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A CLASS OF NONCONVEX FUNCTIONS AND MATHEMATICAL PROGRAMMING

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A class of functions, called pre-invex, is defined. These functions are more general than convex functions and when differentiable are invex. Optimality conditions and duality theorems are given for both scalar-valued and vector-valued programs involving pre-invex functions.

1. Introduction

Let X and Y be real normed spaces of any dimension and let $K \subseteq Y$ be a closed convex cone. Let $S \subset X$. The function $f\colon S \to Y$ is said to be K-*convexlike* (see for example [**10**, **13**, **15**]) if for any x, $y \in S$ and $0 \leqslant \lambda \leqslant 1$ there is a $z \in S$ such that

$$\lambda f(x) + (1-\lambda) f(y) - f(z) \in K. \tag{1.1}$$

If S is a convex set and if f is a K-convex function, then clearly f is K-convexlike. Any real valued function is $\mathbb{R}_+$-convexlike.

Elster and Neshe [**10**] considered convexlike mathematical programs and obtained a saddlepoint optimality condition. Hayashi and Komiya [**13**] also considered convexlike mathematical programs and established a theorem of the alternative involving convexlike functions and considered Lagrangian duality.

Following [**8**], a function $f\colon S \to Y$ is called K-*invex*, with respect to a function $\eta\colon S \times S \to X$, if, for each x, $y \in S$

$$f(x) - f(y) - f'(y)\eta(x,y) \in K, \tag{1.2}$$

where $f'(y)$ denotes the Fréchet derivative of f at y. If $Y = \mathbb{R}$ and $K = \mathbb{R}_+$, then f is called *invex*. Invex functions were first considered by Hanson [**11**] who showed that if, instead of the usual convexity conditions, the objective function and each of the constraints of a nonlinear program are all invex for the same $\eta(x,y)$ then the sufficiency of the Kuhn-Tucker conditions [**17**] and weak (Wolfe[**24**]) duality still holds. Moreover, Craven and Glover [**9**] (also Ben-Israel and Mond [**1**], Martin [**19**]) showed that the class

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of real valued invex functions is equivalent to the class of functions whose stationary points are global minima.

Following Ben-Israel and Mond [**1**] and Hanson and Mond [**12**] consider a function $f\colon S \to Y$ having the property that there exists a function $\eta\colon S \times S \to X$ such that, for each x, $y \in S$ and $0 \leqslant \lambda \leqslant 1$, $y + \lambda\eta(x,y) \in S$ and

$$\lambda f(x) + (1-\lambda)f(y) - f(y + \lambda\eta(x,y)) \in K. \tag{1.3}$$

It is to be observed that if f is Fréchet differentiable and satisfies (1.3) then f also satisfies (1.2). This can be seen by rewriting (1.3) as

$$\lambda(f(x) - f(y)) - [f(y + \lambda\eta(x,y)) - f(y)] \in K$$

and then dividing by $\lambda > 0$ and taking the limit as $\lambda \to 0_+$ gives

$$f(x) - f(y) - f'(y)\eta(x,y) \in K.$$

In view of this observation functions satisfying (1.3) will be called *K-pre-invex.* It is to be noted that the set S should have the "connectedness" property that $y + \lambda\eta(x,y) \in S$ for x, $y \in S$ and $0 \leqslant \lambda \leqslant 1$. Note also that if $\eta(x,y) \equiv \alpha(x,y)(x-y)$ where $0 < \alpha(x,y) \leqslant 1$ then S should be *star-shaped* [**16**].

If $Y = \mathbb{R}$ and $K = \mathbb{R}_+$ and if f satisfies (1.3) then f will be called *pre-invex.* If $\eta(x,y) = x - y$ then clearly f is convex and S is a convex set; however there are functions which are pre-invex but not convex. For example, consider the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = -|x|$. Then f is not convex but is pre-invex with η given by

$$\eta(x,y) = \begin{cases} x - y \text{ if } x \leqslant 0, & y \leqslant 0 \\ x - y \text{ if } x \geqslant 0, & y \geqslant 0 \\ y - x \text{ otherwise.} \end{cases}$$

It is easy to see that a pre-invex function is also $\mathbb{R}_+$-convexlike; however pre-invex functions have some interesting properties that are not generally shared by the wider class of convexlike functions. For example, as for convex functions, every local minimum of a pre-invex function is a global minimum and non-negative linear combinations of pre-invex functions are pre-invex.

