. The second line follows by complex conjugation. □

Corollary 2.7 *The operators* ∂_r, ∂_s *preserve modularity:*

$$\partial_r : \mathcal{M}_{r,s} \longrightarrow \mathcal{M}_{r+1,s-1} \quad \textit{and} \quad \overline{\partial}_s : \mathcal{M}_{r,s} \longrightarrow \mathcal{M}_{r-1,s+1}.$$

Proof This follows immediately from Lemmas 2.5 and 2.6. □

Definition 2.8 Let us define linear operators

$$\partial, \overline{\partial} : \mathcal{M} \longrightarrow \mathcal{M}$$

of bidegrees $(1, -1)$ and $(-1, 1)$, respectively, where ∂ acts on the component $\mathcal{M}_{r,s}$ via ∂_r for all s, and similarly, $\overline{\partial}$ acts on $\mathcal{M}_{r,s}$ via $\overline{\partial}_s$ for any r.

The operator ∂ is a derivation, i.e, $\partial(fg) = \partial(f)g + f\,\partial(g)$ for all $f, g \in \mathcal{M}$, and similarly for $\overline{\partial}$. This follows, component by component, from the formula (2.12). Likewise, it commutes with multiplication by $\mathbb{L}^k$:

$$\partial(\mathbb{L}^k f) = \mathbb{L}^k\,\partial(f)$$

for all k and all $f \in \mathcal{M}$, and similarly for $\overline{\partial}$. This is equivalent to (2.13). We can rewrite the previous equation in the form

$$[\partial, \mathbb{L}] = [\overline{\partial}, \mathbb{L}] = 0,$$

or think of $\mathbb{L}$ as being constant: $\partial(\mathbb{L}) = \overline{\partial}(\mathbb{L}) = 0$.

2.4 Action of $\mathfrak{sl}_2$

The Eq. (2.14) implies that

$$[\partial, \overline{\partial}] = \mathsf{h},$$

where we define the linear map

$$\mathsf{h} : \mathcal{M} \longrightarrow \mathcal{M} \tag{2.16}$$

to be multiplication by $r - s$ on the component $\mathcal{M}_{r-s}$.

Proposition 2.9 *The operators* $\partial, \overline{\partial}$ *generate a copy of the Lie algebra* $\mathfrak{sl}_2$*:*

$$[\mathsf{h}, \partial] = 2\partial, \quad [\mathsf{h}, \overline{\partial}] = -2\overline{\partial}, \quad [\partial, \overline{\partial}] = \mathsf{h} \tag{2.17}$$

acting upon $\mathcal{M}$*. Every element commutes with multiplication by* $\mathbb{L}^k$*.*

Proof Straightforward computation. □

2.5 Almost holomorphic modular forms

The subspace $\widetilde{M}[\mathbb{L}^{\pm}]$ of almost holomorphic modular forms inherits an $\mathfrak{sl}_2$ module structure which is not to be confused with another $\mathfrak{sl}_2$ module structure [34] Sect. 5.3, which involves multiplication by $\mathbb{G}_2$. For the convenience of the reader, we describe the differential structure here.

Let us define a new generator

$$\mathfrak{m} := 4\mathbb{L}\mathbb{G}_2^* = 4\mathbb{L}\mathbb{G}_2 - 1 \in \mathcal{M}_{1,-1}.$$

Then, the ring $M[\mathbb{L}, \mathfrak{m}]$ is an $\mathfrak{sl}_2$-module with the following structure: $\overline{\partial}(\mathbb{L}) = 0$, and

$$\overline{\partial}\, \mathfrak{m} = 1, \quad \overline{\partial} f = 0 \quad \text{for all } f \in M. \tag{2.18}$$

Therefore, $\overline{\partial}\big|_{M[\mathbb{L},\mathfrak{m}]} = \frac{\partial}{\partial \mathfrak{m}}$ is simply differentiation with respect to $\mathfrak{m}$. On the other hand, by looking at their first few Fourier coefficients, we easily verify that:

$$\begin{aligned}
\partial \mathfrak{m} &= -\mathfrak{m}^2 + \tfrac{20}{3}\mathbb{L}^2\mathbb{G}_4,\\
\partial \mathbb{G}_4 &= -4\mathfrak{m}\mathbb{G}_4 + \tfrac{7}{5}\mathbb{L}\mathbb{G}_6,\\
\partial \mathbb{G}_6 &= -6\mathfrak{m}\mathbb{G}_6 + \tfrac{800}{7}\mathbb{L}\mathbb{G}_4^2.
\end{aligned}$$

Since the ring of holomorphic modular forms M is generated by $\mathbb{G}_4$ and $\mathbb{G}_6$, we conclude that $M[\mathbb{L}, \mathfrak{m}]$ is indeed closed under the action of ∂. These formulae are equivalent to a computation due to Ramanujan. In general, for any $f \in M_n$ we have

$$\partial f = -nf\, \mathfrak{m} + 2\,\vartheta(f)\mathbb{L}, \tag{2.19}$$

where $\vartheta(f) \in M_{n+2}$ is the 'Serre derivative' of f [34] (53). The previous formula is compatible with the commutation relation $h = [\partial, \overline{\partial}]$, as the reader may wish to check.

For example, the Hecke normalised cusp form Δ of weight 12 satisfies $\vartheta(\Delta) = 0$. It follows that $\partial(\Delta) = -12\mathfrak{m}\Delta$, which gives another interpretation of $\mathfrak{m}$.

2.6 Bigraded Laplace operator

By taking polynomials in $\mathbb{L}$, ∂ and $\overline{\partial}$ one can define any number of operators acting on the space $\mathcal{M}$. Examples include the Laplace operator, Rankin–Cohen brackets Sect. 6, and the Bol operator (see [4]).

Definition 2.10 For all integers r, s, consider the Laplace operator

$$\begin{aligned}\Delta_{r,s} &= -\overline{\partial}_{s-1}\partial_r + r(s-1)\\ &= -\partial_{r-1}\overline{\partial}_s + s(r-1).\end{aligned} \tag{2.20}$$

The second definition is equivalent to the first by the commutation relation (2.14). These operators are compatible with complex conjugation: $\overline{\Delta_{r,s}f} = \Delta_{s,r}\overline{f}$.

From the definition and the formula $z = x + iy$, one verifies that

$$\begin{aligned}\Delta_{r,s} &= -4y^2 \frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}} + 2iry\frac{\partial}{\partial \overline{z}} - 2isy\frac{\partial}{\partial z}\\ &= \Delta_{0,0} + i(r-s)y\frac{\partial}{\partial x} - (r+s)y\frac{\partial}{\partial y},\end{aligned}$$

where $\Delta_{0,0}$ is the usual hyperbolic Laplacian

$$\Delta_{0,0} = -y^2\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big), \tag{2.21}$$

and $\Delta_{r,s}$ is denoted by $\Omega_{r,s}$ in the theory of Maass [23] **176** and (9). It follows from the previous computation (2.15) that $\Delta_{r,s}$ acts via:

$$\begin{aligned}&\Delta_{r,s}(\mathbb{L}^k q^m \overline{q}^n) =\\ &\Big(-4mn\,\mathbb{L}^2 + 2(kn+km+rn+sm)\,\mathbb{L} - k(k+r+s-1)\Big)\mathbb{L}^k q^m\overline{q}^n,\end{aligned} \tag{2.22}$$

which has integral coefficients. The modular transformation properties of Lemma 2.5 imply that the Laplace operator preserves the transformation law (2.3).

Corollary 2.11 *The operator $\Delta_{r,s}$ defines a linear map*

$$\Delta_{r,s} : \mathcal{M}_{r,s} \longrightarrow \mathcal{M}_{r,s}.$$

In particular, the hyperbolic Laplacian $\Delta_{0,0}$ acts on the modular-invariant space $\mathcal{M}_{0,0}$.

Let $\Delta : \mathcal{M} \to \mathcal{M}$ denote the linear operator which acts by $\Delta_{r,s}$ on $\mathcal{M}_{r,s}$. Let $\mathsf{w} : \mathcal{M} \to \mathcal{M}$ be the linear map which acts by multiplication by $w = r + s$ on $\mathcal{M}_{r,s}$.

Lemma 2.12 *The Laplace operator satisfies the equations*

$$(\Delta + \mathsf{w})\mathbb{L}f = \mathbb{L}\,\Delta f \tag{2.23}$$

i.e. $[\mathbb{L}, \Delta] = \mathsf{w}\mathbb{L}$, *and also* $[\partial, \Delta] = [\overline{\partial}, \Delta] = 0$.

Proof By (2.20), for any f we have

$$\begin{aligned}\mathbb{L}(\Delta_{r,s}f) &= \mathbb{L}(-\overline{\partial}\partial f + r(s-1)f) = (-\overline{\partial}\partial + r(s-1))\mathbb{L}f\\ &= (-\overline{\partial}\partial + (r-1)(s-2))\mathbb{L}f + (r+s-2)\mathbb{L}f = \Delta_{r-1,s-1}\mathbb{L}f + (r+s-2)\mathbb{L}f,\end{aligned}$$

which implies (2.23). Similarly,

$$\partial(\nabla_{r,s}f) = \partial(-\overline{\partial}\partial f + r(s-1)f) = (-\partial\overline{\partial} + (r+1-1)(s-1))\partial f = \nabla_{r+1,s-1}(\partial f)$$

which implies that $[\partial, \nabla] = 0$. By complex conjugating, $[\overline{\partial}, \nabla] = 0$. □

2.7 Real analytic Petersson inner product

Let

$$\mathcal{D} = \left\{ |z| > 1, |\mathrm{Re}(z)| < \tfrac{1}{2} \right\} \quad \text{and} \quad \mathrm{dvol} = \frac{\mathrm{d}x\,\mathrm{d}y}{y^2}$$

be the interior of the standard fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathfrak{H}$, and the $\mathrm{SL}_2(\mathbb{Z})$-invariant volume form on $\mathfrak{H}$ in its usual normalisation. For any r, s, let

$$\mathcal{S}_{r,s} \subset \mathcal{M}_{r,s}$$

denote the subspace of functions f whose constant part f^0 vanishes. If $\mathcal{S} = \bigoplus_{r,s} \mathcal{S}_{r,s}$, there is an exact sequence

$$0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{M} \longrightarrow \mathbb{C}[\mathbb{L}^{\pm}] \longrightarrow 0,$$

where the third map is the 'constant part' $f \mapsto f^0$.

Definition 2.13 For any integer n, consider the pairing

$$\begin{aligned} \mathcal{M}_{r,s} \times \mathcal{S}_{n-s,n-r} &\longrightarrow \mathbb{C} \\ f \quad \times \quad g \quad &\mapsto \langle f, g \rangle := \int_{\mathcal{D}} f(z)\overline{g(z)}\, y^n \,\mathrm{d\,vol}. \end{aligned} \tag{2.24}$$

The function $f(z)\overline{g(z)}y^n$ is modular of weights $(0,0)$ and lies in $\mathcal{S}_{0,0}$. The pairing (2.24) coindices with the usual Petersson inner product when restricted to holomorphic modular forms. To verify that the integral is finite, it suffices to bound the integrand near the cusp. Any element of $\mathcal{M}$ grows at most polynomially in y as $y \to \infty$, but via $q = \exp(2\pi i x)\exp(-2\pi y)$ and $\overline{q} = \exp(-2\pi i x)\exp(-2\pi y)$ it tends to zero in absolute value exponentially fast in y at the cusp since $g(z) \in \mathcal{S}$.

Two spaces $\mathcal{M}_{r,s}$ and $\mathcal{S}_{r',s'}$ can be paired via (2.24) if and only if $r - s = r' - s'$. Equivalently, $\langle f, g \rangle$ exists whenever $h(f) = h(g)$, where h was defined in (2.6).

The pairing (2.24) satisfies

$$\langle \overline{f}, \overline{g} \rangle = \overline{\langle f, g \rangle},$$

and, for any $e, f \in \mathcal{M}$ and $g \in \mathcal{S}$ such that $h(e) + h(f) = h(g)$, we have $\langle fe, g \rangle = \langle f, \overline{e}g \rangle$. Via (2.2), we also have for all $m \in \mathbb{Z}$:

$$\langle f, \mathbb{L}^m g \rangle = \langle \mathbb{L}^m f, g \rangle = (-2\pi)^m \langle f, g \rangle. \tag{2.25}$$

We now consider special cases of this pairing. When $n = r + s$, we have

$$\langle\ ,\ \rangle : \mathcal{M}_{r,s} \times \mathcal{S}_{r,s} \longrightarrow \mathbb{C}$$

which restricts to a positive definite quadratic form on $\mathcal{S}_{r,s}$, since

$$\langle f, f \rangle = \int_{\mathcal{D}} |f(z)|^2 y^n \,\mathrm{dvol} > 0 \quad \text{for } f \in \mathcal{S}_{r,s}.$$

2.8 Holomorphic projections [32]

In the particular case $n = r$, we have

$$\begin{aligned} \mathcal{M}_{r,s} \times \mathcal{S}_{r-s} &\longrightarrow \mathbb{C} \\ f \quad \times \quad g \quad &\mapsto \langle f, g \rangle \end{aligned} \tag{2.26}$$

where $S_{r-s} \subset \mathcal{S}_{r-s,0}$ is the space of holomorphic cusp forms of weight $r-s$. It is non-trivial only if $h(f) = r - s \geq 12$. Similarly, by setting $n = s$ we obtain

$$\begin{array}{cccccc} \mathcal{M}_{r,s} & \times & \overline{S}_{s-r} & \longrightarrow & \mathbb{C} & \\ f & \times & g & \mapsto & \langle f, \overline{g} \rangle, \end{array} \tag{2.27}$$

which is non-trivial only if $h(f) = r - s \leq -12$.

Equivalently, these two maps can be combined into a single linear map

$$\mathcal{M}_{r,s} \longrightarrow \mathrm{Hom}(S_{r-s}, \mathbb{C}) \oplus \mathrm{Hom}(\overline{S}_{s-r}, \mathbb{C}),$$

at least one component of which is zero. Since the classical Petersson inner product restricts to a non-degenerate quadratic form on S_{r-s}, we can identify $\mathrm{Hom}(S_{r-s}, \mathbb{C})$ with S_{r-s}, and similarly for its complex conjugate. Via this identification, the previous map defines a projection

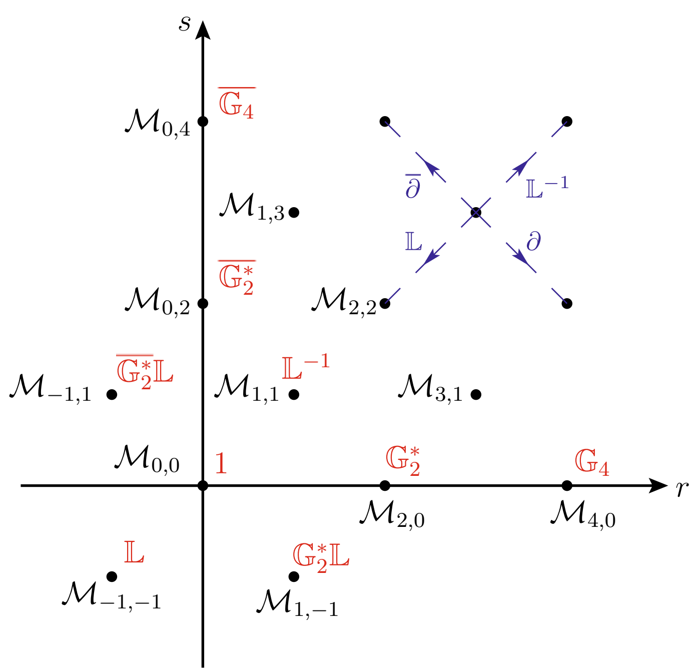
$$p = (p^h, p^a) : \mathcal{M}_{r,s} \longrightarrow S_{r-s} \oplus \overline{S}_{s-r}, \tag{2.28}$$

whose components we call the holomorphic and anti-holomorphic projections. By taking the direct sum over r and s, this defines a linear map

$$p = (p^h, p^a) : \mathcal{M} \longrightarrow S \oplus \overline{S}. \tag{2.29}$$

2.9 A picture of $\mathcal{M}$

The bigraded algebra $\mathcal{M}$ can be depicted as follows.



