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A class of scale mixtures of Gamma(k)-distributions that are generalized gamma convolutions

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Let $k > 0$ be an integer and Y a standard Gamma(k) distributed random variable. Let X be an independent positive random variable with a density that is hyperbolically monotone (HM) of order k. Then $Y \cdot X$ and Y/X both have distributions that are generalized gamma convolutions (GGCs). This result extends a result of Roynette *et al.* from 2009 who treated the case $k = 1$ but without use of the HM-concept. Applications in excursion theory of diffusions and in the theory of exponential functionals of Lévy processes are mentioned.

Keywords: excursion theory; exponential functionals; generalized gamma convolution; hyperbolic monotonicity; Lévy process; products and ratios of independent random variables

1. Introduction

A generalized gamma convolution (GGC) is a limit distribution for sums of independent gamma distributed random variables (r.v.s). The GGCs were introduced by the actuary O. Thorin in 1977 when he tried to prove that the lognormal distribution is infinitely divisible (see [22]). He used a technique that later on led the second author of this paper to introduce in [6] the concept of hyperbolic complete monotonicity (HCM). The simpler concept of hyperbolic monotonicity (HM) was mentioned in [6], pages 101–102, and more carefully studied in [7].

The GGCs have got applications in many different fields including infinite divisibility (e.g., Steutel and van Harn [21]), mathematical analysis (e.g., Schilling *et al.* [20]), stochastic processes (e.g., James *et al.* [13] and Behme *et al.* [3]), and financial mathematics (e.g., Barndorff-Nielsen *et al.* [1]).

In 2009 Roynette *et al.* [17] proved a novel GGC result that has provided stimulus to the present work. In our terminology, they showed that the product of an exponentially distributed r.v. Y and another independent r.v. X has a GGC distribution provided that the density of X is HM. We will give a new and more transparent proof of this result and generalize it considerably to cover gamma distributions.

The paper is organized as follows. In Section 2, the HM, HCM and GGC theory is briefly recalled. In Section 3, the main result that the product of a gamma variable with shape parameter k and an r.v. with HM_k distribution has a GGC distribution is given. This result can be formulated in several alternative ways. It has also an important extension. The proof contains some surprising elements. Applications, analytical as well as stochastic process related ones, are given in Section 4. Finally, in Section 5 some open problems are mentioned.

2. Background

Basic facts on hyperbolic monotonicity (HM) and generalized gamma convolutions (GGCs) are presented here. They are taken from Bondesson [6,7]. Much information about GGCs and hyperbolic complete monotonicity (HCM) can also be found in the book by Steutel and van Harn [21].

2.1. Hyperbolic monotonicity

Let f be a nonnegative function on $(0,\infty)$. Consider, for any fixed $u>0$, the function $f(uv)f(u/v)$, $v>0$. Obviously, it is invariant under the transformation $v\mapsto v^{-1}$. It follows that it is a function $h(w)$ of $w=v+v^{-1}$ since the value of w determines the set $\{v,v^{-1}\}$.

Definition 1. *A nonnegative function f on $(0,\infty)$ is said to be* hyperbolically monotone (HM *or* HM$_1$) *if, for each fixed $u>0$, the function $h(w)=f(uv)f(u/v)$ is non-increasing as a function of $w=v+v^{-1}$. More generally, it is called* hyperbolically monotone of order k (HM$_k$) *if $(-1)^j h^{(j)}(w)\geq 0$, $j=0,1,\ldots,k-1$ and $(-1)^{k-1}h^{(k-1)}(w)$ is non-increasing. If this holds for all $k\geq 1$, f is also called* hyperbolically completely monotone (HCM).

The class of HM$_k$-functions is also denoted HM$_k$. Obviously

$$\mathrm{HCM}=\mathrm{HM}_\infty\subset\cdots\subset\mathrm{HM}_3\subset\mathrm{HM}_2\subset\mathrm{HM}_1=\mathrm{HM}.$$

Simple examples of HCM-functions are provided by (with $\gamma\in\mathbb{R}$, $c\geq 0$): x^γ, e^{-cx}, and $e^{-c/x}$. It is apparent that the HM$_k$-class is closed with respect to multiplication of functions. For $f\in\mathrm{HM}_k$, obviously $f(uv)f(u/v)\leq (f(u))^2$. It easily follows that $\log f(e^x)$ is concave and hence that $f(x)\leq Cx^\gamma$ for some constants $C\geq 0$ and $\gamma\in\mathbb{R}$ (depending on f). Every HM$_k$-function f can therefore be modified to an HM$_k$ probability density function (p.d.f.) by multiplication by a factor $\exp(-\delta_1 x-\delta_2 x^{-1})$ (with $\delta_1>0$ and $\delta_2>0$ arbitrarily small) and a normalizing constant. In this paper, we are mainly concerned with p.d.f.s.

Example 1. Let f be a p.d.f. on $(a,b)\subset(0,\infty)$ of the form $f(x)=C(x-a)^{\alpha-1}(b-x)^{\beta-1}$, where C is a constant. It can be shown that for $\alpha\geq 1$ and $\beta\geq 1$, f is HM$_k$ for $k=\min([\alpha],[\beta])$, where $[\cdot]$ denotes integer part. However, if $a=0$, then for any value of α, f is HM$_k$ for $k=[\beta]$. In particular, the $U(a,b)$ density is HM$_1$. In fact, in this case it is easy to see that $h(w)=f(uv)f(u/v)$ is 0 for all u sufficiently large or small and that for the other values of u, $h(w)$ equals 1 if w is below some bound and otherwise 0.

