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A Class of Simple Hamiltonians with Degenerate Ground State. II
A Model of Nuclear Rotation

—— *Spontaneous Breakdown of Rotation Symmetry and Goldstone Theorem for Finite Dimensional System* ——

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As a direct generalization of our previous work, we construct a manifestly rotation invariant Hamiltonian describing a system composed of a tensor boson with spin $I=2$ — such as, for example, phonon of nuclear quadrupole surface vibration — in which special cubic and quartic self-interactions are taken into account. All the ground state solutions are obtained. It is shown that these solutions can be classified into three classes: The first is the normal vacuum of boson, which is manifestly rotation symmetry. The second class of solutions can be viewed as generated from a single chosen state by rotating it into an arbitrary direction. The third one is Goldstone modes built on this chosen ground state. This chosen state — an exact ground state solution of our rotation invariant Hamiltonian — is not an eigenstate of angular momentum, but contains infinitely many different angular momentum states in it; thus, *typifying* "spontaneous breakdown" of "rotation symmetry". It is further shown that this state can be naturally identified to an "intrinsic" deformed state in nuclear collective model of Bohr and Mottelson.

The results are examined from the standpoint of Goldstone theorem and detailed discussions are given to explore how Goldstone theorem is modified in our finite dimensional system.

§ 1. Introduction

After the advent of quantum mechanics, the symmetry and invariance principle has played an essential role in modern physics.[1] A new insight has been recently introduced into the invariance principle through the concept of *spontaneous breakdown of the symmetry* — as familiarly exemplified by Nambu-Goldstone boson and Goldstone theorem in high energy physics. Another well-known example[2] is the ground state of ferromagnet and associating zero-excitation-energy magnons as Goldstone mode. In all the examples so far treated, one is concerned with the spontaneous breakdown of a given *internal* symmetry of the system with *infinite* degree of freedom.

It is the purpose of the present paper to propose a concrete model of a *finite* degree of freedom which explicitly exhibits *spontaneous breakdown* of *rotation symmetry* and to examine thereby how Goldstone theorem is modified in this finite dimensional system. The model Hamiltonian to be treated here is a direct generalization of the Hamiltonian of the many-boson system composed of a single kind of boson — a scalar boson, treated in our previous paper[3] — to a system composed of a tensor boson with spin $I(=\text{even})$. We shall be concerned mainly with the spin 2 boson system, because the spin 2 boson naturally arises as the phonon of nuclear surface vibration in the framework of Bohr and Mottelson model[4]. Thus, we consider nuclear rotation motion of *intrinsic* deformed nuclei in an ideal limit. Indeed, our Hamiltonian usually appears as a positive definite part of nuclear phenomenological Hamiltonian in nuclear collective model or of nuclear microscopic Hamiltonian when sophisticated boson expansion technique[5] is applied to it.

This paper is organized as follows; the Hamiltonian of our model is presented in the

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next section, which is manifestly rotation invariant and positive definite. Its lowest eigenvalue is given by zero. All the ground state solutions of the system are obtained in § 3. It is shown there that the ground state is infinitely degenerate in the usual sense. For later convenience, we classify the ground state solutions into three classes. The first one is the normal vacuum state of the boson, which is manifestly rotation invariant. The second class of the solutions can be viewed as generated from a single chosen state by rotating it into an arbitrary direction. This single chosen state — an exact ground state of our rotation invariant Hamiltonian — is not an eigenfunction of total angular momentum, but contains infinitely many different angular momentum states in it. Thus, this state typifies *spontaneous breakdown of rotation symmetry* in its literal sense. The third one can be interpreted as Goldstone modes built on this chosen ground state.

We examine in detail the ground state solutions from the standpoint of Goldstone theorem. For this purpose, we briefly summarize in the first part of § 4 the Goldstone theorem in local field theory and, then, discuss the ground state of ferromagnet from the standpoint of Goldstone theorem. We summarize all our results in parallel to the corresponding ones in the case of ferromagnet in order to clarify the formal analogy and difference between them. Our conclusion on the Goldstone theorem for the finite dimensional system is stated at the end of § 4. In § 5, we discuss our ground state from the conventional standpoint of nuclear structure. We perform angular momentum projection out of the spontaneous-symmetry breaking ground state. It is shown that this state contains all the even (including 0) spin states *once and only once* — that is to say, the degenerate *ground state rotation band*. Section 6 is devoted to concluding remarks.

