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A class of quasi-quintic trigonometric Bézier curve with two shape parameters

Uzma Bashir[a,*], **Muhammad Abbas**[a], **Mohd Nain Hj Awang**[b], **Jamaludin Md. Ali**[a]

[a] School of Mathematical Sciences, Universiti Sains Malaysia, 11800 Penang, Malaysia
[b] School of Distance Education, Universiti Sains Malaysia, 11800 Penang,Malaysia

*Corresponding author, e-mail: missheikh92@gmail.com

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ABSTRACT: In this paper, a class of quasi-quintic trigonometric Bézier curve with two shape parameters, based on newly constructed trigonometric basis functions, is presented. The new basis functions share the properties with Bernstein basis functions, so the generated curves inherit many properties of traditional Bézier curves. The presence of shape parameters provides a local control on the shape of the curve which enables the designer to control the curve more than the ordinary Bézier curve.

KEYWORDS: trigonometric basis functions, shape control, open curves, closed curves

INTRODUCTION

In computer aided geometric design (CAGD), it is often necessary to generate curves and surfaces that approximate shapes with some desired shape features. Designing free form curves and surfaces is a prevalent feature of CAGD. The key problem, simply stated, is to enable the designer to generate curves and surfaces which behave as he wants them to. The parametric cubic is a powerful tool which, when properly defined, is capable of representing most geometric entities of practical interest. In recent years, the trigonometric spline with shape parameters has gained more interest in CAGD, in particular curve design. Han[1] presented a class of quadratic trigonometric polynomial curves with a shape parameter. The shape of the curve was more easily controlled by altering the values of shape parameter than the ordinary quadratic B-Spline curves. Han[2] introduced piecewise quadratic trigonometric polynomial curves with C^1 continuity analogous to the quadratic B-Spline curves which have C^1 continuity. Cubic trigonometric polynomial curves with a shape parameter were discussed by Han[3]. In these papers the authors described the trigonometric polynomial with global shape parameter. Single parameter does not provide local control on the curves. To remedy this, Wu et al[4] presented quadratic trigonometric polynomial curves with multiple shape parameters.

Bézier technique is one of the methods of analytic representation of curves and surfaces that has won wide acceptance as a valuable tool in CAD/CAM system. They are used to produce curves which appear reasonably smooth at all scales. Today, many CAGD systems feature Bézier curves as their major building block, since they are very efficient and attain a number of mathematical properties. Both rational and non-rational forms of Bézier curves have been studied by many authors. A cubic trigonometric Bézier curve with two shape parameters was presented by Han et al[5]. It enjoyed all the geometric properties of the ordinary cubic Bézier curve and was used for spur gear tooth design with S-shaped transition curve Abbas, et al[6]. Liu, et al[7] presented a study on class of TC-Bézier curve with shape parameters. Yang, et al[8] discussed trigonometric extension of quartic Bézier curves attaining G^2 and C^2 continuity. A class of general quartic spline curves with shape parameters were introduced by Han[9]. Yang, et al studied a class of quasi-quartic trigonometric Bézier curves and surfaces[10].

A newly constructed quasi-quintic trigonometric Bézier curve with two shape parameters is presented in this paper. The proposed curve inherits all geometric properties of the traditional Bézier curve and is used to construct open and closed curves.

The paper is organized as follows. Firstly, we construct new trigonometric Bernstein-like basis functions with two shape parameters. Secondly, a quasi-quintic trigonometric Bézier curve is constructed from these basis functions. Thirdly, we describe the local shape control of the curve using different values of shape parameters. This is then used to generate open and closed curves. Finally, the conclusion with future

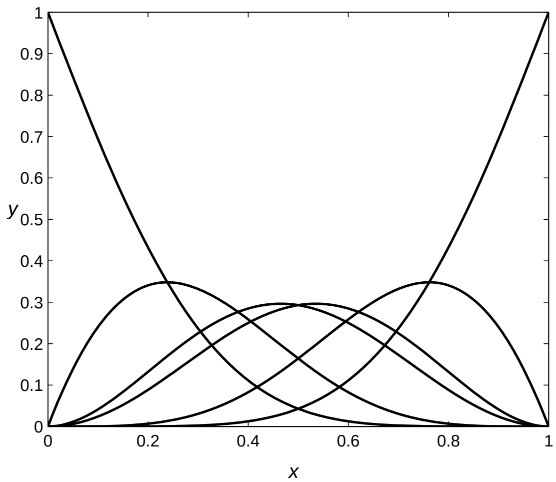

Fig. 1 Trigonometric Bernstein-like basis functions.

work is given.

