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A CLASS OF RIEMANNIAN METRICS ON A MANIFOLD

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0. Introduction

In his suggestive paper [3], R. Bott proved that if geodesics starting from a point p in a riemannian manifold M are all closed geodesics whose length of a lap is constant, then the number of conjugate points of p on a lap of these closed geodesics are constant, counting the multiplicity. This result has been extended recently by Nakagawa [9], who proved that if all geodesics starting from a point p with a constant length c come back to the point p (these are not necessarily closed geodesics), then the number of conjugate points on a lap of these closed geodesic segments are constant, counting the multiplicity.

If a stronger condition is assumed so that the cut point of p with respect to every geodesic starting from p may become a middle point of this closed geodesic segment, then the manifold M has a decomposition $M = D_p \cup_\varphi D_N$, as it is seen in Warner's paper [11], where D_p is a disk, N is a cut locus of p, which becomes a closed submanifold in this case, and D_N is a normal disk bundle of N in M.

In this paper, as an extension of these facts, it will be proved that *if a compact connected real analytic riemannian manifold M has a submanifold N such that the cut point of N with respect to every geodesic, which starts from N and whose initial direction is orthogonal to N has a constant distance π from N, then M has a decomposion $M = D_N \cup_\varphi D_{N'}$, where N' is the cut locus of N and D_N, $D_{N'}$ are normal disk bundles of N, N' respectively* (cf. Theorem 3.1). Of course, manifolds having such a decomposition are very special, but at any rate, it seems interesting to consider some details about that kind of manifold.

On a single manifold M, there are many, various riemannian metrics, which form a convex set. Each of these riemannian metrics, however, ought to be influenced by the topological structures of the manifold. Roughly speaking, one must be able to determine the topological structures of M by using only one riemannian metric, but at least at the present time it seems impossible. Therefore, it seems interesting to consider some useful class of riemannian metrics instead of a single metric or the whole metrics. In this paper, it

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will be considered a class of riemannian metrics satisfying the following conditions:

Let N, N' be connected submanifolds. Suppose there is a riemannian metric G_0 such that (i) the cut locus of N is N', (ii) the cut point for each geodesic starting orthogonally from N has the constant distance π from N. The class $\mathscr{G}(G_0)$ of riemannian metrics to be considered in this paper is the set of riemannian metrics G such that (a) G satisfies (i), (ii) above, (b) normal vector bundles of N, N' under the metric G_0 coincide with those with respect to G as a set and riemannian vector bundle.

Let $S_N(\pi/2)$ be the normal sphere bundle of N each of whose fibre is a sphere of radius $\pi/2$, and let $\mathrm{Diff}(S_N(\pi/2))$ be the set of all diffeomorphisms on $S_N(\pi/2)$. The topology of $\mathscr{G}(G_0)$ and $\mathrm{Diff}(S_N(\pi/2))$ is so-called C^∞-topology. Then, *there is a continuous mapping J of $\mathscr{G}(G_0)$ into* $\mathrm{Diff}(S_N(\pi/2))$, *and letting* $\widetilde{\mathrm{Diff}}(S_N(\pi/2)) =$ *image of J, $\mathscr{G}(G_0)$ is the total space of a fibre bundle over* $\widetilde{\mathrm{Diff}}(S_N(\pi/2))$ (cf. Theorem 2.12), where the fibre may be different at each connected component of $\widetilde{\mathrm{Diff}}(S_N(\pi/2))$. The fibre of this bundle is closely related to the set of all diffeomorphisms on M, which leave the sets N, N' fixed respectively.

Therefore, if one can see the topological structure of $\mathscr{G}(G_0)$ by using techniques of differential geometry, then one can see the topological structures of $\widetilde{\mathrm{Diff}}(S_N(\pi/2))$. However, at least at the present time the author does not know anything about the topological stuructures of $\mathscr{G}(G_0)$.

1. Compact riemannian manifold M with a submanifold N' as the cut locus of $N \subset M$

Let M be a compact C^∞-manifold, and N, N' closed, connected C^∞-submanifolds of M. Consider a riemannian metric G on M. Hereafter, parameters of geodesics are the arc length throughout this paper. For a geodesic $g(t)$, $0 \le t < \infty$, starting from N and orthogonal to N at the starting point, the *cut point* $g(t_0)$ of N is the point such that the geodesic $g(t)$, $0 \le t \le t_0$, attains the distance between $g(t_0)$ and N but $g(t)$ does not for $t > t_0$. For defining the cut point, the geodesic considered ought obviously to be orthogonal to N at the starting point. The *cut locus* is the set of cut points for all geodesics starting orthogonally from N.

In this section, a manifold M having a C^∞-riemannian metric with the following property $P(N, N')$ with respect to the two submanifolds N, N' will be considered.

$P(N, N')$: (i) The cut locus of N is N'. (ii) For every geodesic, starting orthogonally from N, the distance from N to the cut point of N is constant and equal to π.

1.1. Theorem. *If there is a riemannian metric satisfying $P(N, N')$, then M is diffeomorphic to a manifold $\mathscr{D}_N \cup_\varphi \mathscr{D}_{N'}$, where $\mathscr{D}_N$, $\mathscr{D}_{N'}$ are normal disk* *bundles of N, N' in M respectively and φ is an attaching diffeomorphism of $\partial\mathscr{D}_N$ onto $\partial\mathscr{D}_{N'}$.*

Proof. Let G be the riemannian metric satisfying $P(N, N')$ and fix this metric on M. Since N' coincides with the cut locus of N, for every point $q \in N'$, there is a geodesic $g(t)$, $0 \leq t \leq \pi$, such that $g(0) \in N$, $g(\pi) = q$. $\left.\frac{d}{dt}\right|_\pi g(t)$ is orthogonal to N'. In fact, if not, then for small $\delta > 0$, the distance δ' between $g(\pi - \delta)$ and N' satisfies $\delta' < \delta$. Let $\acute{g}(t)$, $0 \leq t \leq \delta'$, be the geodesic attaining this distance. Put $\acute{g}(0) = g(\pi - \delta)$ and $\acute{g}(\delta') \in N'$. Then there is a geodesic $h(t)$, $0 \leq t \leq t_0$, such that $h(0) \in N$, $h(t_0) = g'(\delta')$ and $t_0 < \pi$. This is a contradiction because there is a geodesic $\acute{h}(t)$, $0 \leq t \leq \pi$, such that $\acute{h}(0) \in N$, $\acute{h}(\pi) = h(t_0)$ and $\acute{h}(\pi)$ is a cut point of N. Therefore, the distance between N and N' is π. It follows that every geodesic starting orthogonally from N strikes N' orthogonally when the length becomes π.

Let $\mathscr{D}_{N'}(\varepsilon)$ be the normal disk bundle of N each of whose fibre is a disk of radius ε. For a small $\varepsilon > 0$, $\mathscr{D}_{N'}(\varepsilon)$ is a tubular neighborhood of N, and for each $q \in \partial\mathscr{D}_{N'}(\varepsilon)$, $\mathscr{D}_q(\varepsilon) \cap N' = p(q)$, where $\mathscr{D}_q(\varepsilon)$ is the ε-neighborhood of q and p is the projection of the bundle $\mathscr{D}_{N'}(\varepsilon)$. Let $q \in \partial\mathscr{D}_{N'}(\varepsilon)$ and $g(t)$, $0 \leq t \leq t_0$, be the geodesic such that $g(0) \in N$, $g(t_0) = q$ and attains the distance between q and N.

If $t_0 < \pi - \varepsilon$, then the distance between N and N' becomes $< \pi$. Thus, $t_0 \geq \pi - \varepsilon$. If $t_0 > \pi - \varepsilon$, then $g(\pi)$ is not contained in N'. Therefore, $t_0 = \pi - \varepsilon$. Moreover, letting $\acute{g}(t)$, $0 \leq t \leq \varepsilon$, be the geodesic such that $\acute{g}(0) = q$, $\acute{g}(\varepsilon) = p(q)$, we obtain $\left.\frac{d}{dt}\right|_{\pi-\varepsilon} g(t) = \left.\frac{d}{dt}\right|_0 g'(t)$; otherwise the distance between N and N' is $< \pi$. Thus $g'(t) \equiv g(\pi - \varepsilon + t)$.

Let T_M, V_N and $D_N(r)$ be the tangent bundle of M, normal vector bundle of N and normal disk bundle of N in V_N, each of whose fibre is a disk of radius r. As is well known, the exponential mapping $\mathrm{Exp}: T_M \to M$ is a C^∞-mapping. Put $E = \mathrm{Exp} | V_N$. Then, from the argument above, $E(D_N(\pi)) = M$. Let $D_N^i(r)$ be the set of interior points of $D_N(r)$. Then $E: D_N^i(\pi) \to M - N'$ is diffeomorphic, because $E(\partial D_N(\pi)) = N'$ is the cut locus of N.

Let $V_{N'}$, $D_{N'}(r)$ be the normal vector bundle and normal disk bundle of N' respectively. Let $E' = \mathrm{Exp} | V_{N'}$. Then $M = E(D_N(\pi/2)) \cup E'(D_{N'}(\pi/2))$. Since $E: D_N(\pi/2) \to E(D_N(\pi/2))$, $E': D_{N'}(\pi/2) \to E'(D_{N'}(\pi/2))$ are both diffeomorphism and $E(\partial D_N(\pi/2)) = \partial E(D_N(\pi/2)) = \partial E'(D_{N'}(\pi/2)) = E'(\partial D_{N'}(\pi/2))$, we obtain a desired decomposition of M by putting $\mathscr{D}_N = E(D_N(\pi/2))$, $\mathscr{D}_{N'} = E'(D_{N'}(\pi/2))$ and $\varphi =$ identity.