§ 2. The rotation invariant Hamiltonian

2. 1. *Hamiltonian of spin 2 boson system*

Consider the creation and annihilation operators of boson having spin $I=2$, such as, for example, the phonon of nuclear surface quadrupole vibration — so-called d-boson,

$$[d_m, d_{m'}^\dagger]=\delta(m,m'), \qquad (2\cdot1)$$

where the suffix $m(=2,1,0,-1,-2)$ denotes the z-component of the spin. In other words, the creation operator $d_m{}^\dagger$'s are defined such that they transform among themselves under rotation as a second rank irreducible spherical tensor operator:

$$[J_z, d_m{}^\dagger]=md_m \quad \text{and} \quad [J_\pm, d_m{}^\dagger]=\sqrt{(2\mp m)(3\pm m)}\,d_{m\pm1}^\dagger\,, \qquad (2\cdot2)$$

where $\boldsymbol{J}$ is the angular momentum operator of the system. Let us define the spherical annihilation operator $\tilde{d}_m$ through

$$\tilde{d}_m=(-)^m d_{-m}\,,$$

which transforms correctly as the m-th component of a second rank tensor. We adopt the usual notation of the tensor product, i.g.,

$$[d^\dagger * d^\dagger]_M{}^{(2)}=\sum(22mm'|2M)d_m{}^\dagger d_{m'}^\dagger\,.$$

We, then, define the following second rank tensors:

$$D_M{}^\dagger=d_M{}^\dagger+g[d^\dagger * d^\dagger]_M{}^{(2)}\,, \qquad (2\cdot3a)$$

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$$\tilde{D}_M=(-)^M D_{-M}=\tilde{d}_M+g[\tilde{d}*\tilde{d}]_M^{(2)}. \tag{2·3b}$$

The model Hamiltonian to be considered is the scalar product of these two tensors:

$$H=\sum(-)^M D_M{}^\dagger \tilde{D}_{-M}=\sum D_M{}^\dagger D_M\,, \tag{2·4}$$

which is, by construction, clearly rotation invariant and positive definite. In the context of nuclear collective model, this Hamiltonian describes nuclear quadrupole surface vibration in which the special forms of the cubic and quartic anharmonicity are taken into account. The leading order anharmonicity (the cubic one) can be viewed as the self-interaction of d-boson, while the quartic anharmonicity can be regarded as the repulsive boson-boson interaction in the d-wave channel. It is noted that the system becomes unstable if we take into account only the cubic term alone. The special form of the quartic anharmonicity is added so as to make the Hamiltonian positive definite.

The angular momentum operator in this model is represented, in the spherical basis, as

$$J_M=\sqrt{2\cdot 5}[d^\dagger*\tilde{d}]_M^{(1)}. \tag{2·5}$$

2. 2. *Hamiltonian of spin I boson system*

A generalization of the Hamiltonian (2·4) to the system composed of the spin I bosons can be made straightforwardly. Let $a_m{}^{\dagger(I)}$ and $a_m{}^{(I)}$ be the boson creation and annihilation operators of spin I (=even);

$$[a_m{}^{(I)},\ a_{m'}^{\dagger(I)}]=\delta(m,m').$$

Defining the spherical annihilation opertor by $\tilde{a}_m{}^{(I)}=(-)^m a_{-m}^{(I)}$, we introduce the following spherical tensor operators of the rank I:

$$A_M{}^\dagger=a_M{}^{\dagger(I)}+g[a^{\dagger(I)}*a^{\dagger(I)}]_M{}^{(I)}, \tag{2·6a}$$

$$\tilde{A}_M=\tilde{a}_M{}^{(I)}+g[\tilde{a}^{(I)}*\tilde{a}^{(I)}]_M{}^{(I)}. \tag{2·6b}$$

We then define the Hamiltonian

$$H=\sum(-)^M A_M{}^\dagger \tilde{A}_{-M}=\sum A_M{}^\dagger A_M\,. \tag{2·7}$$

The angular momentum of this system can be written as

$$J_M=\sqrt{I(I+1)(2I+1)/3}\,[a^{\dagger(I)}*\tilde{a}^{(I)}]_M{}^{(1)}. \tag{2·8}$$

We shall discuss later the ground states of this generalized Hamiltonian in the limit of infinitely large I.

§ 3. The ground state solutions

We shall examine in this section the ground state of the Hamiltonian (2·4) of d-boson system. First of all, we note that, since the Hamiltonian is positive definite, all its eigenvalues are non-negative. The energy of the ground state is clearly given by $E=0$. Further, it is obvious from the structure of H that if there exists a state $|\Psi)$ which simultaneously satisfies

$$D_M|\Psi)=0; \qquad \text{for all } M(=2,1,0,-1,-2), \tag{3·1}$$

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$|\Psi)$ is a zero-energy solution (a ground state) of our Hamiltonian. Conversely, any zero-energy solution of H satisfies (3·1), because all D_M's are constructed out of the annihilation operators alone. Let us summarize here the explicit forms of D_M's, together with that of the angular momentum operator in this d-boson model.

$$D_0 = d_0(1-Gd_0)+G(d_1d_{-1}+2d_2d_{-2}), \quad (3\cdot 2a)$$

$$D_{\pm 1} = d_{\pm 1}(1-Gd_0)+\sqrt{6}Gd_{\mp 1}d_{\pm 2}, \quad (3\cdot 2b)$$

$$D_{\pm 2} = d_{\pm 2}(1+2Gd_0)-\sqrt{3/2}\,Gd_{\pm 1}d_{\pm 1}, \quad (3\cdot 2c)$$

where

$$G=-(2200|20)g=\sqrt{2/7}\,g, \quad (3\cdot 2d)$$

$$J_z=2(d_2^\dagger d_2-d_{-2}^\dagger d_{-2})+(d_1^\dagger d_1-d_{-1}^\dagger d_{-1}), \quad (3\cdot 3a)$$

$$J_\pm=2(d_{\pm 2}^\dagger d_{\pm 1}+d_{\mp 1}^\dagger d_{\mp 2})+\sqrt{6}(d_0^\dagger d_{\mp 1}+d_{\pm 1}^\dagger d_0). \quad (3\cdot 3b)$$

After these preparations, we now seek the solutions of (3·1).