The dashed arrows represent the action of $\mathbb{L}, \mathbb{L}^{-1}, \partial, \overline{\partial}$. Each solid circle represents a copy of $\mathcal{M}_{r,s}$ for $r+s$ even. Some examples of modular forms are indicated in red.

3 Primitives and obstructions

In this section, we study the equation

$$\partial F = f, \tag{3.1}$$

where $F, f \in \mathcal{M}$. We say that $f \in \mathcal{M}$ has a modular ∂-primitive if (3.1) holds for some F. We exhibit three obstructions for the existence of modular primitives: the first is combinatorial, the second relates to modularity, and the third is arithmetic.

3.1 Constants

Let us view the operators $\partial_r, \overline{\partial}_s$ as (continuous) linear maps

$$\partial_r, \overline{\partial}_s : \mathbb{Q}[[q, \overline{q}]][\mathbb{L}^{\pm}] \longrightarrow \mathbb{Q}[[q, \overline{q}]][\mathbb{L}^{\pm}]$$

of formal power series, setting aside questions of modularity for the time being.

Lemma 3.1 *The kernels of these maps are*

$$\begin{aligned} \ker \partial_r &= \mathbb{L}^{-r}\mathbb{Q}[[\overline{q}]], \\ \ker \overline{\partial}_s &= \mathbb{L}^{-s}\mathbb{Q}[[q]]. \end{aligned} \tag{3.2}$$

In particular, $\ker \partial_r \cap \ker \overline{\partial}_s$ *vanishes if* $r \neq s$ *and is equal to* $\mathbb{Q}\mathbb{L}^{-r}$ *if* $r = s$.

Proof Since $\partial_r \mathbb{L}^k f = \mathbb{L}^k \partial_{r+k} f$ (2.13), we can assume, by multiplying by $\mathbb{L}^r$, that $r = 0$. The kernel of $\partial_0 = (z - \overline{z})\frac{\partial}{\partial z}$ consists of anti-holomorphic functions. The second formula in (3.2) is the complex conjugate of the first. □

We now consider the kernel of the operator ∂ acting on the space $\mathcal{M}$.

Proposition 3.2 *Let* $F \in \mathcal{M}_{r,s}$ *such that* $\partial_r F = 0$. *Then,*

$$\mathbb{L}^r F \in \overline{M}_{s-r},$$

where $\overline{M}_n$ *denotes the space of anti-holomorphic modular forms of weight n. In the case* $r > s$ *i.e. 'below the diagonal', F vanishes. In the case* $r = s$, *we have*

$$\ker \partial \cap \mathcal{M}_{r,r} = \mathbb{C}\,\mathbb{L}^{-r}.$$

Proof By Lemma 3.1, we can write $\mathbb{L}^r F = \overline{g}$ where $g : \mathfrak{H} \to \mathbb{C}$ is a holomorphic function. Since f (respectively $\mathbb{L}^r$) has weights (r, s) (respectively $(-r, -r)$), it follows that $\overline{g}$ has weights $(0, s-r)$ and transforms like a modular form of weight $s - r$, i.e. $g(\gamma(z)) = (cz + d)^{s-r} g(z)$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ of the form (2.1). Thus, $g \in M_{s-r}$. For the last part, use the well-known fact that there are no nonzero holomorphic modular forms of negative weight. □

Thus, if Eq. (3.1) has a solution, it is unique up to addition by an element of $\mathbb{C}\mathbb{L}^{-r}$ if $h(F) = 0$, and is unique if $h(F) > 0$.

Corollary 3.3 *Let* $F \in \mathcal{M}_{r,s}$ *and let* $f = \partial F$. *There is a solution* $F' \in \mathcal{M}_{r,s}$ *to* (3.1) *whose anti-holomorphic projection* $p^a(F')$ *vanishes. It is unique up to addition by a multiple of* $\mathbb{L}^{-r}\overline{\mathbb{G}}_{s-r}$ *for* $s - r \geq 4$, *where* $\mathbb{G}_n$ *is the Eisenstein series* (2.8).

Proof Since the Petersson inner product is non-degenerate, there exists a unique cusp form $g \in S_{s-r}$ such that $p^a(\overline{g}) = p^a(F)$. Then, $F' = F - (-2\pi)^r \mathbb{L}^{-r}\overline{g}$ has the required properties. The second part follows since the orthogonal complement of S_{s-r} in M_{s-r} is exactly the vector space generated by the Eisenstein series. □

3.2 Combinatorial obstructions

The maps $\partial_r, \overline{\partial}_s$ are far from surjective.

Lemma 3.4 *Suppose that $f \in \mathbb{C}[[q,\overline{q}]][\mathbb{L}^{\pm}]$ satisfies $f = \partial_r F$ for some $F \in \mathbb{C}[[q,\overline{q}]][\mathbb{L}^{\pm}]$. Then, the coefficients in its expansion (2.5) satisfy*

$$a^{(-r)}_{0,n} = 0 \quad \textit{for all } n \geq 0. \tag{3.3}$$

Proof Follows immediately from Lemma 2.6. □

This is not the only constraint: for every $m, n \geq 0$, there is a condition on the $a^{(k)}_{m,n}$, for varying k, in order for f to lie in the image of the map ∂_r. Nonetheless, (3.3) is already sufficient to rule out the existence of primitives in many interesting cases.

Corollary 3.5 *There exists no element $F \in \mathbb{C}[[q,\overline{q}]][\mathbb{L}^{\pm}]$ satisfying $\partial_0 F = \mathbb{L}\mathbb{G}^*_2$.*

Proof By (2.9), the $a^{(0)}_{0,0}$ term in $\mathbb{L}\mathbb{G}^*_2 = \mathbb{L}\mathbb{G}_2 - \frac{1}{4}$ is non-zero. This violates (3.3). □

3.3 A condition involving the pole filtration

Lemma 3.6 *If f satisfies the condition*

$$f \in P^{1-r}\mathcal{M}_{r+1,s-1}, \tag{3.4}$$

then it admits a combinatorial primitive $F \in P^{-r}\mathbb{C}[[q,\overline{q}]][\mathbb{L}^{\pm}]$ such that $\partial_r F = f$.

Proof Denote the coefficients in the expansion of f by $a^{(k)}_{m,n}$. By assumption, they vanish for all $k \leq -r$ and $k \geq N$ for some $N \geq -r$. Denote the coefficients of F by $b^{(k)}_{m,n}$. Equation (2.15) is equivalent to the set of equations

$$a^{(k)}_{m,n} = 2m\, b^{(k-1)}_{m,n} + (r+k)\, b^{(k)}_{m,n} \tag{3.5}$$

for every m, n. Fix an n and an $m \geq 1$. Then, if we set $b^{(k)}_{m,n} = 0$ for all $k \geq N$, (3.5) holds for all $k \geq N+1$. For $k = N$, we can solve it by setting

$$a^{(N)}_{m,n} = 2m\, b^{(N-1)}_{m,n}.$$

Suppose we have determined $b^{(k)}_{m,n}$ for all $k > K$. Then, Eq. (3.5) in the case $k = K+1$ can be solved uniquely for $b^{(K)}_{m,n}$ since $2m \neq 0$. The process terminates at $k = 1-r$, since for $k = -r$ Eq. (3.5) reduces to:

$$0 = a^{(-r)}_{m,n} = 2m\, b^{(-r-1)}_{m,n} + 0.$$

Setting $b^{(k)}_{m,n} = 0$ for all $k < -r$, we therefore obtain a complete solution to (3.5) for all values of k. In the case $m = 0$, the Eqs. (3.5) can be solved trivially, provided that (3.3) holds. This is certainly implied by (3.4). □

The commutation relation $\mathsf{h} = [\partial, \overline{\partial}]$ implies that $hf + \overline{\partial}\partial f$ is in the image of ∂ for all $f \in \mathcal{M}$. This remark, combined with (3.4), enables one to prove the existence of combinatorial primitives in many cases of interest.

3.4 Obstructions from the Petersson inner product

Another obstruction comes from the fact that a formal power series solution to (3.1) is not necessarily modular.

Theorem 3.7 *Let $f \in \mathcal{M}_{r,s}$. If f has a ∂-primitive in $\mathcal{M}$, then*

$$\langle f, g\rangle = 0 \quad \text{for all } g \in S_{r-s} \text{ holomorphic.} \tag{3.6}$$

In particular, f is in the kernel of the holomorphic projection (2.28).

Proof By multiplying by $\mathbb{L}^{r-1}$ and appealing to (2.13), we see that the equation $\partial F' = f$ has a solution if and only if there exists $F \in \mathcal{M}_{0,s-r+2}$ such that

$$\partial_0(i\pi F) = \mathbb{L}^{r-1}f.$$

From the definition of ∂_0, this implies that

$$\frac{\partial F}{\partial z} = \mathbb{L}^{r-2}f.$$

Let $g \in S_{r-s}$ be a holomorphic cusp form and consider the differential form

$$\omega = F\overline{g}\, d\overline{z}.$$

It is $\mathrm{SL}_2(\mathbb{Z})$-invariant, since $F, \overline{g}, d\overline{z}$ are of weights $(0, s-r+2)$, $(0, r-s)$, $(0, -2)$, respectively, and therefore their product is of type $(0, 0)$. It satisfies

$$d\omega = \frac{\partial F}{\partial z}\overline{g(z)}dz \wedge d\overline{z} = \mathbb{L}^{r-2}f(z)\overline{g(z)}\, dz \wedge d\overline{z} = (-2\pi)^{r-2}f(z)\overline{g(z)}y^r \,\mathrm{dvol},$$

which is also of type $(0, 0)$. By Stokes' theorem, we have

$$\int_{\partial\mathcal{D}} \omega = \int_{\mathcal{D}} d\omega = (-2\pi)^{r-2}\,\langle f, g\rangle.$$

Consider the left-hand integral along the boundary $\partial\mathcal{D}$ of the standard fundamental domain. By a classical argument using the modular invariance of ω, it gives zero, since the contributions along the vertical line segments (from ρ to $i\infty$ and $i\infty$ to $-\overline{\rho}$, where $\rho = e^{2\pi i/3}$) cancel due to translation-invariance $\omega(z) = \omega(z+1)$; the contributions along the segments of the circle $|z| = 1$ from ρ to i and from i to $-\overline{\rho}$ cancel due to $\omega(-z^{-1}) = \omega(z)$; and finally the contribution along a path from $i\infty$ to $i\infty + 1$, which corresponds to a small loop in the q-disc, also gives zero because g is cuspidal. □

The statement of the theorem can formally be written $\langle\partial F, g\rangle = 0$ if $g \in S$. By taking its complex conjugate, we also deduce that $\langle\overline{\partial}F, \overline{g}\rangle = 0$ for all $F \in \mathcal{M}_{r,s}$ and all cusp forms $g \in S_{s-r}$. These equations can be written

$$p^h(\partial(F)) = 0 \quad \text{and} \quad p^a(\overline{\partial}(F)) = 0 \quad \text{for all } F \in \mathcal{M}.$$

Corollary 3.8 *For every nonzero cusp form $f \in S_n$, and every $k \in \mathbb{Z}$, the equation $\partial F = \mathbb{L}^k f$ has no solution in $\mathcal{M}$.*

Proof By (2.13), we can assume that $k = 0$. If F were to exist, the previous theorem with $g = f$ would imply that $0 = \langle f, f\rangle$. But this contradicts the fact that the Petersson inner product is positive definite. □

Primitives of cusp forms do exist if one allows poles at the cusp (Sect. 11 and [4]).

3.5 Arithmetic obstructions

Although this is largely irrelevant here, since we work mostly over the complex numbers, the equation $\partial F = f$ involves some subtle questions regarding the field of definition of the coefficients $a^{(k)}_{m,n}$. Fundamentally, complex conjugation is not rationally defined on algebraic de Rham cohomology.

For example, $\partial F = \mathbb{G}_4\mathbb{L}$ has a unique solution given by a real analytic Eisenstein series $\mathcal{E}_{2,0} \in \mathcal{M}_{2,0}$, to be defined in Sect. 4, but it has no solution with rational coefficients. This is because $\mathcal{E}_{2,0}$ involves the value of the Riemann zeta function $\zeta(3)$, which is irrational as shown by Apéry. The examples of functions in $\mathcal{M}$ constructed in this paper arise from iterated integrals of modular forms, and their coefficients $a^{(k)}_{m,n}$ are, in a certain sense, periods. The period conjecture suggests that they are transcendental.

3.6 A class of modular iterated primitives

The functions studied in this paper lie in a special subclass of functions inside $\mathcal{M}$.

Definition 3.9 Consider the largest space of functions

$$\mathcal{MI} \subset \bigoplus_{r,s\geq 0} P^{-r-s}\mathcal{M}_{r,s},$$

equipped with an increasing 'length' filtration $\mathcal{MI}_k \subset \mathcal{MI}$ such that $\mathcal{MI}_k = 0$ if $k < 0$, and every $F \in \mathcal{MI}_k$ satisfies

$$\begin{aligned}\partial F &\in \mathcal{MI}_k + M[\mathbb{L}] \times \mathcal{MI}_{k-1},\\ \overline{\partial} F &\in \mathcal{MI}_k + \overline{M}[\mathbb{L}] \times \mathcal{MI}_{k-1},\end{aligned} \tag{3.7}$$

where M (resp. $\overline{M}$) denotes the ring of holomorphic (anti-holomorphic) modular forms. The conditions (3.7) are stable under the operation of taking sums of vector spaces, and therefore a largest such space exists and is unique.

By replacing $\mathcal{MI}$ with $\mathcal{MI} + \overline{\mathcal{MI}}$, we deduce that $\mathcal{MI}$ is closed under complex conjugation by maximality. This definition is computable: since $\mathcal{MI}_k$ is contained in the first quadrant, Eq. (3.7) implies that any $F \in \mathcal{MI}_k$ of weights $(n, 0)$ with $n \geq 0$ must satisfy

$$\partial F \in M[\mathbb{L}] \times \mathcal{MI}_{k-1}. \tag{3.8}$$

In this region, modular primitives are unique by Proposition 3.2 (up to a possible constant when $n = 0$). Then, for $F \in \mathcal{MI}_k$ of modular weights (r, s) with $r \geq s$, the first equation of (3.7) determines F in terms of previously determined functions by increasing induction on s. The functions in the region $r < s$ are deduced by complex conjugation [or by using the second equation of (3.7), starting from weights $(0, n)$].