THEOREM 1.1. *Let $f\colon S \to \mathbb{R}$ be pre-invex. Then any local minimum of f is a global minimum.*

PROOF: Let f attain a local minimum $p \in S$; assume that $f(x) < f(p)$ for some $x \in S$. Since f is pre-invex there exists $\eta\colon S \times S \to X$ such that

$$\lambda f(x) + (1-\lambda)f(p) \geqslant f(p + \lambda\eta(x,p)), \qquad 0 \leqslant \lambda \leqslant 1.$$

Thus

$$f(p+\lambda\eta(x,p)) - f(p) \leqslant \lambda[f(x) - f(p)] < 0$$

for arbitrarily small $\lambda > 0$, contradicting the local minimum. ∎

THEOREM 1.2. *Let* $f_i\colon S \to \mathbb{R}$ *be pre-invex (with respect to* η*),* $i = 1, 2, \ldots, k$. *Then* $\sum_{i=1}^{k} y_i f_i(x)$ *is pre-invex (with respect to* η*), where* $y_i \geqslant 0$, $i = 1, 2, \ldots, k$.

PROOF:

$$\lambda \sum_{i=1}^{k} y_i f_i(x) + (1-\lambda) \sum_{i=1}^{k} y_i f_i(y)$$

$$= \sum_{i=1}^{k} y_i \{\lambda f_i(x) + (1-\lambda) f_i(y)\} \geqslant \sum_{i=1}^{k} y_i f_i(y + \lambda\eta(x,y)).$$

∎

Consider now a function $f\colon S \to Y$. Then f is directionally differentiable at $a \in S$ if, for each $x \in S$, the limit

$$f'(a,x) = \lim_{\alpha \downarrow 0} \alpha^{-1}[f(a+\alpha x) - f(a)]$$

exists in Y. When $Y = \mathbb{R}$ this reduces to the usual definition of directional differentiability.

THOEREM 1.3. *Let* $f\colon S \to Y$ *be directionally differentiable at each point in each direction, and let* f *be* K*-pre-invex. Then, for all* a, $x \in S$,

$$f(x) - f(a) - f'(a, \eta(x,a)) \in K.$$

PROOF: Since f is K-pre-invex then for all a, $x \in S$ there exists $\eta(x,a)$ such that

$$f(x) - f(a) - \lambda^{-1}[f(a + \lambda\eta(x,a)) - f(a)] \in K.$$

Letting $\lambda \downarrow 0$ gives the desired result. ∎

2. PRE-INVEX FUNCTIONS AND MATHEMATICAL PROGRAMMING

In this section we discuss some applications of pre-invex functions in mathematical programming. The discussion begins with an alternative theorem due to Hayashi and Komiya [**13**] (see also Jeyakumar [**15**]) established for convexlike functions which, of course, must also hold for pre-invex functions. From this alternative theorem we will deduce a saddlepoint theorem and Lagrangian duality theorem. We will also discuss Fritz John and Kuhn-Tucker conditions in terms of directional derivatives of the objective and constraint functions.

THEOREM 2.1. *Let X, Y be real normed linear spaces and let K be a closed convex cone in Y with nonempty interior; let $S \subseteq X$. Suppose that $f\colon S \to Y$ is K-pre-invex. Then exactly one of the following holds:*

(i) $(\exists x \in S)\ -f(x) \in \mathrm{int}\, K$,

(ii) $(\exists 0 \neq p \in K^{\star})\ (pf)(S) \subseteq \mathbb{R}_+$,

where int denotes interior and $K^{\star}$ is the dual cone of K.

