A CLASS OF SOLUTIONS TO THE 3D CUBIC NONLINEAR SCHRÖDINGER EQUATION THAT BLOW-UP ON A CIRCLE

JUSTIN HOLMER AND SVETLANA ROUDENKO

Abstract. We consider the 3d cubic focusing nonlinear Schrödinger equation (NLS) $i\partial_t u + \Delta u + |u|^2 u = 0$, which appears as a model in condensed matter theory and plasma physics. We construct a family of axially symmetric solutions, corresponding to an open set in $H^1_{\text{axial}}(\mathbb{R}^3)$ of initial data, that blow-up in finite time with singular set a circle in xy plane. Our construction is modeled on Raphaël's construction [33] of a family of solutions to the 2d quintic focusing NLS, $i\partial_t u + \Delta u + |u|^4 u = 0$, that blow-up on a circle.

1. Introduction

Consider, in dimension $n \geq 1$, the p-power focusing nonlinear Schrödinger

$$i\partial_t u + \Delta u + |u|^{p-1} u = 0\,. \tag{1.1}$$

This equation obeys the scaling symmetry

$$u(x,t) \text{ solves } (1.1) \implies \lambda^{\frac{2}{p-1}} u(\lambda x, \lambda^2 t) \text{ solves } (1.1)\,,$$

which implies that the homogeneous Sobolev norm $\dot{H}^s$ is scale invariant provided $s = \frac{n}{2} - \frac{2}{p-1}$. The equation (1.1) also obeys mass, energy, and momentum conservation, which are respectively defined as

$$M[u] = \|u\|_{L^2}^2\,, \quad E[u] = \frac{1}{2}\|\nabla u\|_{L^2}^2 - \frac{1}{p+1}\|u\|_{L^{p+1}}^{p+1}\,, \qquad P[u] = \operatorname{Im}\int \nabla u\, \bar{u}\, dx\,.$$

In the $\dot{H}^1$ subcritical setting ($1 < p < 1 + \frac{4}{n-2}$), there exist soliton solutions $u(x,t) = e^{it}Q(x)$, where

$$-Q + \Delta Q + |Q|^{p-1} Q = 0\,. \tag{1.2}$$

We take Q to be the unique, radial, positive smooth solution (in $\mathbb{R}^n$) of this nonlinear elliptic equation of minimal mass. For further properties see, for example, [14]-[15] and references therein.

The local theory in H^1 ($p \leq 1 + \frac{4}{n-2}$) is known from work of Ginibre-Velo [10]. Local existence in time extends to the maximal interval $(-T_*, T^*)$, and if T^* or T_* are finite, it is said that the corresponding solution blows up in finite time. The existence of blow up solutions are known and the history goes back to work of Vlasov-Petrishev-Talanov '71 [40], Zakharov '72 [42] and Glassey '77 [12] who showed that negative energy solutions with finite variance, $\|xu_0\|_{L^2} < \infty$, blow up in finite time. Ogawa-Tsutsumi [31] extended this result to radial solutions by localizing the variance. Martel [22] showed that further relaxation to the nonisotropic finite variance or radiality only in some of the variables guarantees that negative energy solutions blow up in finite time. First result for nonradial infinite variance (negative energy) solutions blowing up in finite time was by Glangetas - Merle [11] using a concentration-compactness method (see also Nawa [30] for a similar result). Positive energy blow up solutions are also known and go back to [40], [42] (see [38, Theorem 5.1] for the precise statement). Turitsyn [39] and Kuznetsov et al [20] extended the blow up criteria for finite variance solutions of $\dot{H}^1$ subcritical NLS and, in particular, for the $3d$ cubic NLS showed that finite variance solutions blow up in finite time provided they are under the 'mass-energy' threshold $M[u_0]E[u_0] < c(\|Q\|_{L^2})$, where Q is the ground state solution of (1.2), and further assumption on the initial size of the gradient $\|\nabla u_0\|_{L^2} > c(M[u_0], \|Q\|_{L^2})$. Independently, the same conditions for finite variance as well as for radial data (for all $\dot{H}^1$ subcritical, L^2 supercritical NLS) were obtained in [14]. [Similar sufficient blow up conditions for finite variance or radial data for the $\dot{H}^1$ critical NLS ($s = 1$ or $p = 1 + \frac{4}{n-2}$) are due to Kenig - Merle [18]; for the situation in the L^2 critical NLS ($s = 0$ or $p = 1 + \frac{4}{n}$) with H^1 data refer to Weinstein [41] for the sharp threshold and Merle [22] for the characterization of minimal mass blow up solutions.] The nonradial infinite variance solutions of the 3d cubic NLS blow up in finite or infinite time provided they are under the 'mass-energy' threshold having the same condition on the size of the gradient as discussed above. This was shown using a variance of the rigidity /concentration-compactness method in [16], thus, extending the result of Glangetas-Merle to positive energy solutions. Further extensions of sufficient conditions for finite time blow up were done by Lushnikov [21] and Holmer-Platte-Roudenko [13] which include blow up solutions above the 'mass-energy' threshold and are given via variance (or its localized version) and its first derivative if the data is not real. It is also possible to construct solutions which blow up in one time direction and scatter or approach the soliton solution (up to the symmetries of the equation) in the other time direction, see [13], [5], [6].

A detailed description of the dynamics of blow-up solutions in the L^2 critical ($p = 1 + \frac{4}{n}$) case has been developed by Merle-Raphael [25, 28, 24, 27, 26, 32]. They show that blow-up solutions are, to leading order in H^1, described by the profile $Q_{b(t)}$, rescaled at rate $\lambda(t) \sim ((T-t)/\log|\log(T-t)|)^{1/2}$, where $b(t) \sim 1/\log|\log(T-t)|$. Here, Q_b is a slight modification of Q – see notational item N5 in §3 for the details of the definition in the 2d cubic case.

In the L^2 supercritical, $\dot{H}^1$ subcritical regime ($1 + \frac{4}{n} < p < 1 + \frac{4}{n-2}$), one has a large family of blow-up solutions as discussed above, but there are fewer results characterizing the dynamical behavior of blow-up solutions – we are only aware of three: Raphaël [33] (quintic NLS in 2d) and the extension by Raphaël-Szeftel [34] (quintic NLS in all dimensions) and Merle-Raphaël-Szeftel [29] (slighly mass-supercritical

NLS). In this paper[1], we consider the 3d cubic equation ($n = 3$, $p = 3$) which is a physically relevant case in condensed matter and plasma physics. We adapt the method introduced by Raphaël [33] to give a construction of a family of finite-time blow-up solutions that blow-up on a circle in the xy plane. Raphaël constructed a family of finite-time blow-up solutions to the 2d quintic equation ($n = 2$, $p = 5$) which is $\dot{H}^{1/2}$ critical, that blow-up on a circle. He accomplished this by introducing a radial symmetry assumption, and at the focal point of blow-up (without loss $r = 1$), the equation effectively reduces to the 1d quintic NLS, which is L^2 critical and for which there is a well-developed theory characterizing the dynamics of blow-up. We employ a similar dimensional reduction scheme – starting from the 3d cubic NLS (which is $\dot{H}^{1/2}$ critical) we impose an axial symmetry assumption, and construct blow-up solutions with a focal point at $(r, z) = (1, 0)$, where the equation in (r, z) coordinates effectively reduces to the 2d cubic NLS equation, which is L^2 critical.

Our main result is

Theorem 1.1. *Let $Q_b = Q_b(\tilde{r}, \tilde{z})$ be as defined in N5 in §3. There exists an open set $\mathcal{P}$ in $H^1_{\text{axial}}(\mathbb{R}^3)$, defined precisely in §4, such that the following holds true. Let $u_0 \in \mathcal{P}$. Then the corresponding solution $u(t)$ to 3d cubic NLS ((1.1) with $n = 3$, $p = 3$) blows-up in finite time $0 < T < \infty$ according to the following dynamics.*

(1) Description of the singularity formation. *There exists $\lambda(t) > 0$, $r(t) > 0$, $z(t)$, and $\gamma(t) \in \mathbb{R}$ such that, if we define*

$$u_{\text{core}}(r, z, t) = \frac{1}{\lambda(t)} Q_b\left(\frac{r - r(t)}{\lambda(t)}, \frac{z - z(t)}{\lambda(t)}\right) e^{i\gamma(t)} ,$$

$$\tilde{u}(t) = u(t) - u_{\text{core}}(t) ,$$

then

$$\tilde{u}(t) \to u^* \text{ in } L^2(\mathbb{R}^3) \text{ as } t \to T , \tag{1.3}$$

$$\|\nabla \tilde{u}(t)\|_{L^2(\mathbb{R}^3)} \lesssim \|\nabla u_{\text{core}}(t)\|_{L^2(\mathbb{R}^3)} \cdot \frac{1}{|\log(T - t)|^c} , \tag{1.4}$$

and the position of the singular circle converges:

$$r(t) \to r(T) > 0 \text{ as } t \to T, \tag{1.5}$$

$$z(t) \to z(T) \text{ as } t \to T . \tag{1.6}$$

[1] A result similar to the one presented in this paper was simultaneously developed by Zwiers [43], see remarks at the end of the introduction.

(2) Estimate on the blow-up speed. *We have, as* $t \nearrow T$,

$$\lambda(t) \sim \left(\frac{T-t}{\log|\log(T-t)|} \right)^{1/2}, \tag{1.7}$$

$$b(t) \sim \frac{1}{\log|\log(T-t)|}, \tag{1.8}$$

$$\gamma(t) \sim |\log(T-t)| \log|\log(T-t)|. \tag{1.9}$$

(3) Structure of the L^2 remainder u^*. *For all* $R > 0$ *small enough,*

$$\int_{|r-r(T)|^2+|z-z(T)|^2 \le R^2} |u^*(r)|^2 \, dr \, dz \sim \frac{1}{(\log|\log R|)^2}, \tag{1.10}$$

and, in particular, $u^* \notin L^p$ *for* $p > 2$.

(4) $H^{1/2}$ gain of regularity outside the singular circle. *For any* $R > 0$,

$$u^* \in H^{1/2}(|r-r(T)|^2 + |z-z(T)|^2 > R^2). \tag{1.11}$$

A key ingredient in exploiting the cylindrical geometry away from the z-axis is the *axially symmetric Gagliardo-Nirenberg* inequality, which we prove in §2. This takes the role of the radial Gagliardo-Nirenberg inequality of Strauss [36] employed by Raphaël. In §3, we collect most of the notation employed, and in §5, we outline the structure of the proof of Theorem 1.1. Most of the argument is a lengthy bootstrap, and in §5, we enumerate the bootstrap input statements (BSI 1–8) and the corresponding bootstrap output statements (BSO 1–8). As the output statements are stronger than the input statements, we conclude that all the BSO assertions hold for the full time interval of existence. The steps involved in deducing BSO 1–8 under the assumptions BSI 1–8 are outlined in §5, and carried out in detail in §7–21. §21 already proves (1.11) in Theorem 1.1, the $H^{1/2}$ gain of regularity outside the singular circle. The proof of Theorem 1.1 is completed in §22–24. In §22, we prove the log-log rate of blow-up (1.7). In §23, we prove the convergence of the position of the singular circle, (1.5) and (1.6). The proof of the size estimates on the remainder profile, (1.10) is the same as for Theorem 3 in [26] (we refer the reader to Sections 3, 4, 5 there). Finally, in §24, we prove the convergence of the remainder in L^2, the estimate (1.3).

A similar result to Theorem 1.1 but for a slightly smaller class of initial data was recently obtained by Zwiers [43] (our H^1_{axial} class of initial data is replaced by a smoother version H^3_{axial}). We mention that the methods to prove our main theorem do not treat the cubic equation in higher dimensions, but it is addressed by Zwiers in [43, Theorem 1.3]. We think that it should be possible to adapt our method to treat the $\dot{H}^{1/2}$ critical ($p = 1 + 4/(n-1)$) case in higher dimensions, proving the existence of solutions blowing up on a circle (dimension 1 blow-up set), however, dealing with fractional nonlinearities is a delicate matter. Zwiers' result treats blow up sets of codimension 2 in all dimensions; our approach might be able to treat the blow up sets of dimension 1 in all dimensions. (In 3 dimensions, of course, codimension 2 equals dimension 1, and thus, the results intersect.)

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2. A Gagliardo-Nirenberg inequality for axially symmetric functions

We begin with an axially symmetric Gagliardo-Nirenberg estimate, analogous to the radial Gagliardo-Nirenberg estimate of Strauss [36]. Consider a function $f = f(x,y,z) = f(r\cos\theta, r\sin\theta, z) = f(r,z)$, independent of θ. We call such a function *axially symmetric.*

Lemma 2.1. *Suppose that f is axially symmetric. Then for each $\epsilon > 0$,*

$$\|f\|_{L^4(r>\epsilon)}^4 \le \frac{1}{\epsilon}\|f\|_{L^2_{xyz}(r>\epsilon)}^2 \|\nabla f\|_{L^2_{xyz}(r>\epsilon)}^2. \tag{2.1}$$

Proof. The proof is modeled on the classical proof of the Sobolev estimates. We use the notation $\nabla = (\partial_x, \partial_y, \partial_z)$ (i.e., not (∂_r, ∂_z)). Note that $\|f(x,y,z)\|_{L^2_{xyz}} = \|f(r,z)\|_{L^2_{rdrdz}}$, and also note $\partial_r f(r,z) = (\partial_x f)(r,z)\cos\theta + (\partial_y f)(r,z)\sin\theta$, and thus, $|\partial_r f| \le |\nabla f|$. Observe that for a fixed $r > \epsilon$, $z \in \mathbb{R}$, by the fundamental theorem of calculus and the Cauchy-Schwarz inequality,

$$\begin{aligned} |f(r,z)|^2 &= |f(+\infty,z)|^2 - |f(r,z)|^2 \\ &\le 2\int_{r'=\epsilon}^{+\infty} |f(r',z)||\partial_r f(r',z)|\, dr \\ &\le \frac{2}{\epsilon}\int_{r'=\epsilon}^{+\infty} |f(r',z)||\partial_r f(r',z)|\, r'dr' \\ &\le \frac{2}{\epsilon}\|f(r',z)\|_{L^2_{r'dr'}(r'>\epsilon)}\|\nabla f(r',z)\|_{L^2_{r'dr'}(r'>\epsilon)}\,, \end{aligned}$$

and also,

$$|f(r,z)|^2 \le 2\int_{z'=-\infty}^{+\infty} |f(r,z')||\partial_z f(r,z')|\, dz' \le \|f(r,z')\|_{L^2_{z'}}\|\nabla f(r,z')\|_{L^2_{z'}}\,.$$

By multiplying the above two inequalities, and then integrating against $rdrdz$, we get

$$\begin{aligned} \int_{r>\epsilon,z} |f(r,z)|^4 r\, dr\, dz \le \frac{4}{\epsilon}&\left(\int_z \|f(r',z)\|_{L^2_{r'dr'}(r'>\epsilon)}\|\nabla f(r',z)\|_{L^2_{r'dr'}(r'>\epsilon)}\, dz\right) \\ &\times\left(\int_{r=\epsilon}^{+\infty} \|f(r,z')\|_{L^2_{z'}}\|\nabla f(r,z')\|_{L^2_{z'}} r\, dr\right). \end{aligned}$$

Following through with Cauchy-Schwarz in each of the two integrals gives (2.1). □

As a corollary of (2.1), we have for $1 \le p \le 3$, that

$$\|f\|^{p+1}_{L^{p+1}_{xyz}(r>\epsilon)} \le \frac{2^{p-1}}{\epsilon^{\frac{p-1}{2}}} \|f\|^2_{L^2_{xyz}(r>\epsilon)} \|\nabla f\|^{p-1}_{L^2_{xyz}(r>\epsilon)} \,. \tag{2.2}$$

This follows by the interpolation estimate (Hölder's inequality)

$$\|f\|^{p+1}_{L^{p+1}} \le \|f\|^{3-p}_{L^2} \|f\|^{2p-2}_{L^4} \,.$$

Before proceeding, we present a simple application of Lemma 2.1.