Let $p: T_M \to M$ be the projection. Two riemannian metrics G_1, G_2 both of which have the property $P(N, N')$ are called *equivalent* (denoted by $G_1 \sim G_2$), if $pX \in N \cup N'$ for $X \in T_M$ is orthogonal to $N \cup N'$ with respect to G_1 implies that X is also so with respect to G_2 and $||X||_{G_1} = ||X||_{G_2}$. Let $\mathscr{G}(G_0)$ be the set of riemannian metrics each of which has the property $P(N, N')$ and is equivalent to G_0. Of course, G_0 is assumed to have the property $P(N, N')$. Let V_N, $V_{N'}$ be normal vector bundles of N, N' with respect to G_0. Normal vector bundles of N, N' with respect to any other metric in $\mathscr{G}(G_0)$ coincide with V_N, $V_{N'}$ respectively as riemannian vector bundles. For every $G \in \mathscr{G}(G_0)$, denote by Exp_G the exponential mapping $T_M \to M$ with respect to G and put $E_G = \mathrm{Exp}_G | V_N$, $E'_G = \mathrm{Exp}_G | V_{N'}$.

Since $M = E_G(D_N(\pi/2)) \cup E'_G(D_{N'}(\pi/2))$ and $\partial E_G(D_N(\pi/2)) = \partial E'_G(D_N(\pi/2))$ for every $G \in \mathscr{G}(G_0)$, we have a C^∞-diffeomorphism $\psi_G = E'^{-1}_G E_G$ of $S_n(\pi/2)$ onto $S_{N'}(\pi/2)$, where $S_N(\pi/2) = \partial D_N(\pi/2)$, $S_{N'}(\pi/2) = \partial D_{N'}(\pi/2)$. Clearly, another definition of ψ_G is given by

$$\psi_G(X) = -\frac{\pi}{2} \frac{d}{dt}\Big|_\pi \mathrm{Exp}_G t\frac{X}{\|X\|} = -\frac{d}{dt}\Big|_2 \mathrm{Exp}_G tX\,.$$

Let $\widetilde{\mathrm{Diff}}(S_N(\pi/2))$ be the set of C^∞-diffeomorphisms such that, denoting by φ an element of $\widetilde{\mathrm{Diff}}(S_N(\pi/2))$, there is a C^∞-diffeomorphism Φ of M onto $D_N(\pi/2) \cup_{\psi_{G_0}\varphi} D_{N'}(\pi/2)$ satisfying $\Phi(N)=N$, $\Phi(N')=N'$ and $d\Phi | V_N : V_N \to V_N$, $d\Phi | V_{N'} : V_{N'} \to V_{N'}$ are both identities, where V_N, $V_{N'}$ are identified with normal vector bundles of N, N' in $D_N(\pi/2)$, $D_{N'}(\pi/2)$ respectively.

Let $\mathrm{Diff}_0(S_N(\pi/2))$ be an arcwise connected component containing the identity of the set of whole diffeomorphisms on $S_n(\pi/2)$. There is no difficulty in verifying that for each $\varphi \in \widetilde{\mathrm{Diff}}(S(\pi/2))$ we have $\varphi\, \mathrm{Diff}_0(S_N(\pi/2)) \subset \widetilde{\mathrm{Diff}}(S_N(\pi/2))$.

1.2. Proposition. *The mapping J defined by $J(G) = \psi_{G_0}^{-1} \cdot \psi_G$ is a mapping of $\mathscr{G}(G_0)$ onto $\widetilde{\mathrm{Diff}}(S_N(\pi/2))$.*

Proof. Since $J(\mathscr{G}(G_0)) \subset \widetilde{\mathrm{Diff}}(S_N(\pi/2))$ is clear, we have only to show that J is surjective. Let $\varphi \in \widetilde{\mathrm{Diff}}(S_N(\pi/2))$. Since there is a diffeomorphism $\Phi: M \to D_N(\pi/2) \cup_{\psi_{G_0}\varphi} D_{N'}(\pi/2)$ having the property mentioned above, it is enough to show that there is a riemannian metric on $D_N(\pi/2) \cup_{\psi_{G_0}\varphi} D_{N'}(\pi/2)$ satisfying $P(N, N')$ and normal bundles of N, N' coincide with V_N, $V_{N'}$ respectively. It will be done in the next three lemmas.

Denote by (M, G) the manifold with riemannian metric G and let $\hat{G}$ be the usual metric on S^{n-1}.

1.3. Lemma. *Let $B(r) = \{(x_1, \cdots, x_n) \in R^n;\ \sum_{i=1}^n x_i^2 \leq r^2\}$. Then there is a C^∞-riemannian metric on $B(r)$ satisfying* (i) *$B(r) - B(r-\varepsilon)$ is isometric to $(S^{n-1}, \hat{G}) \times (r-\varepsilon', r]$, where ε' is some positive number depending only on ε,* (ii) *$\{t(x_1, \cdots, x_n);\ 0 \leq t \leq 1\}$ is a geodesic segment of constant length s_1 for every $(x_1, \cdots, x_n) \in \partial B(r)$ and the parameter t is proportional to the arc length,* (iii) *this riemannian metric is invariant under the natural operation of the orthogonal group $O(n)$ on $B(r)$.*

Proof. Let $h(t)$, $0 \leq t \leq \xi$, be a C^∞-function satisfying a) $h(t) = 1$ for $\xi - \varepsilon' \leq t \leq \xi$, b) $h'(t) > 0$ for $0 < t < \xi - \varepsilon'$, c) $\lim_{t\to 0} h^{(m)}(t) = \infty$ for $m = 1, 2, \cdots$ and $\lim_{t\to 0} h(t) = 0$, d) the length of the graph is s_1, that is,

$$\int_0^\xi \sqrt{1 + (h'(t))^2}\, dt = s_1 .$$

Let $\theta_1, \cdots, \theta_{n-1}$ be local coordinates of $\partial B(r)$, and $\hat{G}_{ij}(\theta_1 \cdots \theta_{n-1})$ the riemannian metric $\hat{G}$ expressed by the local coordinates. Define the metric G on $B(r) - \{0\}$ by

$$(G_{ij}(t, \theta_1, \cdots, \theta_{n-1})) = h\left(\frac{r}{\xi} t\right)\begin{pmatrix} 1, 0 & \cdots & 0 \\ 0 & & \\ \vdots & \hat{G}_{ij}(\theta_1 \cdots \theta_{n-1}) & \\ 0 & & \end{pmatrix},$$

where $(t, \theta_1, \cdots, \theta_{n-1})$ is considered as a polar coordinate. From the property (c), G is considered as a metric on $B(r)$. There is no difficulty in verifying that G is a desired metric. Clearly $\varepsilon = \dfrac{r}{\varepsilon}\varepsilon'$.

1.4. Lemma. *Let $W = M \times [0, 1]$. Given any C^∞-riemannian metrics $\acute{G}_0$, $\acute{G}_1$ on $M \times \{0\}$, $M \times \{1\}$ respectively. Then there is a C^∞-riemannian metric G on W satisfying* (i) *there are ε-neighborhood U_0 of $M \times \{0\}$, isometric to $(M, \acute{G}_0) \times [0, \varepsilon)$, and ε-neighborhood U_1 of $M \times \{1\}$, isometric to $(M, \acute{G}) \times (1 - \varepsilon, 1]$,* (ii) *any $\{(x, t); 0 \leq t \leq 1\}$ for every $x \in M$ is a geodesic segment of constant length s, where the parameter t is proportional to the arc length and s is a positive number previously given.*

Proof. Let $h(t)$ be a C^∞-function such that (i) $h(t) \equiv 0$ for $0 \leq t \leq \dfrac{\varepsilon}{s}$ and $h(t) \equiv 1$ for $1 - \dfrac{\varepsilon}{s} \leq t \leq 1$, (ii) $h'(t) > 0$ for $\dfrac{\varepsilon}{s} < t < 1 - \dfrac{\varepsilon}{s}$. Define the metric G on W by

$$(G_{ij}(t, x_1 \cdots x_n)) = \begin{pmatrix} s^2 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & h(t)\acute{G}_0 + (1 - h(t))\acute{G}_1 & \\ 0 & & & \end{pmatrix},$$

where $x_1, \cdots, x_n$ are local coordinates on M. It is not hard to verify that G has the desired property.

The following lemma is trivial.

1.5. Lemma. *Let $N_1 = M \times (-\varepsilon, 0]$, $N_2 = M \times [0, \varepsilon)$. Given metrics G_1, G_2 on N_1, N_2 such that N_i is isometric to the product of $(M; G_i | M)$ and*

$(-\varepsilon, 0]$ *or* $[0, \varepsilon)$. *Assume there is an isometry* φ: $(M, G_1|M) \to (M, G_2|M)$. *Then* $N_1 \cup_\varphi N_2$ *becomes a riemannian manifold isometric to* $(M, G_1|M) \times (-\varepsilon, \varepsilon)$.

Now, using these three lemmas, a poof of Proposition 1.2 will be given below.

Let G_1, G_1' be arbitrarily fixed riemannian metrics on N, N' respectively. For each point $x \in N$ (resp. $y \in N'$) there exists a neighborhood U of x (resp. U' of y) such that $p^{-1}(U) = U \times D_x(\pi/2)$ (resp. $p^{-1}(U) = U' \times D_y(\pi/2)$) where p is the projection and $D_x(\pi/2)$ (resp. $D_y(\pi/2)$) is the fibre at x (resp. at y). Take the metric G_2 (resp. G_2') on $D_x(\pi/2)$ (resp. $D_y(\pi/4)$) which is defined in Lemma 1.3 and define the riemannian metric on $U \times D_x(\pi/2)$ (resp. $U' \times D_y(\pi/4)$) which is a direct product. Being invariant under the natural operation of orthogonal groups, the metrics on $D_x(\pi/2)$, $D_y(\pi/4)$ define riemannian metrics G_3, G_3' on $D_N(\pi/2)$, $D_{N'}(\pi/4)$ respectively. N, N' are both totally geodesic in $D_N(\pi/2)$, $D_{N'}(\pi/4)$. From the property (i) in Lemma 1.3, we see that $D_N(\pi/2) - D_N(\pi/2 - \varepsilon)$ and $D_{N'}(\pi/4) - D_{N'}(\pi/4 - \varepsilon)$ are isometric to $\partial D_N(\pi/2) \times (\pi/2 - \varepsilon, \pi/2]$, $\partial D_{N'}(\pi/4) \times (\pi/4 - \varepsilon, \pi/4]$ respectively. Remark that $D_{N'}(\pi/2) - D_N^i(\pi/4)$ is diffeomorphic to $\partial D_{N'}(\pi/2) \times [\pi/4, \pi/2]$, where $D_{N'}^i(\pi/4)$ is an interior point of $D_{N'}(\pi/4)$.