This result is a special case of the convexlike results of Hayashi and Komiya [**13**] and Jeyakumar [**15**]. The following saddlepoint and duality theorems follow from the alternative theorem in a manner analogous to those in [**15**] for convexlike programs.

Consider the following programs:

(P) minimise $f(x)$ subject to $-g(x) \in K$,

where X, Y are normed linear spaces, $K \subseteq Y$ is a closed convex cone with nonempty interior; $S \subset X$, $f\colon S \to \mathbb{R}$ is pre-invex (with respect to η) and $g\colon S \to Y$ is K-pre-invex (with respect to η). The hypotheses stated here will be assumed to hold throughout the remainder of this section.

(D) maximise $\varphi(v)$ subject to $v \in K^{\star}$,

where $\varphi(v) = \inf_{x \in S}\{f(x) + vg(x)\}$.

The program (P) is said to satisfy the generalised Slater condition if there is $\bar{x} \in S$ such that $-g(\bar{x}) \in \mathrm{int}\, K$.

THEOREM 2.2. *If (P) attains a minimum at $x = x_0 \in S$ and if the generalised Slater condition is satisfied, then there is a $v_0 \in K^{\star}$ such that the Lagrangian $\psi(x,v) = f(x) + vg(x)$ satisfies the saddlepoint condition at (x_0, v_0):*

$$(2.1) \qquad (\forall x \in S, \quad \forall v \in K^{\star}), \qquad \psi(x_0, v) \leqslant \psi(x_0, v_0) \leqslant \psi(x, v_0).$$

Furthermore, if (2.1) is satisfied for some (x_0, v_0) then x_0 is a minimum for (P).

Remark. The saddlepoint condition (2.1) is sufficient without any pre-invexity assumptions.

THEOREM 2.3. *Assume f is pre-invex (with respect to η) and that g is K-pre-invex (with respect to η). Assume also that (P) satisfies the generalised Slater condition. Then (D) is a dual for (P).*

We now turn our attention to local necessary optimality conditions and in particular the Fritz John and Kuhn-Tucker conditions. We consider the program (P) where now $S \subseteq X$ is an open set and where f and g are directionally differentiable at each point in each direction.

THEOREM 2.4. *For the program (P) let f and g be directionally differentiable. Assume, also, that f and g are pre-invex and K-pre-invex (with respect to η) respectively and that (P) attains a minimum at $x = x_0$. Then there exist $\tau \in \mathbb{R}_+$ and $\lambda \in K^\star$ not both zero such that*

$$(2.2) \qquad (\tau f + \lambda g)'(x_0, x) \geqslant 0 \qquad \forall x \in S,$$

$$(2.3) \qquad \lambda g(x_0) = 0.$$

PROOF: Since $-g(x) \in K$ implies that $f(x_0) - f(x) \leqslant 0$ for all $x \in S$, then there is no solution $x \in S$ to the system

$$-(f(x) - f(x_0), g(x)) \in \operatorname{int}\left(\mathbb{R}^+ \times K\right).$$

Then by Theorem 2.1 there exists $\tau \in \mathbb{R}_+$, $\lambda \in K^\star$, not both zero, such that for all $x \in S$

$$\tau f(x) + \lambda g(x) \geqslant \tau f(x_0).$$

Since $-g(x_0) \in K$, $\lambda g(x_0) = 0$. Therefore, for all $x \in S$,

$$\tau f(x) + \lambda g(x) - [\tau f(x_0) + \lambda g(x_0)] \geqslant 0.$$

This gives, for all $x \in S$,

$$(\tau f + \lambda g)'(x_0, x) \geqslant 0$$

since the functions are directionally differentiable. ∎

The Fritz John conditions (2.2) and (2.3) lead to appropriate Kuhn-Tucker conditions under any assumption that implies $\tau \neq 0$. Moreover, the Kuhn-Tucker conditions are also sufficient.

