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A Class of Solutions of Einstein's Equations which Admit a 3-Parameter Group of Isometries

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Abstract. The Plebański and Stachel and Goenner and Stachel lists of metrics which are solutions of Einstein's field equations, have two double eigenvalues and admit 3-parameter groups of isometries with 2-dimensional spacelike orbits are completed by the addition of metrics which result from the use of a more general metric form.

1. Introduction

Plebański and Stachel [1] state that they have carried out a complete classification of spherically-symmetric metrics whose Einstein tensors have two double eigenvalues. Goenner and Stachel [4] extend Ref. [1] to include the cases in which the two-curvature of the spacelike orbit may be zero or negative. However, as was pointed out by Takeno and Kitamura [2] and Goenner [3] the general form of the metric used by Plebański and Stachel and Goenner and Stachel

$$ds^2 = e^{\nu}(dx^0)^2 - e^{\lambda}\,dr^2 - r^2\,d\omega^2 \tag{1}$$

where

$$d\omega^2 = d\theta^2 + \Sigma^2\,d\phi^2$$

$$\text{and also where } \Sigma = \begin{cases} \sin\theta & \text{for positive two-curvature} \\ \sinh\theta & \text{for negative two-curvature} \\ 1 & \text{for zero two-curvature}. \end{cases}$$

in fact excludes one class of metrics. Here $\lambda = \lambda(x^0, r)$ and $\nu = \nu(x^0, r)$.

Goenner [3] extends Ref. [4] to include the class which had previously been omitted without, however, specifying the additional metrics involved. These metrics are listed in this paper.

From Goenner [3] the general form of a metric which admits a three parameter group of isometries with two-dimensional spacelike orbits (i.e. $G_3(2, s)$) is

$$ds^2 = 2G(u, v)\,du\,dv - M^2(u, v)\,(d\theta^2 + \Sigma^2\,d\phi^2)\,. \tag{2}$$

The condition that two double eigenvalues exists is (see Appendix A)

$$GM_{,vv} - G_{,v}M_{,v} = 0 \tag{3}$$

such that either

$$M_{,v} = AG\,, \tag{4}$$

where A is a function of u only, or

$$M_{,v} = 0\,. \tag{5}$$

The case (5) is the one omitted by Plebański and Stachel and Goenner and Stachel.

In the case (4) A can be made one by a coordinate transformation. Then (2) becomes

$$ds^2 = 2M_{,v}\,du\,dv - M^2(u,v)\,d\omega^2\,. \tag{6}$$

The case examined by Plebański and Stachel is the one for which the surfaces $M=$ constant are spacelike.

We put

$$M(u,v) = r\,, \qquad -2M_{,u} = F \tag{7}$$

and (6) becomes

$$ds^2 = F(u,r)\,du^2 + 2\,du\,dr - r^2\,d\omega^2 \tag{8}$$

where $F > 0$.

A complete list of metrics of the form (8) and of Petrov-Plebański types $[2N-2S]_{[2-1]}(A,B)$, $[2T-2S]_{[1-1]}(A,B)$, $[4N]_{[2]}(A,B)$ and $[4T]_{[1]}(A,B)$ is given in Ref. [1] for $\lambda=-1$ and in Table V of Ref. [4] for $\lambda = 0, \pm 1$. Here

$$\lambda = [d^2\Sigma(\theta)/d\theta^2]/\Sigma\,. \tag{9}$$

In the above description of the metric types, $A=0$ for zero curvature invariant R, and $A=R$ otherwise, B is the Petrov type of the Weyl tensor (types D and O only for metric (2)) and the other symbols give the various Petrov-Plebański types of the Ricci tensor (see Refs. [1] and [4]).

It is to be noted that the constant e^2 which appears in two of the Goenner and Stachel solutions can be negative or positive, the positive e^2 having been chosen for the physical interpretation.

It would appear from Part 4 of Ref. [1] that there are no spherically-symmetric metrics with $(A,B)=(0,0)$ other than the flat solution $[4T]_{[1]}(0,0)$. A similar conclusion for metrics admitting $G_3(2,s)$ would seem to follow from Table V of Ref. [4]. Appropriate metrics of these classes do in fact exist and take the form of metric (2) with (5) holding.

2. Classification of the Metrics of Case (5)

After making coordinate transformations it is clear that the metrics with (5) holding have the form of either

$$ds^2 = 2G(u, v)\, du\, dv - d\omega^2 \tag{10}$$

or

$$ds^2 = 2G(u, v)\, du\, dv - u^2\, d\omega^2 . \tag{11}$$

All such metrics have two double eigenvalues and the Ricci scalar is zero if and only if the Petrov type of the Weyl tensor is 0, i.e. $(A, B) = (R, D)$ or $(0, 0)$. The classification of these metrics follows the method discussed in the Appendix.

The metrics of the form (10) are:

(i) $[4T]_{[1]}(0, 0)$, flat space,

$$ds^2 = 2UV\, du\, dv - d\omega^2 \tag{12}$$

where $\lambda = 0$ (i.e. $d\omega^2 = dx^2 + dy^2$) and U and V are functions of u and v respectively.

(ii) $[2T - 2S]_{[1-1]}(0, 0)$

$$ds^2 = \frac{4U'V'}{\lambda(U+V)^2}\, du\, dv - d\omega^2 \tag{13}$$

where $\lambda \neq 0$.

Here $G = \dfrac{2U'V'}{\lambda(U+V)^2}$ is the solution of Liouville's equation

$$\left(\frac{G_{,u}}{G}\right)_{,v} = \lambda G \tag{14}$$

(see Goursat [5]).