Lemma 3.10 $\mathcal{MI}_0 = \mathbb{C}[\mathbb{L}^{-1}]$.

Proof Since $\mathcal{MI}_{-1} = 0$, any $F \in \mathcal{MI}_0$ of weights $(n, 0)$ satisfies $\partial F = 0$ by (3.8). If $n > 0$, then F vanishes by Proposition 3.2. Continuing in this manner, we see that any $F \in \mathcal{MI}_0$ of weights (r, s) for $r > s$ must also vanish, and in the case $r = s$, it must be of the form $F \in \mathbb{C}\mathbb{L}^{-r}$. Therefore, $\mathcal{MI}_0 \subset \mathbb{C}[\mathbb{L}^{-1}]$. Since $\partial\mathbb{L} = \overline{\partial}\mathbb{L} = 0$, the ring $\mathbb{C}[\mathbb{L}^{-1}]$ indeed satisfies the conditions (3.7) and hence $\mathcal{MI}_0 = \mathbb{C}[\mathbb{L}^{-1}]$. □

Remark 3.11 There are some variants. We can replace the space M of holomorphic modular forms in the Eq. (3.7) with another space of modular forms M' to define a class of functions $\mathcal{MI}(M')$. Some examples:

(1) Replace M with S, the space of cusp forms. Since cusp forms do not admit modular primitives, one deduces by induction that $\mathcal{MI}(S) = \mathbb{C}[\mathbb{L}^{-1}]$.
(2) Replace M with

$$E = \bigoplus_{n\geq 2} \mathbb{G}_{2n}\mathbb{Q} \tag{3.9}$$

the $\mathbb{Q}$-vector space generated by Eisenstein series. We obtain a space

$$\mathcal{MI}(E) \subset \mathcal{MI}.$$

In the sequel to this paper, we construct a subspace $\mathcal{MI}^E \otimes \mathbb{C} \subset \mathcal{MI}(E)$ (Sect. 10) and hope that equality holds, which would have deep consequences. We shall show below that $\mathcal{MI}(E)_k = \mathcal{MI}_k$ for $k = 0, 1$ but not for $k = 2$.

The class of functions $\mathcal{MI}$ has an interesting $\mathfrak{sl}_2$-module structure which could profitably be reformulated in the language of [6].

3.7 Homological interpretations

The following remarks can be skipped. Let $\mathcal{M}^{D+k} = \bigoplus_p \mathcal{M}_{p+k,p}$ denote the subspace of $\mathcal{M}$ upon which h acts by k. It is stable under multiplication by $\mathbb{L}$. Write $\mathcal{M}^D = \mathcal{M}^{D+0}$. Define an operator

$$\partial^D f = \partial(f)\mathrm{d}z + \overline{\partial}(f)\mathrm{d}\overline{z}.$$

Since $\mathrm{d}z$ and $d\overline{z}$ transform, respectively, like modular forms of weights $(-2, 0)$ or $(0, -2)$, it follows that ∂^D defines a linear map of bidegrees $(-1, -1)$:

$$\partial^D : \mathcal{M}^D \longrightarrow \mathcal{M}^{D+2}\,\mathrm{d}z \oplus \mathcal{M}^{D-2}\,\mathrm{d}\overline{z}.$$

It extends in the usual manner via the Leibniz rule to a linear map

$$\partial^D : \mathcal{M}^{D+2}\,\mathrm{d}z \oplus \mathcal{M}^{D-2}\,\mathrm{d}\overline{z} \longrightarrow \mathcal{M}^D\mathrm{d}z \wedge \mathrm{d}\overline{z}\,.$$

It acts by $\partial^D(f\,\mathrm{d}z + g\,\mathrm{d}\overline{z}) = (\partial g - \overline{\partial} f)\mathrm{d}z \wedge \mathrm{d}\overline{z}$. It follows from the fact that $[\partial, \overline{\partial}] = h$ vanishes on $\mathcal{M}^D$, that these operators satisfy $(\partial^D)^2 = 0$. Define the diagonal complex (which generalises to vector-valued modular forms to be considered below) by

$$0 \longrightarrow \mathcal{M}^D \xrightarrow{\partial^D} \mathcal{M}^{D+2}\mathrm{d}z \oplus \mathcal{M}^{D-2}d\overline{z} \xrightarrow{\partial^D} \mathcal{M}^D\mathrm{d}z \wedge \mathrm{d}\overline{z} \longrightarrow 0\,.$$

Denote its cohomology groups to be $H^i(\mathcal{M}^D)$ for $i = 0, 1, 2$. They inherit a grading in even degrees via the total weight grading on $\mathcal{M}$, where $\mathrm{d}z$ and $d\overline{z}$ have weight -2.

It follows from Lemma 3.1 that $H^0(\mathcal{M}^D) \cong \bigoplus_p \mathbb{L}^{-p}\mathbb{C}$. In general, $H^i(\mathcal{M}^D)$ is a free graded $\mathbb{C}[\mathbb{L}^{\pm}]$-module for all i. For example, the one-form of weight zero

$$\omega = \mathbb{G}_2^*\mathrm{d}z + \overline{\mathbb{G}_2^*}d\overline{z}$$

is closed and by Lemma 3.5 defines a non-trivial cohomology class $[\omega] \in H^1(\mathcal{M}^D)$. The obstructions to primitives discussed above can be interpreted in terms of this complex. For example, the proof of Theorem 3.7 can be interpreted as a functional:

$$\begin{aligned} \mathrm{gr}_{2m} H^2_c(\mathcal{M}^D) &\longrightarrow \mathbb{C} \\ \alpha \mathrm{d}z \wedge \mathrm{d}\overline{z} &\mapsto \int_{\mathcal{D}} y^{-m} \alpha \mathrm{d}z \wedge \mathrm{d}\overline{z}, \end{aligned}$$

where H^2_c denotes the subspace of H^2 representable by forms in $S^D \mathrm{d}z \wedge \mathrm{d}\overline{z}$. The obstructions in $H^1(\mathcal{M}^D)$ are purely combinatorial, by the following lemma:

Lemma 3.12 *Let $F \in \mathbb{C}[[q, \overline{q}]][\mathbb{L}^{\pm}]$. If*

$$\partial^D F \in \mathcal{M}^{D+2} dz \oplus \mathcal{M}^{D-2} d\overline{z}$$

is modular, then so is F, i.e. $F \in \mathcal{M}^D$.

Proof Suppose that $\partial F \in \mathcal{M}_{r+1,r-1}$ and $\overline{\partial} F \in \mathcal{M}_{r-1,r+1}$. Then, for every $\gamma \in \mathrm{SL}_2(\mathbb{Z})$,

$$F(\gamma z) - (cz + d)^r (c\overline{z} + d)^r F(z) = C_\gamma \mathbb{L}^{-r},$$

where $C_\gamma \in \mathbb{C}$, since by Lemma 2.5, the left-hand side lies in $\ker \partial \cap \ker \overline{\partial}$. It follows that $\gamma \mapsto C_\gamma \in \mathbb{C}$ is a cocycle. Since $\mathrm{SL}_2(\mathbb{Z})$ acts trivially on $\mathbb{C}$, $Z^1(\mathrm{SL}_2(\mathbb{Z}), \mathbb{C}) = \mathrm{Hom}(\mathrm{SL}_2(\mathbb{Z}); \mathbb{C})$ vanishes, and therefore $C_\gamma = 0$, i.e. F is modular of weights (r, r). □

4 Real analytic Eisenstein series

We consider in some detail the simplest possible family of non-holomorphic functions in $\mathcal{M}$ as a concrete illustration.

4.1 Modular primitives of Eisenstein series

Eisenstein series, unlike cusp forms, admit modular primitives in $\mathcal{M}$. Recall that the real analytic Eisenstein series are defined for $\mathrm{Re}\, s > 1$, $z \in \mathfrak{H}$ by the following function

$$E(z, s) = \frac{1}{2} \sum_{m,n \neq (0,0)} \frac{y^s}{|mz + n|^{2s}}.$$

Proposition 4.1 *For every $w \geq 1$, there exists a unique set of functions*

$$\mathcal{E}_{r,s} \in P^{-w} \mathcal{M}_{r,s}$$

with $r, s \geq 0$ and $r + s = w$, which satisfy the following equations:

$$\begin{aligned} \partial\, \mathcal{E}_{w,0} &= \mathbb{L}\mathbb{G}_{w+2} \\ \partial\, \mathcal{E}_{r,s} - (r+1)\mathcal{E}_{r+1,s-1} &= 0 \quad \text{for all } 1 \leq s \leq w \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} \overline{\partial}\, \mathcal{E}_{0,w} &= \mathbb{L}\overline{\mathbb{G}}_{w+2}, \\ \overline{\partial}\, \mathcal{E}_{r,s} - (s+1)\mathcal{E}_{r-1,s+1} &= 0 \quad \text{for all } 1 \leq r \leq w. \end{aligned} \tag{4.2}$$

These functions can be given explicitly by the following formula;

$$\mathcal{E}_{r,s}(z) = \frac{w!}{(2\pi i)^{w+2}} \frac{1}{2} \sum_{m,n} \frac{\mathbb{L}}{(mz + n)^{r+1} (m\overline{z} + n)^{s+1}}, \tag{4.3}$$

where the sum is over all integers $m, n \in \mathbb{Z}^2 \backslash (0, 0)$.

Proof The uniqueness follows from Proposition 3.2. For the existence, formula (4.3) converges and defines a modular function of weights r, s. We must verify (4.1) and (4.2). These follow from the following identity, which holds for any integers r, s:

$$\partial_r\Big(\frac{z-\overline{z}}{(mz+n)^{r+1}(m\overline{z}+n)^{s+1}}\Big)=(r+1)\Big(\frac{z-\overline{z}}{(mz+n)^{r+2}(m\overline{z}+n)^{s}}\Big).$$

By taking the complex conjugate, we deduce a similar formula for $\overline{\partial}$ on interchanging r and s. It follows from the definition of the holomorphic Eisenstein series as a sum:

$$\mathbb{G}_{w+2}=\frac{(w+1)!}{(2\pi i)^{w+2}}\frac{1}{2}\sum_{(m,n)\neq(0,0)}\frac{1}{(mz+n)^{2w+2}}$$

that $\mathcal{E}_{w,0}$ satisfies the first equation of (4.1). The first equation of (4.2) follows by conjugating. It remains to verify the Expansion (2.5). For this, note that for all $m\geq 1$ the Definition (4.3) implies the identity

$$\mathcal{E}_{m,m}=\frac{i}{(2\pi i)^{2m+1}}\frac{(2m)!}{y^m}E(z,m+1). \tag{4.4}$$

The expansion of the right-hand side is well known and lies in $\mathbb{C}[[q,\overline{q}]][\mathbb{L}^{\pm}]$. The expansions of the functions $\mathcal{E}_{r,s}$ are deduced from $\mathcal{E}_{m,m}$ by applying $\partial,\overline{\partial}$. □

We immediately deduce the following properties:

Corollary 4.2 *For all $r+s=w>0$, the functions $\mathcal{E}_{r,s}$ satisfy $\overline{\mathcal{E}}_{r,s}=\mathcal{E}_{s,r}$,*

$$\Delta\mathcal{E}_{r,s}=-w\,\mathcal{E}_{r,s},$$
$$p(\mathcal{E}_{r,s})=0,$$

where $p=p^h+p^a$ denotes the holomorphic and anti-holomorphic projections.

Proof The compatibility with complex conjugation follows by symmetry of (4.1) and (4.2) and uniqueness. The Laplace equation follows from (2.20), (4.1) and (4.2). The last equation follows from Theorem 3.7 since $\mathcal{E}_{r,s}$ is in the image of either ∂ or $\overline{\partial}$. □

Proposition 4.3 *The constant part of $\mathcal{E}_{r,s}$ is given by*

$$\mathcal{E}^0_{r,s}=\frac{-B_{w+2}}{2(w+1)(w+2)}\mathbb{L}+\frac{(-1)^s}{2}\frac{w!}{2^w}\binom{w}{r}\zeta(w+1)\mathbb{L}^{-w}, \tag{4.5}$$

where $w=r+s>0$ is even. Furthermore, $\mathcal{E}-\mathcal{E}^0_{r,s}$ has rational coefficients.

Proof The statement is well known for $r=s=w$, since it reduces to the Fourier expansion of the real analytic Eisenstein series $E(z,w+1)$. The remaining cases are deduced by applying ∂ via (4.1) and by $\mathcal{E}_{r,s}=\overline{\mathcal{E}}_{s,r}$. An alternative way to prove this theorem is to use the expression for $\mathcal{E}_{r,s}$ as the real part of the single iterated integral of holomorphic Eisenstein series [1] §8, and use the computation of the cocycle of the latter [1], Lemma 7.1, to write down the constant terms directly. See Sect. 8.4.2. □

4.2 Explicit formulae

For all $w\geq 1$, write

$$g^{(k)}_{2w+2}(q)=(-1)^k k!\sum_{n\geq 1}\frac{\sigma_{2w+1}(n)}{(2n)^{k+1}}q^n.$$

Then, for any $a+b=2w$, define

$$R_{a,b}=(-1)^a\binom{2w}{a}\sum_{k=b}^{a+b}\binom{a}{k-b}\frac{g^{(k)}_{2w+2}(q)}{\mathbb{L}^k}. \tag{4.6}$$

Then, the real analytic Eisenstein series are given explicitly by

$$\mathcal{E}_{a,b}=\mathcal{E}^0_{a,b}+R_{a,b}+\overline{R}_{b,a},$$

where $\mathcal{E}^0_{a,b}$ is (4.5). This formula in the case $a=b$ is equivalent to the known Fourier expansion of the real analytic Eisenstein series. One can verify the other cases by checking that they satisfy the differential Eqs. (4.1) and (4.2). See [4] for details.

4.3 Description of $\mathcal{MI}_1$

We already showed that $\mathcal{MI}_0=\mathbb{C}[\mathbb{L}^{-1}]$.