THEOREM 2.5. *For the program (P), let f and g be directionally differentiable at each point in each direction. Assume also that f is pre-invex (with respect to η) and that g is K-pre-invex (with respect to η) and that the generalised Slater condition is satisfied. Then (P) attains a minimum at $x = x_0$ if and only if there exists $\lambda \in K^\star$ such that*

$$(2.4) \qquad (f + \lambda g)'(x_0, x) \geqslant 0 \qquad \forall x \in S$$

$$(2.5) \qquad \lambda g(x_0) = 0.$$

PROOF: ($\Longrightarrow$) Assume that (P) attains a minimum at $x = x_0$. Then the Fritz John conditions (2.2) and (2.3) must be satisfied at $x = x_0$ for some $\tau \in \mathbb{R}_+$, $\lambda \in K^\star$ not both zero. If $\tau = 0$, then $\lambda \neq 0$ and $(\lambda g)'(x_0, x) \geqslant 0$ for all $x \in S$

and $\lambda g(x_0) = 0$. Since g is K-pre-invex it follows that $\lambda g(x) \geqslant \lambda g(x_0) = 0$; this contradicts the generalised Slater condition by Theorem 2.1. Hence $\tau \neq 0$ and we may assume $\tau = 1$; (2.4) and (2.5) then follow directly from (2.2) and (2.3).

($\Longleftarrow$) Let x be feasible and assume that (2.4) and (2.5) are satisfied. Then

$$\begin{aligned} f(x) - f(x_0) &\geqslant f'(x_0, \eta(x, x_0)) \qquad \text{(by Theorem 1.3)} \\ &\geqslant -(\lambda g)'(x_0, \eta(x, x_0)) \qquad \text{(by (2.4))} \\ &\geqslant -\lambda(g(x) - g(x_0)) \qquad \text{(since } g \text{ is } K\text{-pre-invex)} \\ &= -\lambda g(x) \qquad \text{(since } \lambda g(x_0) = 0) \\ &\geqslant 0 \qquad \text{(since } \lambda \in K^*,\ -g(x) \in K). \end{aligned}$$

Hence $f(x) \geqslant f(x_0)$. ∎

It is to be noted that, for a related convexlike program, the Kuhn-Tucker conditions may not be sufficient for a minimum. However, for pre-invex programs the Kuhn-Tucker conditions are both necessary and sufficient. This extends a well-known result in convex programming (see for example Rockafellar [**20**]).

Now, in relation to (P) consider the program

(D1) maximise $f(u) + \lambda g(u)$,
subject to $(f + \lambda g)'(u, x) \geqslant 0$, $\lambda \in K^*$, $u \in S$. $\forall x \in S$.

We show that (D1) is a dual to (P).

THEOREM 2.6. *In (P), let f and g be directionally differentiable at each point in each direction. Let f be pre-invex (with respect to η) and let g be K-pre-invex (with respect to η). Let (P) attain a minimum at $x_0 \in S$, and let the Kuhn-Tucker conditions (2.4) and (2.5) hold at x_0. Then (D1) is a dual to (P).*

PROOF: Let $-g(x) \in K$ and let $\lambda \in K^*$. Then

$$\begin{aligned} f(x) - [f(u) + \lambda g(u)] &\geqslant f'(u, \eta(x, u)) - \lambda g(u) \text{ (by Theorem 1.3)} \\ &\geqslant -\lambda(g(u) + g'(u, \eta(x, u))) \text{ (substituting from the constraint of (D1))} \\ &\geqslant -\lambda g(x) \text{ (since } \lambda g(\cdot) \text{ is pre-invex and by Theorem 1.3)} \\ &\geqslant 0 \text{ since } -g(x) \in K \text{ and } \lambda \in K^*. \end{aligned}$$

This proves weak duality. Now, from the Kuhn-Tucker conditions for (P), there is a $\bar\lambda \in K^*$ with

$$(f + \bar\lambda g)'(x_0, x) \geqslant 0 \text{ and } \bar\lambda g(x_0) = 0;$$

so $(x_0, \bar\lambda)$ satisfies the constraints of (D1) and

$$\max \text{ (D1)} \geqslant f(x_0) + \bar\lambda g(x_0) = f(x_0) = \min \text{ (P)}.$$