Corollary 2.2. *If u is a cylindrically symmetric solution to*

$$i\partial_t u + \Delta u + |u|^{p-1} u = 0$$

for $p < 3$ in $\mathbb{R}^3$ that blows-up at finite time $T > 0$, then blow-up must occur along the z-axis. Specifically for any fixed $\epsilon > 0$,

$$\lim_{t \uparrow T} \|\nabla u\|_{L^2_{xyz}(r<\epsilon)} = +\infty \,. \tag{2.3}$$

Proof. Fix any $\epsilon > 0$. All L^p norms will be with respect to $dxdydz$. For any $t > 0$, by (2.2), we have for $u = u(t)$

$$\begin{aligned} \frac{1}{2}\|\nabla u\|^2_{L^2} &= E + \frac{1}{p+1}\|u\|^{p+1}_{L^{p+1}(r<\epsilon)} + \frac{1}{p+1}\|u\|^{p+1}_{L^{p+1}(r>\epsilon)} \\ &\le E + \frac{1}{p+1}\|u\|^{p+1}_{L^{p+1}(r<\epsilon)} + \frac{C}{\epsilon^{\frac{p-1}{2}}}\|u\|^2_{L^2}\|\nabla u\|^{p-1}_{L^2}. \end{aligned}$$

Using the inequality $\alpha\beta \le \frac{3-p}{2}\alpha^{\frac{2}{3-p}} + \frac{p-1}{2}\beta^{\frac{2}{p-1}}$, we obtain

$$\|\nabla u\|^2_{L^2} \le 4E + \frac{4}{p+1}\|u\|^{p+1}_{L^{p+1}(r<\epsilon)} + \frac{C}{\epsilon^{\frac{p-1}{3-p}}}\|u\|^{\frac{4}{3-p}}_{L^2} \,.$$

Since $\lim_{t\nearrow+\infty} \|\nabla u(t)\|_{L^2} = +\infty$, we obtain that $\lim_{t\nearrow+\infty} \|u(t)\|_{L^{p+1}(r>\epsilon)} = +\infty$. By the (standard) Gagliardo-Nirenberg inequality, we obtain (2.3). □

3. Notation

Recall that we will impose the *axial symmetry* assumption, i.e.,

$$\tilde{u}(r,\theta,z) = u(r\cos\theta, r\sin\theta, z)$$

is assumed independent of θ. The equation (1.1) in cylindrical coordinates (r, z), assuming the axial symmetry, is

$$i\partial_t u + \frac{1}{r}\partial_r u + \partial_r^2 u + \partial_z^2 u + |u|^2 u = 0. \tag{3.1}$$

Denote by $Q = Q(r)$, $r = |x|$, $x \in \mathbb{R}^2$, a ground state solution to the 2d nonlinear elliptic equation (which corresponds to the mass-critical cubic NLS equation in 2d):

$$-Q + \Delta_{\mathbb{R}^2} Q + |Q|^2 Q = 0.$$

We emphasize that Q is a two-dimensional object.

We now enumerate our notational conventions.

N 1. We will adopt the convention that $\nabla = (\partial_x, \partial_y, \partial_z)$ is the full gradient and $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$ is the full Laplacian. When viewing an axially symmetry function as a function of r and z, we will write the corresponding operators as $\nabla_{(r,z)} = (\partial_r, \partial_z)$ and $\Delta_{(r,z)} = \partial_r^2 + \partial_z^2$. Note that under the axial symmetry assumption, $\Delta = \Delta_{(r,z)} + r^{-1}\partial_r$.

N 2. In the argument, parameters $\lambda(t)$, $\gamma(t)$, $r(t)$ and $z(t)$ emerge. Rescaled time, for given b_0, is

$$s(t) = \int_0^t \frac{dt'}{\lambda^2(t')} + s_0, \quad s_0 = e^{\frac{3\pi}{4b_0}}, \tag{3.2}$$

and rescaled position is

$$\tilde{r}(t) = \frac{r - r(t)}{\lambda(t)}, \qquad \tilde{z}(t) = \frac{z - z(t)}{\lambda(t)}.$$

We note that the $\tilde{r}$ and $\tilde{z}$ can both be negative, although there is the restriction that $\lambda(t)\tilde{r} + r(t) \geq 0$. Introduce the full (rescaled) radial variable

$$\tilde{R} = \sqrt{\tilde{r}^2 + \tilde{z}^2}.$$

We have $r = \lambda(t)\tilde{r} + r(t)$. To help avoid confusion between r and $\tilde{r}$, we will use the following notation: given two parameters $\lambda(t)$ and $r(t)$, define

$$\mu_{\lambda(t),r(t)}(\tilde{r}) = (\lambda(t)\tilde{r} + r(t))\mathbf{1}_{\{\lambda(t)\tilde{r}+r(t)\geq 0\}}.$$

Often the $\lambda(t), r(t)$ subscript will be dropped.

N 3. The inner product $(\cdot,\cdot)$ will mean 2d real inner product in the $\tilde{r}$, $\tilde{z}$ variables.

N 4. If $f = f(\tilde{r}, \tilde{z})$, then define $\Lambda f = f + (\tilde{r}, \tilde{z}) \cdot (\partial_{\tilde{r}}, \partial_{\tilde{z}})f$, the *generator for scaling*. Observe that $\Lambda f = f + \tilde{R}\partial_{\tilde{R}} f$.

N 5. For a parameter b, $Q_b(\tilde{r}, \tilde{z})$ is the *2d localized self-similar profile*. Specifically, following Prop. 8 in [27] which is a refinement of Prop. 1 in [28], given b, $\eta > 0$, define

$$\tilde{R}_b = \frac{2}{|b|}\sqrt{1-\eta}\,.$$

There exist universal constants $C > 0$, $\eta^* > 0$ such that the following holds true. For all $0 < \eta < \eta^*$, there exists constants $\epsilon^*(\eta) > 0$, $b^*(\eta) > 0$ going to zero as $\eta \to 0$ such that for all $|b| < b^*(\eta)$, there exists a unique *radial* solution Q_b (i.e., Q_b depends only on $\tilde{R}$) to

$$\begin{cases} \Delta_{(\tilde{r},\tilde{z})} Q_b - Q_b + ib\Lambda Q_b + Q_b|Q_b|^2 = 0 \\ Q_b(\tilde{R})e^{\frac{ib\tilde{R}^2}{4}} > 0 \text{ for } \tilde{R} \in [0, \tilde{R}_b) \\ Q_b(\tilde{R}_b) = 0 \\ Q(0) - \epsilon^*(\eta) < Q_b(0) < Q(0) + \epsilon^*(\eta). \end{cases}$$

N 6. For a parameter b, $\tilde{Q}_b(\tilde{r},\tilde{z}) = \tilde{Q}_b(\tilde{R})$ is a *truncation* of the 2d localized self-similar profile Q_b. Specifically, (following Prop. 8 in [27]), given $b>0$, $\eta>0$ small, let $\tilde{R}_b^- = \sqrt{1-\eta}\tilde{R}_b$ so that $\tilde{R}_b^- < \tilde{R}_b$. Let ϕ_b be a radial smooth cut-off function such that $\phi_b(x)=0$ for $|x|\geq \tilde{R}_b$ and $\phi_b(x)=1$ for $|x|\leq \tilde{R}_b^-$, and everywhere $0\leq \phi(x)\leq 1$, such that

$$\|\phi_b'\|_{L^\infty} + \|\Delta\phi_b\|_{L^\infty} \to 0 \text{ as } |b|\to 0\,.$$

Now set

$$\tilde{Q}_b(\tilde{R}) = Q_b(\tilde{R})\phi_b(\tilde{R})\,.$$

Then

$$\Delta_{(\tilde{r},\tilde{z})}\tilde{Q}_b - \tilde{Q}_b + ib\Lambda\tilde{Q}_b + \tilde{Q}_b|\tilde{Q}_b|^2 = -\Psi_b, \tag{3.3}$$

where

$$-\Psi_b = \Delta\varphi_b\, Q_b + 2\nabla Q_b\cdot\nabla\varphi_b + ibQ_b\,(\tilde{r},\tilde{z})\cdot\nabla\varphi_b + (\varphi_b^3-\varphi_b)Q_b|Q_b|^2 \tag{3.4}$$

with the property that for any polynomial $P(y)$ and $k=0,1$

$$\|P(y)\,\Psi_b^{(k)}\|_{L^\infty} \leq e^{-C_P/|b|}. \tag{3.5}$$

In terms of $\Psi_b(\tilde{r},\tilde{z})$, we define an adjusted $\tilde{\Psi}_b(t,\tilde{r},\tilde{z})$ as

$$\tilde{\Psi}_b(t,\tilde{r},\tilde{z}) = \Psi_b(t,\tilde{r},\tilde{z}) - \frac{\lambda(t)}{\mu_{\lambda(t),r(t)}(\tilde{r})}\partial_{\tilde{r}}\tilde{Q}_b(\tilde{r},\tilde{z}) \tag{3.6}$$

so that $\tilde{Q}_b$ solves

$$\Delta_{(\tilde{r},\tilde{z})}\tilde{Q}_b + \frac{\lambda}{\mu}\partial_{\tilde{r}}\tilde{Q}_b - \tilde{Q}_b + ib\Lambda\tilde{Q}_b + \tilde{Q}_b|\tilde{Q}_b|^2 = -\tilde{\Psi}_b. \tag{3.7}$$

We also split $\tilde{Q}_b$ into real and imaginary parts as

$$\tilde{Q}_b = \Sigma + i\Theta.$$

It is implicitly understood that Σ and Θ depend on b (or $b(t)$), and when we want to emphasize dependence, it will be stated explicitly; this decomposition is done only for the truncated profile $\tilde{Q}_b$ (not for Q_b). Similarly, we denote $\Psi_b = \operatorname{Re}\Psi + i\operatorname{Im}\Psi$ and $\tilde{\Psi}_b = \operatorname{Re}\tilde{\Psi} + i\operatorname{Im}\tilde{\Psi}$.

N 7. The $\tilde{Q}_b$ satisfies the following properties.

QP 1. ((44) in [28]) Uniform closeness to the ground state. For a fixed universal constant $C>0$,

$$\|e^{C\tilde{R}}(\tilde{Q}_b - Q)\|_{C^3} \to 0 \text{ as } b\to 0\,.$$

In particular, this implies that $\|e^{c\tilde{R}}\partial_{\tilde{r}}^k\Theta\|_{L^2(\tilde{r},\tilde{z})}\to 0$ as $b\to 0$ for $0\leq k\leq 3$.

QP 2. ((45) in [28]) Uniform closeness of the derivative $\partial_b Q_b$ to the ground state. For a fixed universal constant $C>0$,

$$\left\|e^{C\tilde{R}}(\partial_b\tilde{Q}_b + i\tfrac{1}{4}\tilde{R}^2 Q)\right\|_{C^2} \to 0 \text{ as } b\to 0\,.$$

QP 3. (Prop. 2(ii) in [28] and Prop. 1(iii) in [24]) Degeneracy of the energy and momentum. Specifically,

$$|E(\tilde{Q}_b)| \le e^{-(1+C\eta)\pi/|b|}, \quad \text{since} \quad 2E(\tilde{Q}_b) = -\operatorname{Re}\int \Lambda\Psi_b\,\bar{\tilde{Q}}_b, \tag{3.8}$$

and

$$\operatorname{Im}\int \nabla_{(\tilde{r},\tilde{z})}\tilde{Q}_b\,\bar{\tilde{Q}}_b = 0, \qquad \operatorname{Im}\int (\tilde{r},\tilde{z})\cdot\nabla_{(\tilde{r},\tilde{z})}\tilde{Q}_b\,\bar{\tilde{Q}}_b = -\frac{b}{2}\|\tilde{R}\,\tilde{Q}_b\|_{L^2}^2. \tag{3.9}$$

QP 4. The profile $\tilde{Q}_b$ has supercritical mass, and more precisely

$$0 < \frac{d}{d(b^2)}\Big|_{b^2=0}\int |\tilde{Q}_b|^2 = d_0 < +\infty.$$

QP 5. Algebraic relations corresponding to Galilean, conformal and scaling invariances:

$$\Delta(\tilde{R}\tilde{Q}_b) - \tilde{R}\tilde{Q}_b + ib\tilde{R}\Lambda\tilde{Q}_b + \tilde{R}\tilde{Q}_b|\tilde{Q}_b|^2 = 2\partial_{\tilde{R}}\tilde{Q}_b - \tilde{R}\Psi_b \tag{3.10}$$

$$\Delta(\tilde{R}^2\tilde{Q}_b) - \tilde{R}^2\tilde{Q}_b + ib\tilde{R}^2\Lambda\tilde{Q}_b + \tilde{R}^2\tilde{Q}_b|\tilde{Q}_b|^2 = 4\Lambda\tilde{Q}_b - \tilde{R}^2\Psi_b \tag{3.11}$$

$$\begin{aligned}\Delta(\Lambda\tilde{Q}_b) - \Lambda\tilde{Q}_b + (\Lambda\tilde{Q}_b)|\tilde{Q}_b|^2 + 2\tilde{Q}_b(\Sigma(\Lambda\Sigma) + \Theta(\Lambda\Theta))\\ = 2(\tilde{Q}_b - ib\Lambda\tilde{Q}_b - \Psi_b) - \Lambda\Psi_b - ib\Lambda^2\tilde{Q}_b.\end{aligned} \tag{3.12}$$

The proof of the above identities are similar to Prop. 2(iii) in [28] adapted to the 2d case. For example, to obtain the third equation from scaling invariance, multiply (3.7) by λ^3 and take argument to be $(\lambda\tilde{r},\lambda\tilde{z})$, then differentiate with respect to λ and evaluate the derivative at $\lambda=1$. Note that $\frac{d}{d\lambda}|_{\lambda=1}\,(\lambda^3\Psi_b(\lambda\tilde{r},\lambda\tilde{z})) = 3\Psi_b + (\tilde{r},\tilde{z})\cdot\nabla\Psi_b \equiv 2\Psi_b + \Lambda\Psi_b$, and thus, we obtain the claimed equation.