3. 1. *The manifestly rotation invariant solution*

The normal vacuum state $|0)$,

$$|0)=\Pi_M\,|0\rangle_M, \quad (3\cdot 4)$$

which trivially satisfies Eq. (3·1), is a ground state solution. The rotation invariance is manifest in this ground state.

3. 2. *Symmetry-breaking ground states*

To show quickly the degeneracy of our ground states, we first prepare the product of the vacuum-state for $M\neq 0$ boson subspaces, $\Pi_{M\neq 0}\,|0\rangle_M$. Our Hamiltonian is then reduced to the Hamiltonian of the $(M=0)$ boson alone, which is identical in form to that treated in our previous paper. Explicitly, $H=(d_0^\dagger - Gd_0^\dagger d_0^\dagger)(d_0-Gd_0d_0)$. We, therefore, obtain the abnormal vacuum solution in addition to the normal vacuum solution (3·4),

$$|\text{Deform})=|1/G\rangle_{M=0}\Pi_{M\neq 0}|0\rangle_M, \quad (3\cdot 5a)$$

where $|1/G\rangle_0$ is the coherent state of d_0-boson; $d_0|1/G\rangle_0=(1/G)|1/G\rangle_0$. Namely, we have

$$|\text{Deform})=\exp(-1/2G^2)\exp\{(1/G)d_0^\dagger\}|0). \quad (3\cdot 5b)$$

This zero-energy solution clearly typifies *"spontaneous breakdown of rotation symmetry"*, because this is not an eigenstate of the angular momentum, but contains infinitely many angular momentum states. Thus, one sees already at this stage the infinite degeneracy of our ground state. In the context of nuclear Bohr-Mottelson model, this wave function can be regarded as describing nuclear intrinsic wave function with prolate deformation (for $G>0$). the deformation parameters of Bohr, β and γ, being given by $\beta=1/G$ and $\gamma=0$. In this connection, it is interesting to note that the wave function (3·5) has been frequently used as a trial wave function in the phenomenological description[6] of nuclear quadrupole excitation of transition nuclei. On the other hand, this wave function appears as an exact ground state of our Hamiltonian. Further note the deforma-

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tion becomes larger and larger, as the coupling constant becomes smaller and smaller, because $\beta=1/G$. This is one of typical features of our system, as has been discussed in detail in I.

Several special solutions of (3·1) other than (3·5) can be immediately obtained by inspection on (3·1). For example, we obtain

$$|\mathrm{Deform}')=|-(1/2G)\rangle_0\,|0\rangle_1\,|0\rangle_{-1}\,|-(\sqrt{3}/2\sqrt{2}G)\rangle_2\,|-(\sqrt{3}/2\sqrt{2}G)\rangle_{-2}\,, \qquad (3\cdot 6)$$

which, at first sight, may be interpreted as describing the intrinsic deformed state with oblate deformation ($\beta=-1/G$, $\gamma=60°$). This interpretaion is, however, wrong for $G>0$, as shown later on.

The above state is described by the product wave function composed of three coherent states in the ($M=2,0,-2$) subspaces and vacua in the ($M=1,-1$) subspace. We, therefore, expect that there should be general solutions of (3·1), whose form can be represented as product of five coherent states of five different subspaces : $\Pi_M|z_M\rangle_M=|\boldsymbol{Z})$. Introducing this product wave function into (3·1), we obtain the set of algebraic equations which determines the five unknown complex numbers z_M. This procedure actually works well to determine the most general form of the solutions under the restriction that the solution should be a product of coherent states. We do not pursue this treatment until the end of this subsection. Instead, in order to quickly present the results, we recall "rotation invariance" of our Hamiltonian:

$$R(\Omega)HR^{-1}(\Omega)=H\,, \qquad (3\cdot 7)$$

where R is the rotation operator

$$R(\Omega)=\exp\{-i\phi J_z\}\exp\{-i\theta J_y\}\,\exp\{-i\psi J_z\}. \qquad (3\cdot 8)$$