Example 2. Let $X=U_1U_2\cdots U_k$, where the random variables (r.v.s) U_i are independent and uniformly distributed on $(0,1)$. Since $-\log X$ has a Gamma$(k,1)$-distribution, $f_X(x)=\frac{1}{(k-1)!}(-\log x)^{k-1}, 0<x<1$. This p.d.f. is HM$_k$. In fact, $h(w)=f(uv)f(u/v)=0$ for $u\geq 1$, whereas, for $u < 1$, $h(w) \propto ((\log u)^2 - (\log v)^2))^{k-1}$ if $u < v < u^{-1}$ (i.e. if $2 \le w < u + u^{-1}$) and otherwise vanishes. The HM_k result then follows from the fact that $d(\log v)^2/dw$ is completely monotone (CM). In fact, this derivative can be shown to be equal to $\int_0^\infty (1 + t^2 + tw)^{-1}\, dt$.

The following result, which concerns powers, products and ratios of r.v.s, is important. Its proof (in [7]) is far from trivial. A main idea in the proof is to use hyperbolic substitutions of the form $x = uv$, $y = u/v$ in certain double integrals.

Proposition 1. *Let X and Y be independent r.v.s with HM_k-densities ($X \sim \mathrm{HM}_k$, $Y \sim \mathrm{HM}_k$). Then, for any $q \in \mathbb{R}$ with $|q| \ge 1$, we have $X^q \sim \mathrm{HM}_k$. Moreover, $X \cdot Y \sim \mathrm{HM}_k$ and $X/Y \sim \mathrm{HM}_k$.*

A simple consequence of Proposition 1 (with one of the r.v.s exponentially distributed) is that the Laplace transform of an HM_k function is HM_k. Let X have the HM_2-density $f(x) = 2\max(0, 1 - x)$ and let $Y \sim U(0, 1)$ (with an HM_1-density). Then it can be shown that $X/Y \not\sim \mathrm{HM}_2$. Thus, there is no trivial extension of Proposition 1.

The HM_1-densities can be identified as follows (see [7]).

Proposition 2. *We have $X \sim \mathrm{HM}_1$ if and only if $Y = \log X$ has a p.d.f. that is logconcave, i.e. $\log f_Y(y)$ is concave. Equivalently, $X \sim \mathrm{HM}_1$ if and only if $f_X(x) = C \exp(-\int_{x_0}^x \frac{\psi(y)}{y}\, dy)$, where ψ is non-decreasing, C a constant, and x_0 is suitably chosen.*

With this, the well-known fact that logconcavity is preserved under convolution (see, e.g., [12], pages 17–23) becomes a simple consequence of Proposition 1 for $k = 1$.

Typical HCM ($= \mathrm{HM}_\infty$) p.d.f.s have the form $f(x) = Cx^{\beta-1} \prod_{i=1}^n (1 + c_i x)^{-\gamma_i}$, where the parameters are positive, or are limits of such densities. In fact, all HCM-densities (and functions) are such limits. An open problem is to find canonical representations for HM_k-densities for $1 < k < \infty$.

The HM_k-class of densities (functions) can alternatively be described by the condition that

$$h(w) = f(uv)f(u/v) = c_u + \int_{(w,\infty)} (\lambda - w)^{k-1} H_u(d\lambda), \tag{1}$$

where $c_u \ge 0$ and $H_u(d\lambda)$ is a nonnegative measure. The simple example $f(x) = x^\gamma$ gives $c_u = u^{2\gamma}$ and $H_u(d\lambda) \equiv 0$. However, for a p.d.f. we must have $c_u = 0$. The representation (1) follows from a representation of the non-increasing function $(-1)^{k-1} h^{(k-1)}(w)$ as an integral over (w, ∞) (or possibly $[w, \infty)$) of a nonnegative measure. For instance, for $k = 2$ we put $-h'(w) = \int \mathbf{1}(w < \lambda) H_u(d\lambda)$. We then get, by a change of the order of integration,

$$h(w) - h(\infty) = -\int_w^\infty h'(\tilde{w})\, d\tilde{w} = \iint \mathbf{1}(w < \tilde{w} < \lambda)\, d\tilde{w}\, H_u(d\lambda) = \int_{(w,\infty)} (\lambda - w) H_u(d\lambda).$$

The representation (1) was derived and used in [7]. For functions with monotone derivatives up to some order it seems to have been first used by Williamson [23].

2.2. Generalized gamma convolutions

Convolving different gamma distributions, Gamma(u,t), with p.d.f.s and Laplace transforms (LTs) of the forms $f(x) = (\Gamma(u))^{-1}x^{u-1}t^u \exp(-xt)$ and $\phi(s) = (\frac{t}{t+s})^u$, respectively, and then taking weak limits, Thorin [22] was led to the following definition.

Definition 2. *A* generalized gamma convolution (GGC) *is a probability distribution on* $[0,\infty)$ *with LT of the form*

$$\phi(s) = \exp\left(-as + \int_{(0,\infty)} \log\left(\frac{t}{t+s}\right)U(dt)\right),$$

where (*the left-extremity*) $a \geq 0$ *and* $U(dt)$ *is a nonnegative measure on* $(0,\infty)$ (*with finite mass for any compact subset of* $(0,\infty)$) *such that* $\int_{(0,1)} |\log t| U(dt) < \infty$ *and* $\int_{(1,\infty)} t^{-1}U(dt) < \infty$.