Let $\varphi \in \widetilde{\mathrm{Diff}}(S_N(\pi/2))$, and $G_3'' = (\psi_{G_0}\varphi)_*(G_3|S_N(\pi/2))$. G_3'' is a metric on $S_{N'}(\pi/2)$. On $S_{N'}(\pi/4)$, there is a riemannian metric $G_3'|S_N(\pi/4)$.

From Lemma 1.4, there is a riemannian metric G_4 on $D_{N'}(\pi/2) - D_{N'}^i(\pi/4)$ satisfying the properties (i), (ii) in this lemma. Therefore Lemma 1.5 shows that $G_3 \cup G_4 \cup G_3'$ is a riemannian metric on $D_N(\pi/2) \cup_{\psi_{G_0}\varphi} D_{N'}(\pi/2)$ satisfying the desired property.

Let $\mathscr{G}(G_0, \mathscr{F})$ be the set of the riemannian metrics on M such that (i) $G \in \mathscr{G}(G_0)$, (ii) if $G \in \mathscr{G}(G_0, \mathscr{F})$, then $E_{G_0}(X) = E_G(X)$ and $E'_{G_0}(Y) = E'_G(Y)$ for $X \in D_N(\pi/2)$, $Y \in D_{N'}(\pi/2)$.

1.6. Lemma. $\mathscr{G}(G_0, \mathscr{F})$ *is a convex subset, that is, for any* $G, G' \in \mathscr{G}(G_0, \mathscr{F})$, $tG + (1-t)G'$ *is contained in* $\mathscr{G}(G_0, \mathscr{F})$ *for* $0 \leq t \leq 1$.

Proof. Let p be the projection from $D_N^i(2\pi/3)$ onto N. Let U be an open neighborhood of x in N such that $p^{-1}(U)$ is diffeomorphic to $U \times D_x^i(2\pi/3)$, where $D_x^i(2\pi/3) = p^{-1}(x)$. This diffeomorphism ξ: $U \times D_x^i(2\pi/3) \to p^{-1}(U)$ can be so chosen that $p\xi(x, y) = x$.

Give a polar coordinate $(t, \theta', \cdots \theta^{r-1})$ on $D_x^i(2\pi/3)$ and an arbitrary coordinate $x^1 \cdots x^q$ on U. Since E_G is a diffeomorphism of $D_N^i(2\pi/3)$ onto $\mathscr{D}_N^i(2\pi/3) = E_G(D_N^i(2\pi/3))$, $(t, \theta^1, \cdots, \theta^{r-1}, x^1, \cdots, x^q)$ is a coordinate on some open subset of M. Since $E_G(X) = E_{G'}(X)$ for any $X \in D_N^i(2\pi/3)$, G and G' are expressed in this coordinate by

$$G = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & G_{ij}(t, \theta, x) & \\ 0 & & & \end{pmatrix}, \qquad G' = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & G'_{ij}(t, \theta, x) & \\ 0 & & & \end{pmatrix},$$

where $\theta = (\theta^1 \cdots \theta^{r-1})$, $x = (x^1 \cdots x^q)$. Clearly, if $t \to 0$, then the $*$-parts of these metrics

$$\left(\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline 0 & * & * \\ \hline 0 & * & \end{array}\right)\begin{array}{l} \left.\begin{array}{l} \\ \\ \end{array}\right\} r \\ \} q \end{array}$$

tend to 0.

For any $s \in [0, 1]$, $G'' = sG + (1 - s)G'$ is obviously a riemannian metric on M and on the open subset $E_G p^{-1}(U)$, G'' is expressed by

$$G'' = \begin{pmatrix} 1 & 0 \cdots 0 \\ 0 & \\ \vdots & G''_{ij}(t, \theta, x) \\ 0 & \end{pmatrix},$$

and $*$-parts of this matrix tend to 0 if $t \to 0$. Considering all coordinate neighborhoods by the same method as above, we can conclude from only the shape of these matrices that $G'' \in \mathscr{G}(G_0, \mathscr{F})$.

1.7. Proposition. *There is a one-to-one correspondence K from $\mathscr{G}(G_0, \mathscr{F}) \times$ Diff(M, N, N') onto J^{-1}(id.), where* Diff(M, N, N') *is the diffeomorphism on M such that if $\xi \in$ Diff(M, N, N') then $\xi(N) = N$, $\xi(N') = N'$ and $d\xi \mid V_N$, $d\xi \mid V_{N'}$ are both identity. Moreover, the correspondence K is given by $K(G, \xi) = \xi_*(G)$.*

Proof. If $J(G) =$ id., then $\xi_*(G) \in J^{-1}$(id.) for every $\xi \in$ Diff(M, N, N'). Since $J(\mathscr{G}(G_0, \mathscr{F})) =$ id., $K(\mathscr{G}(G_0, \mathscr{F}) \times$ Diff$(M, N, N')) \subset J^{-1}$(id.).

Assume $K(G, \xi) = K(G', \xi')$ for $G, G' \in \mathscr{G}(G_0, \mathscr{F})$, $\xi, \xi' \in$ Diff(M, N, N'). Then $\xi'^{-1}_* \xi_*(G) = G'$. Since $d(\xi'^{-1} \cdot \xi) \mid V_N =$ id. and $E_G(X) = E_{G'}(X)$, $E'_G(X) = E'_{G'}(X)$, putting $x = E_G(X)$. $\xi'^{-1}\xi(x) = \xi'^{-1}\xi(E_G(X)) = E_{G'}(d(\xi'^{-1}\xi)X = E_{G'}(X) = x$. It follows that $\xi = \xi'$ and then $G = G'$.

Let $G_1 \in J^{-1}$(id.). Define a mapping ξ as follows:

$$\xi(x) = \begin{cases} E_{G_0}(X)\,, & \text{if } x = E_{G_1}X\,, \quad X \in D_N(\pi/2)\,, \\ E'_{G_0}(Y)\,, & \text{if } x = E'_{G_1}Y\,, \quad Y \in D'_N(\pi/2)\,, \end{cases}$$

Let ξ_0, ξ_1 be diffeomorphisms of $D_N(\pi/2) \cup_{\phi G_0} D_{N'}(\pi/2)$ onto M defined by

$$\xi_i(X) = \begin{cases} E_{G_i}(X)\,, & X \in D_N(\pi/2)\,, \\ E'_{G_i}(X)\,, & X \in D_{N'}(\pi/2)\,. \end{cases} \quad i = 0, 1\,,$$

Then $\xi = \xi_0 \xi_1^{-1}$. It is easy to see that $\xi \in$ Diff(M, N, N'). From the definition of ξ_0, ξ_1, ξ, we see $\xi(E_{G_1}(X)) = E_{G_0}(X)$ for $X \in D_N(\pi/2)$ and $\xi(E'_{G_1}(X)) = E'_{G_0}(X)$ for $X \in D_{N'}(\pi/2)$. It follows that $\xi_* G_1 \in \mathscr{G}(G_0, \mathscr{F})$ and then $G_1 = K(\xi_*(G_1), \xi^{-1})$.

A curve φ_s in $\widetilde{\text{Diff}}(S_N(\pi/2))$ is called differentiable, if, putting $\eta(x, s) = \varphi_s(x)$, η is a C^∞-differentiable mapping of $S_N(\pi/2) \times [0, 1]$ onto $S_N(\pi/2)$.

1.8. Proposition. *If there is a differentiable curve φ_s in $\widetilde{\text{Diff}}(S_N(\pi/2))$, then there is a one-to-one correspondence I from $J^{-1}(\varphi_0)$ onto $J^{-1}(\varphi_1)$.*

Proof. Let $J^{-1}(\varphi_0) \ni G$. There exists a C^∞-diffeomorphism ξ_G of $D_N(\pi/2) \cup_{\varphi_{G_0}\varphi_0} D_{N'}(\pi/2)$ onto M, which is defined by

$$\xi_G(X) = \begin{cases} E_G(X)\,, & X \in D_N(\pi/2)\,, \\ E'_G(X)\,, & X \in D_{N'}(\pi/2)\,, \end{cases}$$

Let $h(t)$ be a C^∞-function such that $h(t) \equiv 0$, $0 \le t \le 1/3$, $h(t) \equiv 1$, $2/3 \le t \le 1$, $h'(t) > 0$, $1/3 < t < 2/3$. Define a diffeomorphism Φ on $D_N(\pi/2)$ by

$$\Phi(X) = \begin{cases} X\,, & X \in D_N(\pi/4)\,, \\ \psi_{h(t)}(X)\,, & X \in S_N((\pi/4)t + \pi/4)\,, \end{cases}$$

where $\psi_s = \varphi_0^{-1}\cdot\varphi_s$. Let $\mathscr{X}$ be a vector field on $D_N^i(\pi/2) - D_N(\pi/5)$ defined by $\mathscr{X}(X) = \frac{d}{dt}\Big|_{\|X\|} t\frac{X}{\|X\|}$. Put $\mathscr{Y} = d\Phi\mathscr{X}$ and $Y = d\xi_G\mathscr{Y}$. Y is a vector field on $\mathscr{D}_N^i(\pi/2) - \mathscr{D}_N(\pi/5)$, where $\mathscr{D}_N(r) = E_G D_N(r)$. Y coincides with $\acute{X} = \frac{d}{dt}\Big|_{\|X\|} E_G t\frac{X}{\|X\|} = d\xi_G\mathscr{X}$ at neighborhoods of $\mathscr{S}_N(\pi/2) = E_G(S_N(\pi/2))$ and $\mathscr{S}_N(\pi/5) = E_G(S_N(\pi/5))$.