TRIGONOMETRIC BERNSTEIN-LIKE BASIS FUNCTION

We define trigonometric basis functions with two shape parameters and discuss some of their properties.

Definition 1 For $u \in [0, \frac{1}{2}\pi]$ trigonometric basis functions with two shape parameters m and n, where $-3 \leqslant m, n \leqslant 1$ are defined as:

$$\begin{aligned} f_0(u) &= (1-\sin u)^3(1-m\sin u) \\ f_1(u) &= \sin u(1-\sin u)^2(3+m(1-\sin u)) \\ f_2(u) &= (-1+\cos u+\sin u)^2(1+\sin u) \\ f_3(u) &= (-1+\cos u+\sin u)^2(1+\cos u) \\ f_4(u) &= \cos u(1-\cos u)^2(3+n(1-\cos u)) \\ f_5(u) &= (1-\cos u)^3(1-n\cos u) \end{aligned} \tag{1}$$

Theorem 1 *The basis functions* (1) *satisfy the following properties:*
(i) *Non-negativity:* $f_i(u) \geqslant 0, i = 0(1)5$
(ii) *Partition of unity:* $\sum_{i=0}^{5} f_i(u) = 1$
(iii) *Monotonicity: For the given value of the shape parameters* m *and* n, $f_0(u)$ *is monotonically decreasing and* $f_5(u)$ *is monotonically increasing.*
(iv) *Symmetry: for* $i = 0, 1, 2, 3, 4, 5$,

$$f_i(u; m, n) = f_{5-i}(\tfrac{1}{2}\pi - u; n, m)$$

Proof:

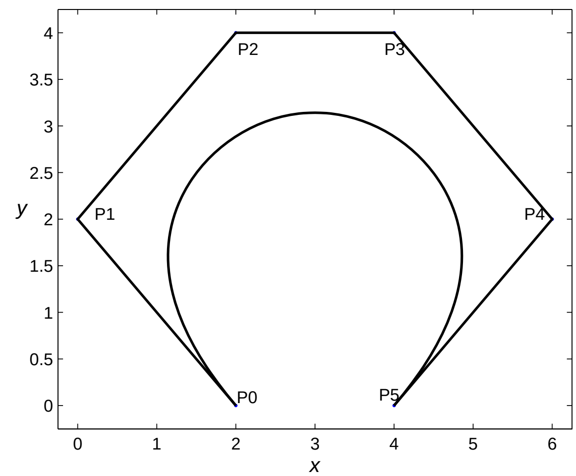

Fig. 2 Quasi-quintic Bézier curve with two shape parameters.

(i) For $u \in [0, \frac{1}{2}\pi]$ and $m, n \in [-3, 1]$,

$$\begin{aligned} (1 \pm \sin u) \geqslant 0,\ (1 - m\sin u) \geqslant 0, \\ (1 \pm \cos u) \geqslant 0(1 - n\cos u) \geqslant 0, \\ \cos(u) \geqslant 0,\ \sin(u) \geqslant 0. \end{aligned}$$

Thus the non-negativity of the basis functions $f_i(u) \geqslant 0, i = 0(1)5$ follows immediately.

(ii) Obvious from the definition of the basis functions.

(iii) For $u_0,\ u_1 \in [0, \frac{1}{2}\pi]$ such that $u_0 \leqslant u_1$, $f_0(u_0) \geqslant f_0(u_1)$ which shows that $f_0(u)$ is monotonically decreasing. Similarly for $u_0 \leqslant u_1$, $f_5(u_0) \leqslant f_5(u_1)$ which shows that $f_5(u)$ is monotonically increasing.

(iv) For $i = 1$,

$$\begin{aligned} f_1(u; m, n) &= \sin u(1-\sin u)^2(3+m(1-\sin u)) \\ &= \cos\left(\frac{\pi}{2} - u\right)\left(1 - \cos\left(\frac{\pi}{2} - u\right)\right)^2 \\ &\quad \times \left(3 + n\left(1 - \cos\left(\frac{\pi}{2} - u\right)\right)\right) \\ &= f_4\left(\frac{\pi}{2} - u; n, m\right) \end{aligned}$$

□

Fig. 1 shows the shapes of trigonometric Bernstein-like basis functions for $m = n = -1$.

QUASI-QUINTIC TRIGONOMETRIC BÉZIER CURVES AND ITS PROPERTIES

We define a class of quasi- quintic trigonometric Bézier curve with two shape parameters as follows:

Definition 2 For the control points $P_i (i = 0, \ldots, 5)$ in $\mathbb{R}^2$, we define the curve as:

$$f(u) = \sum_{i=0}^{5} f_i(u) P_i, u \in \left[0, \frac{\pi}{2}\right], m, n \in [-3, 1] \tag{2}$$

where $f_i(u), i = 0(1)5$ are the trigonometric basis functions defined in (1).