The GGC-class of distributions is closed with respect to (w.r.t.) addition of independent random variables and w.r.t. weak limits. Each GGC is infinitely divisible and each convolution root of a GGC is a GGC as well. The p.d.f. $f(x)$ of a GGC is strictly positive on (a,∞) and, if $a = 0$ and $\beta = \int_{(0,\infty)} U(dt)$ is finite, then $f(x) = x^{\beta-1}h(x)$, where $h(x)$ is completely monotone (see [6], page 49).

The p.d.f. of a GGC need not be HM_1. For instance, for a gamma distribution with shape parameter less than 1 and shifted to have left-extremity $a > 0$ the p.d.f. is not HM_1. An HM_k-density, which may have compact support, is in general not a GGC. However, see [6], Theorem 5.1.2:

Proposition 3. *If the p.d.f. f on* $(0,\infty)$ *is* HCM, *then it is a* GGC. *Thus* HCM $\subset$ GGC.

Many well-known p.d.f.s are HCM and therefore also GGCs and hence infinitely divisible. For instance, gamma densities are HCM. Then it follows from Proposition 1 (for $k = \infty$) that also the power q, $q \geq 1$, of the ratio of two independent gamma variables has a density that is HCM. This density is of the form $f(x) = Cx^{\beta-1}(1 + cx^{\alpha})^{-\gamma}$, $x > 0$, with $\alpha = q^{-1}$. Every lognormal density is also HCM.

The next proposition gives a characterization of the LT of a GGC ([6], Theorem 6.1.1).

Proposition 4. *A function $\phi(s)$ on* $(0,\infty)$ *is the LT of a* GGC *if and only if* $\phi(0+) = 1$ *and ϕ is* HCM.

This result will be our basic tool in Section 4. Since the LT of an HM_k function is HM_k, and this also holds for $k = \infty$, Proposition 3 can be seen as a consequence of Proposition 4. Using another complex characterization of the LT of a GGC, we can get the following result ([6], Theorem 4.2.1).

Proposition 5. *Let* $Y \sim$ Gamma$(1,1)$ *and let* $X > 0$ *be an independent r.v. with a density* $f(x)$ *that is logconcave* (*or only such that* $xf(x)$ *is logconcave*). *Then* $Y/X \sim$ GGC.

One should notice that in Proposition 5 the r.v. X is not assumed to have an HM_1-density.

Proposition 6. *If $f(x)$ is the density of a* GGC *and $x^{-\alpha}f(x)$, where $\alpha \geq 0$, can be normalized to be the p.d.f. $g(x)$ of a probability distribution, then $g(x)$ is also the p.d.f. of a* GGC.

This result is only a limit case of [6], Theorem 6.2.4. The following recent result from [8] needs to be mentioned. It can be proved by the help of Proposition 4.

Proposition 7. *Let $X \sim \mathrm{GGC}$ and $Y \sim \mathrm{GGC}$ be independent r.v.s. Then $X \cdot Y \sim \mathrm{GGC}$.*

Well-known examples of GGC distributions include the log-normal distribution and positive strictly α-stable distributions. Also, each *negative* power of a gamma variable is shown to have a GGC-distribution in [10]. Bosch and Simon [9] and Jedidi and Simon [14] give other novel results on HM, HCM, and GGC distributions.

3. Main result

Here the main result is presented as a theorem in Section 3.1. Moreover, comments are given. The proof is presented in Section 3.2.

3.1. Formulation of the main result and comments

Theorem 1. *Let $k \geq 1$ be an integer. Let $Y \sim \mathrm{Gamma}(k, 1)$ and $X \sim \mathrm{HM}_k$ be independent r.v.s. Then $Y \cdot X \sim \mathrm{GGC}$ and $Y/X \sim \mathrm{GGC}$.*

We give some comments on the above theorem.

Remarks 1. (i) For $k = 1$ Theorem 1 differs from Proposition 5. One should notice that $X \sim \mathrm{HM}_1 \Leftrightarrow X^{-1} \sim \mathrm{HM}_1$ but logconcavity of f_X is not equivalent to logconcavity of $f_{X^{-1}}$. One can also notice that every gamma density is HCM (and thus HM_1) but only logconcave when the shape parameter is ≥ 1.

(ii) In the case $k = 1$ the LT $\phi_1(s) = \int_0^\infty (x+s)^{-1} x f_X(x)\,dx$ of Y/X for independent $Y \sim \mathrm{Gamma}(1, 1)$ and $X \sim \mathrm{HM}_1$ is the Stieltjes transform (or double Laplace transform) of the measure $x f_X(x)\,dx$. For $k > 1$, the LT $\phi_k(s) = \int_0^\infty (x+s)^{-k} x^k f_X(x)\,dx$ coincides with the so-called generalized Stieltjes transform (of order k) of the measure $x^k f_X(x)\,dx$. In that sense the above theorem can be restated as follows: Assume $f_X(x)$ is an HM_k function. Then the kth order generalized Stieltjes transform of $x^k f_X(x)\,dx$ is HCM, that is, it is the LT of a GGC.

(iii) Clearly Theorem 1 remains true if $Y \sim \mathrm{Gamma}(k, \theta)$ for any $\theta > 0$, since in this case $\theta Y \sim \mathrm{Gamma}(k, 1)$. Considering $Y \sim \mathrm{Gamma}(k, k)$ and letting $k \to \infty$ we get that $Y \to 1$ in probability. Hence, for $X \sim \mathrm{HM}_k$ with k fixed it is necessary in the theorem to have a restriction upwards on the shape parameter of Y since otherwise it would incorrectly follow that $\mathrm{HM}_k \subset \mathrm{GGC}$. For instance, if $Y \sim \mathrm{Gamma}(2, 1)$ and $X \sim U(1, 2)$, then f_X is HM_1 but $Y/X \nsim \mathrm{GGC}$.