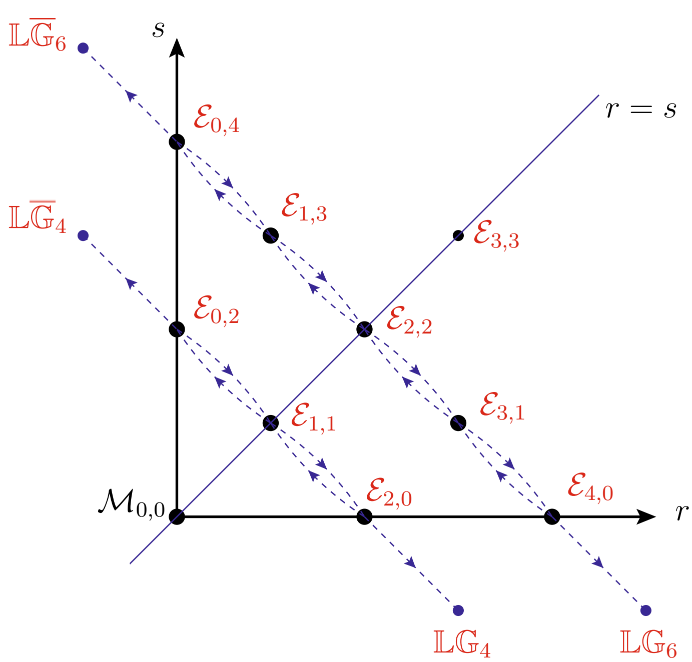
Corollary 4.4 *In length one,*

$$\mathcal{MI}_1=\mathcal{MI}_0\otimes_{\mathbb{C}}\bigoplus_{r,s\geq 0, r+s\geq 2}\mathbb{C}\,\mathcal{E}_{r,s}$$

Proof Let $F\in\mathcal{MI}_1$ of weights $(n,0)$. By (3.8), it satisfies $\partial F\in M\mathbb{L}$. Since ∂F is orthogonal to cusp forms by Theorem 3.7, it must satisfy $\partial F\in\mathbb{C}\mathbb{G}_{n+2}\mathbb{L}$. This equation has the unique family of solutions $F\in\mathbb{C}\mathcal{E}_{n,0}$. By eq. (3.7), the elements $F\in\mathcal{MI}_1$ of weights (r,s) with $r>s$ are iterated primitives of real analytic Eisenstein series and modular forms $M[\mathbb{L}]$, and hence also real analytic Eisenstein series, by a similar argument. We conclude that $\mathcal{MI}_1$ is contained in the $\mathbb{C}[\mathbb{L}^{-1}]$-module generated by the $\mathcal{E}_{r,s}$. Since the latter satisfy (3.7), this proves equality. □

4.4 Picture of the real analytic Eisenstein series

Based on the previous picture of $\mathcal{M}$, the real analytic Eisenstein series can be viewed as follows:



The dashed arrows going up and down the anti-diagonals are ∂ and $\overline{\partial}$. The classical real analytic Eisenstein series are the functions $\mathcal{E}_{n,n}$ lying along the diagonal $r = s$.

5 Eigenfunctions of the Laplacian

This section is not needed for the rest of the paper. We show that the space $\mathcal{M}$ has very limited overlap with the theory of Maass waveforms [23], and determine to what extent the solutions to a Laplace eigenvalue equation are not unique.

Call $F \in \mathcal{M}$ an eigenfunction of Δ if there exists $\lambda \in \mathbb{C}$, the eigenvalue, such that $\Delta F = \lambda F$. It decomposes into a sum of terms $F_{r,s} \in \mathcal{M}_{r,s}$ satisfying $\Delta_{r,s} F = \lambda F$.

Theorem 5.1 *Let F be an eigenfunction of the Laplacian. Then, its eigenvalue is an integer, and F is a linear combination over $\mathbb{C}[\mathbb{L}^{\pm}]$ of real analytic Eisenstein series $\mathcal{E}_{r,s}$, almost holomorphic modular forms and their complex conjugates.*

Let us write $\mathcal{HM} \subset \mathcal{M}$ to denote the space of Laplace eigenfunctions. It follows from Lemma 2.12 that it is stable under the action of $\mathcal{O} = \mathbb{Q}[\mathbb{L}^{\pm}][\partial, \overline{\partial}]$. Furthermore, the subspace $\mathcal{HM}(n)$ of eigenfunctions with eigenvalue n is stable under the action of the Lie algebra $\mathfrak{sl}_2$ generated by $\partial, \overline{\partial}$.

Every holomorphic modular form $f \in M_n$ lies in $\mathcal{HM}(0)$ since $\Delta f = -\partial_{n-1}\overline{\partial}_0 f = 0$. The same is true of $\mathfrak{m}$ defined in Sect. 2.5. More generally, $\mathbb{L}^k f$ is an eigenfunction with eigenvalue $(n-k-1)k$. Since the ring of almost holomorphic modular forms is generated by holomorphic modular forms and $\mathfrak{m}$ by the action of ∂, it follows that any almost holomorphic (or anti-holomorphic) modular form lies in $\mathcal{HM}$.

5.1 Proof of Theorem 5.1

Lemma 5.2 *Let $F \in \mathcal{M}_{r,s}$ such that $\Delta_{r,s} F = \lambda F$. Then, there exists an integer k_0 such that $\lambda = -k_0(k_0 + w - 1)$, where $w = r + s$ is the total weight. We can assume $k_0 = \min\{k_0, 1 - w - k_0\}$. Then, F is of the form*

$$F = \alpha\, \mathbb{L}^{k_0} + \beta\, \mathbb{L}^{1-w-k_0} + \sum_{k_0 \leq k \leq -s} \mathbb{L}^k f_k(q) + \sum_{k_0 \leq k \leq -r} \mathbb{L}^k g_k(\overline{q}), \tag{5.1}$$

where $\alpha, \beta \in \mathbb{C}$, and $f_k(q) \in \mathbb{C}[[q]]$, $g_k(\overline{q}) \in \mathbb{C}[[\overline{q}]]$ have no constant terms.

Proof Assume that F is nonzero and denote the coefficients in its expansion (2.5) by $a^{(k)}_{m,n}$. We first show that $a^{(k)}_{m,n} = 0$ if $mn \neq 0$. Fix m, n such that $a^{(k)}_{m,n} \neq 0$ for some k. Choose k maximal with this property. Taking the coefficient of $\mathbb{L}^{k+2} q^m \overline{q}^n$ in the equation $\Delta_{r,s} F = \lambda F$ implies, via (2.22), that $\lambda a^{(k+2)}_{m,n} = -4mn\, a^{(k)}_{m,n}$, which implies that $mn = 0$. Therefore, all $a^{(k)}_{m,n}$ vanish for $mn \neq 0$. Now, for any m, n, choose k minimal such that $a^{(k)}_{m,n}$ is nonzero. Equation (2.22) implies that $\lambda a^{(k)}_{m,n} = -k(k+w-1) a^{(k)}_{m,n}$, which proves the first part of the lemma. The equation $x^2 + x(w-1) + \lambda = 0$ has two integral solutions k_0 and $1 - w - k_0$, which are distinct since w is even. The assumption that k_0 is the smaller of the two implies that $a^{(k)}_{m,n}$ vanishes for all $k < k_0$.

Now consider a nonzero coefficient of the form $a^{(k)}_{m,0}$ with $m \neq 0$. Let k be maximal. Equation (2.22) implies that $\lambda a^{(k+1)}_{m,0} = 2m(k+s) a^{(k)}_{m,n} - k(k+w-1) a^{(k+1)}_{m,0}$, which implies that $m(k+s) = 0$ since $a^{(k+1)}_{m,0} = 0$. Therefore, $k = -s$. A similar computation with terms of the form $a^{(k)}_{0,n}$ shows that they all vanish if $k > -r$. It remains to determine the constant terms $a^{(k)}_{0,0}$. Equation (2.22) implies that $\lambda a^{(k)}_{0,0} = -k(k+w-1)a^{(k)}_{0,0}$, so by the above $a^{(k)}_{0,0}$ is non-zero only for $k \in \{k_0, 1-w-k_0\}$. □

Lemma 5.3 *Let $F \in \mathcal{M}_{r,s}$ be an eigenfunction of the Laplacian. Then, there exist integers $M, N \geq 0$ such that $\overline{\partial}^M \partial^N F \in \mathbb{C}[\mathbb{L}^{\pm}]$.*

Proof Apply ∂_r to the expansion (5.1). By Lemma 3.1, this annihilates the term $\mathbb{L}^k g_k(\overline{q})$ for $k = -r$. The terms of the form $\mathbb{L}^k g_k(\overline{q})$ are simply multiplied by $k + r$. Its action on terms of the form $\mathbb{L}^k f_k(q)$ increases the degree in $\mathbb{L}$ by at most one, by (2.15). Therefore, $\partial_r F$ has a similar expansion to (5.1), with (r, s) replaced by $(r+1, s-1)$. Applying ∂_{r-1} kills the term $\mathbb{L}^k g_k(\overline{q})$ for with $k = 1 - r$. Proceeding in this manner, every term of the form $\mathbb{L}^k g_k(\overline{q})$ is eventually annihilated (this also follows directly from Lemma 5.2 since $\partial^m F$ are eigenfunctions of the Laplacian with the same eigenvalue λ as F). Now, by a similar argument, repeated application of $\overline{\partial}$ annihilates all the terms of the form $\mathbb{L}^k f_k(q)$. □

Lemma 5.4 *The maps $\overline{\partial} : \widetilde{M} \to \widetilde{M}$ and $\partial : \widetilde{\widetilde{M}} \to \widetilde{\widetilde{M}}$ are surjective.*

Proof Since $\overline{\partial}\, \mathfrak{m} = 1$, any element $f \mathfrak{m}^i$, where $i \geq 0$ and $f \in M[\mathbb{L}^{\pm}]$, is the $\overline{\partial}$-image of $(i+1)^{-1} f \mathfrak{m}^{i+1}$. The second statement follows by complex conjugation. □

Lemma 5.5 *Consider the linear map $\partial : \widetilde{M}[\mathbb{L}^{\pm}] \to \widetilde{M}[\mathbb{L}^{\pm}]$. Then,* $\ker \partial \cong \mathbb{C}$ *and*

$$\mathrm{Coker}\, \partial \cong M[\mathbb{L}^{\pm}] \oplus \mathbb{C}\mathfrak{m}[\mathbb{L}^{\pm}] .$$

Proof The statement about the kernel follows immediately from Lemma 3.1. It follows from the calculations in Sect. 2.5, that for any $f \in M_n$ and $k \geq 0$,

$$\partial \mathfrak{m}^k f = (-k-n)\mathfrak{m}^{k+1} f + \text{terms of degree } \leq k \text{ in } \mathfrak{m}.$$

Since $\mathbb{L}$ commutes with ∂, all terms of the form $f\mathfrak{m}^k \mathbb{L}^r$, where $f \in M_n$, are in the image of ∂ whenever $k \geq 2$ or $k = 1$ and $n > 0$. Conclude using $\widetilde{M}[\mathbb{L}^{\pm}] = M[\mathfrak{m}, \mathbb{L}^{\pm}]$. □

Corollary 5.6 *Let $V \subset \mathcal{M}$ denote the $\mathbb{C}[\mathbb{L}^{\pm}]$-module generated by the real analytic Eisenstein series $\mathcal{E}_{r,s}$, $\widetilde{M}$ and $\widetilde{\widetilde{M}}$. If $F \in \mathcal{M}$ satisfies $\partial F \in V$, then $F \in V$. By complex conjugation, the same statement holds with ∂ replaced with $\overline{\partial}$.*

Proof By Proposition 4.1, the Eisenstein series $\mathbb{G}_{2n}\mathbb{L}^k$, for $n \geq 2$ and the functions $\mathcal{E}_{r,s}$ with $r > 0$ admit ∂-primitives in V. By the above, we can assume that ∂F is a linear combination of

$$\mathfrak{m}\mathbb{L}^k, \quad f\mathbb{L}^k, \quad \mathcal{E}_{0,2n}\mathbb{L}^k,$$

where f is a cusp form. Since these elements have distinct h-degrees, we can treat each case in turn, by linearity. But we showed in corollary 3.5 that $\mathfrak{m}\mathbb{L}^k$ has no ∂-primitive in $\mathcal{M}$, and likewise, in corollary 3.8 that cusp forms have no primitives either. The elements $\mathcal{E}_{0,2n}$ (and hence $\mathbb{L}^k \mathcal{E}_{0,2n}$) have no modular primitives by Lemma 3.4, since the coefficient of $\mathbb{L}$ in $\mathcal{E}^0_{0,2n}$ is nonzero by (4.5). Therefore, none of these cases can arise, and we conclude that if $\partial F \in V$, so too is $F \in V$. □

An eigenfunction of the Laplacian F satisfies $\overline{\partial}^M \partial^N F \in \mathbb{C}[\mathbb{L}^{\pm}] \subset V$. It follows from the previous corollary and induction on N that $F \in V$. This completes the proof.

Remark 5.7 In passing, we have shown that the ring of almost holomorphic modular forms $M[\mathfrak{m}, \mathbb{L}^{\pm}]$ is the subspace of functions $f \in \mathcal{M}$ such that $a^{(k)}_{m,n}(f) = 0$ for all $n > 0$, or equivalently, which satisfy $\overline{\partial}^N f = 0$ for sufficiently large N.

6 Mixed Rankin–Cohen brackets

This section can be skipped. Any operator in $\mathcal{O} = \mathbb{Q}[\mathbb{L}^{\pm}][\partial, \overline{\partial}]$ can be expressed as a polynomial in $\mathbb{L}^{\pm}, \frac{\partial}{\partial z}, \frac{\partial}{\partial \overline{z}}$. We wish to find elements of $\mathcal{O} \otimes \mathcal{O}$, which act via

$$\mathcal{O} \otimes \mathcal{O} : \mathcal{M} \otimes \mathcal{M} \longrightarrow \mathcal{M}\,,$$

which are homogeneous in $\mathbb{L}$ when expressed in terms of $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \overline{z}}$. Since these operators will not be used in this paper, we shall only illustrate how the theory of Rankin–Cohen brackets can be recovered in some basic examples and leave the many possible extensions to the reader.

Example 6.1 (Operators of order 1). Starting with the four operators given by $\partial \otimes \mathrm{id}$, $\mathrm{id} \otimes \partial$, and their complex conjugates, we form the general operator:

$$a_1 \partial \otimes \mathrm{id} + a_2 \mathrm{id} \otimes \partial + a_3 \overline{\partial} \otimes \mathrm{id} + a_4 \mathrm{id} \otimes \overline{\partial},$$

where $a_i \in \mathbb{Q}$. It acts upon $f \otimes g \in \mathcal{M}_{r_1,s_1} \otimes \mathcal{M}_{r_2,s_2}$ by

$$a_1 \left(\mathbb{L}' \frac{\partial f}{\partial z} + r_1 f \right) g + a_2 f \left(\mathbb{L}' \frac{\partial g}{\partial z} + r_2 g \right) + a_3 \left(-\mathbb{L}' \frac{\partial f}{\partial \overline{z}} + s_1 f \right) g + a_4 f \left(-\mathbb{L}' \frac{\partial g}{\partial \overline{z}} + s_2 g \right)$$

where $\mathbb{L}' i\pi = \mathbb{L}$. The terms of degree zero in $\mathbb{L}'$ vanish if and only if

$$a_1 r_1 + a_2 r_2 + a_3 s_1 + a_4 s_2 = 0\,.$$

A basis for its solutions are $(r_2, -r_1, 0, 0)$, $(s_1, 0, -r_1, 0)$ and $(0, 0, -s_2, s_1)$. Dividing by $\mathbb{L}'$, the first and third solutions yield the combinations:

$$\begin{aligned} [f, g]_1 &= r_2 \frac{\partial f}{\partial z} g - r_1 f \frac{\partial g}{\partial z}, \\ [f, g]_{\overline{1}} &= s_2 \frac{\partial f}{\partial \overline{z}} g - s_1 f \frac{\partial g}{\partial \overline{z}}, \end{aligned} \tag{6.1}$$

which are the first Rankin–Cohen bracket and its complex conjugate. The second solution defines an additional element $(Df)g$ of mixed weights, where

$$Df := \frac{1}{\mathbb{L}} (s_1 \partial_{r_1} - r_1 \overline{\partial}_{s_1}) f = s_1 \frac{\partial f}{\partial z} + r_1 \frac{\partial f}{\partial \overline{z}} \in \mathcal{M}_{r_1+2,s_1} \oplus \mathcal{M}_{r_1,s_1+2}. \tag{6.2}$$

It splits into two components of different modular weights in the algebra $\mathcal{M}$. For example, $\frac{\partial f}{\partial z} \in \mathcal{M}_{n+2,0} \oplus \mathcal{M}_{n,2}$ for any holomorphic modular form $f \in M_n$.