Theorem 2 *The curve* (2) *upholds the following properties:*

(i) *End point properties:*

$$f(0) = P_0, f(1) = P_2$$
$$f'(0) = (3+m)(P_1 - P_0)$$
$$f'\left(\frac{\pi}{2}\right) = (3+n)(P_5 - P_4)$$
$$f''(0) = 2[3(1+m)P_0 - 3(2+m)P_1 + P_2 + 2P_3]$$
$$f''\left(\frac{\pi}{2}\right) = 2[3(1+n)P_5 - 3(2+n)P_4 + P_3 + 2P_2]$$

(ii) *Symmetry: The control points P_i and $P_{5-i}, i = 0(1)5$ define the same quasi-quintic trigonometric Bézier curve in different parameterizations, i.e., for $i = 0, 1, 2, 3, 4, 5$*

$$f(u; m, n, P_i) = f\left(\frac{\pi}{2} - u; n, m, P_{5-i}\right).$$

(iii) *Geometric invariance: The shape of the curve* (2) *is independent of the choice of its control points. Equivalently the curve* (2) *satisfies the following two equations for $i = 0, 1, 2, 3, 4, 5$.*

$$f(u; m, n, P_i + q) = f(u; m, n, P_i) + q$$
$$f(u; m, n, P_i * q) = f(u; m, n, P_i) * q,$$

where q is any arbitrary vector in $\mathbb{R}^2$ and T is an arbitrary 2×2 matrix.

(iv) *Convex hull: The entire curve is contained within the convex hull of its defining control points P_i.*

SHAPE CONTROL OF THE QUARTIC TRIGONOMETRIC BÉZIER CURVE

Once the control points are chosen, Bézier curve or surface is generated uniquely by using the Bernstein basis functions and the shape is not adjustable by any means. The introduction of shape parameters has remedied this defect. The values of the shape parameters m and n can be used to attain the local control.

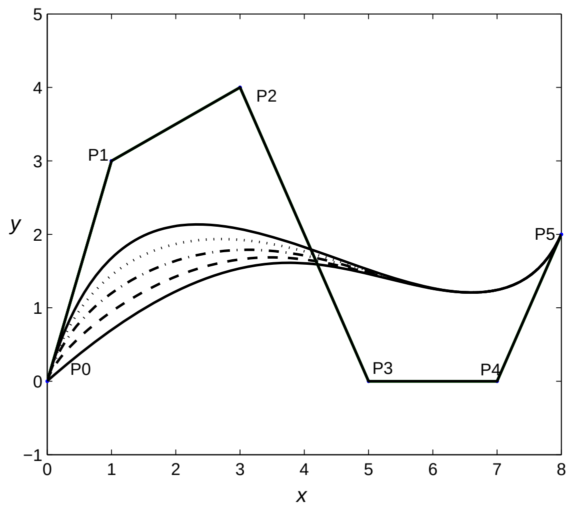

Fig. 3 Effect on the shape of the curve with altering the values of shape parameter m.

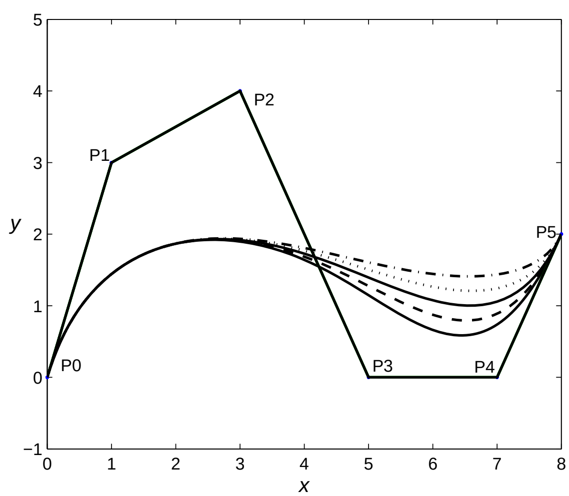

Fig. 4 Effect on the shape of the curve with altering the values of shape parameter n.