(iv) Letting again $k \to \infty$ and so that $Y \to 1$ in probability, we get back Proposition 3 as a limit case of Theorem 1. Since a Gamma$(k, 1)$ density is HCM, it also follows that the class of GGCs provided by Theorem 1 is closed w.r.t. multiplication and division of independent r.v.s. However, if $Z = Y \cdot X$ with $Y \sim \text{Gamma}(k, 1)$ and $X \sim \text{HM}_k$, it is not true that Z^{-1} always has the same representation.

(v) Theorem 1 can also be expressed in the following way. Any scale mixture of Gamma(k) distributions with a scale mixing HM$_k$-density is a GGC. It is well known ([21], Theorem 3.3, page 334) that any scale mixture of Gamma(1) distributions is infinitely divisible (ID). More generally, any scale mixture of Gamma(2) distributions is ID ([15]). However, for $k > 2$ ID fails to hold in general for such mixtures.

There is a nice extension of Theorem 1 which we see as a corollary of it.

Corollary 1. *Let $Y \sim \text{Gamma}(r, 1)$ be independent of $X \sim \text{HM}_k$ where $r > 0$ and k is an integer such that $k \geq r$. Then $Y \cdot X \sim \text{GGC}$ and $Y/X \sim \text{GGC}$.*

Proof. Since $X \sim \text{HM}_k$ if and only if $1/X \sim \text{HM}_k$, it suffices to consider the ratio $Z = Y/X$. Let $\alpha = k - r$ and let $Y' \sim \text{Gamma}(k, 1)$. Then

$$\begin{aligned} f_Z(z) &= \int_0^\infty x f_Y(zx) f_X(x)\, dx = \frac{1}{\Gamma(r)} \int_0^\infty x (zx)^{r-1} e^{-zx} f_X(x)\, dx \\ &= \frac{\Gamma(k)}{\Gamma(r)} z^{-\alpha} \int_0^\infty x f_{Y'}(zx) x^{-\alpha} f_X(x)\, dx. \end{aligned}$$

Here $x^{-\alpha} f_X(x)$ is HM$_k$ and so is, for any $\delta \geq 0$, $x^{-\alpha} e^{-\delta/x} f_X(x)$. Letting if necessary $\delta > 0$ and normalizing this latter function to become the p.d.f. of an r.v. X', we get from Theorem 1 that $Y'/X' \sim \text{GGC}$. Using then Proposition 6 and letting $\delta \to 0$, we conclude that $Y/X \sim \text{GGC}$. $\square$

3.2. Proof of the main result

The proof of Theorem 1 is given in two parts. First, the case $k = 1$ is treated. This proof is short but contains the essential ideas. The proof in the general case becomes more technical. Of course, we use the HCM-characterization of the LT of a GGC and hyperbolic substitutions in the proofs. For the transformation $T = t + t^{-1}$, we avoid to use the inverse transformation $t = T/2 \pm \sqrt{T^2/4 - 1}$. In fact, the HCM-concept was introduced in the early 1990s in order to avoid, at least in presentations, such inverse transformations.

Proof of Theorem 1, k = 1. It suffices to consider the ratio Y/X, where $Y \sim \text{Gamma}(1, 1)$. The LT $\phi(s)$ of the distribution of the ratio is given by, with $f = f_X$,

$$\phi(s) = E\bigl(\exp(-sY/X)\bigr) = \int_0^\infty E\bigl(\exp(-sY/x)\bigr) f(x)\, dx = \int_0^\infty \frac{x}{x+s} f(x)\, dx.$$

For fixed $s > 0$, consider

$$J = \phi(st)\phi(s/t) = \int_0^\infty \int_0^\infty \frac{xy}{(x+st)(y+s/t)} f(x) f(y)\, dx\, dy.$$

In view of Proposition 4, we only have to show that J is completely monotone (CM) w.r.t. $T = t + t^{-1}$. We make the hyperbolic substitution $x = uv$, $y = u/v$ with Jacobian with modulus $2u/v$. Hence,

$$J = \int_0^\infty \int_0^\infty \frac{2u}{v} \frac{u^2}{(uv+st)(u/v+s/t)} f(uv) f(u/v)\, du\, dv.$$

Using the representation $f(uv)f(u/v) = \int_{[w,\infty)} H_u(d\lambda)$, where $H_u(d\lambda)$ is a nonnegative measure and $w = v + v^{-1}$, letting $b = b(\lambda) \geq 1$ be such that $b + b^{-1} = \lambda$, letting $a = u/s$, and changing the order of integration, we get by some simple algebra that

$$J = \int_0^\infty \frac{2u^2}{s} \int_2^\infty \underbrace{\left(\int_{1/b}^b \frac{t}{(v+t/a)(v+at)}\, dv \right)}_{=:J_1} H_u(d\lambda)\, du.$$