This, with weak duality, shows $(x_0, \bar\lambda)$ is optimal for (D1). ∎

3. Pre-invex functions and vector-valued programming

Let X and Y be real normed spaces of any dimension and let $S \subseteq X$. Let $f\colon S \to Y$ and let $Q \subset Y$ be a closed convex cone. Consider the vector valued problem

$$\text{minimise } f(x) \text{ subject to } x \in T \tag{3.1}$$

where $T \subset S$. The problem (3.1) has a *weak minimum* at $x = x_0 \in T$ (see for example [**3**, **5**, **6**]) if there exists no $x \in T$ for which

$$f(x_0) - f(x) \in \operatorname{int} Q,$$

where int denotes interior. Local weak minima may be obtained from the above with $T \cap N$ replacing T where N is a sufficiently small neighbourhood of x_0.

Consider the problem

$$\text{(P1) minimise } f(x) \text{ subject to } -g(x) \in K$$

where X, Y, Z are real normed vector spaces with $S \subseteq X$; $Q \subseteq Y$ and $K \subseteq Z$ are closed convex cones, and $f\colon S \to Y$, $g\colon S \to Z$. The hypotheses stated will be assumed to hold throughout this section.

For vector-valued problems it is natural to study a vector-valued Lagrangian generalising the usual scalar Lagrangian. For convex problems this has been done in finite dimensions for Pareto optima by Tanino and Sarawagi [**21**] and White [**23**] and for weak optima in infinite dimensions by Weir, Mond and Craven [**22**]. Other approaches, using matrix Lagrange multipliers, have been given by Bitran [**2**], Ivanov and Nehse [**14**] for finite dimensions and by Corely [**4**] for infinite dimensions.

In this section we will use the same vector-valued Lagrangian as in [**22**] and regard f and g as Q-pre-invex and K-pre-invex functions respectively. We will establish necessary and sufficient conditions for weak minimisation and duality theorems.

First we need some preliminaries. Let X, Y, Z be real normed spaces and S a subset of X. Let $P \subseteq Z$ be a convex cone and let W be a set in Z. A point $w_0 \in W$ is called an *extreme point* (see for example [**21**]) of W with respect to P if there is no $w \in W$, $w \neq w_0$, such that $w - w_0 \in \operatorname{int} P$. The problem (3.1) may thus be interpreted as that of finding all the extreme points of $-f(T)$ with respect to Q.

For the problem (P1) with $\operatorname{int} Q \neq \phi$ define a Lagrangian $L_r\colon X \times K^* \to Y$ by $L_r(x,v) = f(x) + vg(x)r$, for a fixed $r \in \operatorname{int} Q$. The point (x_0, v_0) will be called a *saddlepoint* of $L_r(x,v)$ if for all $x \in S$, $v \in K^*$,

$$L_r(x_0, v) - L_r(x_0, v_0) \notin \operatorname{int} Q \tag{3.2}$$

$$L_r(x_0, v_0) - L_r(x, v_0) \notin \operatorname{int} Q \tag{3.3}$$

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We will now give sufficient and necessary optimality conditions for (P1) in terms of a vector-valued Lagrangian. As in the case of scalar programming, if (x_0, v_0) is a solution of (3.2) and (3.3) for some $r \in \operatorname{int} Q$ then x_0 is an optimal solution for (P1). This is established in [**22**, Theorem 2]. However, as in[**22**], if x_0 is an optimal solution of (P1) a constraint qualification and convexity is required to assure the existence of v_0 such that (x_0, v_0) is a solution of (3.2) and (3.3). Here we will show that this convexity requirement can be weakened to pre-invexity.