The properties of the brackets (6.1) are well-known. For instance, the bracket is anti-symmetric and satisfies the Jacobi identity [34] §5.2.

Example 6.2 (Operators of order 2). We can easily extend this analysis to operators of higher order. We exclude mixed terms such as (6.2). The operators of order 2 and bidegrees $(2, -2)$ are of the form

$$\partial^2 \otimes \mathrm{id}, \quad \partial \otimes \partial, \quad \mathrm{id} \otimes \partial^2.$$

A similar analysis to the one above produces a one-dimensional family of linear combinations which are homogeneous of degree two in $\mathbb{L}'$. They are generated by the second-order Rankin–Cohen bracket, defined for $f \in \mathcal{M}_{r_1,s_1}$ and $g \in \mathcal{M}_{r_2,s_2}$ by

$$[f,g]_2 = \frac{r_2(r_2+1)}{2}\frac{\partial^2 f}{\partial z^2}g - (r_1+1)(r_2+1)\frac{\partial f}{\partial z}\frac{\partial g}{\partial z} + \frac{r_1(r_1+1)}{2}f\frac{\partial^2 g}{\partial z^2}\,. \tag{6.3}$$

In bidegrees $(-2,2)$, we obtain the complex conjugate bracket. A new feature appears in bidegree $(0,0)$. Indeed, consider the following five terms of this type:

$$\partial\overline{\partial}\otimes \mathrm{id}, \quad \partial\otimes\overline{\partial}, \quad \overline{\partial}\otimes\partial, \quad \mathrm{id}\otimes\partial\overline{\partial}, \quad \mathrm{id}\otimes\mathrm{id}.$$

The commutation relation (2.14) implies that $\partial\overline{\partial}$, $\overline{\partial}\partial$ and id are linearly related, so there are exactly five such operators. The linear combination

$$r_2s_2\,\partial\overline{\partial}\otimes\mathrm{id} - s_1r_2\,\partial\otimes\overline{\partial} - s_2r_1\,\overline{\partial}\otimes\partial + r_1s_1\,\mathrm{id}\otimes\partial\overline{\partial} + r_1r_2(s_1+s_2)\mathrm{id}\otimes\mathrm{id}$$

generates a one-dimensional family of operators which become homogeneous in $\mathbb{L}'$ after rewriting them in terms of $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \overline{z}}$. The coefficient of $(\mathbb{L}')^2$ is the quantity

$$\big(f,g\big)_2 := s_1r_2\,\frac{\partial f}{\partial z}\frac{\partial g}{\partial \overline{z}} + s_2r_1\,\frac{\partial f}{\partial \overline{z}}\frac{\partial g}{\partial z} - r_1s_1 f\,\frac{\partial^2 g}{\partial z\partial\overline{z}} - r_2s_2\,g\,\frac{\partial^2 f}{\partial z\partial\overline{z}}\,, \tag{6.4}$$

which is symmetric in f and g and is an element of $\mathcal{M}_{r_1+r_2,s_1+s_2}$. It can be written more elegantly as a composition of operators, as follows:

$$\big(f,g\big)_2 = \big(\partial_z\otimes r_2 - r_1\otimes\partial_z\big)\big(s_1\otimes\partial_{\overline{z}} - \partial_{\overline{z}}\otimes s_2\big)\,(f\otimes g),$$

where $\partial_z = \partial/\partial z$, or again as a product of commuting determinants

$$\big(f,g\big)_2 = \begin{vmatrix}\partial_z\otimes\mathrm{id} & r_1\\ \mathrm{id}\otimes\partial_z & r_2\end{vmatrix}\begin{vmatrix} s_1 & \partial_{\overline{z}}\otimes\mathrm{id}\\ s_2 & \mathrm{id}\otimes\partial_{\overline{z}}\end{vmatrix}(f\otimes g).$$

Interesting operators of order two in the ring $\mathcal{O}\otimes\mathcal{O}$ therefore include: the Laplace operators $\Delta\otimes\mathrm{id}$ and $\mathrm{id}\otimes\Delta$, the Rankin–Cohen bracket $[f,g]_2$ and its conjugate, and a symmetric product $(f,g)_2$. All this is part of the general study of differential operators on $\mathcal{M}$, which we shall not pursue any further here.

7 Modular forms and equivariant sections

In this section, all tensor products are over $\mathbb{Q}$.

7.1 Reminders on representations of SL_2

For all $n \geq 0$, define

$$V_{2n} = \bigoplus_{r+s=2n} X^rY^s\mathbb{Q},$$

equipped with the right action of $\mathrm{SL}_2(\mathbb{Z})$ given by

$$(X,Y)\big|_\gamma = (aX+bY, cX+dY)$$

for γ of the form (2.1). There is an isomorphism of SL_2-representations

$$V_{2m}\otimes V_{2n} \cong V_{2m+2n}\oplus V_{2m+2n-2}\oplus\cdots\oplus V_{2|m-n|}\,.$$

We shall use the following choice of SL_2-equivariant projector

$$\delta^k : V_{2m} \otimes V_{2n} \longrightarrow V_{2m+2n-2k} \tag{7.1}$$

by setting

$$\delta^k = m \circ \Big(\frac{\partial}{\partial X} \otimes \frac{\partial}{\partial Y} - \frac{\partial}{\partial Y} \otimes \frac{\partial}{\partial X} \Big)^k ,$$

where $m : \mathbb{Q}[X, Y] \otimes \mathbb{Q}[X, Y] \to \mathbb{Q}[X, Y]$ is the multiplication map. For an equivalent formulation and further properties, see [1].

7.2 A characterisation of functions in $\mathcal{M}$

See also [35], Proposition 2.1.

Proposition 7.1 *Let $f : \mathfrak{H} \to V_{2n} \otimes \mathbb{C}$ be real analytic. Then, it can be written in two equivalent manners: either in the form*

$$f = \sum_{r+s=2n} f^{r,s}(z) X^r Y^s \tag{7.2}$$

for some real analytic functions $f^{r,s} : \mathfrak{H} \to \mathbb{C}$, or in the form

$$f = \sum_{r+s=2n} f_{r,s}(z)(X - zY)^r (X - \overline{z}Y)^s , \tag{7.3}$$

where $(z - \overline{z})^{2n} f_{r,s} : \mathfrak{H} \to \mathbb{C}$ are real analytic. The function f is equivariant:

$$f(\gamma(z))\big|_{\gamma} = f(z) \quad \textit{for all } \gamma \in \mathrm{SL}_2(\mathbb{Z}) \tag{7.4}$$

if and only if the coefficients $f_{r,s}$ are modular (2.3) of weight (r, s). Suppose now that the coefficients (7.2) of f admit expansions of the form

$$f^{r,s} \in \mathbb{C}[[q, \overline{q}]][z, \overline{z}].$$

Then, f is equivariant if and only if $f_{r,s} \in P^{-r-s}\mathcal{M}_{r,s}$.

Proof First, observe that the inclusion

$$\mathbb{Z}[z, \overline{z}][(X - zY), (X - \overline{z}Y)] \longrightarrow \mathbb{Z}[z, \overline{z}][X, Y]$$

becomes an isomorphism after inverting $z - \overline{z}$. Indeed, the inverse is given by

$$\begin{aligned} X &\mapsto \frac{z}{z - \overline{z}}(X - \overline{z}Y) - \frac{\overline{z}}{z - \overline{z}}(X - zY), \\ Y &\mapsto \frac{1}{z - \overline{z}}(X - \overline{z}Y) - \frac{1}{z - \overline{z}}(X - zY). \end{aligned} \tag{7.5}$$

This proves that the expansions (7.2) and (7.3) are equivalent. The identity

$$(X - \gamma(z)Y)\big|_{\gamma} = \frac{\det(\gamma)}{(cz + d)}(X - zY)$$

implies that $(X - zY)^r (X - \overline{z}Y)^s$ transforms, under the simultaneous action of $\mathrm{SL}_2(\mathbb{Z})$ on the argument z in the usual manner and on the right of V_{2n}, like a modular function of weights $(-r, -s)$. The coefficient of $(X - zY)^r (X - \overline{z}Y)^s$ for f equivariant is therefore modular of weights (r, s). For the last statement, the assumption on the Fourier expansions of $f^{r,s}$ implies that the coefficients $(z - \overline{z})^{r+s} f_{r,s}$ admit expansions in the ring $\mathbb{C}[[q, \overline{q}]][z, \overline{z}]$ by (7.5). By Lemma 2.2, the $f_{r,s}$ have expansions of the form (2.5). □

We construct equivariant functions f from iterated integrals. These only involve non-negative powers of $\log q$. Their coefficients f_{ij} will have poles in $\mathbb{L}$ of degree at most the total weight, and their modular weights will naturally be located in the first quadrant $r, s \geq 0$.

7.3 Vector-valued differential equations

The operators $\partial, \overline{\partial}$ of Definition 2.4 admit the following interpretation.

Proposition 7.2 *Let $F, A, B : \mathcal{H} \to V_{2n} \otimes \mathbb{C}$ be real analytic. Then, the equation*

$$\frac{\partial F}{\partial z} = \frac{2\pi i}{2} A(z) \tag{7.6}$$

is equivalent to the system of equations for all $r+s=2n$, and $r,s \geq 0$:

$$\begin{aligned} \partial F_{2n,0} &= \mathbb{L} A_{2n,0}, \\ \partial F_{r,s} - (r+1) F_{r+1,s-1} &= \mathbb{L} A_{r,s} \quad \textit{if} \quad s \geq 1. \end{aligned} \tag{7.7}$$

In a similar manner,

$$\frac{\partial F}{\partial \overline{z}} = \frac{2\pi i}{2} B(z) \tag{7.8}$$

is equivalent to the following system of equations:

$$\begin{aligned} \overline{\partial} F_{0,2n} &= \mathbb{L} B_{0,2n}, \\ \overline{\partial} F_{r,s} - (s+1) F_{r-1,s+1} &= \mathbb{L} B_{r,s} \quad \textit{if } r \geq 1. \end{aligned} \tag{7.9}$$

Proof Differentiate the expression (7.3) to obtain

$$\frac{\partial F}{\partial z} = \sum_{r+s=2n} \Big(\frac{\partial F_{r,s}}{\partial z} (X - zY)^r (X - \overline{z} Y)^s - r Y F_{r,s} (X - zY)^{r-1} (X - \overline{z} Y)^s \Big).$$

On replacing Y using the second line of (7.5), the right-hand side becomes

$$\sum_{r+s=2n} \Big(\Big(\frac{\partial F_{r,s}}{\partial z} + \frac{r F_{r,s}}{z - \overline{z}} \Big) (X - zY)^r (X - \overline{z} Y)^s - \frac{r F_{r,s}}{z - \overline{z}} (X - zY)^{r-1} (X - \overline{z} Y)^{s+1} \Big).$$

Multiplying through by $z - \overline{z}$, collecting terms and using Definition 2.4, we see that the Eq. (7.6) is equivalent to the system of equations

$$\partial_r F_{r,s} - (r+1) F_{r+1,s-1} = i\pi (z - \overline{z}) A_{r,s}$$

for $1 \leq s \leq 2n$, and in the case $s = 0$, $\partial_{2n} F_{2n,0} = i\pi (z - \overline{z}) A_{2n,0}$. Conclude using (2.2). The second set of equations can be deduced by conjugation. □

The commutation relation $[\partial, \overline{\partial}] = \mathsf{h}$ of Proposition 2.9 is equivalent to

$$\frac{\partial^2 F}{\partial z \partial \overline{z}} = \frac{\partial^2 F}{\partial \overline{z} \partial z}.$$

Lemma 7.3 *Suppose that $A : \mathfrak{H} \to V_{2n}$ and set*

$$F = \frac{\delta^k}{(k!)^2} \Big((X - zY)^{2m} \otimes A \Big).$$

Then, $F : \mathfrak{H} \rightarrow V_{2m+2n-2k}$ vanishes if $k > 2n$ or $k > 2m$, but otherwise satisfies

$$F_{r,s} = (z - \overline{z})^k \binom{2m}{k}\binom{s+k}{k} A_{r-2m+k,s+k}, \tag{7.10}$$

where we set $A_{p,q} = 0$ for $p < 0$ or $q < 0$. Therefore, $F_{r,s}$ vanishes if $r < 2m - k$, or equivalently, $s + k > 2n$.

Proof By direct application of the definition of δ^k, we find that

$$\begin{aligned}&\frac{\delta^k}{(k!)^2}\Big((X - zY)^{2m} \otimes A(X, Y)\Big)\\ &\quad= (z - \overline{z})^k \sum_{r,s,r+2m\geq k} \binom{2m}{k}\binom{s}{k} A_{r,s}(X - zY)^{r+2m-k}(X - \overline{z}Y)^{s-k},\end{aligned}$$

where

$$A(X, Y) = \sum_{r+s=2n} A_{r,s}(X - zY)^r(X - \overline{z}Y)^s.$$

Equation (7.10) follows on replacing (r, s) with $(r - 2m + k, s + k)$. □

Combining the lemma with Proposition 7.2, we find that if

$$\frac{\partial F}{\partial z} = (\pi i)^{k+1} \frac{\delta^k}{(k!)^2}\Big(f_{2m+2}(z)(X - zY)^{2m} \otimes A(X, Y)\Big)$$

then

$$\partial F_{2n,0} = \binom{2m}{k} \mathbb{L}^{k+1} f_{2m+2}(z) A_{2n-2m+k,k}(z). \tag{7.11}$$

7.4 Example: real analytic Eisenstein series

Let us write, for $w > 0$ even:

$$\mathcal{E}_w(z) = \sum_{r+s=w} \mathcal{E}_{r,s}(X - zY)^r(X - \overline{z}Y)^s.$$

It is equivariant for $\mathrm{SL}_2(\mathbb{Z})$. Consider also the equivariant 1-form

$$\underline{E}_{w+2}(z) = 2\pi i\, \mathbb{G}_{w+2}(z)(X - zY)^w \mathrm{d}z. \tag{7.12}$$

Then, by Proposition (7.6) the systems of Eqs. (4.1) and (4.2) are equivalent to the following differential equation:

$$\begin{aligned}d\mathcal{E} &= \frac{1}{2}\Big(\underline{E}_{w+2}(z) + \overline{\underline{E}_{w+2}(z)}\Big)\\ &= \mathrm{Re}\left(\underline{E}_{w+2}(z)\right).\end{aligned}$$

The real analytic Eisenstein series of Sect. 4 are the real parts of primitives of vector-valued holomorphic Eisenstein series [1], §9.2.2. This motivates a general construction of modular forms in $\mathcal{M}$ via equivariant iterated integrals of modular forms.

8 Modular forms from equivariant iterated integrals

The main idea behind our construction of functions in $\mathcal{M}$ is a modification of the theory of single-valued periods as we presently explain in some simple examples.