Replace the riemann metric G on $\mathscr{D}_N^i(\pi/2) - \mathscr{D}_N(\pi/5)$ by G', where

$$\begin{aligned} \langle Z, Z'\rangle_{G'} = \|p_1Z\|_G\cdot\|p_1\acute{Z}\|_G\cdot\|Y\|_G^2 - \|p_1Z\|_G\langle Y - \acute{X}, p_2\acute{Z}\rangle_G \\ - \|p_1\acute{Z}\|_G\langle Y - \acute{X}, p_2Z\rangle_G + \langle p_2Z, p_2\acute{Z}\rangle_G\,, \end{aligned}$$

$$p_2Z = Z - \langle Z, \acute{X}\rangle_G\acute{X}\,,\quad p_1Z = Z - p_2Z = \langle Z, \acute{X}\rangle_G\acute{X}\,.$$

Since $\|Y\|_G^2 = \|\acute{X}\|_G^2 + \|Y - \acute{X}\|_G^2 = 1 + \|Y - \acute{X}\|_G^2$, it is easy to see that G' is positive definite, $\langle Y, Y\rangle_{G'} = 1$ and $\langle Y, p_2Z\rangle_{G'} = 0$. G' coincides with G at neighborhoods of $\mathscr{S}_N(\pi/2)$, $\mathscr{S}_N(\pi/5)$, and therefore can be naturally extended to be a riemannian metric on M; this extended metric is denoted by G'. Put $I(G) = G'$. An integral curve of Y is a geodesic with respect to G'. Therefore $E_{G'} = E_G\cdot\Phi$, and then $E_{G'}'^{-1}E_{G'}\,|\,S_N(\pi/2) = E_G'^{-1}E_G\cdot\varphi_0^{-1}\varphi_1 = \psi_{G_0}\varphi_0\varphi_0^{-1}\varphi_1 = \psi_{G_0}\varphi_1$. Thus, $I(G) \in J^{-1}(\varphi_1)$.

Conversely, let $G' \in J^{-1}(\varphi_1)$. Replace Φ by Φ^{-1} and make a riemannian metric G″ by the same method as above. Then, putting $I'(G') = G''$, $G'' \in J^{-1}(\varphi_0)$ and it is easy to show that $I'\cdot I =$ identity (the proof of this will be seen in Lemma 2.11).

2. C^∞-topology for $\mathscr{G}(G_0)$, $\widetilde{\mathrm{Diff}}(S_n(\pi/2))$ and $\mathrm{Diff}(M, N, N')$

Let M be a compact C^∞-manifold, and $\mathrm{Diff}^r(M)$, $\mathrm{Diff}(M)$ the sets of the C^r- and C^∞-diffeomorphisms on M respectively. As is well known $\mathrm{Diff}^r(M)$ is a C^∞-Banach manifold compatible with C^r-topology. Let I_j^k, $k \geq j$ be the natural inclusion of $\mathrm{Diff}^k(M)$ into $\mathrm{Diff}^j(M)$. The projective limit topology of $\{\mathrm{Diff}^r(M), I_r^k\}$ for $\mathrm{Diff}(M)$ is called C^∞-topology. Hereafter $\mathrm{Diff}^r(M)$, $\mathrm{Diff}(M)$ imply the groups with C^r-topology, C^∞-topology respectively.

2.1. Lemma. *$\mathrm{Diff}(M)$ is locally arcwise connected.*

Proof. Let I_r^∞ be the inclusion of $\mathrm{Diff}(M)$ into $\mathrm{Diff}^r(M)$. For any neighborhood U of id. in $\mathrm{Diff}(M)$, there are r and a neighborhood V of id. in $\mathrm{Diff}^r(M)$ such that $(I_r^\infty)^{-1}(V) \subset U$. For an arbitrarily fixed C^∞-riemannian metric, if $r \geq 1$, there is a neighborhood W such that (i) $W \subset V$, (ii) for every $\varphi \in W$, there exists uniquely a vector field X on M satisfying $\varphi(x) = \mathrm{Exp}\, X(x)$, (iii) $\varphi_t = \mathrm{Exp}\, tX(x)$, $0 \leq t \leq 1$, is also contained in W. (This is a proof of locally arcwise connectedness of $\mathrm{Diff}^r(M)$.) W is not necessarily convex or open.

Since Exp is a C^∞-mapping, if φ is a C^∞-diffeomorphism, then so also is φ_t. Thus, $(I_r^\infty)^{-1}W$ is arcwise connected.

2.2. Corollary. *$\widetilde{\mathrm{Diff}}(S_N(\pi/2))$ is an open subset of* $\mathrm{Diff}(S_N(\pi/2))$.

It is an immediate conclusion from the fact that $\widetilde{\mathrm{Diff}}(S_N(\pi/2)) \supset \varphi\, \mathrm{Diff}_0(S_N(\pi/2))$ for every $\varphi \in \widetilde{\mathrm{Diff}}(S_N(\pi/2))$.

The following lemma is trivial.

2.3. Lemma. *$\mathrm{Diff}(M, N, N')$ is a closed subgroup of* $\mathrm{Diff}(M)$.

Let $\mathscr{G}^r$, $\mathscr{G}$ be the C^r- and C^∞-riemanian metrics on M, and $\mathscr{S}^r$, $\mathscr{S}$ the C^r- and C^∞-symmetric bilinear forms on M, respectively. $\mathscr{S}^r$ is a Banach space with respect to C^r-norm and $\mathscr{G}^r$ is an open subset of $\mathscr{S}^r$. Then $\mathscr{G}^r \times T_M$ is a C^∞-Banach manifold. $\mathscr{G}^r \times \mathscr{S}^r$ is identified naturally with the tangent bundle of $\mathscr{G}^r$. Then $\mathscr{G}^r \times \mathscr{S}^r \times T(T_M)$ is the tangent bundle of $\mathscr{G}^r \times T_M$, where $T(T_M)$ is the tangent bundle of T_M. A vector field on $\mathscr{G}^r \times T_M$ is a cross section of this bundle. Using a local coordinate neighborhood U on M, the tangent bundles T_U, $T(T_U)$ are identified with $U \times \boldsymbol{R}^n$, $U \times \boldsymbol{R}^n \times \boldsymbol{R}^n \times \boldsymbol{R}^n$ respectively. Therefore the tangent bundle of $\mathscr{G}^r \times T_U$ is identified with $\mathscr{G}^r \times U \times \boldsymbol{R}^n \times \mathscr{S}^r \times \boldsymbol{R}^n \times \boldsymbol{R}^n$ and a vector field on $\mathscr{G}^r \times T_U$ is a mapping of $\mathscr{G}^r \times U \times \boldsymbol{R}^n$ into $\mathscr{S}^n \times \boldsymbol{R}^n \times \boldsymbol{R}^n$.

Define a vector field $\mathscr{X}$ as follows: Letting $x^1, \cdots, x^n$ be a coordinate on U, and $p^1, \cdots, p^n$ the natural coordinates of $\boldsymbol{R}^n$, $\mathscr{X}(G, x^1, \cdots, x^n, p^1, \cdots, p^n) = (0, p^1, \cdots, p^n, -\left\{ {1 \atop ij} \right\}_G p^i p^j, \cdots, -\left\{ {n \atop ij} \right\}_G p^i p^j)$, where $\left\{ {i \atop jk} \right\}_G$ is the Christoffel's symbol of G with respect to the coordinate system $x^1, \cdots, x^n$.

2.4. Lemma. *$\mathscr{X}$ is a C^{r-1}-vector field on $\mathscr{G}^r \times T_M$.*

Proof. We have only to show that

$$(G, x^1, \cdots, x^n, p^1, \cdots, p^n) \to \begin{Bmatrix} i \\ jk \end{Bmatrix}_G$$

is a C^{r-1}-function. For this, it suffices to prove that for an arbitrarily fixed $(x, p) = (x^1, \cdots, x^n, p^1, \cdots, p^n)$, the function $G \to \begin{Bmatrix} i \\ jk \end{Bmatrix}_G (x, p)$ is a C^∞-function. Therefore, we have only to show that for a fixed (x, p) and i, j, k, the mappings $G \to G_{ij,k}$, $G \to G^{ij}$ are both C^∞-differentiable. Let $\eta: S \to S_{ij,k}$ be a function on $\mathscr{S}^r$. Then η is a linear function and thus C^∞-differentiable. It is easy to show that $G \to G^{ij}$ is C^∞-differentiable.

2.5. Lemma. *The mapping $\Phi_s: \mathscr{G}^r \times T_M \to \mathscr{G}^r \times T_M$ defined by*

$$\Phi_s(G, X) = \left(G, \frac{d}{dt}\bigg|_s \mathrm{Exp}_G tX\right)$$

is a C^{r-1}-diffeomorphism for $r > 1$.

Proof. Since $\mathrm{Exp}_G tX$ is a geodesic, it is easy to see that $\left(G, \frac{d}{dt}\Big|_t \mathrm{Exp}_G tX\right)$ is the integral curve of $\mathscr{X}$ whose intial point is (G, X). Thus, from the well-known theorem concerning the integral curve of a differentiable vector field on Banach manifolds (cf. [10]), we have that the mapping Φ_s is a C^{r-1}-diffeomorphism.

2.6. Corollary. *$\mathscr{G}(G_0)$, $\mathscr{G}(G_0, \mathscr{F})$ are both closed subsets of $\mathscr{G}$.*

Proof. From the above lemma, the mappings $E_G: D_N(\pi/2) \to M$, $E'_G: D_N(\pi/2) \to M$ depend continuously on $G \in \mathscr{G}^r$. Therefore $\mathscr{G}^r(G_0)$ is closed for any $r > 1$, where $\mathscr{G}^r(G_0)$ is the set of the C^r-riemannian metrics having the property $P(N, N')$ and being equivalent to G_0. Thus, $\mathscr{G}(G_0)$ is closed in $\mathscr{G}$.