THEOREM 3.2. *Let f be Q-pre-invex and g K-pre-invex. Suppose x_0 is an optimum solution for (P1) such that $\tau f(x_0) \leqslant \tau f(x)$ for some $0 \neq \tau \in Q^\star$ and all feasible $x \in S$. If the generalised Slater condition is satisfied then there exists $v_0 \in K^\star$ such that the saddlepoint conditions (3.2) and (3.3) hold for some $r \in \operatorname{int} Q$ and $v_0 g(x_0) = 0$.*

Remark. A sufficient condition guaranteeing the existence of $0 \neq \tau \in Q^\star$ such that $\tau f(x_0) \leqslant \tau f(x)$ for all feasible $x \in S$ is that f is Q-pre-invex, g is K-pre-invex, (P1) attains a weak local minimum at $x = x_0$ and that for some sufficiently small neighbourhood N of x_0 the set

$$C = \{\beta(f(x) - f(x_0)) \colon \beta \in \mathbb{R}_+, \quad x \in F \cap N\}$$

is convex, where $F = \{x \colon -g(x) \in K\}$ [**7**].

PROOF: From the assumptions x_0 is a solution of the scalar minimisation problem

$$\text{minimise } \tau f(x) \text{ subject to } -g(x) \in K$$

and, since τf is pre-invex and g is K-pre-invex, Theorem 2.2 gives $\tau(f(x_0) + v_0 g(x_0) r) \leqslant \tau(f(x_0) + v_0 g(x_0) r) \leqslant \tau(f(x) + v_0 g(x) r)$ for some $r \in \operatorname{int} Q$ chosen such that $\tau r = 1$. If (3.2) and (3.3) did not hold then

$$\tau(f(x_0) + v g(x_0) r - (f(x_0) + v_0 g(x_0) r)) > 0 \text{ and}$$
$$\tau(f(x_0) + v_0 g(x_0) r - (f(x) + v_0 g(x) r)) > 0,$$

a contradiction. ∎

Consider the two problems

(A) minimise $\Psi(x)$ (weakly with respect to some cone C) subject to $x \in F$

and

(B) maximise $\Phi(y)$ (weakly with respect C) subject to $y \in G$.

Problem (B) will be called a *dual* of (A) if ([6]) there holds.

(i) (weak duality) $\Psi(x) - \Phi(y) \notin -\operatorname{int} C$ whenever $x \in F$ and $y \in G$; and

(ii) (strong duality) if (A) attains a weak minimum at some point $x = a$, then (B) attains a weak maximum at some point $y = b \in G$ and $\Psi(a) = \Phi(b)$.

In relation to (P1) consider the problem

$$(\mathrm{D}') \text{ maximise } \Xi = \{\xi \in Y : (\exists 0 \neq \tau \in Q^\star, \quad v \in S^\star), \quad \tau\xi = \inf\{\tau f(z) : z \in S_0\}\}.$$

The maximisation problem (D′) is the problem of finding the extreme points of Ξ with respect to the cone Q.

THEOREM 3.3. *(Weak Duality) Let x be feasible for (P1) and let $\eta \in \Xi$. Then $f(x) - \eta \notin -\operatorname{int} Q$*

PROOF: For some $0 \neq \tau \in Q^\star$, $v \in S^\star$, $\tau\eta = \inf\{\tau f(z) + vg(z) : z \in X_0\}$. Hence $\tau f(x) \geqslant \tau f(x) + vg(x) \geqslant \inf\{\tau f(z) + vg(z) : z \in S_0\} = \tau\eta$; so $\tau(f(x) - \eta) \geqslant 0$; thus $f(x) - \eta \notin \operatorname{int} Q$. ∎

THEOREM 3.4. *(Strong Duality). Let f be Q-pre-invex and g K-pre-invex. Let x_0 be a solution to (P1) such that $\tau f(x_0) \leqslant \tau f(x)$ for some $0 \neq \tau \in Q^\star$ and all $x \in S$. If the generalised Slater condition is satisfied then there is $\xi_0 \in \Xi$ such that $f(x_0) = \xi_0$ and ξ_0 is an extreme point of Ξ.*