8.1 Single-valued functions

The logarithm

$$\log z = \int_1^z \frac{\mathrm{d}t}{t} \tag{8.1}$$

is a multivalued analytic function on $\mathbb{C}^\times$. This means that its pull-back to the universal covering space $\mathbb{C}$ of $\mathbb{C}^\times$ based at 1 is an analytic function. Indeed, via the covering map $\exp : \mathbb{C} \to \mathbb{C}^\times$, it simply corresponds to the function z on $\mathbb{C}$. The fundamental group of $\mathbb{C}^\times$ at the point 1 is isomorphic to $\mathbb{Z}$ and is generated by a simple loop around the origin. Analytic continuation around this loop creates a discontinuity

$$\log z \mapsto \log z + 2\pi i\,. \tag{8.2}$$

Since the monodromy period $2\pi i$ is purely imaginary, the multivaluedness of the logarithm can be eliminated by taking its real part:

$$\log |z| = \mathrm{Re}\,(\log z).$$

This is the 'single-valued' version of the logarithm. It is a well-defined function on $\mathbb{C}^\times$, invariant under the left action of $\mathbb{Z} = \pi_1(\mathbb{C}^\times, 1)$. The dilogarithm

$$\mathrm{Li}_2(z) = \sum_{k\geq 1} \frac{z^k}{k^2}$$

defined for $|z| < 1$ and analytically continued to a multivalued function on $\mathbb{C}^\times \backslash \{0, 1\}$, satisfies the equation $d\mathrm{Li}_2(z) = -\log(1-z)\frac{\mathrm{d}z}{z}$, and has a single-valued version:

$$D(z) = \mathrm{Im}\left(\mathrm{Li}_2(z) + \log|z| \log(1-z)\right)$$

called the Bloch–Wigner function. In the following sections, we construct modular analogues of the functions $\log|z|$ and $D(z)$.

There is a general way to associate single-valued functions to any period integrals [2] §4, §8.3 generalising (8.1). The latter can depend on parameters, or even be constant. A variant of this construction, applied to iterated integrals of modular forms, yields a class of functions in $\mathcal{M}$. This follows from Proposition 7.1 since a real analytic section $f : \mathfrak{H} \to V_n \otimes \mathbb{C}$ is equivariant if and only if the coefficients $f_{r,s}$ in the expansion $f = \sum_{r,s} f_{r,s}(X - zY)^r (X - \overline{z}Y)^s$ are modular of weights (r, s). The equivariance

$$f(\gamma z)\big|_\gamma = f(z) \quad \text{for all } \gamma \in \mathrm{SL}_2(\mathbb{Z})$$

can be interpreted as single-valuedness of the vector-valued function $f(z)$ on the orbifold quotient of $\mathfrak{H}$ by the action of $\mathrm{SL}_2(\mathbb{Z})$.

8.2 Notation

For f a holomorphic modular form of weight n, let us denote by

$$\underline{f}(z) = 2\pi i f(z)(X - zY)^{n-2}\mathrm{d}z\,.$$

It is invariant under the action of $\mathrm{SL}_2(\mathbb{Z})$ on z and X, Y. We shall write

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

8.3 The modular function Im(*z*)

Denote by

$$F(\tau) = \int_\tau^{i\infty} \underline{1} = 2\pi i \int_\tau^{i\infty} (X - zY)^{-2} \mathrm{d}z = \frac{2\pi i}{Y(X - zY)} .$$

We obtain in this manner a $\mathbb{Q}(X, Y)$-valued cocycle

$$F(\gamma(\tau))\big|_\gamma - F(\tau) = C_\gamma,$$

where $C : \mathrm{SL}_2(\mathbb{Z}) \to \mathbb{Q}(X, Y)$ is the function

$$C_\gamma = -\frac{c\, 2\pi i}{Y(Xc + Yd)} \quad \text{where } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} .$$

This cocycle is cuspidal (vanishes for $\gamma = T$). Since C_γ is imaginary, the real part $\mathrm{Re}\, F(\tau)$ is modular equivariant. Indeed, we have

$$\mathrm{Re}\, F(\tau) = \frac{\mathbb{L}}{(X - \tau Y)(X - \overline{\tau} Y)},$$

which is $\mathrm{SL}_2(\mathbb{Z})$-invariant since $\mathbb{L}$ is modular of weights $(-1, -1)$.

8.4 Primitives of holomorphic modular forms

Now, we construct, or fail to construct, equivariant versions of classical Eichler integrals in the same vein.

8.4.1 Cusp forms

Let $f \in S_{2n}$ be a cusp form with rational Fourier coefficients. Let

$$F(\tau) = \int_\tau^{i\infty} \underline{f}(z) = 2\pi i \int_\tau^{i\infty} f(z)(X - zY)^{2n-2} \mathrm{d}z . \tag{8.3}$$

It satisfies, by invariance of $\underline{f}$, the following: monodromy equation

$$F(\gamma(\tau))\big|_\gamma \;=\; F(\tau) + C_\gamma , \tag{8.4}$$

for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ and $\tau \in \mathfrak{H}$. It is the analogue of (8.2). It follows from (8.4) that the function $C : \mathrm{SL}_2(\mathbb{Z}) \to \mathbb{C}[X, Y]$ is a cocycle for $\mathrm{SL}_2(\mathbb{Z})$, and indeed that

$$C_S = C_S^+ + i\, C_S^-,$$

where $C_S^+, C_S^- \in \mathbb{R}[X, Y]$ are the even and odd real period polynomials of f. By the Eichler–Shimura theorem, the classes of C^+ and C^- are independent in group cohomology $H^1(\mathrm{SL}_2(\mathbb{Z}), \mathbb{C}[X, Y])$, and so there is no way, by taking real and imaginary parts, that we can kill the right-hand side of (8.4) to obtain a single-valued function. Therefore, a single iterated integral, or primitive, of a cusp form yields nothing new. Indeed, by Proposition 7.2, such a function, if it existed, would provide a solution to the equation $\partial F = f$, in $\mathcal{M}$, which would contradict 3.8. This obstruction can be circumvented by introducing poles; cusp forms do have primitives in $\mathcal{M}^!$ (see §11).

8.4.2 Eisenstein series

If $f = \mathbb{G}_{2k+2}$, the corresponding integral (8.3) diverges, but can be regularised in the manner of [1] §4, yielding a primitive

$$F(\tau) = \int_{\tau}^{\vec{1}_\infty} \underline{E_{2k+2}}(z),$$

which satisfies (8.4). Here, $\vec{1}_\infty$ denotes the unit tangential basepoint at the cusp. The associated Eisenstein cocycle C satisfies ([1], §7):

$$C_\gamma = \frac{(2k)!}{2} \frac{\zeta(2k+1)}{(2\pi i)^{2k}} \, Y^{2k}\Big|_{\gamma-1} + (2\pi i) e^0_{2k+2}(\gamma),$$

where $e^0_{2k+2}(\gamma) \in \mathbb{Q}[X, Y]$. Now, if we consider the real part of $F(\tau)$, it satisfies the analogue of (8.4) with C replaced with $\mathrm{Re}\, C$. The key point is that the real part of C_γ only involves the first term in the previous equation, which is a coboundary for $\mathrm{SL}_2(\mathbb{Z})$. Therefore, the function $\mathrm{Re}\, F(\tau)$ can be modified in the following manner to define a vector-valued real analytic function

$$\mathcal{E}_{2k}(X, Y)(\tau) \; = \mathrm{Re}\, F(\tau) \; + \; \frac{(2k)!}{2} \frac{\zeta(2k+1)}{(2\pi i)^{2k}} Y^{2k},$$

which is now invariant under the action of $\mathrm{SL}_2(\mathbb{Z})$. This function can be rewritten

$$\mathcal{E}_{2k}(X, Y)(z) = \sum_{r+s=2k} \mathcal{E}_{r,s}(z)(X - zY)^r (X - \bar{z}Y)^s,$$

where the functions $\mathcal{E}_{r,s} : \mathfrak{H} \to \mathbb{C}$ are modular, and lie in $\mathcal{M}_{r,s}$. As the notation suggests, the coefficients are precisely the real analytic functions $\mathcal{E}_{r,s}$ defined in Sect. 4. They are the analogues of the single-valued functions $\log |z|$ on $\mathbb{C}^\times$.

Remark 8.1 The systematic use of tangential basepoints to regularise period integrals associated with Eisenstein series clarifies and simplifies many constructions in the literature. Since this was only recently introduced [1] §4, we provide some commentary:

- The period polynomial of the Eisenstein series is equivalent to formulae which must have been known to Ramanujan and are given by [1], §9:

$$e^0_{2k}(S) = \frac{(2k-2)!}{2} \sum_{i=1}^{k-1} \frac{B_{2i}}{(2i)!} \frac{B_{2k-2i}}{(2k-2i)!} X^{2i-1} Y^{2k-2i-1} \, ,$$

$$e^0_{2k}(T) = \frac{(2k-2)!}{2} \frac{B_{2k}}{(2k)!} \Big(\frac{(X+Y)^{2k-1} - X^{2k-1}}{Y} \Big) \, .$$

  However, I could not find this precise formulation elsewhere. The literature tends to focus on period polynomials (value of a cocycle on S) which only determine the cocycle in the cuspidal case. Zagier's approach is to introduce poles in X, Y to force the Eisenstein cocycle to be cuspidal.
- It is often stated that $X^{2n} - Y^{2n}$ is the period polynomial of an Eisenstein series, but is in fact the value of the cuspidal coboundary cocycle at S and vanishes in cohomology. It is, however, nonzero in *relative* cohomology and is dual to the Eisenstein cocycle under the Petersson inner product (which pairs cocycles and compactly supported cocycles). This is discussed in [1] §9.
- The 'extra' relation satisfied by period polynomials of cusp forms [21] expresses the orthogonality of the cocycle of a cusp form to the Eisenstein cocycle with respect to the Haberland-Petersson inner product.

9 Equivariant double iterated integrals

We now define equivariant versions of double Eisenstein integrals, which are modular analogues of the Bloch–Wigner function $D(z)$.

9.1 Double Eisenstein integrals

Recall that

$$\mathcal{E}_{2n} : \mathfrak{H} \longrightarrow V_{2n} \otimes \mathbb{C}$$

is the modular-invariant real analytic function which satisfies

$$d\,\mathcal{E}_{2n} = \mathrm{Re}\,\underline{E}_{2n+2}\,. \tag{9.1}$$

For every $a, b \geq 2$, consider the family of one-forms

$$\mathcal{D}_{2a,2b} \;:\; \mathfrak{H} \longrightarrow \big(V_{2a-2} \otimes V_{2b-2}\big) \otimes \big(\mathbb{C}\,\mathrm{d}z + \mathbb{C}\,\mathrm{d}\overline{z}\big)$$
$$\mathcal{D}_{2a,2b} \;=\; \underline{E}_{2a} \otimes \mathcal{E}_{2b-2} + \mathcal{E}_{2a-2} \otimes \overline{\underline{E}_{2b}}.$$

They are modular invariant: for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, we have

$$\mathcal{D}_{2a,2b}(\gamma z)\big|_{\gamma} = \mathcal{D}_{2a,2b}(z)\,.$$

Lemma 9.1 *The family of forms $\mathcal{D}_{2a,2b}$ are closed:*

$$d\mathcal{D}_{2a,2b} = 0.$$

Proof By (9.1) and writing $\overline{\mathrm{d}z} \wedge \mathrm{d}z = -\mathrm{d}z \wedge \overline{\mathrm{d}z}$, we find that

$$d\,\mathcal{D}_{2a,2b} = -\underline{E}_{2a} \otimes \overline{\underline{E}}_{2b} + \underline{E}_{2a} \otimes \overline{\underline{E}}_{2b} = 0\,.$$

□

By the previous lemma, it makes sense to consider the indefinite integral

$$K_{2a,2b}(z) \;=\frac{1}{2}\int_z^{\vec{1}_\infty} \mathcal{D}_{2a,2b}(z)\,, \tag{9.2}$$

since the integrand is closed, and the integral only depends on the homotopy class of the chosen path. This function can be written in terms of real and imaginary parts of products of iterated integrals of Eisenstein series. Indeed, we have

$$K_{2a,2b}(z) \equiv \frac{1}{2i}\,\mathrm{Im}\left(\int_z^{\vec{1}_\infty} \underline{E}_{2a}\underline{E}_{2b}\right) - \frac{1}{2}\mathrm{Re}\left(\int_z^{\vec{1}_\infty} \underline{E}_{2a}\right) \times \int_z^{\vec{1}_\infty} \overline{\underline{E}_{2b}}$$

modulo iterated integrals of length one (we integrate from left to right).

The real analytic function

$$K_{2a,2b}(z) : \mathfrak{H} \longrightarrow V_{2a-2} \otimes V_{2b-2} \otimes \mathbb{C}$$

satisfies the pair of differential equations

$$\frac{\partial}{\partial z}K_{2a,2b} = \frac{2\pi i}{2}\mathbb{G}_{2a}(z)(X - zY)^{2a-2} \otimes \mathcal{E}_{2b-2}(z),$$
$$\frac{\partial}{\partial \overline{z}}K_{2a,2b} = \frac{2\pi i}{2}\mathcal{E}_{2a-2}(z) \otimes \overline{\mathbb{G}_{2b}(z)}(X - \overline{z}Y)^{2b-2}.$$

Recall the normalised projection

$$(\pi i)^k \frac{\delta^k}{(k!)^2} : V_{2a-2} \otimes V_{2b-2} \otimes \mathbb{C} \longrightarrow V_{2a+2b-4-2k} \otimes \mathbb{C}.$$

Definition 9.2 For any $a, b \geq 1$ and $0 \leq k \leq \min\{2a, 2b\}$, define

$$K^{(k)}_{2a+2,2b+2}(z) = (\pi i)^k \frac{\delta^k}{(k!)^2} K_{2a+2,2b+2}(z)\,.$$

By Eq. (7.11), its lowest weight components satisfy

$$\partial\left(K^{(k)}_{2a+2,2b+2}\right)_{2a+2b-2k,0} = \binom{2a}{k} \mathbb{L}^{k+1} \mathbb{G}_{2a+2} \mathcal{E}_{2b-k,k}. \tag{9.3}$$

9.2 Equivariant versions of double Eisenstein integrals

Since $\mathcal{D}_{2a,2b}$, and hence $\mathrm{d}\,K^{(k)}_{2a,2b}$, is modular equivariant, it follows that

$$C_\gamma = K^{(k)}_{2a,2b}(\gamma\tau)\big|_\gamma - K^{(k)}_{2a,2b}(\tau)$$

is constant (does not depend on τ), and defines a cocycle

$$C_\gamma \in Z^1(\mathrm{SL}_2(\mathbb{Z}), V_{2a+2b-4-2k} \otimes \mathbb{C}).$$

By the Eichler–Shimura theorem, any such cocycle can be expressed as a linear combination of cocycles of cusp forms or their complex conjugates, Eisenstein series, and a coboundary $c|_{\gamma-\mathrm{id}}$ for some $c \in V_{2a+2b-4-2k} \otimes \mathbb{C}$. Define a modified function

$$M^{(k)}_{2a,2b} = K^{(k)}_{2a,2b} - c - \frac{1}{2}\int_z^{\vec{1}_\infty} (\underline{f} + \overline{\underline{g}}), \tag{9.4}$$

where f is a holomorphic modular form, and g a cusp form, both of weight $2a+2b-2-2k$, which is modular equivariant:

$$M^{(k)}_{2a,2b}(\gamma\tau)\big|_\gamma = M^{(k)}_{2a,2b}(\tau)\,.$$

This equation uniquely determines the functions f, g, c and $M^{(k)}_{2a,2b}$, except in the case when $2a + 2b - 4 - 2k = 0$ since we can add an arbitrary constant $c \in \mathbb{C}$. Extracting the coefficients of $M^{(k)}_{2a,2b}$ via (7.3) yields a class of functions in $\mathcal{M}$.