N 8. The linear operator close to $\tilde{Q}_b$ in (3.7) is $M = (M_+, M_-)$, where $(\epsilon = \epsilon_1 + i\epsilon_2)$

$$M_+(\epsilon) = -\Delta_{(\tilde{r},\tilde{z})}\epsilon_1 - \frac{\lambda}{\mu}\partial_{\tilde{r}}\epsilon_1 + \epsilon_1 - \left(\frac{2\Sigma^2}{|\tilde{Q}_b|^2} + 1\right)|\tilde{Q}_b|^2\epsilon_1 - 2\Sigma\Theta\epsilon_2,$$

$$M_-(\epsilon) = -\Delta_{(\tilde{r},\tilde{z})}\epsilon_2 - \frac{\lambda}{\mu}\partial_{\tilde{r}}\epsilon_2 + \epsilon_2 - \left(\frac{2\Theta^2}{|\tilde{Q}_b|^2} + 1\right)|\tilde{Q}_b|^2\epsilon_2 - 2\Sigma\Theta\epsilon_1.$$

N 9. For a given parameter b, the function $\zeta_b(\tilde{r},\tilde{z})$ is the *2d linear outgoing radiation.* Specifically, following Lemma 15 in [27], there exist universal constants $C > 0$ and $\eta^* > 0$ such that for all $0 < \eta < \eta^*$, there exists $b^*(\eta) > 0$ such that for all $0 < b < b^*(\eta)$, the following holds true: There exists a unique radial solution ζ_b to

$$\begin{cases} (\partial_{\tilde{r}}^2 + \partial_{\tilde{z}}^2)\zeta_b - \zeta_b + ib\Lambda\zeta_b = \Psi_b \\ \displaystyle\int |(\partial_{\tilde{r}},\partial_{\tilde{z}})\zeta_b|^2\,d\tilde{r}d\tilde{z} < +\infty, \end{cases}$$

where Ψ_b is the error in the $\tilde{Q}_b$ equation above. The number Γ_b is defined as the *radiative asymptotic*, i.e.,

$$\Gamma_b = \lim_{\tilde{R}\to+\infty} |\tilde{R}|^2 |\zeta_b(\tilde{R})|^2.$$

The $\zeta_b(\tilde{r},\tilde{z})$ and Γ_b have the following properties:

ZP 1. Control (and hence, smallness by ZP2) of ζ_b in $\dot{H}^1$

$$\int |\nabla_{(\tilde{r},\tilde{z})}\zeta_b|^2 \le \Gamma_b^{1-C\eta}.$$

ZP 2. Smallness of the radiative asymptotic

$$\forall\, |\tilde{R}| > R_b, \qquad e^{-(1+C\eta)\pi/b} \le \frac{4}{5}\Gamma_b \le |\tilde{R}|^2|\zeta_b(\tilde{R})|^2 \le e^{-(1-C\eta)\pi/b}.$$

N 10. We will make the *spectral assumption* made in [25, 28, 27, 24, 26]. We note that it involves Q and not $\tilde{Q}_b$. A numerically assisted proof[2] is given in Fibich-Merle-Raphaël [9]. Let (see **N** 4)

$$\Lambda Q = Q + (\tilde{r},\tilde{z})\cdot\nabla Q, \qquad \Lambda^2 Q = \Lambda Q + (\tilde{r},\tilde{z})\cdot\nabla(\Lambda Q).$$

Recall that $(\cdot,\cdot)$ denotes the 2d inner product in $(\tilde{r},\tilde{z})$. Consider the two Schrödinger operators

$$\mathcal{L}_1 = -\Delta + 3Q[(\tilde{r},\tilde{z})\cdot\nabla Q],$$
$$\mathcal{L}_2 = -\Delta + Q[(\tilde{r},\tilde{z})\cdot\nabla Q],$$

and the real-valued quadratic form for $\epsilon = \epsilon_1 + i\epsilon_2 \in H^1$:

$$H(\epsilon,\epsilon) = (\mathcal{L}_1\epsilon_1,\epsilon_1) + (\mathcal{L}_2\epsilon_2,\epsilon_2)\,.$$

Then there exists a universal constant $\tilde{\delta}_1 > 0$ such that $\forall\,\epsilon \in H^1$, if the following orthogonality conditions hold:

$$(\epsilon_1, Q) = 0, \quad (\epsilon_1,\Lambda Q) = 0, \quad (\epsilon_1, yQ) = 0,$$
$$(\epsilon_2,\Lambda Q) = 0, \quad (\epsilon_2,\Lambda^2 Q) = 0, \quad (\epsilon_2,\nabla Q) = 0,$$

then

$$H(\epsilon,\epsilon) \ge \tilde{\delta}_1\left(\int |\nabla\epsilon|^2 + \int |\epsilon|^2 e^{-|y|}\right).$$

N 11. The *ring cutoffs* are the following. The *tight external cutoff* is

$$\chi_0(r) = \begin{cases} 1 & \text{for } 0 \le r \le \frac{13}{14} \text{ and for } r \ge \frac{14}{13} \\ 0 & \text{for } \frac{15}{16} \le r \le \frac{16}{15}. \end{cases}$$

[2]The spectral property was proved in 1d and in dimensions 2-4 has a numerically assisted proof.

The *wide external cutoff* is

$$\chi_1(r) = \begin{cases} 1 & \text{for } 0 \le r \le \frac{1}{4} \text{ and for } r \ge 4 \\ 0 & \text{for } \frac{1}{2} \le r \le 2. \end{cases}$$

The *internal cutoff* is

$$\psi(r) = \begin{cases} 0 & \text{for } 0 \le r \le \frac{1}{4} \text{ and for } r \ge 3 \\ r & \text{for } \frac{1}{2} \le r \le 2. \end{cases}$$

4. Description of the set $\mathcal{P}$ of initial data

We now write down the assumptions on the initial data set $\mathcal{P}$. Let $\mathcal{P}$ be the set of axially symmetric $u_0 \in H^1(\mathbb{R}^3)$ of the form

$$u_0(r,z) = \frac{1}{\lambda_0}\tilde{Q}_{b_0}\left(\frac{r-r_0}{\lambda_0}, \frac{z-z_0}{\lambda_0}\right)e^{i\gamma_0} + \tilde{u}_0(r,z)\,,$$

and define the rescaled error

$$\epsilon_0(\tilde{r},\tilde{z}) = \lambda_0\tilde{u}_0(\lambda_0(\tilde{r},\tilde{z}) + (r_0,z_0))e^{-i\gamma_0}\,,$$

with the following controls:

IDA 1. Localization of the singular circle:

$$|r_0 - 1| < \alpha^*\,, \quad |z_0| < \alpha^*\,.$$

IDA 2. Smallness of b_0, or closeness of Q_{b_0} to Q on the singular circle:

$$0 < b_0 < \alpha^*\,.$$

IDA 3. Orthogonality conditions on ϵ_0:

$$(\operatorname{Re}\epsilon_0, |\tilde{R}|^2\Sigma_{b_0}) + (\operatorname{Im}\epsilon_0, |\tilde{R}|^2\Theta_{b_0}) = 0\,,$$

$$(\operatorname{Re}\epsilon_0, (\tilde{r},\tilde{z})\Sigma_{b_0}) + (\operatorname{Im}\epsilon_0, (\tilde{r},\tilde{z})\Theta_{b_0}) = 0\,,$$

$$-(\operatorname{Re}\epsilon_0, \Lambda^2\Theta_{b_0}) + (\operatorname{Im}\epsilon_0, \Lambda^2\Sigma_{b_0}) = 0\,,$$

$$-(\operatorname{Re}\epsilon_0, \Lambda\Theta_{b_0}) + (\operatorname{Im}\epsilon_0, \Lambda\Sigma_{b_0}) = 0\,.$$

IDA 4. Smallness condition on ϵ_0:

$$\mathcal{E}(0) \equiv \int |\nabla_{(\tilde{r},\tilde{z})}\epsilon_0|^2 \mu_{\lambda_0,r_0}(\tilde{r})\, d\tilde{r}d\tilde{z} + \int_{|(\tilde{r},\tilde{z})|\le 10/b_0} |\epsilon_0(\tilde{r},\tilde{z})|^2 e^{-|\tilde{R}|}\, d\tilde{r}d\tilde{z} \le \Gamma_{b_0}^{\frac{6}{7}}\,.$$

IDA 5. Normalization of the energy and localized momentum (recall ψ from **N 11**):

$$\lambda_0^2|E_0| + \lambda_0\left|\operatorname{Im}\int \nabla\psi\cdot\nabla u_0\,\bar{u}_0\right| \le \Gamma_{b_0}^{10}\,.$$

IDA 6. Log-log regime:

$$0 < \lambda_0 < \exp\left(-\exp\left(\frac{8\pi}{9b_0}\right)\right).$$

IDA 7. Global L^2 smallness:

$$\|\tilde{u}_0\|_{L^2} \leq \alpha^*.$$

IDA 8. $H^{1/2}$ smallness outside the singular circle:

$$\|\chi_0 \tilde{u}_0\|_{H^{1/2}} \leq \alpha^*.$$

IDA 9. Closeness to the 2d ground state mass:

$$\|u_0\|^2_{L^2(dxdydz)} \leq \|Q\|^2_{L^2(d\tilde{r}d\tilde{z})} + \alpha^*.$$

IDA 10. Negative energy:

$$E(u_0) < 0.$$

IDA 11. Axial symmetry of u_0.

Lemma 4.1. *The set $\mathcal{P}$ is nonempty.*

Proof. This follows as in Remark 3 of Raphaël [33]. □

5. Outline

Now let $u(t)$ be the solution to NLS with initial data from the above set $\mathcal{P}$, and let $T > 0$ be its maximal time of existence (which at this point could be $+\infty$). Because u_0 is axially symmetric and the Laplacian is rotationally invariant in the xy plane, the solution $u(t)$ will be axially symmetric. Thus, we occasionally write $u(r, z, t)$. The first step is to obtain a "geometrical description" of the solution. Since we are not truly in the L^2 critical setting and cannot appeal to the variational characterization of Q, we need to incorporate this geometrical description into the bootstrap argument. The lemma that we need, which follows from the implicit function theorem, is:

Lemma 5.1 (cf. Merle-Raphaël [28] Lemma 2, [25] Lemma 1)**.** *If for $0 \leq t \leq t_1$, there exist parameters $(\bar{\lambda}(t), \bar{\gamma}(t), \bar{r}(t), \bar{z}(t), \bar{b}(t))$ such that $\|\bar{\epsilon}(t)\|_{H^1} \leq \alpha_* \ll 1$ on $0 \leq t \leq t_1$, where*

$$\bar{\epsilon}(\tilde{r}, \tilde{z}) = e^{i\bar{\gamma}} \bar{\lambda} u(\bar{\lambda}\tilde{r} + \bar{r}, \bar{\lambda}\tilde{z} + \bar{z}) - \tilde{Q}_{\bar{b}}(\tilde{r}, \tilde{z}),$$

then there exist modified parameters $(\lambda(t), \gamma(t), r(t), z(t), b(t))$ such that ϵ defined by

$$\epsilon(\tilde{r}, \tilde{z}) = e^{i\gamma} \lambda u(\lambda\tilde{r} + r, \lambda\tilde{z} + z) - \tilde{Q}_b(\tilde{r}, \tilde{z})$$

satisfies the following orthogonality conditions (with $\epsilon = \epsilon_1 + i\epsilon_2$):

ORTH 1. $(\epsilon_1, |\tilde{R}|^2 \Sigma_{b(t)}) + (\epsilon_2, |\tilde{R}|^2 \Theta_{b(t)}) = 0$

ORTH 2. $(\epsilon_1, (\tilde{r}, \tilde{z})\Sigma_{b(t)}) + (\epsilon_2, (\tilde{r}, \tilde{z})\Theta_{b(t)}) = 0$

ORTH 3. $-(\epsilon_1, \Lambda^2\Theta_{b(t)}) + (\epsilon_2, \Lambda^2\Sigma_{b(t)}) = 0$

ORTH 4. $-(\epsilon_1, \Lambda\Theta_{b(t)}) + (\epsilon_2, \Lambda\Sigma_{b(t)}) = 0$ *and*

$$\left|1 - \lambda(t)\frac{\|\nabla u(t)\|_{L^2}}{\|\nabla Q\|_{L^2}}\right| + \|\epsilon(t)\|_{H^1} + |b(t)| \leq \delta(\alpha_0),$$

where $\delta(\alpha_0) \to 0$ *as* $\alpha_0 \to 0$.

Note that the condition **ORTH 2** is a vector equation.

We also use the notation

$$\tilde{u}(r, z, t) = \frac{1}{\lambda(t)}\,\epsilon\left(\frac{r - r(t)}{\lambda(t)}, \frac{z - z(t)}{\lambda(t)}, t\right) e^{i\gamma(t)}.$$

It is important that we consider, by default, u and $\tilde{u}$ as *3d objects* in the spatial variables. Thus, when we write $\|\tilde{u}\|_{L^2}$, we mean $\|\tilde{u}(x, y, z, t)\|_{L^2(dxdydz)} = \|\tilde{u}(r, z, t)\|_{L^2(rdrdz)}$. On the other hand, we consider, by default, $\tilde{Q}_{b(t)}$ and ϵ to be *2d objects* in the spatial variables, and thus, if we were to write $\|\tilde{Q}_{b(t)}\|_{L^2}$, we would just mean $\|\tilde{Q}_{b(t)}(\tilde{r}, \tilde{z})\|_{L^2(d\tilde{r}, d\tilde{z})}$. However, if working with the $\tilde{r}$, $\tilde{z}$ variables and the function $\tilde{Q}_{b(t)}$ and ϵ, we will write the integrals out to help avoid confusion.

We would like to know that the geometric description holds for $0 \leq t < T$, and in addition that properties BSO 1–8 listed below hold for all $0 \leq t < T$. To show this, we do a bootstrap argument. By IDA 1–11 and continuity of the flow $u(t)$ in H^1, we know that for some $t_1 > 0$, Lemma 5.1 applies giving $(\lambda(t), \gamma(t), r(t), z(t), b(t))$ and ϵ such that ORTH 1–4 hold on $0 \leq t < t_1$ with initial configuration $\lambda(0) = \lambda_0$, etc. Again by the continuity of $u(t)$ (and $\epsilon(t)$) in H^1 we know that BSI 1–8 hold on $0 \leq t < t_1$ by taking t_1 smaller, if necessary. Now take t_1 to be the *maximal* time for which Lemma 5.1 applies and BSI 1–8 hold on $0 \leq t < t_1$. (By the above reasoning, we must have $t_1 > 0$.) Under these hypotheses, we show that BSO 1–8 hold on $0 \leq t < t_1$. Since these properties are all *strictly* stronger than those of BSI 1–8, we must have $t_1 = T$.