(iii) $[4T]_{[1]}(R, D)$

$$ds^2 = \frac{-4U'V'}{\lambda(U+V)^2}\, du\, dv - d\omega^2 . \tag{15}$$

(iv) $[2T - 2S]_{[1-1]}(R, D)$

$$ds^2 = 2G(u, v)\, du\, dv - d\omega^2 \tag{16}$$

where

(a) if $\lambda = 0$, $G(u, v)$ is not the product of a function of u only and a function of v only:

(b) if $\lambda \neq 0$, $G(u, v)$ is not a solution of Liouville's Eq. (14).

The metrics of the form (11) are

(v) $[4T]_{[1]}(0, 0)$, flat space,

$$ds^2 = 2V\, du\, dv - u^2(dx^2 + dy^2)\,. \tag{17}$$

(vi) $[2T-2S]_{[1]}(R, D)$

$$ds^2 = 2V\, du\, dv - u^2\, d\omega^2 \tag{18}$$

with $\lambda \neq 0$.

(vii) $[4N]_{[2]}(0, 0)$

$$ds^2 = 2UV\, du\, dv - u^2(dx^2 + dy^2) \tag{19}$$

with $U' \neq 0$.

(viii) $[4N]_{[2]}(R, D)$

$$ds^2 = \frac{-4U'V'u^2}{\lambda(U+V)^2}\, du\, dv - u^2\, d\omega^2\,. \tag{20}$$

(ix) $[2N-2S]_{[2-1]}(0, 0)$

$$ds^2 = \frac{4U'V'u^2}{\lambda(U+V)^2}\, du\, dv - u^2\, d\omega^2\,. \tag{21}$$

For metrics (viii) and (ix) we also have

$$\{U''V'(U+V) - U'^2V'\}\, u + 2U' \neq 0\,. \tag{22}$$

(x) $[2N-2S]_{[2-1]}(R, D)$

$$ds^2 = 2G(u, v)\, du\, dv - u^2\, d\omega^2 \tag{23}$$

where G does not take the forms required for (vii)–(ix). Discussion of simplification of some of these metrics is to be found in Appendix B.

Appendix A

The metric (2) can be written in terms of a pseudoorthonormal tetrad (see Newman and Penrose [6]) as

$$\begin{aligned} l^\mu &= \left(0, \frac{1}{G}, 0, 0\right) \\ n^\mu &= (1, 0, 0, 0) \\ m^\mu &= -\frac{1}{M\sqrt{2}}\left(0, 0, 1, \frac{i}{\Sigma}\right). \end{aligned} \tag{A 1}$$

For this tetrad we have the non-zero Newman-Penrose quantities

$$\Psi_2 = \frac{M_{,uv}}{MG} - 2\Lambda\,, \tag{A 2}$$

$$\Lambda = \frac{1}{12}\left(\frac{G_{,uv}}{G^2} - \frac{G_{,u}G_{,v}}{G^3} - \frac{\lambda}{M^2}\right) + \frac{1}{3}\left(\frac{M_{,uv}}{MG} + \frac{M_{,u}M_{,v}}{2GM^2}\right), \tag{A 3}$$

$$\Phi_{00} = \frac{M_{,v}G_{,v}}{MG^3} - \frac{M_{,vv}}{MG^2}\,, \tag{A 4}$$

$$\Phi_{22} = \frac{M_{,u}G_{,u}}{MG} - \frac{M_{,uu}}{M}\,, \tag{A 5}$$

$$\Phi_{11} = \frac{1}{4}\left(\frac{G_{,u}G_{,v}}{G^3} - \frac{G_{,uv}}{G^2} - \frac{\lambda}{M^2}\right) + \frac{M_{,u}M_{,v}}{2GM^2} \tag{A 6}$$

where Ψ_2 is related to the Weyl tensor, $\Lambda = R/24$, Φ_{AB} are related to the Ricci tensor and λ is defined by (9) in this paper.

From a comparison of Ludwig and Scanlan [7] and Ref. [1] it follows that the metric has two double eigenvalues if

$$\Phi_{00} = 0$$

(or $\Phi_{22} = 0$) which leads to Eq. (3).

Then the metric is of type

$[2N-2S]_{[2-1]}$	when	Φ_{11}, Φ_{22}	are non-zero
$[2T-2S]_{[1-1]}$	when	Φ_{11}	is non-zero
$[4N]_{[2]}$	when	Φ_{22}	is non-zero
$[4T]_{[1]}$	when	all Φ_{AB}	are zero .

and (0, 0) implies $\Psi_2 = 0$ and $\Lambda = 0$.

The above classification applied to metrics (10) and (11) gives the metrics (i)–(x).

Appendix B

In metrics (v), (vi) and (vii) V can be made unity by a coordinate transformation. The transformation

$$W = v + \tfrac{1}{2}x^2 u + \tfrac{1}{2}y^2 u\,, \qquad U = u\,, \qquad X = xu\,, \qquad Y = yu \tag{B 1}$$

takes metric (v) into the Minkowski metric

$$ds^2 = 2\,dU\,dW - dX^2 - dY^2\,. \tag{B 2}$$

Metric (vii) is a purely non-gravitational plane wave metric. Penrose [8] has discussed such a metric in the case of a purely electromagnetic wave but it should be noted that the wave does not have to be a null electromagnetic one but may be a wave of some other energy form such as a scalar wave (see Lun and McIntosh [9]).

Metrics (ii), (iii), (viii) and (ix) can also be simplified. For example (iii) is immediately

$$ds^2 = \frac{4\,dU\,dV}{\lambda(U+V)^2} - d\omega^2 .$$

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