It is now evident that it suffices to show that for each $b \geq 1$ and each $a > 0$ the interior v-integral J_1 is CM w.r.t. to $T = t + t^{-1}$. For $b = 1$, $J_1 = 0$, so it suffices to consider the case $b > 1$. The integral J_1 is a function of T since the change $t \mapsto t^{-1}$ leaves J_1 invariant which is shown by the substitution $v = 1/v'$. Now J_1 can be calculated explicitly. In fact, by a partial fraction expansion we have for $a \neq 1$,

$$\frac{t}{(v+t/a)(v+at)} = \frac{1}{a-a^{-1}} \left(\frac{1}{v+t/a} - \frac{1}{v+at} \right)$$

and hence, for $a \neq 1$, by an integration and some simplification,

$$J_1 = \frac{1}{a-a^{-1}} \log\left(\frac{(t+ab)(t+(ab)^{-1})}{(t+a/b)(t+b/a)} \right) = \frac{1}{a-a^{-1}} \log \frac{T+A}{T+B},$$

where $A = ab + (ab)^{-1}$, $B = a/b + b/a$. For $a = 1$, $J_1 = (b - b^{-1})/(T + b + b^{-1})$. Since $a \mapsto a^{-1}$ leaves J_1 unchanged, we may without restriction assume that $a > 1$ (and as earlier $b > 1$), and then $A > B$ and $J_1 > 0$. Moreover, we get that the kth derivative of J_1, that is, here its first derivative, has the form

$$\frac{dJ_1}{dT} = \frac{1}{a-a^{-1}} \left(\frac{1}{T+A} - \frac{1}{T+B} \right) = \frac{1}{a-a^{-1}} \frac{B-A}{(T+A)(T+B)}$$

and this derivative is negative. Since $(T + A)^{-1}(T + B)^{-1}$ is CM, we get as desired that $(-1)^j J_1^{(j)}(T) \geq 0$, $j = 0, 1, 2, \ldots$, and the proof is complete. $\square$

We now proceed with the general proof of Theorem 1 for any integer $k \geq 1$. We shall see that the above proof needs some complementary arguments.

Proof of Theorem 1, general k. Let $Y \sim \text{Gamma}(k, 1)$ and $X \sim \text{HM}_k$ be independent. Then the LT $\phi(s)$ of the distribution of Y/X is given by

$$\phi(s) = \int_0^\infty \left(\frac{x}{x+s}\right)^k f(x)\,dx.$$

Hence, using (1)

$$\begin{aligned} J = \phi(st)\phi(s/t) &= \int_0^\infty \int_0^\infty \frac{x^k y^k}{(x+st)^k (y+s/t)^k} f(x) f(y)\,dx\,dy \\ &= \int_0^\infty \int_0^\infty \frac{2u}{v} \frac{u^{2k}}{(uv+st)^k (u/v+s/t)^k} f(uv) f(u/v)\,du\,dv \\ &= \int_0^\infty \int_0^\infty \frac{2u}{v} \frac{u^{2k}}{(uv+st)^k (u/v+s/t)^k} \int_w^\infty (\lambda - w)^{k-1} H_u(d\lambda)\,du\,dv, \end{aligned}$$

where $w = v + v^{-1}$ and $H_u(d\lambda)$ is a nonnegative measure. Again we let $b = b(\lambda) \geq 1$ be such that $b + b^{-1} = \lambda$ and put $a = u/s$. After a change of the order of integration with the v-integral as the inner integral and noticing that $b + b^{-1} - v - v^{-1} = (b - v)(v - b^{-1})/v$, we see by some algebraic manipulations that it suffices to show that the integral

$$J_k = \int_{1/b}^b I_k\,dv, \qquad \text{where} \qquad I_k = \frac{t^k((b-v)(v-b^{-1}))^{k-1}}{(v+t/a)^k (v+at)^k}, \tag{2}$$

is CM w.r.t. $T = t + t^{-1}$. The same argumentation as in the case $k = 1$ shows that J_k is a function of T. An important fact is that J_k can be calculated explicitly for all integers k although J_k becomes complicated for k large. Since I_k is a rational function of v, we can get an expression for J_k by using first a partial fraction expansion of I_k w.r.t. v. However, it is more efficient to use an alternating generating function:

$$\begin{aligned} GF(z) = \sum_{k=1}^\infty (-z)^{k-1} J_k &= \int_{1/b}^b \sum_{k=1}^\infty (-z)^{k-1} I_k\,dv \\ &= \int_{1/b}^b \frac{1}{(v+t/a)(v+at)/t + z(b-v)(v-b^{-1})}\,dv. \end{aligned}$$

Minimizing over v and t, we see that the series is absolutely convergent at least if $|z| \leq \frac{a}{(1+a)^2}\frac{(b-1)^2}{b}$. Since the denominator in the integrand is a quadratic function of v and as such a function can be factorized into two real linear factors for $z \geq 0$, we get by integration and considerable simplification with the notation $\alpha = a + a^{-1}$, $\beta = b + b^{-1}$ that

$$GF(z) = \frac{1}{\sqrt{\Delta}} \log R,$$

where

$$\Delta = (\alpha + \beta z)^2 - 4 - 4z^2 + 4zT \quad \text{and} \qquad R = \frac{T - 2z + \frac{1}{2}\beta(\alpha + \beta z) + \frac{1}{2}(b - b^{-1})\sqrt{\Delta}}{T - 2z + \frac{1}{2}\beta(\alpha + \beta z) - \frac{1}{2}(b - b^{-1})\sqrt{\Delta}}.$$