We now outline this bootstrap argument. We have the following "bootstrap inputs" that we enumerate as BSI 1, etc. We assume that all the following properties hold for times $0 \leq t < t_1 < T$.

BSI 1. Proximity of $r(t)$ to 1, or localization of the singular circle

$$|r(t) - 1| \leq (\alpha^*)^{\frac{1}{2}}$$

and proximity of $z(t)$ to 0

$$|z(t)| \leq (\alpha^*)^{\frac{1}{2}}\,.$$

BSI 2. Smallness of $b(t)$, or closeness of Q_b to Q on the singular circle

$$0 < b(t) < (\alpha^*)^{\frac{1}{8}}.$$

BSI 3. Control of $\epsilon(t)$ error by radiative asymptotic $\Gamma_{b(t)}$

$$\mathcal{E}(t) \equiv \int |\nabla_{(\tilde{r},\tilde{z})}\epsilon(\tilde{r},\tilde{z},t)|^2 \mu_{\lambda(t),r(t)}\, d\tilde{r}d\tilde{z} + \int_{|\tilde{R}|<\frac{10}{b(t)}} |\epsilon(\tilde{r},\tilde{z},t)|^2 e^{-|\tilde{R}|}\, d\tilde{r}d\tilde{z} \leq \Gamma_{b(t)}^{\frac{3}{4}}.$$

BSI 4. Control of the scaling parameter $\lambda(t)$ by the radiative asymptotic $\Gamma_{b(t)}$

$$\lambda^2(t)|E_0| \leq \Gamma_{b(t)}^2.$$

BSI 5. Control of the r-localized momentum by the radiative asymptotic

$$\lambda(t)\left|\operatorname{Im}\int \nabla\psi \cdot \nabla u(t)\, \bar{u}(t)\, dxdydz\right| \leq \Gamma_{b(t)}^2.$$

BSI 6. Control of the scaling parameter $\lambda(t)$ by $b(t)$

$$0 < \lambda(t) \leq \exp\left(-\exp\frac{\pi}{10b(t)}\right).$$

BSI 7. Global L^2 bound on $\tilde{u}(t)$

$$\|\tilde{u}(t)\|_{L^2} \leq (\alpha^*)^{\frac{1}{10}}.$$

BSI 8. Smallness of $\tilde{u}(t)$ outside the singular circle

$$\|\chi_1(r)\tilde{u}(t)\|_{\dot{H}^{\frac{1}{2}}} \leq (\alpha^*)^{\frac{1}{4}}.$$

Assuming that properties BSI 1–8 hold for $0 \leq t < t_1 < T$, we prove that the following "bootstrap outputs," labeled BSO 1–9 hold.

BSO 1. Proximity of $r(t)$ to 1, or localization of the singular circle

$$|r(t) - 1| \leq (\alpha^*)^{\frac{2}{3}}$$

and proximity of $z(t)$ to 0

$$|z(t)| \leq (\alpha^*)^{\frac{2}{3}}.$$

BSO 2. Smallness of $b(t)$, or closeness of Q_b to Q on the singular circle

$$0 < b(t) < (\alpha^*)^{\frac{1}{5}}.$$

BSO 3. Control of $\epsilon(t)$ error by radiative asymptotic $\Gamma_{b(t)}$

$$\mathcal{E}(t) \equiv \int |\nabla_{(\tilde{r},\tilde{z})}\epsilon(\tilde{r},\tilde{z},t)|^2 \mu_{\lambda(t),r(t)}\, d\tilde{r}d\tilde{z} + \int_{|\tilde{R}|<\frac{10}{b(t)}} |\epsilon(\tilde{r},\tilde{z},t)|^2 e^{-|\tilde{R}|}\, d\tilde{r}d\tilde{z} \leq \Gamma_{b(t)}^{\frac{4}{5}}.$$

BSO 4. Control of the scaling parameter $\lambda(t)$ by the radiative asymptotic $\Gamma_{b(t)}$

$$\lambda^2(t)|E_0| \leq \Gamma_{b(t)}^4.$$

BSO 5. Control of the r-localized momentum by the radiative asymptotic

$$\lambda(t)\left|\mathrm{Im}\int \nabla\psi\cdot\nabla u(t)\,\bar{u}(t)\,dxdydz\right| \leq \Gamma_{b(t)}^4.$$

BSO 6. Control of the scaling parameter $\lambda(t)$ by $b(t)$ (which will imply an upper bound on the blow-up rate)

$$0<\lambda(t)\leq \exp\left(-\exp\frac{\pi}{5b(t)}\right).$$

BSO 7. Global L^2 bound on $\tilde{u}(t)$

$$\|\tilde{u}(t)\|_{L^2} \leq (\alpha^*)^{\frac{1}{5}}.$$

BSO 8. $H^{\frac{1}{2}}$ smallness of $\tilde{u}(t)$ outside the singular circle

$$\|\chi_1(r)\tilde{u}(t)\|_{\dot{H}^{\frac{1}{2}}} \leq (\alpha^*)^{\frac{3}{8}}.$$

The bootstrap argument proceeds in the following steps. The nontrivial steps are detailed in the remaining sections of the paper.

Step 1. Relative sizes of the parameters $\lambda(t)$, $\Gamma_{b(t)}$, α^*. Using BSI 2, BSI 6, and ZP 2, we have

$$\lambda(t)\leq \Gamma_{b(t)}^{10}. \tag{5.1}$$

Step 2. Application of mass conservation. Using BSI 1, BSI 3, BSI 6, and L^2 conservation, we obtain BSO 2 and BSO 7. In other words, mass conservation reinforces the smallness of $b(t)$ and also the smallness of the L^2 norm of $\tilde{u}$. This is carried out in §7.

Step 3. ϵ interaction energy is dominated by ϵ kinetic energy. That is, the ϵ energy behaves as if it were L^2 critical and subground state. We obtain by splitting the L^4 term in the energy of ϵ into inner and outer radii, using the axial Gagliardo-Nirenberg for outer radii and the usual 3d Gagliardo-Nirenberg for inner radii

$$\int |\epsilon|^4\mu(\tilde{r})d\tilde{r}d\tilde{z} \leq \delta(\alpha^*)\int |\nabla_{(\tilde{r},\tilde{z})}\epsilon|^2\mu(\tilde{r})\,d\tilde{r}d\tilde{z}. \tag{5.2}$$

This states that in the ϵ-energy, the interaction energy term is suitably dominated by the kinetic energy term.

A useful statement that comes out of this computation is

$$\int_{\tilde{R}<10/b} |\epsilon|^4 d\tilde{r}d\tilde{z} \le \|\tilde{u}\|_{L_x^2}^2 \, \mathcal{E}(t). \tag{5.3}$$

The proof of this statement only uses the 2d Gagliardo-Nirenberg inequality (since we have the localization $\tilde{R} < 10/b$) and does not use the $H^{1/2}$ assumption. This is carried out in §8.

Step 4. Energy conservation of u recast as an ϵ statement. Using BSI 3, (5.1), BSI 1, BSI 6, BSI 7, BSI 4, BSI 8, energy conservation, and properties of $\tilde{Q}_b$, we obtain

$$\begin{aligned} &\Big| 2(\epsilon_1, \Sigma) + 2(\epsilon_2, \Theta) - \int |\nabla_{(\tilde{r},\tilde{z})}\epsilon|^2 \mu(\tilde{r})\, d\tilde{r}d\tilde{z} \\ &+3\int_{|\tilde{R}|\le 10/b} Q^2 \epsilon_1^2 \, d\tilde{r}d\tilde{z} + \int_{|\tilde{R}|<10/b} Q^2\epsilon_2^2 \, d\tilde{r}d\tilde{z} \Big| \le \Gamma_b^{1-C\eta} + \delta(\alpha^*)\mathcal{E}(t). \end{aligned} \tag{5.4}$$

It results from plugging the representation of u in terms of $\tilde{Q}_{b(t)}$ and ϵ into the energy conservation equation for u. The result is basically

- the energy of $\tilde{Q}_{b(t)}$, which is small and shows up on the right side as the $\Gamma_{b(t)}^{1-C\eta}$ term,
- the energy of ϵ, which shows up as the $-\int |\nabla_{(\tilde{r},\tilde{z})}\epsilon|^2\mu(\tilde{r})\, d\tilde{r}d\tilde{z}$ term on the left (from Step 3, the interaction component of the energy is small and is put on the right),
- cross terms resulting from $|u|^4$ which are linear, quadratic, and cubic in ϵ. The linear terms are kept on the left as $2(\epsilon_1, \Sigma) + 2(\epsilon_2, \Theta)$. The quadratic terms are kept on the left as well, while the cubic term is estimated away.

This is carried out in §9.

Step 5. Momentum control assumption (BSI 5) recast as an ϵ statement. Using BSI 5, (5.1), properties of $\tilde{Q}_b$, BSI 1, BSI 2, we obtain

$$|(\epsilon_2, \nabla_{(\tilde{r},\tilde{z})}\Sigma)| \le \delta(\alpha^*)\mathcal{E}(t)^{1/2} + \Gamma_b^2. \tag{5.5}$$

The term that we keep on the left side comes from the cross term. This is carried out in §10.

Step 6. Application of the orthogonality conditions. In this step, the orthogonality assumptions are used to deduce "laws" for time evolution of the parameters $\lambda(t)$, $b(t)$, $\gamma(t)$. Using the orthogonality conditions ORTH 1-4, BSI 3, and (5.1), we obtain the following estimates on the modulation parameters:

$$\left|\frac{\lambda_s}{\lambda} + b\right| + |b_s| \le c\,\mathcal{E}(t) + \Gamma_b^{1-C\eta}, \tag{5.6}$$

$$\left|\tilde{\gamma}_s - \frac{1}{\|\Lambda Q\|_{L^2}^2}(\epsilon_1, L_+(\Lambda^2 Q))\right| + \left|\frac{(r_s, z_s)}{\lambda}\right| \leq \delta(\alpha^*)\mathcal{E}(t)^{1/2} + \Gamma_b^{1-C\eta}, \tag{5.7}$$

where, we recall, $L_+ = -\Delta + 1 - 3Q^2$. This is carried out in §11.

Step 7. Deduction of BSO 4 from (5.6). Using (5.6), BSI 4, BSI 6, properties of Γ_b, IDA 3, we obtain

$$\frac{d}{ds}(\lambda^2 e^{5\pi/b}) \leq 0,$$

which, upon integrating in time, gives BSO 4. This is carried out in §12.

Step 8. Deduction of a local virial inequality. Using (5.4), (5.6), (5.7), the coercivity property, and the spectral property for $d = 2$, the orthogonality conditions, we obtain a "local virial inequality"

$$b_s \geq \delta_0 \mathcal{E}(t) - \Gamma_{b(t)}^{1-C\eta}, \tag{5.8}$$

This is carried out in §13.

Step 9. Lower bound on $b(s)$ ($\implies$ upper bound on blow-up rate) Using (5.8), (5.6), BSI 4, BSI 6, BSI 2, and IDA 1, IDA 4, we obtain

$$b(s) \geq \frac{3\pi}{4\log s} \tag{5.9}$$

$$\lambda(s) \leq \sqrt{\lambda_0} e^{-\frac{\pi}{3}\frac{s}{\log s}} \tag{5.10}$$

which, together imply BSO 6:

$$(5.9) + (5.10) \quad \implies \quad \text{BSO 6} \iff \frac{\pi}{5} \leq b(t)\log|\log\lambda(t)|.$$

This will later be used to give an upper bound on the blow-up rate. (5.9)–(5.10) are consequences of a careful integration of (5.8) and an application of the law for the scaling parameter (5.6). This is carried out in §14.

Step 10. Control on the radius of concentration. Using (5.7), (5.10), IDA 2, IDA 4, IDA 1, we obtain BSO 1. This is carried out in §15.

Step 11. Momentum conservation implies momentum control estimate (BSO 5). Using BSI 3, BSI 7, proof of BSO 6, (5.6), BSI 4, BSI 6, IDA 3, we obtain BSO 5. This is carried out in §16.

Step 12. Refined virial inequality in the radiative regime. Here we prove a refinement of (5.8) in the radiative regime. Let $\phi_3(\tilde{R}) = 1$ for $\tilde{R} \leq 1$ and $\phi_3(\tilde{R}) = 0$ for $\tilde{R} \geq 2$ be a radial cutoff. With $\tilde{\zeta}_b = \phi_3(\tilde{R}/A)\zeta_b$, $A = e^{2a/b}$ for some small constant $0 < a \ll 1$, we define $\tilde{\epsilon} = \epsilon - \tilde{\zeta}$. In this step we establish

$$\partial_s f_1(s) \geq \delta_1 \tilde{\mathcal{E}}(t) + c\Gamma_b - \frac{1}{\delta_1}\int_{A \leq \tilde{R} \leq 2A} |\epsilon|^2 d\tilde{r} d\tilde{z}, \tag{5.11}$$

where

$$f_1(s) \equiv \frac{b}{4}\|\tilde{R}\tilde{Q}_b\|^2_{L^2_{\tilde{r}\tilde{z}}} + \frac{1}{2}\operatorname{Im}\int (\tilde{r},\tilde{z})\cdot\nabla\tilde{\zeta}\;\zeta + (\epsilon_2, \Lambda\tilde{\zeta}_{\rm re}) - (\epsilon_1, \Lambda\tilde{\zeta}_{\rm im})$$

and

$$\tilde{\mathcal{E}}(t) = \int |\nabla_{(\tilde{r},\tilde{z})}\tilde{\epsilon}(\tilde{r},\tilde{z},t)|^2 \mu_{\lambda(t),r(t)}\,d\tilde{r}d\tilde{z} + \int_{|\tilde{R}|<\frac{10}{b(t)}} |\tilde{\epsilon}(\tilde{r},\tilde{z},t)|^2 e^{-|\tilde{R}|}\,d\tilde{r}d\tilde{z}.$$

This is carried out in §17.