By the method similar to the above, we have that $\mathscr{G}(G_0, \mathscr{F})$ is closed in $\mathscr{G}(G_0)$.

2.6. Proposition. *The mapping J is continuous from $\mathscr{G}(G_0)$ onto $\widetilde{\mathrm{Diff}}(S_n(\pi/2))$.*

Proof. We have only to show the continuity of J and it suffices to prove that $J^r: \mathscr{G}^r(G_0) \to \mathrm{Diff}^{r-1}(S_N(\pi/2))$ is continuous for all $r > 1$, where the mapping J^r is defined in the same way as J, that is, J^r is defined as follows:

$$J^r(G)(X) = \psi_{G_0}^{-1}\left(-\frac{d}{dt}\bigg|_2 \mathrm{Exp}_G tX\right), \qquad X \in S_N(\pi/2)\,.$$

From Lemma 2.5, Φ_2 is a C^{r-1}-diffeomorpnism of $\mathscr{G}^r \times T_M$ onto itself, and $\Phi_2(\mathscr{G}^r(G_0) \times S_N(\pi/2)) = \mathscr{G}^r(G_0) \times S_{N'}(\pi/2)$. Let q be the projection from $\mathscr{G}^r \times T_M$ onto T_M. Then $q\Phi_2$ is a C^{r-1}-differentiable mapping, and $q\Phi_2(G, X) = \frac{d}{dt}\Big|_2 \mathrm{Exp}_G tX$. If $G \in \mathscr{G}^r(G_0)$, then $\Psi_G: X \to \frac{d}{dt}\Big|_2 \mathrm{Exp}_G tX$ is a C^{r-1} diffeomorphism of $S_N(\pi/2)$ onto $S_{N'}(\pi/2)$. Let $\mathrm{Diff}^{r-1}(S_N(\pi/2), S_{N'}(\pi/2))$ be the

C^{r-1}-diffeomorphisms of $S_N(\pi/2)$ onto $S_{N'}(\pi/2)$ with C^{r-1}-topology. Since $S_N(\pi/2)$ is compact, the differentiability of $q\Phi_2$ implies that the mapping $\acute{J}: \mathscr{G}^r(G_0) \to \mathrm{Diff}^{r-1}(S_N(\pi/2), S_{N'}(\pi/2))$ is continuous, where $\acute{J}(G) = \Psi_G$. Since $\psi_G = -\Psi_G$ and $J^r(G) = \psi_{G_0}^{-1}\psi_G = \psi_{G_0}^{-1}(-\Psi_G)$, we have that J^r is continuous for all $r > 1$.

2.7. Lemma. *The mapping* $\tilde{K}: \mathscr{G} \times \mathrm{Diff}(M) \to \mathscr{G}$ *defined by* $K(G, \varphi) = \varphi_*(G)$ *is continuous, where all the topologies are* C^∞*-topology.*

It is not hard to verify this lemma, since we have only to show that $\tilde{K}^r: \mathscr{G}^r \times \mathrm{Diff}^{r+1}(M) \to \mathscr{G}^r$ is continuous for $r \geq 0$, where $\tilde{K}^r(G, \varphi) = \varphi_*(G)$.

2.8. Proposition. *The mapping* $K: \mathscr{G}(G_0, \mathscr{F}) \times \mathrm{Diff}(M, N, N') \to J^{-1}(\mathrm{id.})$ *is a homeomorphism.*

Proof. Clearly K is continuous. Let $G_1 \in J^{-1}(\mathrm{id.})$. As in the proof of Proposition 1.7, define a diffeomorphism ξ_{G_1} as follows:

$$\xi_{G_1}(x) = \begin{cases} E_{G_0}E_{G_1}^{-1}(x)\,, & x \in E_{G_1}D_N(\pi/2)\,, \\ E'_{G_0}E'^{-1}_{G_1}(x)\,, & x \in E'_{G_1}D_N(\pi/2)\,. \end{cases}$$

ξ_{G_1} is a diffeomorphism contained in $\mathrm{Diff}(M, N, N')$. First of all, it will be proved that the mapping $\acute{K}: J^{-1}(\mathrm{id.}) \to \mathrm{Diff}(D_N(\pi/2) \cup_{\psi_{G_0}} D_{N'}(\pi/2), M)$ defined by

$$\acute{K}(G)(X) = \begin{cases} E_G X\,, & X \in D_N(\pi/2)\,, \\ E'_G X\,, & X \in D_{N'}(\pi/2) \end{cases}$$

is continuous, where $\mathrm{Diff}(D_N(\pi/2) \cup_{\psi_{G_0}} D_{N'}(\pi/2), M)$ is the C^∞-diffeomorphisms from $D_N(\pi/2) \cup_{\psi_{G_0}} D_{N'}(\pi/2)$ onto M with C^∞-topology. Let p be the projection $T_M \to M$. Then from Lemma 2.5, the mapping $\eta: \mathscr{G}^r \times T_M \to M$ defined by $\eta(G, X) = p\left(\frac{d}{dt}\Big|_1 \mathrm{Exp}_G tX\right) = \mathrm{Exp}_G X$ is a C^{r-1}-mapping. If $X \in D_N(\pi/2)$ (resp. $X \in D_{N'}(\pi/2)$), then $\eta(G, X) = E_G X$ (resp. $E'_G X$). Since $D_N(\pi/2) \cup_{\psi_{G_0}} D_{N'}(\pi/2)$ is compact, we see that $\acute{K}^r: J^{-1}(\mathrm{id.})^r \to \mathrm{Diff}^{r-1}(M, N, N')$ is continuous, where $\acute{K}^r$ and $J^{-1}(\mathrm{id.})^r$ are those which are naturally defined on $\mathscr{G}^r(G_0)$. Therefore, $\acute{K}$ is continuous. Since $\xi_{G_1}(x) = E_{G_0}\acute{K}(G_1)^{-1}(x)$ or $E'_{G_0}\acute{K}(G_1)^{-1}(x)$, the mapping $\acute{\acute{K}}: J^{-1}(\mathrm{id.}) \to \mathrm{Diff}(M, N, N')$ defined by $\acute{\acute{K}}(G) = \xi_G$ is continuous.

Since $\acute{\acute{K}}(G)_*(G) \in \mathscr{G}(G_0, \mathscr{F})$, the mapping $\kappa: J^{-1}(\mathrm{id.}) \to \mathscr{G}(G_0, \mathscr{F})$, $\kappa(G) = \acute{\acute{K}}(G)_*(G)$, is continuous. It is easy to show that $K(\kappa(G), \acute{\acute{K}}(G)^{-1}) = G$. Therefore K is a homeomorphism.

2.9. Proposition. *Under the same assumption as in Proposition* 1.8, *I is a homeomorphism.*

Proof. It suffices to prove that I is continuous, because I^{-1} is constructed by the same method. Take a local coordinate system (t, θ, x) on $D_N^i(2\pi/3)$ which is used in the proof of Lemma 1.6. Let $\mathscr{M}$ be the set of the C^∞-metrics on $D_N(\pi/2)$ such that if $\widehat{G} \in \mathscr{M}$, then

$$\widehat{G} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \widehat{G}_{ij}(t, \theta, x) & \\ 0 & & & \end{pmatrix}$$

with respect to this coordinate system. Let

$$I'_{\mathscr{Y}}(\widehat{G}) = \begin{pmatrix} 1 + \sum_{2\le a,b\le n} \widehat{G}_{ab}Y^aY^b\,, & -\sum_{2\le a\le n} \widehat{G}_{aj}Y^a \\ -\sum_{2\le a\le n} \widehat{G}_{ja}Y^a\,, & \widehat{G}_{ij} \end{pmatrix}, \quad 2 \le i\,,\ j \le n\,,$$

where Y^i is the i-th coefficient of $\mathscr{Y}$. Letting $\mathscr{G}(D_N(\pi/2))$ be the set of C^∞-metrics on $D_N(\pi/2)$ with C^∞-topology, $\acute{I}_{\mathscr{Y}}\colon \mathscr{M} \to \mathscr{G}(D_N(\pi/2))$ is a continuous mapping, where the topology for $\mathscr{M}$ is a C^∞-topology. It is not hard to verify that $I(G) \equiv (E_G)_*\acute{I}_{\mathscr{Y}}(E_G^{-1})_*(G)$ on $\mathscr{D}_N(\pi/2)$. $I(G) \equiv G$ at the neighborhoods of $\mathscr{S}_N(\pi/2)$ and $\mathscr{D}_N(\pi/4)$, where $\mathscr{S}_N(\pi/2) = E_G(S_N(\pi/2))$, $\mathscr{D}_N(\pi/4) = E_G(D_N(\pi/4))$. Since $E_G\colon D_N(\pi/2) \to M$ is continuous with respect to G, we see that I is continuous.

In the above proposition, the vector field $\mathscr{Y}$ is fixed, but need not leave $\mathscr{Y}$ fixed, that is, let $\mathrm{Diff}_{\mathscr{s}}(D_N(\pi/2))$ be the set of the C^∞-diffeomorphisms on $D_N(\pi/2)$ such that if $\Phi \in \mathrm{Diff}_{\mathscr{s}}(D_N(\pi/2))$, then (i) $\Phi(S_N(r)) = S_N(r)$, (ii) $\Phi \mid D_N(\pi/4) = \mathrm{id.}$, (iii) $\Phi(tX) = t\Phi(X)$ for $X \in S_N(1)$, $(\pi/4)(5/3) \le t \le \pi/2$. The topology is a C^∞-topology.

Replacing $\mathscr{Y}$ in the above proof by $d\Phi\mathscr{X}$, $\Phi \in \mathrm{Diff}_{\mathscr{s}}(D_N(\pi/2))$, the mapping $I''(\widehat{G}, \Phi) = I'_{d\Phi\mathscr{X}}(\widehat{G})$ is a continuous mapping of $\mathscr{M} \times \mathrm{Diff}_{\mathscr{s}}(D_N(\pi/2))$ into $\mathscr{G}(D_N(\pi/2))$. It is clear that $I''(\widehat{G}, \Phi) \equiv \widehat{G}$ on $D_N(\pi/4)$ and $D_N(\pi/2) - D_N^i((\pi/4)(5/3))$.