PROOF: From the assumptions x_0 is a solution of the scalar minimisation problem:

$$\text{minimise } \tau f(x) \text{ subject to } -g(x) \in K.$$

From Theorem 2.3 there exists $v_0 \in K^\star$ such that $v_0 g(x_0) = 0$ and for all $x \in S$

$$\tau f(x_0) + v_0 g(x_0) \leqslant \tau f(x) + v_0 g(x).$$

Thus,

$$\tau f(x_0) \leqslant \inf\{\tau f(x) + v_0 g(x)\} = \tau\xi$$

for some $\xi \in Y$. From weak duality it follows that $\tau f(x_0) = \tau\xi$. If there was no $\xi_0 \in \Xi$ being an extreme point of Ξ such that $f(x_0) = \xi_0$ then there would be $\hat{\xi}_0 \in \Xi$ such that $\hat{\xi} - f(x_0) \in \operatorname{int} Q$; hence for all $0 \neq \tau \in Q^\star$, $\tau\hat{\xi} > \tau f(x_0)$. Thus, since $\hat{\xi} \in \Xi$, for some $\hat{\tau} \in Q^\star$, $\hat{v} \in K^\star$, $\inf\{\hat{\tau} f(x) + \hat{v} g(x) : x \in S_0\} = \hat{\tau}\hat{\xi} > \hat{\tau} f(x_0) \geqslant \hat{\tau} f(x_0) + \hat{v} g(x_0)$ which is a contradiction. ∎

We now turn our attention to the problem (P1) where f and g are directionally differentiable on the open set S and discuss necessary and sufficient optimality conditions.

THEOREM 3.5. *For the program (P1), let f and g be directionally differentiable at each point in each direction. Assume that f and g are Q-pre-invex and K-pre-invex respectively, and that (P1) attains a weak minimum at $x = x_0$. Then there exist $\tau \in Q^\star$ and $\lambda \in K^\star$, not both zero, such that*

$$(3.4) \qquad (\tau f + \lambda g)'(x_0, x) \geqslant 0 \qquad \forall x \in S,$$

$$(3.5) \qquad \lambda g(x_0) = 0.$$

PROOF: Since $-g(x) \in K$ implies that $f(x_0) - f(x) \notin \operatorname{int} Q$ for all $x \in S$, then there is no solution $x \in S$ to the system

$$-(f(x) - f(x_0), g(x)) \in \operatorname{int}(Q \times K).$$

Then by Theorem 2.1 there exists $\tau \in Q^\star$ and $\lambda \in K^\star$, not both zero, such that for all $x \in S$

$$\tau f(x) + \lambda g(x) \geqslant \tau f(x_0).$$

Since $-g(x_0) \in K$, $\lambda g(x_0) = 0$. Therefore, for all $x \in S$,

$$\tau f(x) + \lambda g(x) - [\tau f(x_0) + \lambda g(x_0)] \geqslant 0.$$

This gives that, for all $x \in S$,

$$(\tau f + \lambda g)'(x_0, x) \geqslant 0$$

since the functions are directionally differentiable. ∎

The Fritz John conditions (3.4) and (3.5) will lead to appropriate Kuhn-Tucker necessary conditions under any assumption giving $\tau \neq 0$. Moreover, the Kuhn-Tucker conditions are also sufficient.

THEOREM 3.6. *For the program (P1), let f and g be directionally differentiable at each point in each direction. Assume also that f is Q-pre-invex and g K-pre-invex and that the generalised Slater condition is satisfied. Then (P1) attains a weak minimum at $x = x_0$ if and only if there exists $0 \neq \tau \in Q^\star$ $\lambda \in K^\star$ such that:*

$$(3.6) \qquad (\tau f + \lambda g)'(x_0, x) \geqslant 0, \qquad \forall x \in S,$$

$$(3.7) \qquad \lambda g(x_0) = 0.$$

PROOF: ($\Longrightarrow$). Assume that (P) attains a weak minimum at $x = x_0$. Then the Fritz John conditions (3.4) and (3.5) must be satisfied at $x = x_0$, for some $\tau \in Q^\star$, $\lambda \in K^\star$ not both zero. If $\tau = 0$, then $\lambda \neq 0$ and $(\lambda g)'(x_0, x) \geqslant 0$ for all $x \in S$, and

$\lambda g(x_0) = 0$. Since g is K-pre-invex it follows that $\lambda g(x) \geqslant \lambda g(x_0) = 0$ for all $x \in S$; this contradicts the generalised Slater condition by Theorem 2.1. Hence, $\tau \neq 0$, and (3.6) and (3.7) follows.