Here Ω denotes the set of Euler angles (ϕ,θ,ψ). From (3·7), it is obvious that, if $|\Psi)$ is an eigenstate of H, $R(\Omega)|\Psi)$ is also an eigenstate — equivalent or non-equivalent, depending upon the system under consideration. By taking the above $|\Psi)$ our abnormal ground state, |Deform), in (3·5), we have

$$R(\Omega)|\mathrm{Deform})\equiv|\Omega;\mathrm{Deform})$$
$$=\exp(-1/2G^2)\exp\{(1/G)\Sigma_M d^{(2)}_{M0}(\theta)e^{-iM\phi}d_M{}^\dagger\}|0), \qquad (3\cdot 9\mathrm{a})$$

where we have used the transformation property of $d_0{}^\dagger$ under the rotation $R(\Omega)$. This wave function is clearly represented as the product of five coherent states, each in a different M-subspace;

$$|\Omega;\mathrm{Deform})=|(1/G)d^{(2)}_{00}(\theta)\rangle_0|(1/G)d^{(2)}_{10}(\theta)e^{-i\phi}\rangle_1|(1/G)d^{(2)}_{-10}(\theta)e^{i\phi}\rangle_{-1}$$
$$\times|(1/G)d^{(2)}_{20}(\theta)e^{-2i\phi}\rangle_2|(1/G)d^{(2)}_{-20}(\theta)e^{2i\phi}\rangle_{-2}\,. \qquad (3\cdot 9\mathrm{b})$$

Needless to say, $|\Omega$; Deform) is our exact ground state solution for any fixed value of Euler angle Ω. If we take ($\theta=\pi/2, \phi=\pi/2$ and $\psi=$arbitrary), state (3·9) reduces to |Deform$'$) obtained by (3·6). Namely, we have $|\mathrm{Deform}')=\exp\{-i(\pi/2)J_z\}\exp\{-i(\pi/2)J_y\}|\mathrm{Deform})$ for a fixed value of G, from which we conclude that |Deform$'$) is unitary equivalent to |Deform) — described by the same intrinsic state. Further, it is noted that our Hamiltonian is invariant under the simultaneous transformation; $G\to-G$, $d_m{}^\dagger\to-d_m{}^\dagger$

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and $d_m \to -d_m$, so that we can always take G positive. From the nuclear structure point of view, however, it is better to start from $|\mathrm{Deform}')$ in Eq. (3·6) when G is negative, because the state $|\mathrm{Deform}')$ can be naturally identified to the nuclear intrinsic state of oblate deformation with ($\beta=-1/G$ and $\gamma=60°$; when $G<0$).

It remains to show that solution (3·9) is the most general solution of (3·1) under the restriction that the solution be a single product of five coherent states. To this end, we adopt here the spherical form of Eq. (3·1) — i.e. $\tilde{D}_M|\psi)$, and restrict $|\psi)$ within a product function of coherent states $|\tilde{Z})=\Pi|\tilde{z}_M\rangle_{M'}$ where $\tilde{z}_M$ denotes the eigenvalue of $\tilde{d}_M$. We thus obtain the set of algebraic equations to determine $\tilde{z}_M$. The first and the most important step to solve the set of equations is to introduce suitable parametrizations for $\tilde{z}_M$. For this purpose, we recall the well-known fact that the eigenvalue of boson annihilation operator can be viewed as a classical dynamical variable; its real part corresponds to the coordinate, while its imaginary part to the conjugate momentum. It is now natural to adopt Bohr's parametrization: $\tilde{z}_M=\alpha_M+i\pi_{M'}$ where α_M's are coordinates of Bohr model. As usual, we put $\alpha_M=\sum_{M'}D^{(2)}_{M'M}(\Omega)a_{M'}$,where $a_0=\beta\cos\gamma$, $a_{\pm1}=0$ and $a_{\pm2}=(\beta/\sqrt{2})\sin\gamma$, with $\alpha_M=(-)^M\ \alpha_{-M}$. π_M's are conjugate momenta to $\alpha_{M'}$ which can be represented in terms of three angular velocities and two momenta associating to β- and γ-vibrations. Since we are treating the ground states of our Hamiltonian, we may expect that all the five momenta will vanish in our solution. Under this assumption, we obtain the set of algebraic equations,

$$\tilde{D}_M|\boldsymbol{\alpha})=\sum_{M'}D^{(2)}_{M'M}(\Omega)F_{M'}=0\,; \qquad (\text{for all } M)$$

where $F_0=\beta(\cos\gamma-G\beta\cos2\gamma)$,$F_{\pm1}=0$ and $F_{\pm1}=\beta(\sin\gamma+G\beta\sin2\gamma)$. We, therefore, get the general solutions, ($\beta=1/G,\gamma=0$) or ($\beta=-1/G,\gamma=60°$) for an arbitrary Ω, which are precisely the same as those previously obtained.

Next, we let all $\tilde{z}_M$'s complex numbers. It is not too difficult to obtain the most general solution. It turns out that the most general solution is nothing but the form of (3·9a), in which all three Euler angles are now taken to be complex numbers. This result is the expected one, because $R(\Omega)\ H\ R^{-1}(\Omega)=H$ is "formally" valid even for the complex extension of Euler angles. By the same reason, this type of solution is obviously not "physically interesting" solution. Thus, we conclude that the solution obtained in (3·9) is the most general relevant solution of our problem under the restriction that the solution can be represented as a single product of coherent states.