Define a mapping $\hat{I}\colon \mathscr{G}(G_1) \times \mathrm{Diff}_{\mathscr{s}}(D_N(\pi/2)) \to \mathscr{G}(G_0)$ as follows:

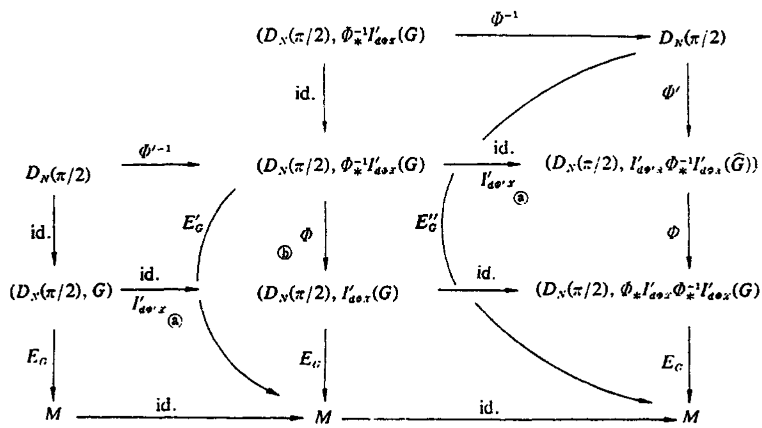
$$\hat{I}(G, \Phi)(x) = \begin{cases} (E_G)_*I''(G, \Phi)(E_G^{-1})_*(G)(x)\,, & x \in E_GD_N(\pi/2)\,, \\ G(x)\,, & x \in E_GD_{N'}(\pi/2)\,. \end{cases}$$

Thus, the following lemma can be easily proved.

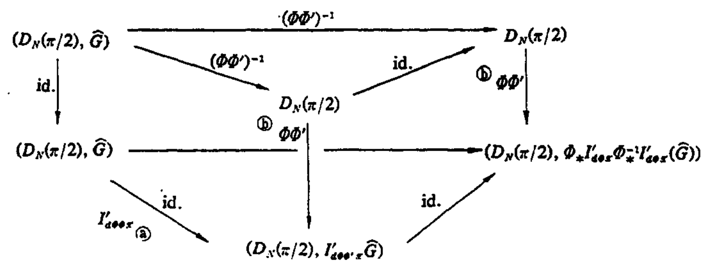
2.10. Lemma. *$\hat{I}$ is a continuous mapping.*

2.11. Lemma. *$\hat{I}(\hat{I}(G, \Phi), \Phi') = \hat{I}(G, \Phi\Phi')$.*

Proof. First of all, it will be proved that $I''(G, \Phi\Phi') = \Phi_*I'_{d\Phi'\mathscr{X}}\Phi_*^{-1}I'_{d\Phi\mathscr{X}}(G)$. We have the following commutative diagram:



$$\widehat{G} = E_{G_*}^{-1}G\,, \quad G' = E_{G_*}I'_{d\Phi x}E_{G_*}^{-1}G\,, \quad G'' = E_{G_*}I_{d\Phi' x}\Phi_*^{-1}I'_{d\Phi x}(\widehat{G})\,,$$



where $(D_N(\pi/2)$, metric) implies $D_N(\pi/2)$ having the riemannian metric, and ⓐ implies the change of metrics. There is no difficulty in verifying that the mappings ⓑ work as exponential mappings, that is, for instance, the mapping Φ: $(D_N(\pi/2),\ \Phi_*^{-1}I'_{d\Phi x}(\widehat{G})) \to (D_N(\pi/2),\ I'_{d\Phi x}(\widehat{G}))$ satisfies $\Phi\left(t\dfrac{X}{\|X\|}\right)$ is a geodesic with respect to $I'_{d\Phi x}(\widehat{G})$, whose parameter is the arc length. The second diagram is obtained from the first one.

Since the exponential mappings $D_N(\pi/2) \to (D_N(\pi/2), I'_{d\Phi\Phi' x}(\widehat{G}))$, $D_N(\pi/2) \to (D_N(\pi/2), \Phi_* I'_{d\Phi' x}\Phi^{-1}I'_{d\Phi x}(\widehat{G}))$ coincide, both metrics are expressed in the same local coordinate as above as follows:

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & * & \\ 0 & & & \end{pmatrix}.$$

Immediately, we see that $*$-parts of both matrices coincide. Therefore, $I'_{d\Phi\Phi' x}(\widehat{G}) = \Phi_* I'_{d\Phi' x}\Phi_*^{-1}I_{d\Phi x}(\widehat{G})$. Since $\hat{I}(G, \Phi) = E_{G_*}I'_{d\Phi}E_{G_*}^{-1}$,

$$\begin{aligned}\hat{I}(G, \Phi\Phi') &= E_{G_*}\Phi_* I'_{d\Phi' x}\Phi_*^{-1}E_{G_*}^{-1}I_{G_*}I'_{d\Phi x}E_{G_*}^{-1}(G) \\ &= E_{\hat{I}(G,\Phi)_*}I'_{d\Phi' x}E_{\hat{I}(G,\Phi)_*}^{-1}E_{G_*}I'_{d\Phi x}E_{G_*}^{-1}(G) \\ &= \hat{I}(\hat{I}(G, \Phi), \Phi') .\end{aligned}$$

2.12. Theorem. *$(\mathscr{G}(G_0), J, \widetilde{\mathrm{Diff}}(S_N(\pi/2)))$ is a fibre bundle where the fibre may be different at each connected component of $\widetilde{\mathrm{Diff}}(S_N(\pi/2))$.*

Proof. We have only to prove the local triviality. Let W be a neighborhood of id. in $\widetilde{\mathrm{Diff}}(S_N(\pi/2))$ such that (i) for every $\varphi \in W$, there exists uniquely a vector field X on $S_N(\pi/2)$ satisfying $\varphi(x) = \mathrm{Exp}\, X(x)$, (ii) $\varphi_t(x) = \mathrm{Exp}\, tX(x)$, $0 \le t \le 1$, is also contained in W, where a C^∞-riemannian metric is fixed on $S_N(\pi/2)$. It suffices to prove that $(J^{-1}(W), J, W)$ is a trivial bundle.

Let $h(t)$ be a C^∞-function such that $h(t) \equiv 0$ for $0 \le t \le 1/3$, $h(t) \equiv 1$ for $2/3 \le t \le 1$, and $h'(t) > 0$ for $1/3 < t < 2/3$. Define a mapping $\eta\colon W \to \mathrm{Diff}_\mathscr{F}(D_N(\pi/2))$ by

$$\eta(\varphi)(x) = \begin{cases} x\,, & x \in D_N(\pi/4)\,, \\ \varphi_{h(t)}(x)\,, & x \in S_N((\pi/4)t + \pi/4)\,. \end{cases}$$

It is clear that η is continuous.

Define $\xi\colon J^{-1}(\mathrm{id.}) \times W \to J^{-1}(W)$ by $\xi(G, \varphi) = \hat{I}(G, \eta(\varphi))$. ξ is a continuous one-to-one mapping. Putting $\zeta(G) = \hat{I}(G, \eta(J(G))^{-1})$, $\zeta\colon G \to \zeta(G)$ is a continuous mapping of $J^{-1}(W)$ onto $J^{-1}(\mathrm{id.})$.

Since $\xi(\zeta(G), J(G)) = \hat{I}(\hat{I}(G, \eta(J(G))^{-1}), \eta(J(G))) = G$, we see that $(J^{-1}(W), J, W)$ is a trivial bundle.

3. Riemannian manifolds having the Property $P(N)$

From Lemmas 1.3-1.5, it is easily seen that *if M is diffeomorphic to $D_N \cup_\varphi D_{N'}$, then there is a riemannian metric having the property $P(N, N')$*, where N, N' are C^∞-manifolds and D_N, $D_{N'}$ are disk bundles over N, N' respectively.

In this section the following theorem will be proved:

3.1. Theorem. *If an analytic riemannian manifold M satisfies the property that there is an analytic submanifold N such that for any geodesics starting orthogonally from N the distance from N to the cut point is the constant π, then the cut locus of N is an analytic submanifold N' and M satisfies $P(N, N')$.*

The assumed property in the above theorem is briefly called $P(N)$.

a) Properties of geodesic spheres

It is well-known that there is a positive number r_0 depending only on M such that if $r < r_0$, then $\mathrm{Exp}_p\colon D_p(r) \to \mathscr{D}_p(r)$ is diffeomorphic for every point of M; this is due to the compactness of M. Fix a real number $r < r_0/2$ and take two points p_1, p_2 such that the distance between p_1 and p_2 is strictly smaller than 2r. Clearly, $\mathscr{D}^i_{p_1}(r) \cap \mathscr{D}^i_{p_2}(r) \neq \phi$. Let $\mathscr{S}_N(\varepsilon)$ denote the normal geodesic sphere bundle of radius ε of the submanifold N. If $N = \{p\}$, then $\mathscr{S}_N(\varepsilon)$ is the geodesic sphere of radius ε. Given $q \in \mathscr{S}_{p_1}(r) \cup \mathscr{S}_{p_2}(r)$ and let $g_i(t)$, $0 \leq t \leq r$, $i = 1, 2$, are geodesic segments such that $g_i(r) = p_i$ and $g_i(0) = q$.

3.2. Lemma. *If a unit vector Y is expressed by $Y = aX_1 + bX_2$, $a \geq 0$, $b \geq 0$, then for sufficiently small $s > 0$, we have*

$$\mathscr{D}_{\mathrm{Exp}_q sY}(s) - \{q\} \subset \mathscr{D}^i_{p_1}(r) \cup \mathscr{D}^i_{p_2}(r) ,$$

where $X_i = \left.\frac{d}{dt}\right|_0 g_i(t)$.