Step 13. L^2 dispersion at infinity in space. Let $\phi_4(\tilde{R})$ be a (nonstrictly) increasing radial cutoff to large radii. Specifically, we require that $\phi_4(\tilde{R}) = 0$ for $0 \le \tilde{R} \le \frac{1}{2}$ and $\phi_4(\tilde{R}) = 1$ for $\tilde{R} \ge 3$, with $\frac{1}{4} \le \phi_4'(\tilde{R}) \le \frac{1}{2}$ for $1 \le \tilde{R} \le 2$. We next prove, via a flux type computation, an estimate giving control on the term $\int_{A\le|\tilde{R}|\le 2A} |\epsilon|^2\,d\tilde{r}d\tilde{z}$:

$$\partial_s\left(\frac{1}{r(s)}\int \phi_4\left(\frac{\tilde{R}}{A}\right)|\epsilon|^2\mu(\tilde{r})d\tilde{r}d\tilde{z}\right) \ge \frac{b}{400}\int_{A\le|\tilde{R}|\le 2A}|\epsilon|^2 - \Gamma_b^{a/2}\mathcal{E}(t) - \Gamma_b^2. \tag{5.12}$$

This is carried out in §18.

Step 14. Lyapunov functional in H^1. By combining (5.11) and (5.12), we define a Lyapunov functional $\mathcal{J}$ and show

$$\partial_s\mathcal{J} \le -cb\left(\Gamma_b + \tilde{\mathcal{E}}(t) + \int_{A\le|\tilde{R}|\le 2A}|\epsilon|^2\right), \tag{5.13}$$

where $\mathcal{J}$ (defined later) can be shown to satisfy

$$|\mathcal{J} - d_0 b^2| \le \tilde{\delta}_1 b^2 \tag{5.14}$$

for some universal constant $0 < \tilde{\delta}_1 \ll 1$, and a more refined control

$$\mathcal{J}(s) - f_2(b(s)) \begin{cases} \ge -\Gamma_b^{1-Ca} + \frac{1}{C}\mathcal{E}(s) \\ \le +\Gamma_b^{1-Ca} + CA^2\mathcal{E}(s) \end{cases} \tag{5.15}$$

with f_2 given by

$$f_2(b) = \left(\int|\tilde{Q}_b|^2 - \int|Q|^2\right) - \frac{\delta_1}{800}\left(b\tilde{f}_1(b) - \int_0^b \tilde{f}_1(v)dv\right)$$

and

$$\tilde{f}_1(b) = \frac{b}{4}\|\tilde{R}\tilde{Q}_b\|^2_{L^2} + \frac{1}{2}\operatorname{Im}\int(\tilde{r},\tilde{z})\cdot\nabla\tilde{\zeta}\;\overline{\tilde{\zeta}}d\tilde{r}d\tilde{z}.$$

Here, $0 < \frac{df_2}{db^2}\Big|_{b^2=0} < +\infty$, and hence, (5.15) refines (5.14). This is carried out in §19.

Step 15. Deduction of control on $\mathcal{E}(t)$ and upper bound on b ($\implies$ lower bound on blow-up rate). By integrating (5.13) and applying (5.14), we prove

$$b(s) \le \frac{4\pi}{3\log s} \tag{5.16}$$

and

$$\int_{s_0}^{s} \Big(\Gamma_{b(\sigma)} + \tilde{\mathcal{E}}(\sigma)\Big)\, d\sigma \le c\,\alpha^* . \tag{5.17}$$

Using (5.13) and (5.15), we prove BSO 3, the key dispersive control on the remainder term ϵ. This is carried out in §20.

Step 16. $H^{1/2}$ interior smallness. Using a local smoothing estimate and (5.17), we prove BSO 8. This is carried out in §21.

This concludes the outline of the bootstrap argument, and we now know that BSO 1–8 hold for all times $0 < t < T$. The remainder of the proof of Theorem 1.1 is carried out in §22–24.

6. The equation for ϵ

Recall (3.1) and write

$$u(t,r,z) = \frac{1}{\lambda(t)}\tilde{Q}_{b(t)}\left(\frac{r-r(t)}{\lambda(t)}, \frac{z-z(t)}{\lambda(t)}\right)e^{i\gamma(t)} + \frac{1}{\lambda(t)}\epsilon\left(\frac{r-r(t)}{\lambda(t)}, \frac{z-z(t)}{\lambda(t)}, t\right)e^{i\gamma(t)}. \tag{6.1}$$

In the remainder of this section, Q will always mean $\tilde{Q}_{b(t)}$. Recall

$$\tilde{r} = \frac{r-r(s)}{\lambda(s)}, \qquad \tilde{z} = \frac{z-z(s)}{\lambda(s)},$$

and thus,

$$\partial_s\tilde{r} = -\frac{r_s}{\lambda} - \tilde{r}\frac{\lambda_s}{\lambda}, \qquad \partial_s\tilde{z} = -\frac{z_s}{\lambda} - \tilde{z}\frac{\lambda_s}{\lambda}.$$

Direct computation when substituting (6.1) into (3.1) gives

$$\begin{aligned}
&e^{-i\gamma}\lambda^3 i\partial_t u = e^{-i\gamma}\lambda i\partial_s u \\
&\quad= -i\frac{\lambda_s}{\lambda}Q + i\left(-\frac{r_s}{\lambda} - \tilde{r}\frac{\lambda_s}{\lambda}\right)\partial_{\tilde{r}}Q + i\left(-\frac{z_s}{\lambda} - \tilde{z}\frac{\lambda_s}{\lambda}\right)\partial_{\tilde{z}}Q + ib_s\partial_b Q - \gamma_s Q \\
&\qquad+ i\partial_s\epsilon - i\frac{\lambda_s}{\lambda}\epsilon + i\left(-\frac{r_s}{\lambda} - \tilde{r}\frac{\lambda_s}{\lambda}\right)\partial_{\tilde{r}}\epsilon + i\left(-\frac{z_s}{\lambda} - \tilde{z}\frac{\lambda_s}{\lambda}\right)\partial_{\tilde{z}}\epsilon - \gamma_s\epsilon \\
&\quad= -i\left(\frac{\lambda_s}{\lambda} + b\right)\Lambda Q - \frac{i}{\lambda}(r_s, z_s)\cdot\nabla_{(\tilde{r},\tilde{z})}Q + ib_s\partial_b Q - \tilde{\gamma}_s Q + (ib\Lambda Q - Q) \\
&\qquad+ i\partial_s\epsilon - i\left(\frac{\lambda_s}{\lambda} + b\right)\Lambda\epsilon - \frac{i}{\lambda}(r_s, z_s)\cdot\nabla_{(\tilde{r},\tilde{z})}\epsilon - \tilde{\gamma}_s\epsilon + (ib\Lambda\epsilon - \epsilon),
\end{aligned}$$

where $\tilde{\gamma}(s) = \gamma(s) - s$.

Also,

$$\lambda^3 e^{-i\gamma}\Delta_{(r,z)}u = \Delta_{(\tilde{r},\tilde{z})}Q + \Delta_{(\tilde{r},\tilde{z})}\epsilon$$

and

$$\lambda^3 e^{-i\gamma}\frac{1}{r}\partial_r u = \frac{\lambda}{\mu}\partial_{\tilde{r}}Q + \frac{\lambda}{\mu}\partial_{\tilde{r}}\epsilon.$$

Finally the nonlinear term:

$$\begin{aligned}\lambda^3 e^{-i\gamma}|u|^2 u &= |Q+\epsilon|^2(Q+\epsilon)\\ &= |Q|^2Q + \underbrace{2(\operatorname{Re} Q\bar\epsilon)Q + |Q|^2\epsilon}_{\text{linear}} + \underbrace{Q|\epsilon|^2 + 2(\operatorname{Re} Q\bar\epsilon)\epsilon}_{\text{quadratic}} + \underbrace{|\epsilon|^2\epsilon}_{\text{cubic}}\\ &= |Q|^2Q + [(2\Sigma^2+|Q|^2)\epsilon_1 + 2\Sigma\Theta\epsilon_2] + i[(2\Theta^2+|Q|^2)\epsilon_2 + 2\Sigma\Theta\epsilon_1] + R_1(\epsilon) + iR_2(\epsilon),\end{aligned}$$

where the quadratic and cubic terms are put into $R(\epsilon)$.

Adding up all of the above in (3.1), and taking the real part, we get:

$$\begin{aligned}b_s\partial_b\Theta + \partial_s\epsilon_2 + M_+(\epsilon) + b\Lambda\epsilon_2 = \;&\left(\frac{\lambda_s}{\lambda}+b\right)\Lambda\Theta + \frac{1}{\lambda}(r_s,z_s)\cdot\nabla_{(\tilde r,\tilde z)}\Theta - \tilde\gamma_s\Sigma\\ &+\left(\frac{\lambda_s}{\lambda}+b\right)\Lambda\epsilon_2 + \frac{1}{\lambda}(r_s,z_s)\cdot\nabla_{(\tilde r,\tilde z)}\epsilon_2 - \tilde\gamma_s\epsilon_1\\ &-\operatorname{Re}\tilde\Psi + R_1(\epsilon),\end{aligned}\tag{6.2}$$

where

$$M_+(\epsilon) = \epsilon_1 - \Delta_{(\tilde r,\tilde z)}\epsilon_1 - \frac{\lambda}{\mu}\partial_{\tilde r}\epsilon_1 - \left(\frac{2\Sigma^2}{|Q|^2}+1\right)|Q|^2\epsilon_1 - 2\Sigma\Theta\epsilon_2.\tag{6.3}$$

Taking the imaginary part, we get

$$\begin{aligned}b_s\partial_b\Sigma + \partial_s\epsilon_1 - M_-(\epsilon) + b\Lambda\epsilon_1 = \;&\left(\frac{\lambda_s}{\lambda}+b\right)\Lambda\Sigma + \frac{1}{\lambda}(r_s,z_s)\cdot\nabla_{(\tilde r,\tilde z)}\Sigma + \tilde\gamma_s\Theta\\ &+\left(\frac{\lambda_s}{\lambda}+b\right)\Lambda\epsilon_1 + \frac{1}{\lambda}(r_s,z_s)\cdot\nabla_{(\tilde r,\tilde z)}\epsilon_1 + \tilde\gamma_s\epsilon_2\\ &+\operatorname{Im}\tilde\Psi - R_2(\epsilon),\end{aligned}\tag{6.4}$$

where

$$M_-(\epsilon) = \epsilon_2 - \Delta_{(\tilde r,\tilde z)}\epsilon_2 - \frac{\lambda}{\mu}\partial_{\tilde r}\epsilon_2 - \left(\frac{2\Theta^2}{|Q|^2}+1\right)|Q|^2\epsilon_2 - 2\Sigma\Theta\epsilon_1.\tag{6.5}$$

7. Bootstrap Step 2. Application of mass conservation

In this section we prove BSO 2 and BSO 7 are consequences of the L^2 conservation of $u(t)$ and several other BSI's. The assumed smallness of $b(t)$, $\epsilon(t)$, and $\lambda(t)$, initial smallness of $\tilde u$ (at $t=0$) combined with L^2 norm conservation for $u(t)$ reinforces the smallness of $\tilde u(t)$ in L^2 for $0<t<t_1$ and smallness of $b(t)$ for $0<t<t_1$.

Recall (6.1) and denote

$$\tilde u(r,z,t) = \frac{1}{\lambda(t)}\,\epsilon\left(\frac{r-r(t)}{\lambda(t)},\frac{z-z(t)}{\lambda(t)},t\right)e^{i\gamma(t)}.$$

Substitute (6.1) into the mass conservation law

$$\int |u(r,z,t)|^2 r dr dz = \|u_0\|_{L^2}^2$$

and rescale to obtain

$$\begin{aligned}\|u_0\|_{L^2}^2 = & \int |\tilde{Q}_{b(t)}(\tilde{r},\tilde{z})|^2 \mu(\tilde{r})\, d\tilde{r} d\tilde{z} \\ & + 2\,\mathrm{Re} \int \tilde{Q}_{b(t)}(\tilde{r},\tilde{z})\bar{\epsilon}(\tilde{r},\tilde{z})\, \mu(\tilde{r})\, d\tilde{r} d\tilde{z} \\ & + \int |\tilde{u}(r,z,t)|^2 r dr dz.\end{aligned} \tag{7.1}$$

In the first term, write out $\mu(\tilde{r}) = \lambda\tilde{r} + (r(t)-1) + 1$, which splits the integral into three pieces. The third piece we keep; the first two we estimate using

$$[\text{BSI 6} : \lambda \le \exp(-\exp \pi/10b(t)) \text{ and BSI 2} : b(t) \le (\alpha^*)^{1/8}] \implies \lambda \le (\alpha^*)^{1/2}$$

and BSI 1: $|r(t)-1| \le (\alpha^*)^{1/2}$. The second term in (7.1) we estimate using Cauchy-Schwarz and the assumed control on $\epsilon(t)$ given by BSI 3, ZP 2, and BSI 2: $\mathcal{E}(t) \le \Gamma_{b(t)}^{3/4} \le (\alpha^*)^{1/2}$. Collecting the results, we now have

$$\int |\tilde{Q}_{b(t)}|^2 d\tilde{r} d\tilde{z} + \|\tilde{u}\|_{L^2}^2 = \|u_0\|_{L^2}^2 + \mathcal{O}((\alpha^*)^{1/2}).$$

Finally, we use QP3:

$$d(b^2) \equiv \frac{d}{db^2}\|\tilde{Q}_b\|_{L^2_{\tilde{r}\tilde{z}}}^2, \quad d_0 = d(0) > 0, \quad d \text{ differentiable},$$

which gives

$$\|\tilde{Q}_b\|_{L^2_{\tilde{r}\tilde{z}}}^2 = \|Q\|_{L^2_{\tilde{r}\tilde{z}}}^2 + d_0 b^2 + \mathcal{O}(b^4 \sup_{0\le\sigma\le b^2} |d'(\sigma)|).$$

Since $b^4 \le (\alpha^*)^{1/2}$, we have

$$d_0 b(t)^2 + \|\tilde{u}(t)\|_{L^2_{xyz}}^2 = \|u_0\|_{L^2}^2 - \|Q\|_{L^2_{\tilde{r}\tilde{z}}}^2 + \mathcal{O}((\alpha^*)^{1/2})$$

and conclude by IDA 9 to get

$$d_0 b(t)^2 + \|\tilde{u}(t)\|_{L^2_{xyz}}^2 \le c(\alpha^*)^{1/2},$$

which implies BSO 2 and BSO 7.

8. Bootstrap Step 3. Interaction energy $\ll$ kinetic energy for ϵ

Here we deduce (5.2) (or (3.94) in [33]), which controls $|\epsilon|^4$ by $|\nabla\epsilon|^2$, and is useful later on.