3. 3. *Goldstone-mode solutions*

Although we have obtained the general solution of (3·1), we have restricted the form of the solution within a single product of the coherent states. There are other solutions, which cannot be represented by such a product form. For example, an inspection on (3·1) immediately reveals that $d_1^\dagger|\mathrm{Deform})$ is a solution of (3·1). This wave function can, in turn, be written as $J_+|\mathrm{Deform})$, because of the explict form of J_+ in (3·3). By repeating applications of J_+ on $|\mathrm{Deform})$, we obtain

$$J_+|\mathrm{Deform})=(\sqrt{6}/G)d_1^\dagger|\mathrm{Deform})$$

$$=(\sqrt{6}/G)\exp\{-1/2G^2\}d_1^\dagger\exp\{(1/G)d_0^\dagger\}|0), \qquad (3\cdot10\mathrm{a})$$

$$(J_+)^2|\mathrm{Deform})=(\sqrt{6}/G)\{2d_2^\dagger+(\sqrt{6}/G)(d_1^\dagger)^2\}|\mathrm{Deform}), \qquad (3\cdot10\mathrm{b})$$

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$$(J_+)^n|\text{Deform}) = \sum_{k=0}^{[n/2]} (\sqrt{6}/G)^{n-k} [n!/(n-2k)!k!] (d_2^\dagger)^k (d_1^\dagger)^{n-2k} |\text{Deform}), \tag{3·10c}$$

all of which are exact ground states of our Hamiltoinan, as can be verified either by directly substituting these into Eq. (3·1), or by simply appealing 'rotation invariance' of the Hamiltonian,

$$[H, J_z] = [H, J_\pm] = 0. \tag{3·11}$$

Likewise, $(J_-)^n|\text{Deform})$ is also a ground state solution. The explicit form of this state can be obtained from (3·10c) by replacing $d_m^\dagger$ to $d_{-m}^\dagger$,

$$(J_-)^n|\text{Deform}) = \sum_{k=0}^{[n/2]} (\sqrt{6}/G)^{n-k} [n!/(n-2k)!k!] (d_{-2}^\dagger)^k (d_{-1}^\dagger)^{n-2k} |\text{Deform}). \tag{3·12}$$

All the states in (3·10) and (3·12) are not only orthogonal to the chosen ground state |Deform), but also mutually orthogonal. Indeed, we have

$$(\text{Deform}|(J_-)^{n'}(J_+)^n|\text{Deform}) = \delta(n,n') \sum_{k=0}^{[n/2]} [(n!)^2/k!(n-2k)!] (6/G^2)^{n-k} \equiv \delta(n,n') N(n). \tag{3·13}$$

We note, further, that

$$J_z\{(J_\pm)^n|\text{Deform})\} = (\pm) n \{(J_\pm)^n|\text{Deform})\}, \tag{3·14}$$

from which, together with (3·13), we obtain the expectation value of J^2;

$${}_N[(\text{Deform}|(J_-)^n] J^2 [(J_+)^n|\text{Deform})]_N = [N(n+1)/N(n)] + n(n+1). \tag{3·15}$$

Here, $N(n)$ has been defined by the last line of Eq. (3·13) and $[(J_+)^n|\text{Deform})]_N$ denotes the normalized wave function.

§ 4. Goldstone theorem for finite dimensional system

We shall show that the mutually orthogonal states obtained in § 3.3 can be interpreted as Goldstone modes built on the chosen ground state |Deform). For this purpose and, at the same time, to examine how Goldstone theorem is modified in our finite dimensional system, we shall first summarize Goldstone theorem in relativistic field theory. Since our problem is non-relativistic one, we next briefly discuss the ground state property of Heisenberg ferromagnet from the standpoint of Goldstone theorem as the simplest non-relativistic example of Goldstone theorem. We summarize all our results obtained in the preceding section in parallel to the corresponding ones in ferromagnet, in order to clarify the formal analogy and difference between them.

GOLDSTONE THEOREM in Relativistic Field Theory Consider a field theory governed by a Lagrangian L, which is assumed to be invariant under a given internal continuous symmetry. Associated with this symmetry, there exists a conserved current — the space integral of the time-component of the current defines a "charge" Q. The commutator of each component of field, ϕ_α, with this charge can be written down explicitly; symbolically, we have a set of commutators

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$$[Q, \phi_a] = i\delta\phi_a . \tag{4·1}$$

We summarize Goldstone theorem in somewhat inexpert way as follows:

(I) There are possibilities that the vacuum expectation values of some of the commutators in set (4·1) are non-vanishing. We call "Goldstone commutator" the above-mentioned commutator whose vacuum expectation value is non-vanishing.
When the "Goldstone commutator" exists, the internal symmetry is said to be *spontaneously broken* in this vacuum. The charge cannot annihilate the vacuum, in contrast to the normal case.

(II) Associated with this spontaneous breakdown of symmetry, there arises massless spin 0^+ boson — Nambu-Goldstone boson. The action of "charge" on this abnormal vacuum can be interpreted as creations of Nambu-Goldstone boson out of the vacuum.