It is far from obvious but some calculation shows that

$$\frac{d}{dz}\log R = \frac{2(b - b^{-1})}{\sqrt{\Delta}}.$$

In fact, the product of the numerator and the denominator in R does not depend on z so the derivative above is just twice the derivative of the logarithm of the numerator in R. Of course, $J_k = J_k(T) = \frac{(-1)^{k-1}}{(k-1)!}GF^{(k-1)}(0)$. Now it is not difficult to see that with, as earlier, $A = ab + (ab)^{-1}$ and $B = a/b + b/a$, we get

$$J_k(T) = P_k(T) + Q_k(T)\log\left(\frac{T + A}{T + B}\right), \tag{3}$$

where $P_k(T)$ and $Q_k(T)$ are polynomials in T of degrees $k - 2$ and $k - 1$, respectively. For $k = 1$, $P_k(T)$ vanishes. For $k = 1, 2$, and 3, we have

$$P_1(T) = 0, \qquad P_2(T) = -2\frac{b - b^{-1}}{(a - a^{-1})^2}, \qquad P_3(T) = -3\frac{b - b^{-1}}{(a - a^{-1})^4}(2T + A + B),$$

$$Q_1(T) = \frac{1}{a - a^{-1}}, \qquad Q_2(T) = \frac{2T + A + B}{(a - a^{-1})^3},$$

$$Q_3(T) = \frac{6T^2 + 6(A + B)T + (A + B)^2 + 2AB}{(a - a^{-1})^5}.$$

By using the above expressions for $P_k(T)$ and $Q_k(T)$ one can easily verify that at least for $k = 1, 2, 3$ we have somewhat surprisingly

$$\frac{d^k J_k}{dT^k} = (-1)^k(k - 1)!\frac{(b - b^{-1})^{2k-1}}{(T + A)^k(T + B)^k}. \tag{4}$$

To see that (4) is completely general, some additional argumentation is needed. Since $P_k(T)$ has degree $k - 2$, it has no influence at all on the kth derivative of J_k. Since $Q_k(T)$ has degree $k - 1$ and hence $Q_k^{(k)}(T) \equiv 0$, it also follows from (3) after some reflection that

$$\frac{d^k J_k}{dT^k} = \frac{R_k(T)}{(T + A)^k(T + B)^k}, \tag{5}$$

where $R_k(T)$ is a polynomial of degree at most $2k - 1$. To see that really $R_k(T)$ is a constant, $(-1)^k(k - 1)!(b - b^{-1})^{2k-1}$, we look at the case when $t \to \infty$. Then $T = t + t^{-1}$ is very close to t. From (2), we get that

$$J_k \sim \frac{1}{t^k}\int_{1/b}^{b}\big((b-v)(v-b^{-1})\big)^{k-1}\,dv \sim B(k,k)\frac{(b-b^{-1})^{2k-1}}{T^k} \qquad \text{as } t\to\infty,$$

where $B(k,k)=(k-1)!(k-1)!/(2k-1)!$. Since $k(k+1)\cdots(2k-1)B(k,k)=(k-1)!$, we also get that

$$\frac{d^kJ_k}{dT^k}\sim(-1)^k(k-1)!\frac{(b-b^{-1})^{2k-1}}{T^{2k}} \qquad \text{as } t\to\infty.$$

Since $R_k(T)$ in (5) is a polynomial, this asymptotic relation can only hold when the polynomial is a constant and hence (4) holds for all k. To complete the proof, we use that $(T+A)^{-k}(T+B)^{-k}$ is CM and hence $(-1)^jJ_k^{(j)}(T)\geq 0$ for $j=k,k+1,\ldots$. Then it only remains to verify that these inequalities also hold for $j=0,1,\ldots,k-1$. Using the same argumentation as above we have that, for each $j\geq 0$, $J_k^{(j)}(T)=O(T^{-k-j})$ as $T\to\infty$. In particular $J_k^{(j)}(T)$ vanishes at $T=\infty$. It follows that

$$J_k^{(j)}(T)=-\int_T^\infty J_k^{(j+1)}(\tilde{T})\,d\tilde{T}, \qquad j=0,1,2,\ldots.$$

We see that the sign of $J_k^{(k-1)}(T)$ is opposite to that of $J_k^{(k)}(T)$. The same then holds for the sign of $J_k^{(k-2)}(T)$ compared with that of $J_k^{(k-1)}(T)$, etc. This shows that $J_k(T)$ is CM w.r.t. T as desired. □

Remark 2. In the general case, some technical details have been omitted in the proof. However, it is easy to check all statements by using a program for symbolic algebra. In fact, the simple form in (4) for the derivative $J_k^{(k)}(T)$ was discovered in that way.

4. Applications

Theorem 1 and Corollary 1 have a wide range of possible applications. We will discuss a few in this section.

4.1. Excursion theory

The random process foundations for the research carried out in this article have been laid by Roynette *et al.* [17] and Salminen *et al.* [18]. In these articles, the authors study excursion times of recurrent linear diffusions on $\mathbb{R}_+$. More precisely, given an $\mathbb{R}_+$-valued recurrent diffusion $(X_t)_{t\geq 0}$ and defining the last and the next visit in 0 via

$$g_t := \sup\{s\leq t, X_s=0\}, \qquad d_t := \inf\{s\geq t, X_s=0\}$$

they are interested in the r.v.s

$$Y_p^{(1)} = Z_p - g_{Z_p}, \qquad Y_p^{(2)} = d_{Z_p} - Z_p, \qquad Y_p^{(3)} = d_{Z_p} - g_{Z_p}, \tag{6}$$

where Z_p denotes an exponential r.v. with density pe^{-pz}, $z > 0$, independent of $(X_t)_{t\geq 0}$. In [18] it is shown, that all $Y^{(i)}$ are infinitely divisible, while in [17] the authors give conditions for $Y^{(i)}$ to have GGC distributions. These conditions are stated in terms of the Krein measure of the Lévy measure of the inverse local time at 0 of $(X_t)_{t\geq 0}$.

For their proof of the GGC property, Roynette *et al.* first show, for $k = 1$, a reformulation of Theorem 1 ([17], Theorem 2). They do not use the HM-concept but define a class $\mathcal{C}$ of functions which essentially coincides with the class HM_1. The proof of [17], Theorem 2, then relies on the HCM-characterization of the LT of a GGC (Proposition 4). Although also our proof for $k = 1$ uses Proposition 4 it is shorter than theirs because of our use of a suitable hyperbolic substitution in a double integral and the avoidance of certain inverse transformations.