Theorem 9.3 *Let $a, b \geq 1$ and $0 \leq k \leq \min\{2a, 2b\}$, and set $w = a + b - k$. There exists a family of elements $F_{r,s} \in \mathcal{MI}_2 \cap \mathcal{M}_{r,s}$ of total modular weight $2w = r + s$ where $r, s \geq 0$, which satisfy the equations*

$$\begin{aligned} \partial F_{r,s} - (r+1)F_{r+1,s-1} &= \binom{2a}{k}\binom{k+s}{k} \mathbb{L}^{k+1} \mathbb{G}_{2a+2} \mathcal{E}_{2b-k-s,k+s} \quad \textit{if } s \geq 1 \\ \partial F_{2w,0} &= \binom{2a}{k} \mathbb{L}^{k+1} \mathbb{G}_{2a+2} \mathcal{E}_{2b-k,k} + \mathbb{L} f, \end{aligned} \tag{9.5}$$

where f is the unique cusp form of weight $2w + 2$ satisfying

$$p^h\left(\binom{2a}{k} \mathbb{L}^{k+1} \mathbb{G}_{2a+2} \mathcal{E}_{2b-k,k} + \mathbb{L} f\right) = 0 \tag{9.6}$$

and $\mathcal{E}_{m,n}$ is understood to be zero if either of m, n are negative.

Proof The function $M^{(k)}_{2a+2,2b+2} : \mathfrak{H} \to V_{2w} \otimes \mathbb{C}$ is equivariant by definition. Let $M_{r,s}$ denote its modular components obtained from Proposition 7.1. From the Definition (9.4), and the differential equation for $K^{(k)}_{2a+2,2b+2}$, we can apply Proposition 7.2 to deduce the equations satisfied by $M_{r,s}$. The first line of (9.5) follows since the correction terms in (9.4) only affect the case $s = 0$. For $s = 0$, (9.4) and (9.3) imply that

$$\partial M_{2w,0} = \binom{2a}{k} \mathbb{L}^{k+1} \mathbb{G}_{2a+2} \mathcal{E}_{2b-k,k} + \mathbb{L} f \,.$$

By modifying $M^{(k)}_{2a+2,2b+2}$ by a suitable multiple of $\mathcal{E}_{2w+2}$, we can assume that f is a cusp form. It is uniquely determined by Theorem 3.7, which gives Eq. (9.6).

The quantities $\overline{\partial} M_{r,s}$ can be computed from (9.4), and satisfy:

$$\begin{aligned} \overline{\partial} M_{r,s} - (s+1) M_{r-1,s+1} &= \binom{2b}{k}\binom{k+s}{k} \mathbb{L}^{k+1} \mathbb{G}_{2b+2} \mathcal{E}_{2a-k-s,k+s}, \\ \overline{\partial} M_{0,2w} &= \binom{2b}{k} \mathbb{L}^{k+1} \mathbb{G}_{2b+2} \mathcal{E}_{2a-k,k} + \mathbb{L}\overline{g}. \end{aligned} \tag{9.7}$$

This proves that the $M_{r,s}$ lie in $\mathcal{MI}_2$. The cusp form g is uniquely determined from the anti-holomorphic projection $p^a(\overline{\partial} M_{0,2w}) = 0$. □

Remark 9.4 The antisymmetrization of $K^{(k)}_{2a,2b}$ is related to the function $I^{(k)}_{2a,2b}$ defined in [1], and its holomorphic projection is related to the double Eisenstein series of [9].

The Eqs. (9.5) and (9.7) uniquely determine $F_{r,s}$ when $r + s > 0$, and determine it up to a constant when $r = s = 0$. We can show that the functions $F_{r,s}$ are linearly independent for distinct values of a, b and k.

9.3 Example

Since there are no cusp forms in weights ≤ 10, it follows that the functions defined above, for $2w = 2a+2b-2k \leq 8$ only involve iterated integrals of Eisenstein series. The simplest possible example is the case $a = 1$, $b = 1$ and $k = 0, 1, 2$. The equivariant iterated integral $M^{(k)}_{4,4}$ of $\mathbb{G}_4$ and $\mathbb{G}_4$ solves the equation

$$\frac{\partial F^{(k)}}{\partial z} = (i\pi)^{k+1} \frac{\delta^{(k)}}{(k!)^2} \left(\underline{\mathbb{G}_4} \otimes \mathcal{E}_2 \right)$$

for $k = 0, 1, 2$, corresponding to the three components of

$$\delta^0 \oplus \delta^1 \oplus \delta^2 : V_2 \otimes V_2 \xrightarrow{\sim} V_4 \oplus V_2 \oplus V_0 \,.$$

By Proposition 7.2, this equation is equivalent to the following three families of equations, which we spell out for concreteness. In the case $k = 0$, we have

$$\begin{aligned} \partial F^{(0)}_{0,4} - F^{(0)}_{1,3} &= 0, \\ \partial F^{(0)}_{1,3} - 2F^{(0)}_{2,2} &= 0, \\ \partial F^{(0)}_{2,2} - 3F^{(0)}_{3,1} &= \mathbb{L}\mathbb{G}_4 \mathcal{E}_{0,2}, \\ \partial F^{(0)}_{3,1} - 4F^{(0)}_{4,0} &= \mathbb{L}\mathbb{G}_4 \mathcal{E}_{1,1}, \\ \partial F^{(0)}_{4,0} &= \mathbb{L}\mathbb{G}_4 \mathcal{E}_{2,0}. \end{aligned} \tag{9.8}$$

In the case $k = 1$, we have

$$\begin{aligned} \partial F^{(1)}_{0,2} - F^{(1)}_{1,1} &= 0, \\ \partial F^{(1)}_{1,1} - 2F^{(1)}_{2,0} &= 4\mathbb{L}^2 \mathbb{G}_4 \mathcal{E}_{0,2}, \\ \partial F^{(1)}_{2,0} &= 2\mathbb{L}^2 \mathbb{G}_4 \mathcal{E}_{1,1}. \end{aligned} \tag{9.9}$$

Finally, in the case $k = 2$ we have the single equation

$$\partial F^{(2)}_{0,0} = \mathbb{L}^3 \mathbb{G}_4 \mathcal{E}_{0,2}\,. \tag{9.10}$$

Since there are no cusp forms in weight 4, the statement of Theorem 3.7 is vacuous and there is no obstruction to the existence of a solution of these equations. The constant terms can be computed from the double integral of the Eisenstein series E_4 and E_4. The latter involves at most $\zeta(3)$, $\zeta(5)$ and $\zeta(3)^2$.

The shuffle product formula for iterated integrals implies that

$$\begin{aligned} F^{(0)}_{0,4} &= \tfrac{1}{2}\mathcal{E}^2_{0,2} & F^{(0)}_{4,0} &= \tfrac{1}{2}\mathcal{E}^2_{2,0} \\ F^{(0)}_{1,3} &= \mathcal{E}_{0,2}\mathcal{E}_{1,1} & F^{(0)}_{3,1} &= \mathcal{E}_{2,0}\mathcal{E}_{1,1} \end{aligned}$$

$$F^{(0)}_{2,2} = \mathcal{E}_{2,0}\mathcal{E}_{0,2} + \tfrac{1}{2}\mathcal{E}^2_{1,1}$$

are linear combinations of products of real analytic Eisenstein series as is

$$F^{(2)}_{0,0} = \mathbb{L}^2(\mathcal{E}_{2,0}\mathcal{E}_{0,2} - \tfrac{1}{4}\mathcal{E}^2_{1,1})$$

but this is not the case for the functions $F^{(1)}$, which are new. The above identities can be verified from their differential equations. Observe that the functions $\mathcal{E}_{2,0}\mathcal{E}_{0,2}$ and $\mathcal{E}^2_{1,1}$ can be expressed as linear combinations of $F^{(0)}_{2,2}$ and $\mathbb{L}^{-2}F^{(2)}_{0,0}$. By Theorem 9.3

$$\begin{aligned} \overline{\partial} F^{(1)}_{0,2} &= 2\mathbb{L}^2 \overline{\mathbb{G}_4}\mathcal{E}_{1,1}, \\ \overline{\partial} F^{(1)}_{1,1} - 2F^{(1)}_{0,2} &= 4\mathbb{L}^2 \overline{\mathbb{G}_4}\mathcal{E}_{2,0}, \\ \overline{\partial} F^{(1)}_{2,0} - F^{(1)}_{1,1} &= 0\,, \end{aligned}$$

and $\overline{\partial} F^{(2)}_{0,0} = \mathbb{L}^3\overline{\mathbb{G}_4}\mathcal{E}_{2,0}$. Now, from the above equations and the Definition 2.20 of the Laplacian, we deduce the equations:

$$\begin{aligned} (\Delta+2)F^{(1)}_{0,2} &= -4\mathbb{L}^2, \overline{\mathbb{G}_4}\mathcal{E}_{2,0} & (\Delta+4)F^{(0)}_{0,4} &= -\mathbb{L}\overline{\mathbb{G}_4}\mathcal{E}_{1,1}, \\ (\Delta+2)F^{(1)}_{1,1} &= -4\mathbb{L}^3\mathbb{G}_4\overline{\mathbb{G}_4} & (\Delta+4)F^{(0)}_{1,3} &= -2\mathbb{L}\overline{\mathbb{G}_4}\mathcal{E}_{2,0}, \\ (\Delta+2)F^{(1)}_{2,0} &= -4\mathbb{L}^2\mathbb{G}_4\mathcal{E}_{0,2} & (\Delta+4)F^{(0)}_{2,2} &= -\mathbb{L}^2\mathbb{G}_4\overline{\mathbb{G}_4}, \\ & & (\Delta+4)F^{(0)}_{3,1} &= -2\mathbb{L}\mathbb{G}_4\mathcal{E}_{0,2}, \\ \Delta F^{(2)}_{0,0} &= -\mathbb{L}^4\,\mathbb{G}_4\overline{\mathbb{G}_4} & (\Delta+4)F^{(0)}_{4,0} &= -\mathbb{L}\mathbb{G}_4\mathcal{E}_{1,1}. \end{aligned}$$

In fact, we note that

$$(\Delta+2)\mathcal{E}^2_{1,1} = -8\,\mathcal{E}_{0,2}\mathcal{E}_{2,0} \quad \text{and} \quad \Delta\,\mathcal{E}_{2,0}\mathcal{E}_{0,2} = -\mathcal{E}^2_{1,1} - \mathbb{L}^2\mathbb{G}_4\overline{\mathbb{G}_4}\,.$$

We have therefore generated three modular-invariant functions

$$\mathbb{L}^2\mathcal{E}_{2,0}\mathcal{E}_{0,2}, \quad \mathbb{L}^2\mathcal{E}^2_{1,1}, \quad \mathbb{L}F^{(1)}_{1,1} \in \mathcal{M}_{0,0}$$

out of which one can construct solutions to inhomogeneous Laplace equations:

$$(\Delta+2)\big(\mathbb{L}F^{(1)}_{1,1} - 4\mathbb{L}^2\mathcal{E}_{2,0}\mathcal{E}_{0,2}\big) = 4\mathbb{L}^2\mathcal{E}^2_{1,1}.$$

By comparing with (1.1), this suggests that the modular graph function $C_{2,1,1}(z)$ of the introduction can be expressed in terms of $\mathbb{L}F^{(1)}_{1,1} - 4\mathbb{L}^2\mathcal{E}_{2,0}\mathcal{E}_{0,2}$, $\mathbb{L}^3\mathcal{E}_{3,3}$ and a constant.

Remark 9.5 The coefficients in the expansion (2.5) of these functions are easily determined from the formulae for the action (2.15) of ∂, $\overline{\partial}$ and the above differential equations, up to the sole exception of a constant term $\alpha\mathbb{L}^{-w}$. When $w > 0$, it is uniquely determined by modularity. If $w = 0$, we can assume this coefficient is zero.

9.4 *L*-functions and constant terms

All expansion coefficients (2.5) of an element $f \in \mathcal{MI}_k$ are uniquely determined by those of functions in $\mathcal{MI}_{k-1}$ of lower length by the defining Eq. (3.7) and Lemma 2.6 *except for* a single constant term of the form $\alpha\mathbb{L}^{-w}$, where α is typically transcendental. This missing constant (when $w > 0$) can be determined from the others by analytic continuation using an L-function [27].

To define this, note that the functions $f \in \mathcal{MI}$ have the property that their coefficients $a^{(k)}_{m,n}$ have polynomial growth, i.e. there exist $K, \mu \in \mathbb{R}$ such that $|a^{(k)}_{m,n}| \leq KN^{\mu}$ for all $m, n \leq N$. This property is stable under sums, products and taking primitives. For any such $f \in \mathcal{M}_{\alpha,\beta}$ with an expansion (2.5), define

$$c^{(k)}_N = \sum_{m+n=N} a^{(k)}_{m,n}$$

for all $N \geq 1$, and consider the Dirichlet series

$$L^{(k)}(f, s) = \sum_{N\geq 1} \frac{c^{(k)}_N}{N^s},$$

which converges absolutely for $\mathrm{Re}(s)$ sufficiently large.

Definition 9.6 The completed L-function is

$$\Lambda(f, s) = \sum_k (-1)^k (2\pi)^{-s} \Gamma(s + k) L^{(k)}(f, s + k)\,.$$

Theorem 9.7 *The function $\Lambda(f, s)$ has a meromorphic continuation to $s \in \mathbb{C}$. It satisfies the functional equation*

$$\Lambda(f, s) = i^h \Lambda(f, w - s)$$

and has at most simple poles at integers $s \in \mathbb{Z}$. Its polar part is

$$\sum_k (-2\pi)^k a^{(k)}_{0,0} \Big(\frac{i^h}{s - w - k} - \frac{1}{s + k} \Big),$$

where $w = w(f)$ and $h = h(f)$.

Since the definition of Λ only involves $a^{(k)}_{m,n}$ for $mn > 0$, the theorem gives a means to deduce the constant part of f by analytic continuation. Indeed, we have

$$\Lambda(f, s) = \int_0^\infty \big(f(iy) - f^0(iy)\big) y^s \frac{dy}{y}$$

for $\mathrm{Re}(s)$ large. The theorem is easily proved by decomposing the range of integration into a piece from 0 to 1 and 1 to ∞ and invoking the functional equation, in the usual manner. In particular, for any convergent modular graph function, we can associate an L-function. Similarly, one can assign (a family of) L-functions to universal mixed elliptic motives [18]. This will be discussed elsewhere.

9.5 Orthogonality conditions

We now wish to consider the problem of finding linear combinations of equivariant iterated integrals which only involve Eisenstein series, i.e. in which all integrals of cusp forms cancel out. This is equivalent to finding linear combinations of the $M^{(k)}_{2a,2b}$ which are orthogonal to all cusp forms under the Petersson inner product. Since this problem is discussed in [1], §22 in an essentially equivalent form, we illustrate with a simple example.