(III) All the vacuum states definable within this field theory other than the chosen vacuum in (I) are *unitary non-equivalent* to this chosen vacuum.

The conventional proof of (II) utilizes the spectrum representation of Green function and, then, by examining its behavior in the limit of $(E, \boldsymbol{p})=0$, shows that the spectrum function in this limit is just proportional to the non-vanishing vacuum expectation value of the Goldstone commutator. Lorentz invariance of the theory restricts this massless particle appearing at this pole of Green function to 0^+-boson. In the non-relativistic theories such as ours and the ferromagnet, there are no such restrictions on the spin-statistics of this zero-excitation energy mode. Although statement (III) may be regarded as an axiom in the constructive field theory, it can be easily proved in the case of ferromaget, as will be shown later. We think of it as heart of Goldstone theorem for any infinite dimensional system. Indeed, it is this statement (III) that is modified in our finite dimensional system.

Since the above theorem in local field theory is rather abstract, it seems better to use a simple non-relativistic example, in which the essential points of the theorem can easily be understood. We now discuss the ground states of Heisenberg ferromagnet as such an example. All our results obtained in §3 will be compared to the results of the ferromagnet is each appropriate position in order to make the formal correspondence clear. Consider the Heisenberg Hamiltonian

$$H = -\Sigma_{ll'} J(l, l')\boldsymbol{S}^{(l)}\boldsymbol{S}^{(l')} , \tag{4·2}$$

where $\boldsymbol{S}^{(l)}$ is the spin operator for the atom on the l-th lattice site, $J(l, l')$ being the nearest neighbor exchange interaction. A ground state of the system at the absolute zero temperature can be taken to be the state in which all the spins are aligned into a definite direction — the z-direction. Let |Ferro) denote the wave function of this state. Obviously, we have $S_+|\text{Ferro})=0$, where

$$S_z = \Sigma_l S_z^{(l)} \quad \text{and} \quad S_\pm = \Sigma_l S_\pm^{(l)} \tag{4·3}$$

is total spin operator of the system. From

$$[H, S_z] = [H, S_\pm] = 0 , \tag{4·4}$$

it follows that all the states constructed by repeating applications of S_- on the chosen ground state |Ferro), $(S_-)^n|\text{Ferro})$, are degenerate to this ground state and that they are all mutually orthogonal.

The fact that these states are Goldstone modes built on |Ferro) can be proved in just

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the same way as in the field theory. Namely, first set up Green function of the system — in this case, by employing an approximation; such as, Tyablikov decoupling=a kind of RPA — and, then, examine its behavior at $E=0$. We need not pursue this standard procedure. Instead, suffice it to say that the Goldstone mode is the zero-energy limit of the elementary excitation mode of the system — i.e., the spin-density wave (magnon), and that $S_-|\mathrm{Ferro})$ is nothing but the long wavelength limit of this spin-density wave. The Goldstone commutator is identified to

$$[S_+, S_-]=2S_z\,. \tag{4·5}$$

Its expectation value with respect to our chosen ground state $|\mathrm{Ferro})$ is, of course, non-vanishing. We note that this non-vanishing expectation value can be used as the *order parameter* to specify the phase of this ground state.

It is now obvious that our ground state solutions obtained in § 3.3. can be naturally identified as Goldstone modes built on the chosen ground state $|\mathrm{Deform})$; $|\mathrm{Deform})$ corresponds to $|\mathrm{Ferro})$ in the ferromagnet. Further, the Goldstone commutator in our model is identified to

$$[J_\pm, \tilde{d}_\mp]=\sqrt{6}d_o\,. \tag{4·6}$$

Indeed, we have

$$(\mathrm{Deform}|[J_\pm, \tilde{d}_{\mp 1}]|\mathrm{Deform})=\sqrt{6}/G\,. \tag{4·7}$$

It is interesting to note that this expectation value is just $\sqrt{6}$ times the value of the "intrinsic quadrupole moment" of this state in the context of Bohr-Mottelson model. Further, all the states in §3.3, $(J_\pm)^n|\mathrm{Deform})$ — when correctly normalized — give the same expectation value of this Goldstone commutator. Namely, these mutually orthogonal states can be specified by the same value of this order parameter. Needless to say, the normal ground state of our Hamiltonian in §3.1 gives us the vanishing expectation value of the commutator (4·6).

Returning to the ferromagnet, we now discuss another ground state in which all spins are directed in another direction, (θ, ϕ) measured from the z-axis. For simplicity, we take $\phi=0$ and denote the wave function of this ground state as $|\theta;\mathrm{Ferro})$. In terms of the previously chosen ground state $|\mathrm{Ferro})$, $|\theta;\mathrm{Ferro})$ is "formally" represented as

$$|\theta;\mathrm{Ferro})=\exp\{-i\theta S_y\}|\mathrm{Ferro}). \tag{4·8}$$

Our problem is now to prove the unitary non-equivalence of these two chosen ground states. For this purpose, we rewrite the spin-rotation operator in an ordered form;

$$\begin{aligned}\exp\{-i\theta S_y\}&=\exp\{-(\theta/2)(S_+-S_-)\}\\ &=\exp\{\tan(\theta/2)S_-\}\exp\{2\log\cos(\theta/2)S_z\}\exp\{-\tan(\theta/2)S_+\}.\end{aligned} \tag{4·9}$$