For the given control points $P_i(x_i, y_i), i = 0(1)5$ of quasi- quintic trigonometric Bézier curve, for a fixed value of the shape parameter n the curve moves towards the edge P_0P_1 as the value m increases in the range $[-3, 1]$. Likewise, the curve bents towards the edge P_4P_5 as the value of the shape parameter n increases in the same range while m is kept fixed. Consequently, the curve approaches the control polygon as the values of m and n increase simultaneously. Fig. 3, Fig. 4 and Fig. 5 show the curves drawn for

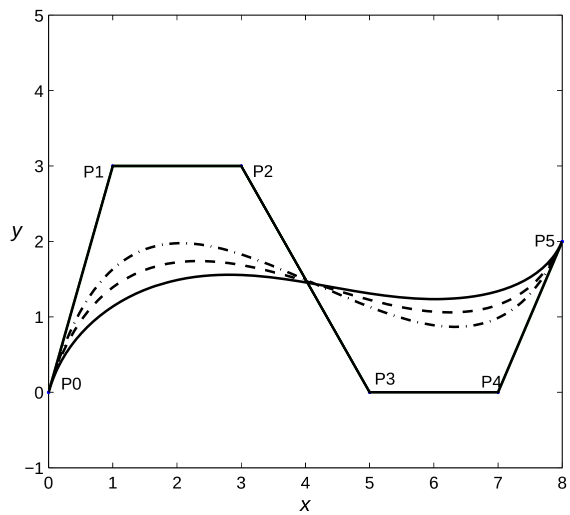

Fig. 5 Effect on the shape of the curve with altering the values of shape parameters m and n simultaneously.

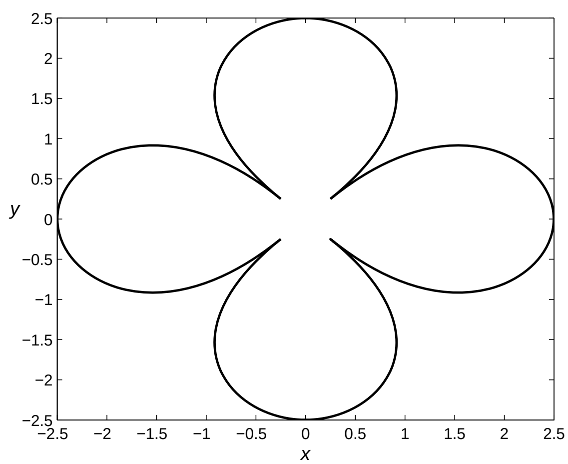

Fig. 6 Open curves.

different values of m and n. In Fig. 3 the curves are drawn by increasing the value of m gradually as $m = -3$ (solid lines), $m = -2$ (dashed lines), $m = -1$ (dashed dotted lines), $m = 0$ (dotted lines) and $m = 1$ (solid lines), keeping n fixed. Whereas in Fig. 4 the curves are drawn with m fixed and gradually increasing n as $n = -3$ (solid lines) $n = -2$ (dashed lines) $n = -1$ (dashed dotted lines), $n = 0$ (dotted lines) and $n = 1$ (solid lines). Accordingly, Fig. 5 shows the curves drawn by increasing the values of both m and n simultaneously as $m = n = -1$ (solid lines), $m = n = 0$ (dashed lines) and $m = n = 1$ (dashed dotted lines).

The presence of shape parameters provides an intuitive control on the shape of the curve. This property is used to generate open and closed curves. Fig. 6 and Fig. 7 show the open and closed curves, respectively, generated by using the quasi-quintic trigonometric Bézier curves. For the closed curves we set $P_5 = P_0$.

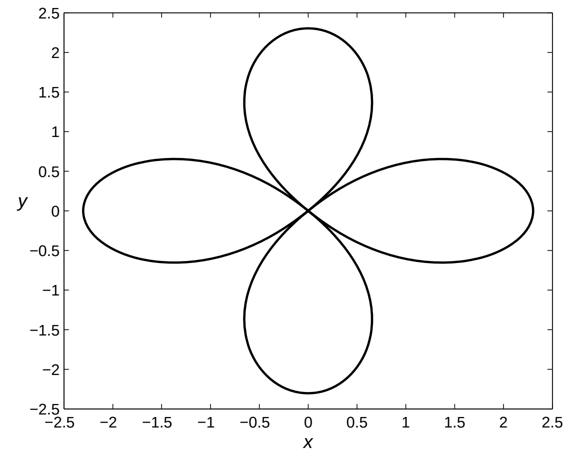

Fig. 7 Closed curves.

CONCLUDING REMARKS AND FUTURE WORK

In this paper, a newly constructed quasi-quintic trigonometric Bézier curve with two shape parameters is presented which inherits most of the geometric properties of the traditional Bézier curve. The shape of the curve can be adjusted using the values of the shape parameters. The proposed curve can be used to generate open and closed curves. In future, the curve can be used to generate the some trigonometric curves like arcs of circle, ellipse or parabola under suitable conditions. Further, it can be extended to tensor product surfaces.

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