Further in [17] the LTs of the $Y^{(i)}$s are shown to be Stieltjes transforms of measures whose densities are HM_1 (compare with Remark 1(ii)).

In the following, we will indicate via an example how one can also use our main theorem in the case $k = 2$ to prove the GGC property of $Y_p^{(3)}$ as defined in (6). Therefore, we briefly recall some notation from [18] and [17].

Let $(L_t)_{t\geq 0}$ be the continuous local-time of $(X_t)_{t\geq 0}$ at 0 and $(\tau_u)_{u\geq 0}$ its right-continuous inverse. Then $(\tau_u)_{u\geq 0}$ is a subordinator and as such has a Lévy exponent ψ and a Lévy density ν, i.e.

$$E\left[e^{-\lambda\tau_u}\right] = e^{-u\psi(\lambda)} = \exp\left(-u\int_0^\infty \left(1 - e^{-\lambda x}\right)\nu(x)\,dx\right),$$

where further ν has the Krein representation

$$\nu(x) = \int_0^\infty e^{-xz} K(dz)$$

with Krein measure K of ν.

Proposition 8. *Assume that the Krein measure K is such that the function $f(u)$ defined via*

$$f\left(u^{-1}\right) = \int_{(u-p)\vee 0}^{u} K(dz)$$

is an HM_2 *function. Then* $Y_p^{(3)} \sim \mathrm{GGC}$.

Proof. It was shown in [18], that the distribution of $Y_p^{(3)}$ is a Gamma(2)-mixture. In particular, it can be deduced from [18], equations (46), (49) and (50), that the density of $Y_p^{(3)}$ is given by

$$f_{Y_p^{(3)}}(u) = \frac{1}{\psi(p)}\int_0^\infty ue^{-ux}\int_{x-p}^{x} K(dz)\,dx$$

which shows that $Y_p^{(3)} = Y \cdot X$ where $Y \sim \text{Gamma}(2, 1)$ and X is independent of Y with density f_X defined via $f_X(u^{-1}) = \frac{1}{\psi(p)} \int_{u-p}^{u} K(dz)$. Thus, the claim follows from Theorem 1 in the case $k = 2$. □

4.2. Exponential functionals of Lévy processes

Let $\xi = (\xi_t)_{t\geq 0}$ be a Lévy process such that $\xi_t \to -\infty$ as $t \to \infty$. Then the exponential functional of ξ is defined as

$$I_\xi := \int_{(0,\infty)} e^{\xi_t}\, dt.$$

Such exponential functionals appear as stationary distributions of generalized Ornstein–Uhlenbeck processes and they have attracted a lot of interest throughout the last years (see, e.g., [11], the survey paper [5] or the more recent contributions [2,3,16] to name just very few references).

It is known, that $I_\xi \sim \text{GGC}$ in several cases. For example, Dufresne (e.g., [5], equation (16)) showed that $I_\xi \overset{d}{=} \frac{2}{\sigma^2} G^{-1}_{2a/\sigma^2}$ where $G_\gamma \sim \text{Gamma}(\gamma, 1)$, whenever ξ is a Brownian motion with variance σ^2 and drift $a < 0$. Concerning processes ξ with jumps, one has for example the following proposition.

Proposition 9. *Suppose that ξ is a compound Poisson process, that is, $\xi_t = \sum_{i=1}^{N_t} X_i$ with i.i.d. jump heights $X_i, i = 1, 2, \dots,$ such that $-\infty < E[X_1] < 0$ and $e^{X_1} \sim \text{GGC}$. Then $I_\xi \sim \text{GGC}$.*

Proof. The proof can be carried out along the lines of the proof of [3], Proposition 3.2, using the more recent Proposition 7. □

Still, assuming that $\xi_t = at - N_t, t \geq 0$, for $a < 0$ and a subordinator (i.e., a nondecreasing Lévy process) $(N_t)_{t\geq 0}$, one easily observes that I_ξ has bounded support and therefore cannot be infinitely divisible so that in particular $I_\xi \nsim \text{GGC}$.

In [16], based on the Wiener–Hopf factorization of Lévy processes, the authors obtain factorizations of exponential functionals. In particular, in case of a spectrally negative process ξ with $\xi_t \to -\infty$, they prove that

$$I_\xi \overset{d}{=} \frac{I_H}{G_\gamma},$$

where $H = (H_t)_{t\geq 0}$ is the descending ladder height process of ξ and $G_\gamma \sim \text{Gamma}(\gamma, 1)$, with γ depending on the characteristics of ξ, is independent of H. We refer to [19] or [4] for any further information on Lévy processes, their characteristics and their Wiener–Hopf factorizations.

Since $-H = (-H_t)_{t\geq 0}$ is a subordinator with drift a_H and Lévy jump measure ν_H, say, it follows from [11], Example B, that if H is non-trivial, then I_H admits a density $f(s)$ which fulfills the integro-differential equation

$$(1 - a_H s) f(s) = \int_s^\infty \bar{\nu}_H\big(\log(t/s)\big) f(t)\, dt,$$

where $\bar{\nu}_H(x) = \nu_H((x, \infty))$. In particular, if $\bar{\nu}_H(s) = ce^{-bs}$, $b, c > 0$, and $a_H > 0$, the authors prove that

$$I_H \overset{d}{=} \frac{1}{a_H} Z_{b+1, c/a_H},$$

where $Z_{\alpha,\beta}$ is a Beta random variable on (0, 1) with parameters $\alpha, \beta > 0$. Hence, $I_H \sim \mathrm{HM}_k$ for $k \leq \min([b+1], [c/a_H])$.