Proof. Since $\mathrm{Exp}_q \colon D_q(2r) \to \mathscr{D}_q(2r)$ is a diffeomorphism, $\mathrm{Exp}_q^{-1} \mathscr{S}_{p_i}(r)$, $i = 1, 2$, are both analytic submanifolds of $T_q(M)$. It follows that there is a small $\acute{r} > 0$ such that

$$\{\acute{Y}; \|\acute{Y} - \acute{r}X_j\| \leq \acute{r}\} - \{0\} \subset \mathrm{Exp}_q^{-1} \mathscr{D}^i_{p_j} , \quad j = 1, 2 .$$

From an elementary calculation, we see that for a small $\acute{s} > 0$

$$\{\acute{Z}; \|\acute{Z} - \acute{s}Y\| \leq \acute{s}\} - \{0\} \subset \bigcup_{j=1,2} \{\acute{Y}; \|\acute{Y} - \acute{r}X_i\| < \acute{r}\} ,$$

that is, the lemma is true for a flat manifold. It follows that

$$\mathrm{Exp}_q \{\acute{Z}; \|\acute{Z} - \acute{s}Y\| \leq \acute{s}\} - \{q\} \subset \mathscr{D}^i_{p_1}(r) \cup \mathscr{D}^i_{p_2}(r) .$$

Since $\mathrm{Exp}_q \{\acute{Z}; \|\acute{Z} - \acute{s}Y\| \leq \acute{s}\}$ is an analytic submanifold of M and Y is a normal vector at q, there exists a small $s > 0$ such that $\mathscr{D}_{\mathrm{Exp}_q sY}(s) \subset \mathrm{Exp}_q \{Z; \|Z - \acute{s}Y\| \leq \acute{s}\}$. This completes the proof.

3.3. Lemma. *Let k be the distance between N and the cut locus of N (denoted by $(C(N))$. For any point $q \in \mathscr{S}_N(k)$ and for any geodesic segment $g(t)$, $0 \leq t \leq k$, such that $g(0) \in N$, $g(k) = q$, we have $\mathscr{D}_{g(s)}(k - s) - \{q\} \subset \mathscr{D}^i_N(k)$ for any $0 < s < k$.*

Conversely, if there is a point $p' \in \mathscr{D}^i_N(k)$ such that $\mathscr{D}_{p'}(k - s) \subset \mathscr{D}_p(k)$, letting s be the distance between N and p', then (i) *$\mathscr{S}_{p'}(k - s) \cap \mathscr{S}_N(k)$ consists of only one point q,* (ii) *there is a geodesic segment of length k joining N and q and through p'.*

Proof. Assume $\mathscr{D}_{g(s)}(k - s) - \{q\} \not\subset \mathscr{D}^i_N(k)$. Then there are a point $\acute{q} \in \mathscr{D}_{g(s)}(k - s) \cap \mathscr{S}_N(k)$ such that $q \neq q'$, and a geodesic segment $g'(t)$, $0 \leq t \leq k - s$, joining $g(s)$ to $\acute{q}$. It follows that

$$\rho(N, \acute{q}) < \rho(N, g(s)) + \rho(g(s), \acute{q}) = k ,$$

because $g(t)$ and $g'(t)$ intersect at $g(s)$ with the angle $<\pi$, where ρ is the distance function defined on M from the riemannian metric. The inequality $\rho(N, \acute{q}) < k$ contradicts the assumption $k = \rho(N, C(N))$, proving the first part of this lemma.

Assume $\mathscr{S}_{p'}(k-s) \cap \mathscr{S}_N(k) = \phi$. Then $\mathscr{D}_{p'}(k-s) \subset \mathscr{D}^i_N(k)$. Taking a geodesic $g(t)$, $0 \leq t < \infty$, such that $g(0) \in N$ and $g(s) = p'$, we see that $g(k) \in \mathscr{D}_{p'}(k-s)$ and then $g(k) \in \mathscr{D}^i_N(k)$, contradicting the assumption.

Let $q, \acute{q} \in \mathscr{S}_N(k) \cap \mathscr{S}_{p'}(k-s)$ and assume $q \neq \acute{q}$. Then we have a contradiction $\rho(N, \acute{q}) < k$ by the same argument as in the proof of the first part of this lemma. Thus, $\mathscr{S}_N(k) \cap \mathscr{S}_{p'}(k-s) = \{q\}$. Moreover, the geodesic $g(t)$, $0 \leq t < \infty$, such that $g(0) \in N$ and $g(s) = p'$ satisfies at the same time that $g(k) = q$, since $g(k) \in \mathscr{S}_N(k)$.

3.4. Proposition. *With the same notation as above, assume that there is $q \in \mathscr{S}_N(k)$ such that there exist at least two geodesic segments $g_1(t), g_2(t)$, $0 \leq t \leq k$, joining N to q. On putting $g_1(k) = g_2(k) = q$ and $X_i = \left.\frac{d}{dt}\right|_k g_i(t)$, we obtain* (i) $X_1 = -X_2$ *or* (ii) $\mathrm{Exp}_q kY \in N$ *for any unit vector Y contained in the 2-plane spanned by X_1, X_2.*

Proof. Clearly $X_1 \neq X_2$. Assume $X_1 \neq -X_2$. Then

$$\mathscr{D}_{\mathrm{Exp}_q sY}(s) - \{q\} \subset \mathscr{D}^i_{g_1(k-r)}(r) \cup \mathscr{D}_{g_2(k-r)}(r),$$

where r, s are small numbers determined in Lemma 3.2, for a unit vector Y satisfying $Y = aX_1 + bX_2$, $a \geq 0$, $b \geq 0$. From Lemma 3.3, we have $\mathscr{D}_{g_1(k-r)}(r) \cup \mathscr{D}_{g_2(k-r)}(r) - \{q\} \subset \mathscr{D}^i_N(k)$. Therefore $\mathscr{D}_{\mathrm{Exp}_q sY}(s) - \{q\} \subset \mathscr{D}^i_N(k)$. From Lemma 3.3 again, there is a geodesic segment g joining N to q and passing through $\mathrm{Exp}_q sY$. It is easy to show that $g = \{\mathrm{Exp}_q tY; 0 \leq t \leq k\}$. Therefore $\mathrm{Exp}_q kY \in N$ for any unit vector Y such that $Y = aX_1 + bX_2$, $a \geq 0$, $b \geq 0$.

Let $L = \{Y; \|Y\| = 1, Y = aX_1 + bX_2, a \geq 0, b \geq 0\}$ and $S = \{Y; \|Y\| = 1, Y = aX_1 + bX_2\}$. S is an analytic submanifold of the tangent space $T_q(M)$ at q and $\mathrm{Exp}_q(L) \subset N$. Since N is analytic, Exp_q is an analytic mapping, L is an open subset of S, and Exp_q maps the set S into N.

3.5. Corollary. *With the same assumptions and notation as above, there is a geodesic starting orthogonally from N and striking N orthogonally when the length becomes $2k$. The subset $\{X \in T_q(M); \|X\| = k, \mathrm{Exp}_q X \in N\}$ is contained in $\lambda+1$-dimensional vector subspace of $T_q(M)$ for some λ, where $T_q(M)$ is the tangent space of M at q.*

b) Manifold having the property $P(N)$

Let $E = \mathrm{Exp} \mid V_N$. Since M is analytic, E is an analytic mapping. Denote by $K(X)$ the kernel of dE at X. It is well-known that if $K(X) \neq \{0\}$, then $E(X)$ is a focal point of N with respect to the geodesic defined by X. Therefore, from the assumed property $P(N)$, $K(X)$ is contained in the tangent space $T_X S_N(\pi)$ of S_N at X for every $X \in S_N(\pi)$. Of course, $K(X)$ may be $\{0\}$ at some point of $S_N(\pi)$.

3.6. Lemma. *Let $C_j = \{X \in S_N(\pi); \dim K(X) = j\}$ and C'_j be the set of interior points of C_j in $S_N(\pi)$. Then, for any open subset U in $S_N(\pi)$, $U \cap C_{j_0+1}$ is an* *open subset of* $U - \cup\{C_j; j = 0, 1, \cdots, j_0\}$. *Therefore,* $\cup\{C'_j; j = 0, 1, \cdots \cdots, n-1\}$ *is an open and dense subset of* $S_N(\pi)$.

Proof. It is easy to see that if a sequence $\{X_n\}$ contained in a single C_j converges to $X_0 \in S_N(\pi)$ then $\dim K(X_0) \geq j$. Since $j_0 + 1$ is the possible minimal dimension of $K(X)$, if X is in $U - \cup\{C_j; j = 0, 1, \cdots, j_0\}$, then $U \cap C_{j_0+1}$ is an open subset in it.

If there were an open subset of $S_N(\pi)$ in $S_N(\pi) - \cup\{C'_j; j = 0, 1, \cdots, n-1\}$, putting this open subset to be U, we see that $U \cap C_k$ is open in $S_N(\pi)$, where k is the minimal integer such that $C_k \cap U \neq \phi$. Therefore, $U \cap C_k$ is contained in $\cup\{C'_j, j = 0, 1, \cdots, n-1\}$. This is a contradiction.

If $C'_j \neq \phi$ for some $j \geq 1$, then $K(X)$ is an involutive distribution on C'_j and is also an analytic distribution since E is analytic. Let I be any integral manifold of K in C. Then we see easily that $E(I)$ is a point in M, that is, there are many geodesics starting from N orthogonally to N and shrinking to a point $E(I)$ when the length becomes π.

3.7. Lemma. *Every geodesic starting orthogonally from N strikes N orthogonally when the length becomes* 2π.

Proof. Let $X \in S_N(\pi)$. Assume at first that $X \in C_0$. Since $S_N(\pi)$ is the cut locus of N, there exists an element $Y \in S_N(\pi)$ different from X such that $E(X) = E(Y)$. Let $g_X(t)$, $g_Y(t)$ be geodesics defined by $E(tX)$, $E(tY)$ respectively. If $-\frac{d}{dt}\Big|_\pi g_X(t) \neq \frac{d}{dt}\Big|_\pi g_Y(t)$, then from Proposition 3.4, we have that $X \in C_j, j \geq 1$. This is a contradiction. Thus, $g_X(2\pi - t) = g_Y(t)$.