By change of variables, we have

$$\int |\epsilon|^4 \mu(\tilde{r})\, d\tilde{r} d\tilde{z} = \lambda^2(t) \int_{|r|\le\frac{1}{2}} |\tilde{u}|^4\, r\, dr dz + \lambda^2(t) \int_{|r|\ge\frac{1}{2}} |\tilde{u}|^4\, r\, dr dz.$$

For the first term, we use the (r-localized) 3d Gagliardo-Nirenberg estimate. For the second term, we use the axially symmetric exterior Gagliardo-Nirenberg estimate Lemma 2.1. Together, they give the bound

$$\leq \lambda^2(t)\|\tilde{u}\|^2_{H^{\frac{1}{2}}_{\{xyz,\ |r|<\frac{1}{4}\}}}\|\nabla\tilde{u}\|^2_{L^2_{xyz}} + \lambda^2(t)\|\tilde{u}\|^2_{L^2_{\{xyz,\ |r|>\frac{1}{4}\}}}\|\nabla\tilde{u}\|^2_{L^2_{xyz}}.$$

We then apply BSI 8 (smallness of $\|\tilde{u}\|_{\dot{H}^{1/2}}$ for $|r| < \frac{1}{4}$) and BSO 7 (smallness of $\|\tilde{u}\|_{L^2}$ globally in r) to get the estimate

$$\leq (\alpha^*)^{1/4}\lambda^2(t)\|\nabla_{xyz}\tilde{u}\|^2_{L^2_{xyz}}.$$

For axially symmetric functions, we have $|\partial_r u|^2 = |\partial_x u|^2 + |\partial_y u|^2$, so we may replace ∇_{xyz} by ∇_{rz}, then rescale to obtain

$$\leq (\alpha^*)^{1/4}\int_{\tilde{r},\tilde{z}} |\nabla_{(\tilde{r},\tilde{z})}\epsilon|^2\mu(\tilde{r})\,d\tilde{r}d\tilde{z}$$

and this is (5.2).

9. Bootstrap Step 4. Energy conservation

In this step, we prove (5.4) as a consequence of various bootstrap assumptions and the conservation of energy.

Plug (6.1) into the energy conservation identity

$$2\lambda^2 E_0 = \lambda^2\int|\nabla_{(r,z)}u|^2 r\,dr\,dz - \frac{1}{2}\lambda^2\int|u|^4 r\,dr\,dz.$$

The result for the first term on the right side is

$$\begin{aligned}\lambda^2\int|\nabla_{(r,z)}u|^2 r\,dr\,dz = &\int|\nabla_{(\tilde{r},\tilde{z})}\tilde{Q}_{b(t)}|^2\mu(\tilde{r})\,d\tilde{r}d\tilde{z}\\ &+2\operatorname{Re}\int\nabla_{(\tilde{r},\tilde{z})}\tilde{Q}_{b(t)}\cdot\nabla_{(\tilde{r},\tilde{z})}\bar{\epsilon}\;\mu(\tilde{r})d\tilde{r}d\tilde{z}\\ &+\int|\nabla_{(\tilde{r},\tilde{z})}\epsilon|^2\,\mu(\tilde{r})d\tilde{r}d\tilde{z}\\ =\ &\mathrm{I.1}+\mathrm{I.2}+\mathrm{I.3}.\end{aligned}$$

For the second term, we obtain

$$\begin{aligned}-\frac{1}{2}\lambda^2\int|u|^4 r\,dr\,dz = {} & -\frac{1}{2}\int|\tilde{Q}_{b(t)}|^4\mu(\tilde{r})\,d\tilde{r}d\tilde{z} - 2\int|\tilde{Q}_{b(t)}|^2\operatorname{Re}(\tilde{Q}_{b(t)}\bar{\epsilon})\mu(\tilde{r})\,d\tilde{r}d\tilde{z} \\ & - 2\int(\operatorname{Re}\tilde{Q}_{b(t)}\bar{\epsilon})^2\mu(\tilde{r})\,d\tilde{r}d\tilde{z} - \int|\tilde{Q}_{b(t)}|^2|\epsilon|^2\mu(\tilde{r})\,d\tilde{r}d\tilde{z} \qquad \leftarrow \text{quadratic} \\ & - 2\int|\epsilon|^2\operatorname{Re}(\tilde{Q}_{b(t)}\bar{\epsilon})\mu(\tilde{r})\,d\tilde{r}d\tilde{z} \qquad \leftarrow \text{cubic} \\ & - \frac{1}{2}\int|\epsilon|^4\mu(\tilde{r})\,d\tilde{r}d\tilde{z} \\ = {} & \mathrm{II.1} + \mathrm{II.2} + \mathrm{II.3} + \mathrm{II.4} + \mathrm{II.5} + \mathrm{II.6}.\end{aligned}$$

In all of these terms except I.3 and II.6, we can write out $\mu(\tilde{r}) = \lambda\tilde{r} + r(t)$ and "discard" the $\lambda\tilde{r}$ term by estimation, since $\lambda(t) \leq \Gamma_{b(t)}^{10}$. We should thus hereafter in this section replace $\mu(\tilde{r})$ with $r(t)$ for all terms but I.3 and II.6. We have the $\tilde{Q}_{b(t)}$ energy terms:

$$r(t)^{-1}(\mathrm{I.1} + \mathrm{II.1}) = 2\,E(\tilde{Q}_{b(t)}).$$

We also have the linear in ϵ terms that we combine, and then substitute the equation for $\tilde{Q}_{b(t)}$ to get

$$\begin{aligned}r(t)^{-1}(\mathrm{I.2} + \mathrm{II.2}) &= -2\operatorname{Re}\int(\Delta_{(\tilde{r},\tilde{z})}\tilde{Q}_{b(t)} + |\tilde{Q}_{b(t)}|^2\tilde{Q}_{b(t)})\bar{\epsilon}\,d\tilde{r}d\tilde{z} \\ &= -2\operatorname{Re}\int\tilde{Q}_{b(t)}\bar{\epsilon}\,d\tilde{r}d\tilde{z} - 2b\operatorname{Im}\int\Lambda\tilde{Q}_{b(t)}\,\bar{\epsilon}\,d\tilde{r}d\tilde{z} + 2\operatorname{Re}\int\Psi_{b(t)}\bar{\epsilon}\,d\tilde{r}d\tilde{z}.\end{aligned}$$

The middle term is zero by ORTH 4, and the first term can be rewritten to obtain

$$r(t)^{-1}(\mathrm{I.2} + \mathrm{II.2}) = -2(\Sigma, \epsilon_1) - 2(\Theta, \epsilon_2) + 2\operatorname{Re}\int\Psi_{b(t)}\,\bar{\epsilon}\,d\tilde{r}d\tilde{z}.$$

Next, for the quadratic in ϵ terms in II, replace $\tilde{Q}_{b(t)}$ by Q and use the proximity estimates for $\tilde{Q}_{b(t)}$ to Q to control the error

$$r(t)^{-1}(\mathrm{II.3} + \mathrm{II.4}) = -3\int Q^2\epsilon_1^2\,d\tilde{r}d\tilde{z} - \int Q^2\epsilon_2^2 d\tilde{r}d\tilde{z} + \text{errors}.$$

For the cubic term in II, we estimate as (using that $1/b \ll 1/\lambda$)

$$\begin{aligned}r(t)^{-1}|\mathrm{II.5}| &\leq 2\int|\epsilon|^3|\tilde{Q}_{b(t)}|d\tilde{r}d\tilde{z} \\ &\leq 2\left(\int_{\tilde{R}\leq 1/10\lambda}|\epsilon|^4 d\tilde{r}d\tilde{z}\right)^{1/2}\left(\int|\epsilon|^2|\tilde{Q}|^2 d\tilde{r}d\tilde{z}\right)^{1/2}.\end{aligned}$$

By 2d Gagliardo-Nirenberg applied to the first term (where ϕ is a smooth cutoff to $\tilde R \le 1/10\lambda$)

$$\lesssim 2\left(\int_{\tilde R \le 1/10\lambda} |\epsilon|^2 d\tilde r d\tilde z\right)^{1/2} \left(\int |\nabla_{(\tilde r,\tilde z)}[\phi(\tilde R\lambda)\epsilon(\tilde R)]|^2 d\tilde r d\tilde z\right)^{1/2} \left(\int |\epsilon|^2 |\tilde Q|^2 d\tilde r d\tilde z\right)^{1/2}.$$

Note that if $\tilde R \le 1/10\lambda$, then $\mu(\tilde r) \sim 1$. The first term above is controlled by some α^* power (using the $\tilde R$ restriction to reinsert a $\mu(\tilde r)$ factor) by rescaling back to $\tilde u$ by BSI 7. The second term is controlled as

$$\begin{aligned}&\left(\lambda \int_{|\tilde R| \ll \frac{1}{\lambda}} |\epsilon|^2 \, d\tilde r \, d\tilde z + \int_{|\tilde R| \ll \frac{1}{\lambda}} |\nabla \epsilon|^2 \, d\tilde r \, d\tilde z\right)^{1/2} \\ &\qquad \le \lambda \|\tilde u\|_{L^2} + \left(\int_{|\tilde R| \ll \frac{1}{\lambda}} |\nabla\epsilon|^2 \mu(\tilde r) \, d\tilde r \, d\tilde z\right)^{1/2} \\ &\qquad \le \lambda + \mathcal{E}(t)^{1/2}.\end{aligned}$$

For the third term, we use $|\tilde Q_{b(t)}| \le e^{-\tilde R/2}$. These considerations give

$$r(t)^{-1}|\mathrm{II}.5| \le (\alpha^*)^{1/5}(\lambda^2 + \mathcal{E}(t)).$$

The quartic term in ϵ in II is controlled by (5.2). We next note that

$$\text{BSI 1}: \ |r(t) - 1| \le (\alpha^*)^{1/2} \implies \frac{1}{r(t)} \sim 1.$$

Collecting all of the above estimates and manipulations, we obtain (5.4).

10. Bootstrap Step 5. Momentum control assumption recast as an ϵ statement

In this section we prove (5.5). Recall BSI 5:

$$\lambda(t) \left| \mathrm{Im} \int \nabla\psi \cdot \nabla u(t) \; \bar u(t) \, dxdydz \right| \le \Gamma_{b(t)}^2.$$

A basic calculus fact states that for axially symmetric functions f and g,

$$\nabla_{(x,y,z)} f \cdot \nabla_{(x,y,z)} g = \nabla_{(r,z)} f \cdot \nabla_{(r,z)} g.$$

Plug in (6.1) into the left side of BSI 5 and change variables (r, z) to $(\tilde r, \tilde z)$ to get

$$\begin{aligned}&\lambda(t)\, \mathrm{Im} \int \nabla\psi \cdot \nabla u(t) \; \bar u(t) \, dxdydz \\ &\quad = \mathrm{Im} \int \Big[(\nabla_{(r,z)}\psi)(r,z) \cdot \Big(\nabla_{(\tilde r,\tilde z)}(\tilde Q_{b(t)} + \epsilon) \Big)(\tilde r, \tilde z) \Big] \; (\overline{\tilde Q_{b(t)}} + \bar\epsilon)(\tilde r, \tilde z) \, \mu(\tilde r) d\tilde r \, d\tilde z,\end{aligned}$$

then using that $\psi(r) = (1,0)$ on the support of $\tilde{Q}_{b(t)}$, we continue as

$$\begin{aligned} &= \operatorname{Im} \int \partial_{\tilde r} \tilde{Q}_{b(t)} \, \overline{\tilde{Q}_{b(t)}} \, \mu(\tilde r) \, d\tilde r d\tilde z \\ &\quad + 2 \operatorname{Im} \int \partial_{\tilde r} \tilde{Q}_{b(t)} \, \bar\epsilon \mu(\tilde r) \, d\tilde r d\tilde z \\ &\quad + \operatorname{Im} \int \partial_{\tilde r} \epsilon \, \bar\epsilon \, \mu(\tilde r) \, d\tilde r d\tilde z \\ &= \mathrm{I} + \mathrm{II} + \mathrm{III}. \end{aligned}$$

In term I, we expand out $\mu(\tilde r) = \lambda(t)\tilde r + r(t)$ which gives two terms: the first of these we estimate out and use $\lambda(t) \le \Gamma_{b(t)}^2$, and the second of these is 0 by QP 3. In term II, we expand out $\mu(\tilde r)$ to get

$$\mathrm{II} = 2\lambda(t) \operatorname{Im} \int \partial_{\tilde r} \tilde{Q}_{b(t)} \, \bar\epsilon \, \tilde r \, d\tilde r \, d\tilde z + 2r(t)(\partial_{\tilde r}\Theta, \epsilon_1) - 2r(t)(\partial_{\tilde r}\Sigma, \epsilon_2). \tag{10.1}$$

The first of these is estimated away using $\lambda(t) \le \Gamma_{b(t)}^2$, the third term we keep, and for the second we note

$$\|e^{\tilde R/2} \partial_{\tilde r} \Theta\|_{L^2_{(\tilde r, \tilde z)}} \to 0 \quad \text{as} \quad b \to 0$$

by QP 1, and thus, we can estimate $(\partial_{\tilde r}\Theta, \epsilon_1)$ by Cauchy-Schwarz. For term III, we first estimate by Cauchy-Schwarz to get

$$|\mathrm{III}| \le \left(\int |\partial_{\tilde r} \epsilon|^2 \mu(\tilde r) \, d\tilde r d\tilde z \right)^{1/2} \left(\int |\epsilon|^2 \mu(\tilde r) \, d\tilde r d\tilde z \right)^{1/2}.$$

For the second factor, we convert ϵ back in terms of $\tilde u$:

$$\epsilon(\tilde r, \tilde z) = \lambda(t) \tilde u(\lambda(t)\tilde r + r(t), \lambda(t)\tilde z + z(t), t)$$

to obtain

$$\int |\epsilon|^2 \mu(\tilde r) \, d\tilde r d\tilde z = \|\tilde u\|_{L^2}^2,$$

which is $\le \alpha^{2/5}$ by BSO 2. Combining all of the above information, we get

$$r(t)(\partial_{\tilde r}\Sigma, \epsilon_2) \le \delta(\alpha^*) \mathcal{E}(t)^{1/2} + \Gamma_{b(t)}^2. \tag{10.2}$$

We finish by using that

$$\text{BSI 1}: \ |r(t) - 1| \le (\alpha^*)^{1/2} \implies \frac{1}{r(t)} \le 2$$

and then (5.5) follows.

11. Bootstrap Step 6. Application of orthogonality conditions

11.1. **Computation of $\lambda_s/\lambda + b$.** Here we explain how to obtain (5.6).

Multiply the equation for ϵ_1 (6.4) by $\tilde{R}^2\Sigma$ and the equation for ϵ_2 (6.2) by $\tilde{R}^2\Theta$ and add. We study the resulting terms one by one.