($\Longleftarrow$). Let x be feasible and assume that (3.6) and (3.7) are satisfied. Since $0 \neq \tau \in Q^\star$ and f is Q-pre-invex, then τf is pre-invex. Then

$$\begin{aligned} \tau f(x) - \tau f(x_0) &\geqslant (\tau f)'(x_0, \eta(x, x_0)) \text{ (by Theorem 1.3)} \\ &\geqslant -(\lambda g)'(x_0, \eta(x, x_0)) \text{ (by (3.6))} \\ &\geqslant \lambda(g(x) - g(x_0)) \text{ (since } g \text{ is } K\text{-pre-invex)} \\ &= -\lambda g(x) \text{ (since } \lambda g(x_0) = 0) \\ &\geqslant 0 \text{ (since } \lambda \in K^\star,\ -g(x) \in K). \end{aligned}$$

Hence $f(x) - f(x_0) \notin -\operatorname{int} Q$. ∎

Using the Kuhn-Tucker conditions for (P1) we will be able to establish a duality theorem for (P1) and the problem

$$\begin{aligned} (\text{D1}') \text{ maximise } \quad & f(u) + \lambda g(u) r, \\ \text{subject to } & (\tau f + \lambda g)'(u, x) \geqslant 0 \quad \forall x \in S, \\ & \tau \in Q^*,\ \lambda \in K^*,\ u \in S,\ \tau r = 1, \end{aligned}$$

and r is any fixed element of $\operatorname{int} Q$.

THEOREM 3.7. *In (P1) let f and g be directionally differentiable at each point in each direction. Let f be Q-pre-invex (with respect to η) and let g be K-pre-invex (with respect to η). Let (P1) attain a weak minimum at $x_0 \in S$ and let Kuhn-Tucker conditions (3.6) and (3.7) hold at x_0. Then (D1$'$) is a dual to (P1).*

PROOF: Let $-g(x) \in K$ and let $\tau \in Q^\star$, $\lambda \in K^\star$ and $\tau r = 1$. Then

$$\begin{aligned} \tau f(x) - \tau[f(u) + \lambda g(u) r] &= \tau f(x) - \tau f(u) - \lambda g(u) \\ &\geqslant (\tau f)'(u, \eta(x, u)) - \lambda g(u) \text{ (by Theorem 1.3)} \\ &\geqslant -(\lambda g)'(u, \eta(x, u)) - \lambda g(u) \\ &\quad \text{(substituting from the constraints of (D1}')) \\ &\geqslant -\lambda g(x) \text{ (since } \lambda g \text{ is pre-invex and by Theorem 1.3)} \\ &\geqslant 0 \text{ (since } -g(x) \in K \text{ and } \lambda \in K^\star). \end{aligned}$$

Hence $f(x) - [f(u) + \lambda g(u) r] \notin -\operatorname{int} Q$. This proves weak duality. Now, from Kuhn-Tucker conditions for (P1), there is $0 \neq \bar{\tau} \in Q^\star$, $\bar{\lambda} \in K^\star$ such that $\bar{\tau} r = 1$ and $(\bar{\tau} f + \bar{\lambda} g)'(x, x_0) \geqslant 0$ and $\bar{\lambda} g(x_0) = 0$; so $(x_0, \bar{\tau}, \bar{\lambda})$ satisfies the constraints of (D1$'$) and the values of (P1) and (D1$'$) are equal. This establishes strong duality. ∎

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