Because $S_+|\mathrm{Ferro})=0=(\mathrm{Ferro}|S_-$, we obtain

$$(\mathrm{Ferro}|\theta;\mathrm{Ferro})=\Pi_l[\cos(\theta/2)]^{2m^{(l)}} \tag{4·10}$$

($m^{(l)}$being the eigenvalue of $S_z{}^{(l)}$)

which obviously vanishes in the limit of infinitely many-spin system. We, thus, complete

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the proof of the *unitary non-equivalence* of these two differently chosen ground states. Physically, this means that it is absolutely impossible to bring all spins into a different direction from the state |Ferro) to obtain |θ; Ferro), how many Goldstone modes are *mobilized* onto |Ferro).

Again, it becomes clear that our ground state |Ω; Deform) obtained in § 3.2 can be compared to the ground state |θ; Ferro) of the ferromagnet. Namely, we have correspondence; |Ω; Deform) to |θ; Ferro). It is also clear that all the |Ω; Deform) are *unitary equivalent* to the chosen ground state |Deform), because of the unitarity of the finite dimensional representation of the rotation operator. From this unitary equivalence, we can transform the chosen state |Deform) into an arbitrary direction by appropriately *mobilizing* the Goldstone modes.

Let us summarize the results in the following table:

	Ferromagnet	Our Model
Hamiltonian	Heisenberg Hamiltonian (4·2)	d-boson model (2·4)
Symmetry of H	$[H, \boldsymbol{S}]=0$ spin symmetry	$[H, \boldsymbol{J}]=0$ rotation symmetry
A chosen symmetry breaking state	|Ferro) all spins in z-axis	|Deform) in (3·5) d_0 -boson condensates
Goldstone mode	S_- |Ferro)	J_+|Deform), J_-|Deform)
Goldstone Commutator	(4·5)	(4·6)
Order Parameter	magnetization in z	intrinsic quadrupole moment
Another Vacuum	|θ; Ferro) all spins in θ direction	|Ω; Deform) in (3·9) bose-condensates in five M
Unitary Equivalence or Non-eqivalence	*non-equivalent*	*equivalent*

From the above, one now sees how Goldstone theorem is modified in the finite dimensional system. Namely, in the last statement (III) of the theorem, "unitary non-equivalent" should be replaced by "unitary equivalence", while first two statements (I) and (II) remain unchanged.

To show more precisely the above statement, we finally discuss the ground states of the generalized Hamiltonian (2·7) composed of the spin I tensor boson. By letting the spin I of the boson infinity, the system becomes infinite dimensional, because infinitely many kinds of boson $a_M^{(I)}$ ($M=0$ to infinity) take part in this model. All the ground state solutions of the model can be obtained in the same way as in § 3. We have the chosen vacuum which is a condensate of $a_{M=0}^{(I)}$-boson;

$$|\text{Deform}, I)=\exp\{-(1/2G^2)\}\exp\{(1/G)a_{M=0}^{\dagger(I)}\}|0),$$

where $G=-(II00|I0)g$. Another symmetry-breaking vacuum corresponding to |Ω; Deform) can be "formally" represented as |Ω; Deform, I)$=R(\Omega)$ |Deform, I).

The overlap between these two vacua is given by

$$(\text{Deform}, I|\Omega; \text{Deform}, I)=\exp\{-1/G^2\}\exp\{(1/G^2)\cdot P_I(\cos\theta)\}, \qquad (4\cdot 11)$$

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which can be easily proved to vanish by making use of asymptotic limits of $P_I(\cos\theta)$ and of the Clebsch-Gordan coefficients $(II00|I0)$ for a large I. Thus, one observes the "unitary nonequivalence" between these two vacua in the limit of infinitely large I.

An additional REMARK: In the case of ferromagnet treated in this section, finite dimensional model can be constructed by restricting the number of the spins finite. In this model, the degeneracy of the ground states is finite and, moreover, the chosen state |Ferro) becomes an eigenstate of the total spin operator $\boldsymbol{S}$. Thus, the model is trivial from our standpoint. On the other hand, in our d-boson model, the ground state is infinitely degenerate and that the chosen state |Deform) — an exact ground state solution of the rotation invariant Hamiltonian — is not an eigenstate of angular momentum, but contains infinitely many different eigenstates in it. Hence, our model *typifies* "Spontaneous Breakdown" of rotation symmetry in its literal sense.

Our conclusions on Goldstone theorem in this section may be stated as follows. When speaking of the spontaneous breakdown of symmetry, it is usually said that the Goldstone mode appears so as to restore the broken symmetry. From our results, we would say — ironically — that the Goldstone mode appears as a manifestation of Nature's effort to restore the symmetry. But — alas! — Her effort is endless — the complete restoration is impossible in infinite dimensional system (unitary non-equivalence); so remains Goldstone on the ground state forever. While, in a finite dimensional system, her effort is finally requited leading to the complete restoration of the symmetry. As a result, after completing his duty, he — poor Goldstone! — fades away into not heaven, but into "underground" — "under" graduate level of physics, as will be shown in the next section.