Now by Corollary 1, I_ξ is the reciprocal of a GGC if $k \geq \gamma$. Notice that in general such inverses of GGCs are not GGCs themselves. However, in this case $I_\xi \sim \mathrm{HM}_k$, since $G_\gamma \sim \mathrm{HCM}$.

Conversely, if again $\bar{\nu}_H(s) = ce^{-bs}$, $b, c > 0$, but $a_H = 0$, then I_H itself is Gamma distributed and so $I_\xi \sim \mathrm{GGC}$.

4.3. Constructing GGCs

Using Theorem 1, we can construct explicit densities and LTs of GGCs as we shall do in the following.

Examples 3. (i) Let $Y \sim \mathrm{Gamma}(1, 1)$ and $X = U \sim U(0, 1)$ be independent. Then we have the following LTs and p.d.f.s for YU and Y/U, respectively:

$$\phi_{YU}(s) = \frac{\log(1+s)}{s}, \qquad \phi_{Y/U}(s) = 1 + s\log\left(\frac{s}{1+s}\right),$$

$$f_{YU}(x) = Ei(x) = \int_1^\infty y^{-1} e^{-yx}\, dy, \qquad f_{Y/U}(x) = \frac{1}{x^2}\big(1 - (1+x)e^{-x}\big).$$

By Example 2, we have $U \sim \mathrm{HM}_1$ and hence by Theorem 1 the above p.d.f.s are GGCs and the LTs are HCM.

(ii) Now let $Y \sim \mathrm{Gamma}(2, 1)$ and $X = \min(U_1, U_2)$, with $U_1, U_2 \sim U(0, 1)$ independent and independent of Y. Then $f_X(x) = 2(1-x), 0 < x < 1$, which belongs to HM_2. We can also represent X as $X \overset{d}{=} U_1 U_2^{1/2}$. We get the following LTs and p.d.f.s:

$$\phi_{YX}(s) = \frac{2}{s}\left(1 - \frac{\log(1+s)}{s}\right), \qquad \phi_{Y/X}(s) = 1 + 6s + \big(6s^2 + 4s\big)\log\left(\frac{s}{1+s}\right),$$

$$f_{YX}(x) = 2e^{-x} - 2x\,Ei(x), \qquad f_{Y/X}(x) = \frac{1}{x^3}\big(-12 + 4x + \big(2x^2 + 8x + 12\big)e^{-x}\big).$$

Again by Theorem 1 the p.d.f.s are GGCs and the LTs are HCM.

Many similar examples can be obtained from Corollary 1 by letting $Y \sim \mathrm{Gamma}(r, 1)$ with a real r.

5. Final comments

There are reasons to believe that Theorem 1 (as well as Proposition 1) can be extended to cover the case that k is any real number ≥ 1. Maybe it can even be extended to all real $k > 0$. As a definition of an HM_k function in the real case, the integral representation (1) can be used. For any real j and k such that $0 < j < k$, we have $\mathrm{HM}_k \subset \mathrm{HM}_j$. To see this, one can use (1) together with the simple formula

$$(\lambda - w)^{k-1} = \frac{1}{B(j, k-j)} \int_w^{\lambda} (\tilde{\lambda} - w)^{j-1} (\lambda - \tilde{\lambda})^{k-j-1} \, d\tilde{\lambda}.$$

For $k \geq 1$, the HM_k class is closed w.r.t. multiplication of functions. However, it is not closed for $k < 1$ which the example $f(x) = (1-x)^{-1/2}$ illustrates. The technique which we have used to prove Theorem 1 for integers k cannot be applied in the general real case since it much depends on an explicit calculation of the integral J_k in (2). However, numerical experiments indicate that J_k is CM as a function of $T = t + t^{-1}$ for all $k > 0$. An important problem for the future is to prove that so is the case.

Let $\mathcal{A}$ and $\mathcal{B}$ denote classes of probability distributions. We denote by $\mathcal{A} \times \mathcal{B}$ the class of distributions generated by $Y \cdot X$ for $Y \sim \mathcal{A}$ and $X \sim \mathcal{B}$ with Y and X independent. Theorem 1 and Proposition 7 can then be formulated as $\mathrm{Gamma}(k) \times \mathrm{HM}_k \subseteq \mathrm{GGC}$ and $\mathrm{GGC} \times \mathrm{GGC} \subseteq \mathrm{GGC}$, respectively.

One may wonder about the largest class $\mathcal{H}_k$ such that $\mathrm{Gamma}(k) \times \mathcal{H}_k \subset \mathrm{GGC}$. Apparently, because of Theorem 1 and Proposition 7, $\mathcal{H}_k \supset \mathrm{HM}_k \times \mathrm{GGC}$. One may also wonder about the largest class $\mathcal{G}_k$ such that $\mathcal{G}_k \times \mathrm{HM}_k \subseteq \mathrm{GGC}$. Of course, $\mathcal{G}_k \supseteq \mathrm{Gamma}(k) \times \mathrm{GGC}$. Could possibly $\mathcal{G}_k \supset \mathrm{GGC}(k)$, where $\mathrm{GGC}(k)$ denotes all GGCs with left-extremity 0 and total U-measure at most k? To prove this possible result, it suffices to show that $Y \cdot X \sim \mathrm{GGC}$ when $X \sim \mathrm{HM}_k$ and Y is a finite sum of independent gamma variables with a shape parameter sum not exceeding k and varying scale parameters.

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