Term 1.

$$b_s\left[(\partial_b\Sigma, \tilde{R}^2\Sigma) + (\partial_b\Theta, \tilde{R}^2\Theta)\right] = b_s \operatorname{Re}(\partial_b\tilde{Q}_b, \tilde{R}^2\tilde{Q}_b). \tag{11.1}$$

Using the QP properties:

$$\left\| e^{\tilde{R}/2}(\partial_b\tilde{Q}_b - i\tfrac{1}{4}\tilde{R}^2Q)\right\|_{C^2} \to 0 \text{ as } b \to 0$$

and

$$\left\| e^{\tilde{R}/2}(\tilde{Q}_b - Q)\right\|_{C^3} \to 0 \text{ as } b \to 0$$

to show that (11.1) $\sim b_s \operatorname{Re}(\frac{1}{4}i\tilde{R}^2Q, \tilde{R}^2Q) = 0$. More specifically, we can show that (11.1) is

$$\leq |b_s|\delta(b), \qquad \text{where } \delta(b) \to 0 \text{ as } b \to 0.$$

Term 2. Using ORTH 1 condition, we have

$$\begin{aligned}
&(\partial_s\epsilon_1, \tilde{R}^2\Sigma) + (\partial_s\epsilon_2, \tilde{R}^2\Theta) \\
&\quad = \partial_s\left[(\epsilon_1, \tilde{R}^2\Sigma) + (\epsilon_2, \tilde{R}^2\Theta)\right] - b_s\left[(\epsilon_1, \tilde{R}^2\partial_b\Sigma) + (\epsilon_2, \tilde{R}^2\partial_b\Theta)\right] \\
&\quad = -b_s\left[(\epsilon_1, \tilde{R}^2\partial_b\Sigma) + (\epsilon_2, \tilde{R}^2\partial_b\Theta)\right],
\end{aligned} \tag{11.2}$$

which we then estimate as

$$\leq |b_s|\left(\int_{\tilde{R}\leq\frac{10}{b}} |\epsilon|^2 e^{-\tilde{R}}d\tilde{r}d\tilde{z}\right)^{1/2} \leq |b_s|\mathcal{E}(t)^{1/2}.$$

Term 3.

$$\begin{aligned}
&(-M_-(\epsilon) + b\Lambda\epsilon_1, \tilde{R}^2\Sigma) + (M_+(\epsilon) + b\Lambda\epsilon_2, \tilde{R}^2\Theta) \\
&\quad = \; -(\epsilon_2, \tilde{R}^2\Sigma) + (\epsilon_1, \tilde{R}^2\Theta) \\
&\qquad + (\Delta_{(\tilde{r},\tilde{z})}\epsilon_2, \tilde{R}^2\Sigma) - (\Delta_{(\tilde{r},\tilde{z})}\epsilon_1, \tilde{R}^2\Theta) \\
&\qquad + \frac{\lambda}{\mu}(\partial_{\tilde{r}}\epsilon_2, \tilde{R}^2\Sigma) - \frac{\lambda}{\mu}(\partial_{\tilde{r}}\epsilon_1, \tilde{R}^2\Theta) \\
&\qquad + \left(\left(\frac{2\Theta^2}{|\tilde{Q}_b|^2} + 1\right)|\tilde{Q}_b|^2\epsilon_2, \tilde{R}^2\Sigma\right) - \left(\left(\frac{2\Sigma^2}{|\tilde{Q}_b|^2} + 1\right)|\tilde{Q}_b|^2\epsilon_1, \tilde{R}^2\Theta\right) \\
&\qquad + (2\Sigma\Theta\epsilon_1, \tilde{R}^2\Sigma) - (2\Sigma\Theta\epsilon_2, \tilde{R}^2\Theta) \\
&\qquad + b[(\Lambda\epsilon_1, \tilde{R}^2\Sigma) + (\Lambda\epsilon_2, \tilde{R}^2\Theta)] \\
&\quad = \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV} + \mathrm{V} + \mathrm{VI}.
\end{aligned} \tag{11.3}$$

By integration by parts,

$$\mathrm{I}+\mathrm{II} = -(\epsilon_2, \tilde{R}^2\Sigma) + (\epsilon_1, \tilde{R}^2\Theta) + (\epsilon_2, \Delta_{(\tilde{r},\tilde{z})}(\tilde{R}^2\Sigma)) - (\epsilon_1, \Delta_{(\tilde{r},\tilde{z})}(\tilde{R}^2\Theta)).$$

Computation gives, for any function f, that $\Delta_{(\tilde{r},\tilde{z})}(\tilde{R}^2 f) = 4\Lambda f + \tilde{R}^2\Delta_{(\tilde{r},\tilde{z})}f$. Thus, using ORTH 4,

$$\mathrm{I}+\mathrm{II} = \mathrm{Im}(\epsilon, \tilde{R}^2(-\tilde{Q}_b + \Delta\tilde{Q}_b)).$$

Substituting the equation for $\tilde{Q}_b$ gives

$$\mathrm{I}+\mathrm{II} = \mathrm{Im}(\epsilon, \tilde{R}^2(-ib\Lambda\tilde{Q}_b - \tilde{Q}_b|\tilde{Q}_b|^2 - \Psi_b)).$$

By examining IV+V and rearranging terms,

$$\mathrm{IV}+\mathrm{V} = \mathrm{Im}(\epsilon, \tilde{R}^2\tilde{Q}_b|\tilde{Q}_b|^2),$$

and thus,

$$\mathrm{I}+\mathrm{II}+\mathrm{IV}+\mathrm{V} = \mathrm{Im}(\epsilon, \tilde{R}^2(-ib\Lambda\tilde{Q}_b - \Psi_b)).$$

In fact, adding VI cancels the middle term:

$$\mathrm{I}+\mathrm{II}+\mathrm{IV}+\mathrm{V}+\mathrm{VI} = -\,\mathrm{Im}(\epsilon, \tilde{R}^2\Psi_b)).$$

Properties (3.5) and ZP 2 imply

$$\|P(y)\Psi^{(k)}(y)\|_{L^\infty} \le C_{P,k}\Gamma_b^{\frac{1}{2}(1-C\eta)}, \tag{11.4}$$

thus, we get

$$|(\epsilon, \tilde{R}^2\Psi_b)| \le \Gamma_b^{\frac{1}{2}(1-C\eta)} \left(\int_{\tilde{R}<10/b} |\epsilon|^2 e^{-\tilde{R}}\, d\tilde{r}d\tilde{z}\right)^{1/2} = \Gamma_b^{\frac{1}{2}(1-C\eta)}\mathcal{E}(t)^{1/2}.$$

For III, use $\lambda \le \Gamma_b^{10}$ and $\mu \sim 1$ and Cauchy-Schwarz to obtain

$$|\mathrm{III}| \le \Gamma_b^{10}\left(\int_{\tilde{R}<10/b} |\epsilon|^2 e^{-\tilde{R}}\, d\tilde{r}d\tilde{z}\right)^{1/2}.$$

Together we have

$$|(11.3)| \le \Gamma_b^{\frac{1}{2}(1-C\eta)} \left(\int_{\tilde{R}<10/b} |\epsilon|^2 e^{-\tilde{R}} d\tilde{r}d\tilde{z}\right)^{1/2}.$$

Term 4.

$$\left(\frac{\lambda_s}{\lambda} + b\right)[(\Lambda\Sigma, \tilde{R}^2\Sigma) + (\Lambda\Theta, \tilde{R}^2\Theta)] \tag{11.5}$$

By integration by parts, for any function f, we have $(\Lambda f, \tilde{R}^2 f) = -\int \tilde{R}^2 f^2$, and thus, the above expression becomes

$$-\left(\frac{\lambda_s}{\lambda} + b\right)\|\tilde{R}\tilde{Q}_b\|_{L^2}^2.$$

Term 5.

$$\left(\frac{1}{\lambda}(r_s, z_s)\cdot\nabla_{(\tilde r,\tilde z)}\Sigma, \tilde R^2\Sigma\right) + \left(\frac{1}{\lambda}(r_s, z_s)\cdot\nabla_{(\tilde r,\tilde z)}\Theta, \tilde R^2\Theta\right) \tag{11.6}$$

Each of these two terms, it turns out, is zero. For example,

$$\begin{aligned}(\partial_{\tilde r}\Theta, \tilde R^2\Theta) &= \frac{1}{2}\int \tilde R^2 \partial_{\tilde r}(\Theta^2)d\tilde r d\tilde z\\ &= -\frac{1}{2}\int 2\tilde r\,\Theta^2\, d\tilde r d\tilde z\\ &= 0,\end{aligned}$$

since it is the integral of an odd function (recall $\tilde r$ goes from $-\infty$ to $+\infty$).

Term 6.

$$\tilde\gamma_s[(\Theta, \tilde R^2\Sigma) - (\Sigma, \tilde R^2\Theta)] = 0. \tag{11.7}$$

Term 7.

$$\left(\frac{\lambda_s}{\lambda}+b\right)[(\Lambda\epsilon_2, \tilde R^2\Theta) + (\Lambda\epsilon_1, \tilde R^2\Sigma)] \le \left|\frac{\lambda_s}{\lambda}+b\right|\mathcal{E}(t)^{1/2}, \tag{11.8}$$

by Cauchy-Schwarz.

Term 8.

$$\left(\frac{(r_s, z_s)}{\lambda}\cdot\nabla_{(\tilde r,\tilde z)}\epsilon_1, \tilde R^2\Sigma\right) + \left(\frac{(r_s, z_s)}{\lambda}\cdot\nabla_{(\tilde r,\tilde z)}\epsilon_2, \tilde R^2\Theta\right) \tag{11.9}$$

we estimate by

$$\frac{|(r_s, z_s)|}{\lambda}\mathcal{E}(t)^{1/2},$$

similar to *Term 7.*

Term 9.

$$\tilde\gamma_s[(\epsilon_2, \tilde R^2\Sigma) - (\epsilon_1, \tilde R^2\Theta)] \le |\tilde\gamma_s|\mathcal{E}(t)^{1/2}. \tag{11.10}$$

Term 10.

$$(\operatorname{Im}\tilde\Psi, \tilde R^2\Sigma) - (\operatorname{Re}\tilde\Psi, \tilde R^2\Theta) = \operatorname{Im}(\tilde\Psi_b, \tilde R^2\tilde Q_b). \tag{11.11}$$

It turns out that this term is merely zero ($\le \mathcal{E}(t)$), which is shown by substituting the equation for $\tilde\Psi_b$ in terms of $\tilde Q_b$ into (11.11) and applying the property (3.9), i.e.,

$$\operatorname{Im}\int (\tilde r, \tilde z)\cdot\nabla_{(\tilde r,\tilde z)}Q_b\,\overline{\tilde Q_b}\,d\tilde r d\tilde z + \frac{b}{2}\|\tilde R\tilde Q_b\|^2_{L^2} = 0$$

and

$$\frac{\lambda}{\mu}(\partial_r\tilde Q_b, \tilde R^2\tilde Q_b) \le \Gamma_b^{10}.$$

Term 11.

$$(R_1(\epsilon), \tilde R^2\Theta) - (R_2(\epsilon), \tilde R^2\Sigma) \tag{11.12}$$

Recall that $R_1(\epsilon)$ and $R_2(\epsilon)$ consist of quadratic and cubic terms. A typical quadratic term has the form

$$\int |\epsilon|^2|\tilde{Q}_b|^2\, d\tilde{r}d\tilde{z} \le \left(\int_{\tilde{R}<10/b} |\epsilon|^2 e^{-\tilde{R}}\, d\tilde{r}d\tilde{z}\right) \|e^{+\tilde{R}/2}\tilde{Q}_b\|_{L^\infty}^2 \le \mathcal{E}(t).$$

For the cubic terms, we use Cauchy-Schwarz, (5.3) and properties of $\tilde{Q}_b$. A typical term is

$$\begin{aligned}\int |\epsilon|^3|\tilde{Q}_b| d\tilde{r}d\tilde{z} &\le \left(\int_{\tilde{R}<10/b} |\epsilon|^4 d\tilde{r}d\tilde{z}\right)^{1/2}\left(\int |\epsilon|^2|\tilde{Q}_b|^2\right)^{1/2}\\ &\le \left(\int_{\tilde{R}<10/b} |\epsilon|^4 d\tilde{r}d\tilde{z}\right)^{1/2}\left(\int |\epsilon|^2 e^{-\tilde{R}}\right)^{1/2} \|e^{+\tilde{R}/2}\tilde{Q}_b\|_{L^\infty}\\ &\le \mathcal{E}(t).\end{aligned}$$

Collecting the above estimates on terms (11.1)–(11.12), keeping only (11.5), we obtain (11.13)

$$\left|\frac{\lambda_s}{\lambda}+b\right| \lesssim \delta(b)|b_s| + \mathcal{E}(t)^{1/2}\left(\left|\frac{\lambda_s}{\lambda}+b\right| + \frac{|(r_s,z_s)|}{\lambda} + |\tilde{\gamma}_s|\right) + \mathcal{E}(t)^{1/2}\Gamma_b^{\frac{1}{2}(1-C\eta)} + \mathcal{E}(t)\,.$$

11.2. **Computation of** $(r_s,z_s)/\lambda$**.** Multiply the equation for ϵ_1 (6.4) by $(\tilde{r},\tilde{z})\Sigma$ and the equation for ϵ_2 (6.2) by $(\tilde{r},\tilde{z})\Theta$ and add. Note we will have now a vectorial equation and again study each term separately.
Term 1.

$$b_s\left[(\partial_b\Sigma,(\tilde{r},\tilde{z})\Sigma) + (\partial_b\Theta,(\tilde{r},\tilde{z})\Theta)\right] = b_s\,\mathrm{Re}(\partial_b\tilde{Q}_b,(\tilde{r},\tilde{z})\tilde{Q}_b). \tag{11.14}$$

Recall that $\partial_b\tilde{Q}_b \sim \frac{i}{4}\tilde{R}^2Q$, and $\left\|e^{\tilde{R}/2}(\tilde{Q}_b-Q)\right\|_{C^3} \to 0$ as $b\to 0$, hence, we have

$$b_s\,\mathrm{Re}(\partial_b\tilde{Q}_b,(\tilde{r},\tilde{z})\tilde{Q}_b) \approx \mathrm{Re}(\frac{i}{4}\,\tilde{R}^2Q,(\tilde{r},\tilde{z})Q) = 0,$$

or similar to *Term 1* in (11.1), $\sim |b_s|\,\delta(b)$ with $\delta(b)\to 0$ as $b\to 0$.
Term 2. Using ORTH 2 condition, we have

$$\begin{aligned}&(\partial_s\epsilon_1,(\tilde{r},\tilde{z})\Sigma) + (\partial_s\epsilon_2,(\tilde{r},\tilde{z})\Theta)\\ &\quad= \partial_s\left[(\epsilon_1,(\tilde{r},\tilde{z})\Sigma) + (\epsilon_2,(\tilde{r},\tilde{z})\Theta)\right] - b_s\left[(\epsilon_1,(\tilde{r},\tilde{z})\partial_b\Sigma) + (\epsilon_2,(\tilde{r},\tilde{z})\partial_b\Theta)\right]\\ &\quad= -b_s\,\mathrm{Re}(\epsilon,(\tilde{r},\tilde{z})\partial_b\tilde{Q}_b)\\ &\quad\le |b_s|\,\delta(b),\end{aligned} \tag{11.15}$$

since again $\partial_b\tilde{Q}_b \approx \frac{i}{4}\tilde{R}^2Q$ and $\mathrm{Re}(\epsilon,(\tilde{r},\tilde{z})\partial_b\tilde{Q}_b)\approx 0$.