§ 5. Discussion

We have already discussed in some detail in § 3.2 the ground states of our system from the standpoint of nuclear collective model. We now treat the same problem one step further from the conventional standpoint of nuclear structure. Namely, we shall perform the angular momentum projection out of the chosen ground state, |Deform) in (3·5), which typifies the spontaneous breakdown of rotation symmetry. The method of angular momentum projection to be employed is essentially the same as that of Peierls and Yoccoz; namely, first suppose to expand the state, |Deform), into the angular momentum eigenstates. Then, (1), firstly by rotating whole wave function, each eigenstate of angular momentum will transform definitely as the basis of rotation group. Next, (2), we read off the angular momentum states contained in it, by examining the transformation properties of the rotated wave function. Usually, step (2) mentioned above is done by multiplying representation function of the rotation group to the rotated wave function and, then, performing the integration over whole group manifold — i.e. over all Euler angles. Because of the simple structure of our wave function |Deform), this second step can be performed much simpler in our case, as will be done shortly.

The result of rotation $R(\Omega)$ on |Deform) has been already given by Eq. (3·9). In order to read off the transformation properties of this rotated state, we take the scalar product of the rotated state and the unrotated state to obtain

$$(\text{Deform}|R(\Omega)|\text{Deform})=\exp\{-(1/G^2)\}\exp\{(1/G^2)\cdot P_2(\cos\theta)\}. \qquad (5\cdot1)$$

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Expansion of the above formula into spherical harmonics can be made straightforwardly:

$$(\text{Deform}|R(\Omega)|\text{Deform})=\Sigma_L a_L Y_{L0}(\theta,\phi), \tag{5·2}$$

where

$$a_L=\sqrt{4\pi(2L+1)}\exp\{-3/2)(1/G^2)\}\{\sqrt{3/2}(1/G)\}^L$$
$$\times\Sigma_p^\infty\{(3/2)(1/G^2)\}^p[(L+2p)!/\{(L/2)+p\}!]/[(2p)!!(2L+2p+1)!!]$$
$$;\ \text{if } L \text{ is even}$$

and

$$a_L=0 \qquad ;\ \text{if } L \text{ is odd}.$$

Thus, one observes that the state |Deform) consists of all the even spin states including $L=0$ to infinity.

Several remarks are now in order. (i) As already shown, the action of $J_\pm$ on the state |Deform) can be regarded as creation of Goldstone modes built on this state. Once the state is expanded into eigenstates of angular momentum, the results of the operation of $J_\pm$ on "each" eigenstate are now apparently contained in any textbook of elementary quantum mechanics — namely, undergraduate level of physics, as stated in the last sentence of the preceding section. (ii) By comparing the above formula (4·11) to (5·1), one sees immediately that the angular momentum projection is impossible in the case of infinite dimensional system. (iii) The method of angular momentum projection usually employed in nuclear structure theory including ours can be used without any care, only if the state under consideration is multiplicity-free in angular momentum states. To show that our state |Deform) is multiplicity-free, we recall the fact that |Deform) can be identified to the "intrinsic" deformed state of Bohr and Mottelson model with the definite values of deformation parameters $\beta=1/G$ and $\gamma=0°$,$(G>0)$. Further, the K-quantum number is clearly given by $K=0$. We, thus, conclude the state |Deform) contains all L =even states from $L=0$ to infinite once and only once.

Further projection of |Deform) into a definite number state of d-bosons is possible, which has been done by Gheorghe et al.[7] in group theoretical context. In this case, we have all the states contained in the full dynamical group of the d-boson system: $Sp(10R)\supset O(5)\times SU(1,1)$.[8] It is, however, noted that each number-projected state is no longer an eigenstate of our Hamiltonian.

§ 6. Concluding remarks

In discussing nuclear structure, in which rotational spectra appear very regularly, one usually starts by presupposing that the nucleus has well-defined *intrinsic deformed shape.* Although there are no problems in defining *deformed shape* in classical physics, the concept of "intrinsic deformation" is not necessarily clear[9] within the framework of quantum mechanics. It seems to the authors that this concept of "intrinsic deformation" within quantum mechanical framework is intimately connected to the concept of the "spontaneous breakdown of rotation symmetry". Indeed, such a point of view has been emphasized by the very inventor of the concept of the "intrinsic deformation" — Bohr, who in his

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Nobel lecture[10]) deliberately stated:

"In a general theory of rotation, symmetry plays a central role. Indeed, the very occurence of collective rotational degree of freedom may be said to originate in the breaking of rotational invariance, which introduces a "deformation" that makes it possible to specify an orientation of the system. Rotation represents the collective mode associated with such a spontaneous symmetry breaking (Goldstone boson)."

In the present paper, we have established a specific model example within quantum mechanical framework, which explicitly exhibits essential point of such an idea.

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