Example 9.8 Let $k = 0$, and let Δ denote the Hecke normalised cusp form of weight 12. Consider the four 'lowest weight' functions $F^{2a+2,2b+2}_{10,0} \in \mathcal{M}_{10,0}$ for $a+b=5$ and $1 \leq a, b$ which are described in Theorem 9.3, and satisfy the equations

$$\begin{aligned} \partial F^{2,8} &= \mathbb{L}\mathbb{G}_4\mathcal{E}_{8,0} - \alpha^{2,8}\mathbb{L}\Delta \quad , \ \partial F^{8,2} = \mathbb{L}\mathbb{G}_{10}\mathcal{E}_{2,0} - \alpha^{10,4}\mathbb{L}\Delta \\ \partial F^{4,6} &= \mathbb{L}\mathbb{G}_6\mathcal{E}_{6,0} - \alpha^{4,6}\mathbb{L}\Delta \quad , \ \partial F^{6,4} = \mathbb{L}\mathbb{G}_8\mathcal{E}_{4,0} - \alpha^{6,4}\mathbb{L}\Delta \end{aligned}$$

where $\alpha^{2a,2b}$ are determined by the Petersson inner product:

$$\alpha^{2a,2b}\langle \Delta, \Delta\rangle = \langle \mathbb{G}_{2a+2}\mathcal{E}_{2b,0}, \Delta\rangle \, .$$

The right-hand side can be computed by the Rankin–Selberg method [1] §9 and implies that $\alpha^{2a,2b}$ is proportional, by some explicit factors, to $L(\Delta, 2a+1)L(\Delta, 12)$. On the other hand, it is well known that the quantities $L(\Delta, k)$ for $1 \leq k \leq 11$ satisfy the period polynomial relations over $\mathbb{Q}$, and we deduce that the quantity

$$X = 9\big(F^{2,8} - F^{8,2}\big) + 14\big(F^{4,6} - F^{6,4}\big)$$

has the property that all terms involving Δ drop out of ∂X. One can show, furthermore, that X is dual to the relations between double zeta values in weight 12.

Viewed in this manner, it might seem hopeless to find iterated integrals of Eisenstein series of higher lengths which are equivariant. Already in length three, the Rankin–Selberg method can no longer be applied in any obvious manner to find the necessary linear combinations of triple Eisenstein integrals. Fortunately, using the theory of the motivic fundamental group of the Tate curve, we can find an infinite class and, conjecturally all, solutions to this problem. This is summarised below.

10 A space of equivariant Eisenstein integrals

Recall that E is the graded $\mathbb{Q}$ vector space generated by Eisenstein series (3.9). Let $\mathcal{Z}^{\mathrm{sv}}$ denote the ring of single-valued multiple zeta values.

Theorem 10.1 *There exists a space $\mathcal{MI}^E \subset \mathcal{M}$ with the following properties:*

(1) *It is the $\mathcal{Z}^{\mathrm{sv}}$-vector space generated by certain (computable) linear combinations of real and imaginary parts of regularised iterated integrals of Eisenstein series.*
(2) *The space $\mathcal{MI}^E[\mathbb{L}^{\pm}]$ is stable under multiplication and complex conjugation.*
(3) *It carries an even filtration (conjecturally a grading) by M-degree, where $\mathbb{L}$ has M-filtration 2, and the $\mathcal{E}_{r,s}$ have M-filtration 2. It is also filtered by the length (number of iterated integrals), which we denote by $\mathcal{MI}^E_k \subset \mathcal{MI}^E$.*
(4) *The subspace of elements of $\mathcal{MI}^E$ of total modular weight w and M-filtration $\leq m$ is finite-dimensional for every m, w.*
(5) *Every element of $\mathcal{MI}^E$ admits an expansion in the ring*

$$\mathcal{Z}^{\mathrm{sv}}[[q, \overline{q}]][\mathbb{L}^{\pm}]\, ,$$

i.e. its coefficients are single-valued multiple zeta values. An element of total modular weight w has poles in $\mathbb{L}$ of order at most w. An element of M-filtration $2m$ has terms in $\mathbb{L}^k$ for $k \leq m$.

(6) *The space $\mathcal{MI}^E$ has the following differential structure:*

$$\begin{aligned}\partial\big(\mathcal{MI}^E_k\big) &\subset \mathcal{MI}^E_k + E[\mathbb{L}] \times \mathcal{MI}^E_{k-1}\\ \overline{\partial}\big(\mathcal{MI}^E_k\big) &\subset \mathcal{MI}^E_k + \overline{E}[\mathbb{L}] \times \mathcal{MI}^E_{k-1}\end{aligned}$$

where E is (3.9). The operators $\partial, \overline{\partial}$ respect the M-filtration, i.e. $\deg_M \partial = \deg_M \overline{\partial} = 0$, *where the generators $\mathbb{G}_{2n+2}$ of E are placed in M-degree* 0.

(7) *Every element $F \in \mathcal{MI}^E_k$ of total modular weight w satisfies an inhomogeneous Laplace equation of the form:*

$$(\Delta + w)F \in (E + \overline{E})[\mathbb{L}] \times \mathcal{MI}^E_{k-1} + E\overline{E}[\mathbb{L}] \times \mathcal{MI}^E_{k-2}\,.$$

Explicitly, we have $\mathcal{MI}^E_0 = \mathcal{Z}^{\mathrm{sv}}$, and

$$\mathcal{MI}^E_1 = \mathcal{Z}^{\mathrm{sv}} \oplus \bigoplus_{r,s\geq 0} \mathcal{E}_{r,s}\mathcal{Z}^{\mathrm{sv}}.$$

In length 2, we can show that $\mathcal{MI}^E_2$ is generated by the coefficients of linear combinations of $M^{(k)}_{2a,2b}$ which do not involve any cusp forms, and in particular the example of Sect. 9.3. The previous theorem can be compared with Sect. 1.1: the class $\mathcal{MI}^E$ satisfies most, if not all, the conjectural properties of modular graph functions.

Remark 10.2 A more precise statement about the Laplace equation (7) can be derived from the differential equations (6). In fact, the differential equations with respect to $\partial, \overline{\partial}$ are the more fundamental structure. This simplicity is obscured when looking only at the Laplace operator. Recently, a generalisation of modular graph functions called modular graph forms were introduced in [11]. These define functions in $\mathcal{M}$ of more general modular weights (r, s), and, up to scaling by $\mathbb{L}^{\pm}$, are closed under the action of $\partial, \overline{\partial}$. It suggests that one should try to find systems of differential equations, with respect to $\partial, \overline{\partial}$, satisfied by modular graph forms using partial fraction identities (see [11], (2.30)), and match their solutions with elements in $\mathcal{MI}^E[\mathbb{L}^{\pm}]$.

We briefly explain how the previous theorem relates to a recent observation in [12] for modular graph functions. Suppose that $f \in \mathcal{MI}^E$ of modular weights (w, w). Then, $\mathbb{L}^w f$ is modular invariant, and by (7) and repeated application of (2.23) it satisfies an inhomogenous Laplace eigenvalue equation with eigenvalue

$$-(2w + 2w - 2 + 2w - 4 + \cdots + 2 + 0) = -2\binom{w}{2} = -w(w-1)\,.$$

It was observed in [12] that dihedral modular graph functions satisfy an inhomogeneous Laplace equation with eigenvalue $-s(s-1)$, where s is a positive integer, and the same statement was proved in [14] for two-loop modular graphs functions using the representation theory of $\mathrm{SO}(2,1)$.

The M-filtration can be made more precise. If $F \in \mathcal{MI}^E$ of M-filtration $\leq 2m$, then the coefficient of $\mathbb{L}^{-k}$ in the constant part F^0 of F is a single-valued multiple zeta value of weight $\leq k + m$. If one assumes (for example, by replacing multiple zeta values with their motivic versions) that multiple zeta values are graded, rather than filtered, by weight, then this filtration would also be a grading. For example, the elements $\mathcal{E}_{r,s}$ have constant parts

$$\mathcal{E}^0_{r,s} \quad \in \quad \mathbb{L}\,\mathbb{Q} + \mathbb{L}^{-r-s}\,\zeta(r+s+1)\mathbb{Q}\,.$$

The (MZV)-weight of $\zeta(r+s+1)$ is (conjecturally) $r+s+1$, and the weight of a rational number is 0. This is entirely consistent with $\deg_M \mathcal{E}_{r,s} = 2$.

This theorem and further properties of $\mathcal{MI}^E$ will be proved in the sequel.

11 Meromorphic primitives of cusp forms

We revisit the problem of finding primitives of cusp forms. If we allow poles at the cusp, then we can indeed construct modular equivariant versions of cusp forms [4].

11.1 Weakly analytic variant of $\mathcal{M}$

Let $\mathcal{M}^!_{r,s}$ denote the vector space of functions $f : \mathfrak{H} \to \mathbb{C}$ which are real analytic modular of weights $(r,s) \in \mathbb{Z}^2$ admitting an expansion of the form

$$f(q) = \sum_{k=-N}^{N} \mathbb{L}^k \left(\sum_{m,n \geq -M} a^{(k)}_{m,n} q^m \overline{q}^n \right)$$

for some integers $M, N \in \mathbb{N}$, i.e. with poles in $q, \overline{q}$ at 0. Let

$$\mathcal{M}^! = \bigoplus_{r,s} \mathcal{M}^!_{r,s}\,.$$

It is a bigraded algebra and satisfies $\mathcal{M}^! = \mathcal{M}[\Delta(z)^{-1}, \overline{\Delta(z)}^{-1}]$ where $\Delta(z)$ denotes the Hecke normalised cusp form of weight 12. This ring of functions satisfies similar properties to $\mathcal{M}$, and is equipped with operators $\partial, \overline{\partial}, \Delta$ as defined earlier.

Definition 11.1 Define a space of modular iterated integrals $\mathcal{MI}^! \subset \mathcal{M}$ as follows. Let $\mathcal{MI}^!_{-1} = 0$ and let $\mathcal{MI}^!_k \subset \mathcal{M}$ be the largest subspace which is contained in the positive quadrant (modular weights (r,s) with $r,s \geq 0$), such that

$$\begin{aligned}\partial \mathcal{MI}^!_k &\subset \mathcal{MI}^!_k + M^![\mathbb{L}] \times \mathcal{MI}^!_{k-1},\\ \overline{\partial} \mathcal{MI}^!_k &\subset \mathcal{MI}^!_k + \overline{M^!}[\mathbb{L}] \times \mathcal{MI}^!_{k-1}\,.\end{aligned}$$

We now give some examples of elements in $\mathcal{MI}^!_k$ for $k \leq 2$.

11.2 Primitives of cusp forms

The following theorem is proved in [4].

Theorem 11.2 *For every cusp form $f \in S_n$, there exists a canonical family of functions $\mathcal{H}(f)_{r,s}$ for all $r,s \geq 0$, with $r+s=n$ satisfying*

$$\begin{aligned}\partial\, \mathcal{H}(f)_{n,0} &= \mathbb{L} f\\ \partial\, \mathcal{H}(f)_{r,s} &= (r+1)\mathcal{H}(f)_{r+1,s-1} \quad \textit{for all } 1 \leq s \leq w\end{aligned}$$

and

$$\begin{aligned}\overline{\partial}\, \mathcal{H}(f)_{0,n} &= \mathbb{L}\, \overline{\mathbf{s}(f)},\\ \overline{\partial}\, \mathcal{H}(f)_{r,s} &= (s+1)\mathcal{H}(f)_{r-1,s+1} \quad \textit{for all } 1 \leq r \leq w,\end{aligned}$$

where $\mathbf{s}(f) \in S^!_n$ is a weakly holomorphic modular form canonically associated with f. The $\mathcal{H}(f)_{r,s}$ are eigenfunctions of the Laplacian with eigenvalue $-n$.

If we write

$$\mathcal{H}(f) = \sum_{r+s=n} \mathcal{H}(f)_{r,s}(X - zY)^r(X - \overline{z}Y)^s,$$

then the system of equations above are equivalent to

$$d\mathcal{H}(f) = \pi i f(z)(X - zY)^n \mathrm{d}z + \pi i \,\overline{\mathbf{s}(f)(z)}(X - \overline{z}Y)^n \mathrm{d}\overline{z}. \tag{11.1}$$

In [1] §18, these formulae were generalised to all higher order iterated integrals. We show in [4] that $\mathcal{MI}_0^! = \mathbb{C}[\mathbb{L}^{-1}]$, and prove:

Theorem 11.3 *$\mathcal{MI}_1^!$ is the free $\mathbb{C}[\mathbb{L}^{-1}]$-module generated by the $\mathcal{H}(f)_{r,s}$.*

11.3 New elements in $\mathcal{M}_{r,s}$

By multiplying by a suitable power of $\Delta(z)\overline{\Delta(z)}$ to clear the poles at the cusp, we obtain elements in $\mathcal{M}$. For every cusp form $f \in S_n$,

$$\overline{\Delta(z)}^N \mathcal{H}(f)_{r,s} \in \mathcal{M}$$

for sufficiently large N (in fact, $N = \dim S_n$ will do). In particular,

$$\overline{\Delta(z)}\mathcal{H}(\Delta(z))_{r,s} \in \mathcal{M}_{r,s+12}\,.$$

This provides further evidence that the space of modular forms $\mathcal{M}$ contains potentially interesting elements.

11.4 Double integrals

Having defined the weakly holomorphic modular primitives of cusp forms, we can use them to construct equivariant double integrals of an Eisenstein series and a cusp form, or two cusp forms. The definition is along very similar lines to Sect. 9: consider the indefinite integrals of the one-forms:

$$\underline{f} \otimes \overline{\mathcal{H}(g)} + \mathcal{H}(f) \otimes \underline{\overline{g}} \quad \text{or} \quad \underline{f} \otimes \mathcal{E}_n + \mathcal{H}(f) \otimes \underline{\overline{E}}_{n+2}.$$

They are closed by (11.1), and so their indefinite integrals are well-defined (homotopy invariant). The general strategy is always the same: let

$$\Omega = \sum_{r+s=n} \omega_{r,s}(X - zY)^r(X - \overline{z}Y)^s$$

with $d\Omega = 0$, and $\omega_{r,s} \in \mathcal{M}^!_{r+2,s}\mathrm{d}z + \mathcal{M}^!_{r,s+2}\mathrm{d}\overline{z}$ (which implies that $\Omega(\gamma z)\big|_\gamma = \Omega(z)$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$). Consider the indefinite integral

$$\mathcal{F}(z) = \int_z^{z_0} \Omega$$

where $z_0 \in \mathfrak{H}$ is any point. Then,

$$\gamma \mapsto \mathcal{F}(\gamma z)\big|_\gamma - \mathcal{F}(z) \quad \in \quad Z^1(\mathrm{SL}_2(\mathbb{Z}); V_n \otimes \mathbb{C})$$

is a cocycle. By the Eichler–Shimura theorem, we can add primitives of holomorphic modular forms, anti-holomorphic cusp forms, and constants to $\mathcal{F}$ to make this cocycle vanish (this is a generalisation of the proof of Lemma 3.12). The resulting function is therefore modular equivariant. Extracting the coefficients in the manner of Proposition 7.1, we obtain non-trivial functions in $\mathcal{MI}^!_2$.

As above, by multiplying by sufficiently large powers of $\Delta(z)\overline{\Delta}(z)$, we can clear poles in the denominators to obtain yet more elements in $\mathcal{M}$, and so on.

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