Assume $X \in \cup\{C'_j; j \geq 1\}$. Let I_X be the integral manifold through X in some $C'_j, j \geq 1$. Since $E(I_X)$ is a point, there is $Y \in I$ such that $-\frac{d}{dt}\Big|_\pi g_X(t) \neq \frac{d}{dt}\Big|_\pi g_Y(t)$. From Proposition 3.4 again, we see that $g_X(2\pi) \in N$ and $g(t)$ strikes N orthogonally.

Since $\cup\{C'_j; j \geq 0\}$ is dense in $S(N)$ and every geodesic with initial direction contained in $\cup\{C'_j; j \geq 0\}$ strikes N orthogonally when the length becomes 2π, we have this lemma, using a continuity property of geodesics.

3.8. Corollary. *Let* $\mathscr{D}^i_N(\pi)$ *be the normal, open disk bundle of N of radius* π *in M. For any point* $p \in \mathscr{D}^i_N(\pi)$, *every geodesic segment g starting orthogonally from N and ending at p is contained in the geodesic* $g_X(t)$, $-\infty < t < \infty$, *where* $X = E^{-1}(p)$ *contained in* $D^i_N(\pi)$. *Moreover, p is not a focal point of N with respect to g.*

Proof. The first part of this corollary is immediately verified from Lemma 3.7, that is, if not, there exists a geodesic segment $\acute{g}$ of length $<\pi$ and not contained in $g_X(t)$, contradicting the assumed property $P(N)$.

Assume there is a geodesic segment $g(t)$, $u \leq t \leq v$, such that $g(u) \in N$, $g(v) = p$ and p is a focal point of N with respect to this geodesic segment. Let t_0 be the maximum of the numbers $\{t'; g(t') \in N, t' < v\}$. Then, using

Lemma 3.7, we obtain that if $v - t_0 < \pi$ (resp. $v - t_0 > \pi$), p is a focal point of N with respect to the geodesic $g(t)$, $t_0 \leq t \leq v$ (resp. $v \leq t \leq 2\pi + t_0$). That is due to the fact, an immediate consequence of Lemma 3.7, that the Jacobi field $Y(t)$ on $g(t)$, $u \leq t \leq v$, satisfying $Y(v) = 0$, $Y(u) \in T_{g(u)}(N)$ and $\left.\frac{d}{dt}\right|_u Y = S_{g(u)}(Y(u))$ also satisfies $Y(u \pm 2\pi) \in T_{g(u\pm 2\pi)}(N)$ and $\left.\frac{d}{dt}\right|_{u\pm 2\pi} Y = S_{g(u\pm 2\pi)}(Y(u \pm 2\pi))$, where $S_{g(u)}(Y)$ is the second fundamental form of N at $g(u)$ regarded as a linear transformation on $T_{g(u)}(N)$ with respect to a normal vector $\left.\frac{d}{dt}\right|_u g(t)$.

Let $\Omega(N, p)$ be the set of absolutely continuous mappings $\sigma\colon I \to M$ such that σ' is square integrable, $\sigma(0) \in N$, $\sigma(1) = p$. The definitions of the absolute continuity of mappings and the square integrability of σ' have been done in §13 of [10]. By a method similar to Theorem (10) in [10], we see that $\Omega(N, p)$ is a C^∞-Hilbert manifold without boundary. Since σ is absolutely continuous, there is an integrable mapping $\sigma'\colon I \to T_M$ and

$$\sigma(t) = \sigma(0) + \int_0^t \sigma'(t)dt .$$

Let J_p be a function on $\Omega(N, p)$ defined by

$$J_p(\sigma) = \int_0^1 \|\sigma'(t)\|^2 dt .$$

This function J_p is differentiable, and if $p \in \mathscr{D}_N^i(\pi)$, then the critical values of J_p are discrete and critical points on the same critical level are finitely many, since p is not a focal point of any geodesic starting orthogonally from N. The proof of this is done by a parallel argument of the proof of condition (C) in [10].

Since a critical point of J_p is a geodesic segment starting orthogonally from N and ending at p with a parameter proportionate to the arc length, the critical values of J_p are $d^2, (2\pi \pm d)^2, \cdots, (2n\pi \pm d)^2, \cdots$, where $d = \rho(N, p)$.

Let p' be a point in $\mathscr{D}_N^i(\pi)$ such that $d = \rho(N, p')$. Put $\varepsilon = \rho(p, p')$. If ε is sufficiently small there exists uniquely a geodesic segment $g(t)$, $0 \leq t \leq \varepsilon$, joining p and p'.

3.9. Lemma. *There is a natural imbedding ι of $\Omega(N, p)$ into $\Omega(N, p')$. Moreover $\iota(J_p^{-1}(-\infty, l]) \subset J_{p'}^{-1}\left(-\infty, \frac{1}{1-\varepsilon} l + \varepsilon\right]$ for all l, where $0 < \varepsilon < 1$ is assumed.*

Proof. For any $\sigma \in \Omega(N, p)$, we define

$$\iota(\sigma)(t) = \begin{cases} \sigma\left(\dfrac{1}{1-\varepsilon}t\right), & 0 \leq t \leq 1-\varepsilon, \\ g(t+\varepsilon-1), & 1-\varepsilon < t \leq 1. \end{cases}$$

Easily, we see $J_{p'}(\iota(\sigma)) = (1-\varepsilon)J(\sigma) + \varepsilon$.

Similarly, there is a natural imbedding ι' of $\Omega(N, p')$ into $\Omega(N, p)$ such that

$$\iota'\left(J_{p'}^{-1}\left(-\infty, \frac{1}{1-\varepsilon}l + \varepsilon\right]\right) \subset J_p^{-1}\left(-\infty, \frac{1}{(1-\varepsilon)^2}l + \frac{\varepsilon(2-\varepsilon)}{1-\varepsilon}\right].$$

Put $l = (2\pi)^2$ and take ε so that it may satisfy

$$\frac{1}{1-\varepsilon}(2\pi)^2 + \varepsilon < (2\pi + d)^2, \quad \frac{1}{(1-\varepsilon)^2}(2\pi)^2 + \frac{\varepsilon(2-\varepsilon)}{1-\varepsilon} < (2\pi + d)^2.$$

Putting $l' = \dfrac{1}{(1-\varepsilon)^2}(2\pi)^2 + \dfrac{\varepsilon(2-\varepsilon)}{1-\varepsilon}$, we have two mappings id. and $\iota'\iota$ of $J_p^{-1}(-\infty, (2\pi)^2]$ into $J_p^{-1}(-\infty, l']$ and it is easy to show that $\iota'\iota$ is homotopic to id. Therefore

$$\iota_*: H_*(J^{-1}(-\infty, (2\pi)^2]) \to H_*\left(J_{p'}^{-1}\left(-\infty, \frac{1}{1-\varepsilon}(2\pi)^2 + \varepsilon\right]\right)$$

is injective, where H_* implies singular homology group.

3.10. Theorem. *Assume a compact connected analytic riemannian manifold has an analytic, connected and closed submanifold N and satisfies $P(N)$, then $K(X)$ is constant on $S_N(\pi)$.*

Proof. Since $S_N(\pi)$ is connected, it suffices to prove that C_j is open for each j. Let $X \in C_j$, $p = E\left(\dfrac{d}{\pi}X\right)$ and take p' as above. Using the handle body decomposition theorem [10] with respect to the function J_p, we have

$$H_\lambda(J_p^{-1}(-\infty, (2\pi)^2]) = \begin{cases} \mathbf{Z}, & \lambda = 0, \\ \mathbf{Z}, & \lambda = j, \\ 0, & \text{otherwise}, \end{cases}$$

$$H_\lambda\left(J_{p'}^{-1}\left(-\infty, \frac{1}{1-\varepsilon}(2\pi)^2 + \varepsilon\right]\right) = \begin{cases} \mathbf{Z}, & \lambda = 0, \\ \mathbf{Z}, & \lambda = j', \\ 0, & \text{otherwise}, \end{cases}$$

where j' is defined as follows:

Since $p' \in \mathscr{D}_N^1(\pi)$, $\rho(N, p') = d$, there is $X' \in S_N(\pi)$ such that $E\left(\frac{d}{\pi} X'\right) = p$. Therefore there is $C_{j'}$ such that $X' \in C_{j'}$.

On the other hand, ι_* is injective. Thus, we have $j = j'$. Therefore C_j is an open subset of $S_N(\pi)$.

From this theorem, K is an involutive distribution on $S_N(\pi)$. From Corollary 3.5, if $\lambda \geq 1$, then $E^{-1}(E(X))$ is the maximum integral manifold through X of this distribution K. Therefore $E(S_N(\pi))$ is an analytic submanifold of M. Clearly $E(S_N(\pi))$ is the cut locus of N, and putting $N' = E(S_N(\pi))$, $P(N, N')$ is satisfied.

If $\lambda = 0$, then, for any $X \in S_N(\pi)$, there is exactly one element $Y \in S_N(\pi)$ such that $E(X) = E(Y)$ and $\left.\frac{d}{dt}\right|_1 E(tX) = -\left.\frac{d}{dt}\right|_1 E(tY)$. This is an immediate conclusion from Corollary 3.5 and the fact that $E(X)$ is a cut point and not focal point of N. Therefore, there is an involutive diffeomorphism φ on $S_N(\pi)$ defined by $\varphi(X) = Y$. Obviously φ is fixed point free and $S_N(\pi)/\{\text{id.}\ \varphi\}$ is diffeomorphic to $E(S_N(\pi))$. Thus, $E(S_N(\pi))$ is a manifold and putting $N' = E(S_N(\pi))$, $P(N, N')$ is satisfied,

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