Term 3.

$$
\begin{aligned}
&(-M_-(\epsilon) + b\Lambda\epsilon_1, (\tilde r, \tilde z)\Sigma) + (M_+(\epsilon) + b\Lambda\epsilon_2, (\tilde r, \tilde z)\Theta) \\
&\quad = \; -\operatorname{Im}(\epsilon, (\tilde r, \tilde z)\tilde Q_b) + \operatorname{Im}(\Delta\epsilon, (\tilde r, \tilde z)\tilde Q_b) + \frac{\lambda}{\mu}\operatorname{Im}(\partial_r\epsilon, (\tilde r, \tilde z)\tilde Q_b) \\
&\qquad + \operatorname{Im}(\epsilon, (\tilde r, \tilde z)|\tilde Q_b|^2\tilde Q_b) + \operatorname{Im}(\epsilon, (\tilde r, \tilde z)\, ib\Lambda\tilde Q_b) \\
&\qquad = \operatorname{Im}(\epsilon, (\tilde r, \tilde z)\left[-\tilde Q_b + \Delta Q_b + |\tilde Q_b|^2\tilde Q_b + ib\Lambda\tilde Q_b\right]) \\
&\qquad + 2\operatorname{Im}(\epsilon, \nabla_{(\tilde r,\tilde z)}\tilde Q_b) + \frac{\lambda}{\mu}\operatorname{Im}(\partial_r\epsilon, (\tilde r, \tilde z)\tilde Q_b) \\
&\qquad = \operatorname{Im}\left[-(\epsilon, (\tilde r, \tilde z)\Psi_b) + 2(\epsilon, \nabla_{(\tilde r,\tilde z)}\tilde Q_b) + \frac{\lambda}{\mu}(\partial_r\epsilon, (\tilde r, \tilde z)\tilde Q_b)\right] \\
&\quad = \mathrm{I} + \mathrm{II} + \mathrm{III}.
\end{aligned}
\tag{11.16}
$$

For term I, we use the estimate (11.4) for Ψ_b to get

$$|(\epsilon, (\tilde r, \tilde z)\Psi_b)| \le \Gamma_b^{\frac12(1-C\eta)}\mathcal{E}(t)^{1/2}.$$

The term II (for example, the first coordinate of the vector with $\partial_r\tilde Q_b$) we write as

$$2[(\epsilon_2, \partial_r\Sigma) - (\epsilon_1, \partial_r\Theta)] + (\epsilon_2, \tilde r\Delta\Sigma) - (\epsilon_1, \tilde r\Delta\Theta)$$

and note that $\partial_r\Theta \approx 0$ as $b \to 0$, thus, $|(\epsilon_1, \partial_r\Theta)| \le \delta(b)\mathcal{E}(t)^{1/2}$, and the term $|(\epsilon_2, \partial_r\Sigma)| \le \delta(\alpha^*)\mathcal{E}(t)^{1/2} + \Gamma_b^2$, which we estimated as in (10.2). The last two terms are estimated out by Cauchy-Schwarz and properties QP.

For III, use $\lambda \le \Gamma_b^{10}$ and $\mu \sim 1$ and Cauchy-Schwarz to obtain $|\mathrm{III}| \le \Gamma_b^{10}\mathcal{E}(t)^{1/2}$. Together we have

$$|(11.3)| \le \Gamma_b^{\frac12(1-C\eta)}\mathcal{E}(t)^{1/2} + (\delta(b) + \delta(\alpha^*))\mathcal{E}(t)^{1/2} + \Gamma_b^2 + \Gamma_b^{10}\mathcal{E}(t)^{1/2}.$$

Term 4.

$$\left(\frac{\lambda_s}{\lambda} + b\right)[(\Lambda\Sigma, (\tilde r, \tilde z)\Sigma) + (\Lambda\Theta, (\tilde r, \tilde z)\Theta)] = \left(\frac{\lambda_s}{\lambda} + b\right)\operatorname{Re}(\Lambda\tilde Q_b, (\tilde r, \tilde z)\tilde Q_b) = 0. \tag{11.17}$$

Term 5.

$$
\begin{aligned}
&\left(\frac{1}{\lambda}(r_s, z_s)\cdot\nabla_{(\tilde r,\tilde z)}\Sigma, (\tilde r, \tilde z)\Sigma\right) + \left(\frac{1}{\lambda}(r_s, z_s)\cdot\nabla_{(\tilde r,\tilde z)}\Theta, (\tilde r, \tilde z)\Theta\right) \\
&= \operatorname{Re}\left(\frac{1}{\lambda}(r_s, z_s)\cdot\nabla_{(\tilde r,\tilde z)}\tilde Q_b, (\tilde r, \tilde z)\tilde Q_b\right) = -\frac{(r_s, z_s)}{\lambda}\,\|\tilde Q_b\|^2_{L^2(\tilde r,\tilde z)}.
\end{aligned}
\tag{11.18}
$$

Term 6.

$$\tilde\gamma_s[(\Theta, (\tilde r, \tilde z)\Sigma) - (\Sigma, (\tilde r, \tilde z)\Theta)] = 0. \tag{11.19}$$

Term 7.

$$\left(\frac{\lambda_s}{\lambda}+b\right)[(\Lambda\epsilon_2,(\tilde r,\tilde z)\Theta)+(\Lambda\epsilon_1,(\tilde r,\tilde z)\Sigma)]\le\left|\frac{\lambda_s}{\lambda}+b\right|\mathcal{E}(t)^{1/2}, \tag{11.20}$$

applying ORTH 2 and by Cauchy-Schwarz.
Term 8.

$$\left(\frac{(r_s,z_s)}{\lambda}\cdot\nabla_{(\tilde r,\tilde z)}\epsilon_1,(\tilde r,\tilde z)\Sigma\right)+\left(\frac{(r_s,z_s)}{\lambda}\cdot\nabla_{(\tilde r,\tilde z)}\epsilon_2,(\tilde r,\tilde z)\Theta\right)\le\frac{|(r_s,z_s)|}{\lambda}\mathcal{E}(t)^{1/2}, \tag{11.21}$$

similar to *Term 7.*
Term 9.

$$\tilde\gamma_s[(\epsilon_2,(\tilde r,\tilde z)\Sigma)-(\epsilon_1,(\tilde r,\tilde z)\Theta)]\le|\tilde\gamma_s|\mathcal{E}(t)^{1/2}. \tag{11.22}$$

Term 10.

$$(\operatorname{Im}\tilde\Psi,(\tilde r,\tilde z)\Sigma)-(\operatorname{Re}\tilde\Psi,(\tilde r,\tilde z)\Theta)=\operatorname{Im}(\tilde\Psi_b,(\tilde r,\tilde z)\tilde Q_b). \tag{11.23}$$

Substituting (3.7) for $\tilde\Psi_b$ in terms of $\tilde Q_b$, one can see that all terms are zero (integration by parts or by degeneracy of the momentum QP 3, first property in (3.9)) except for $\frac{\lambda}{\mu}\operatorname{Im}(\partial_r\tilde Q_b,(\tilde r,\tilde z)\tilde Q_b)$ which is bounded by Γ_b^{10} since $\lambda<\Gamma_b^{10}$ (and localization of $\tilde Q_b$ implies $\mu\sim 1$ and boundedness of the inner product).
Term 11.

$$(R_1(\epsilon),(\tilde r,\tilde z)\Theta)-(R_2(\epsilon),(\tilde r,\tilde z)\Sigma)\le\mathcal{E}(t), \tag{11.24}$$

in the same fashion as *Term 11* in (11.12).

Collecting the above estimates on terms (11.14)–(11.24), keeping only (11.18), we obtain

$$\frac{|(r_s,z_s)|}{\lambda}\|\tilde Q_b\|_{L^2}^2\le\delta(b)|b_s|+\mathcal{E}(t)^{1/2}\left(\left|\frac{\lambda_s}{\lambda}+b\right|+\frac{|(r_s,z_s)|}{\lambda}+|\tilde\gamma_s|\right)+\mathcal{E}(t)^{1/2}\Gamma_b^{\frac12(1-C\eta)}+\mathcal{E}(t)\,. \tag{11.25}$$

11.3. **Computation of b_s.** Multiply the equation for ϵ_1 (6.4) by $-\Lambda\Theta$ and the equation for ϵ_2 (6.2) by $\Lambda\Sigma$ and add.
Term 1.

$$b_s\left[(\partial_b\Theta,\Lambda\Sigma)-(\partial_b\Sigma,\Lambda\Theta)\right]=b_s\operatorname{Im}(\partial_b\tilde Q_b,\Lambda\tilde Q_b). \tag{11.26}$$

Recalling that $\partial_b\tilde Q_b\sim\frac{i}{4}\tilde R^2Q$ and $\Lambda\tilde Q_b=\tilde Q_b+\tilde R\partial_R\tilde Q_b$, we estimate (11.26) as

$$b_s\operatorname{Im}(\partial_b\tilde Q_b,\Lambda\tilde Q_b)\approx\frac14\|\tilde R\,Q\|_{L^2(\tilde r,\tilde z)}^2+\frac14\int\tilde R^3Q\,\partial_R\tilde Q_b)\approx|b_s|\,\|\tilde R\,Q\|_{L^2(\tilde r,\tilde z)}^2.$$

Term 2. Using ORTH 4 condition, we have

$$\begin{aligned}
&(\partial_s\epsilon_2,\Lambda\Sigma)-(\partial_s\epsilon_1,\Lambda\Theta)\\
&\quad=\partial_s\left[(\epsilon_2,\Lambda\Sigma)+(\epsilon_1,\Lambda\Theta)\right]+b_s\left[(\epsilon_1,\Lambda\partial_b\Sigma)-(\epsilon_2,\Lambda\partial_b\Theta)\right]\\
&\quad=-b_s\operatorname{Im}(\epsilon,\Lambda\partial_b\tilde{Q}_b)\\
&\quad\le |b_s|\,\mathcal{E}(t)^{1/2},
\end{aligned}\tag{11.27}$$

by Cauchy-Schwarz and using properties of $\tilde{Q}_b$ (e.g., see (95) in [28]).

Term 3.

$$\begin{aligned}
&(M_+(\epsilon)+b\Lambda\epsilon_2,\Lambda\Sigma)+(M_-(\epsilon)-b\Lambda\epsilon_1,\Lambda\Theta)\\
&\quad=\ \operatorname{Re}(\epsilon,\Lambda\tilde{Q}_b)-\operatorname{Re}(\Delta\epsilon,\Lambda\tilde{Q}_b)-\frac{\lambda}{\mu}\operatorname{Re}(\partial_r\epsilon,\Lambda\tilde{Q}_b)\\
&\qquad-\operatorname{Re}(\epsilon,|\tilde{Q}_b|^2\Lambda\tilde{Q}_b+2|\tilde{Q}_b|(\Sigma\Lambda\Sigma+\Theta\Lambda\Theta))\\
&\qquad+b[(\Lambda\epsilon_2,\Lambda\Sigma)-(\Lambda\epsilon_1,\Lambda\Theta)]\\
&\quad=\mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}+\mathrm{V}.
\end{aligned}\tag{11.28}$$

For term III, use $\lambda\le\Gamma_b^{10}$, localization of $\tilde{Q}_b$, thus, $\mu\sim 1$, and Cauchy-Schwarz to obtain $|\mathrm{III}|\le\Gamma_b^{10}\mathcal{E}(t)^{1/2}$. For term IV we use ORTH 4 to obtain

$$\mathrm{IV}=b[((\tilde{r},\tilde{z})\cdot\nabla\epsilon_2,\Lambda\Sigma)-((\tilde{r},\tilde{z})\cdot\nabla\epsilon_1,\Lambda\Theta)]=\operatorname{Re}\Big(-i\,b((\tilde{r},\tilde{z})\cdot\nabla\epsilon,\Lambda\tilde{Q}_b)\Big).$$

For terms I, II and IV we use QP 5 scaling invariance property (3.12):

$$\mathrm{I}+\mathrm{II}+\mathrm{IV}=-\operatorname{Re}(\epsilon,2(\tilde{Q}_b-ib\Lambda\tilde{Q}_b-\Psi_b)-\Lambda\Psi_b-ib\Lambda^2\tilde{Q}_b).$$

Terms with Ψ_b we estimate by (11.4), terms with ib we combine with the term IV above and estimate using BSI 2 (smallness of b), Cauchy-Schwarz and localization properties of $\tilde{Q}_b$, thus, Terms 3 is bounded by $\Gamma_b^{\frac{1}{2}(1-C\eta)}\mathcal{E}(t)^{1/2}$.

Term 4.

$$\left(\frac{\lambda_s}{\lambda}+b\right)[(\Lambda\Theta,\Lambda\Sigma)-(\Lambda\Sigma,\Lambda\Theta)]=0.\tag{11.29}$$

Term 5.

$$\begin{aligned}
&\left(\frac{1}{\lambda}(r_s,z_s)\cdot\nabla_{(\tilde{r},\tilde{z})}\Theta,\Lambda\Sigma\right)-\left(\frac{1}{\lambda}(r_s,z_s)\cdot\nabla_{(\tilde{r},\tilde{z})}\Sigma,\Lambda\Theta\right)\\
&=\operatorname{Im}\left(\frac{1}{\lambda}(r_s,z_s)\cdot\nabla_{(\tilde{r},\tilde{z})}\tilde{Q}_b,\Lambda\tilde{Q}_b\right)=0,
\end{aligned}\tag{11.30}$$

by the degeneracy of the momentum QP 3.

Term 6.

$$-\tilde{\gamma}_s[(\Sigma,\Lambda\Sigma)+(\Theta,\Lambda\Theta)]=-\tilde{\gamma}_s\operatorname{Re}(\tilde{Q}_b,\Lambda\tilde{Q}_b)=0.\tag{11.31}$$

Term 7.

$$\left(\frac{\lambda_s}{\lambda}+b\right)[(\Lambda\epsilon_2,\Lambda\Sigma)-(\Lambda\epsilon_1,\Lambda\Theta)] = \left(\frac{\lambda_s}{\lambda}+b\right)\mathrm{Im}(\Lambda\epsilon,\Lambda\tilde{Q}_b) \leq \left|\frac{\lambda_s}{\lambda}+b\right|\mathcal{E}(t)^{1/2}, \tag{11.32}$$

by Cauchy-Schwarz and closeness of $\tilde{Q}_b$ to Q and properties of Q (as in *Term 2* above).
Term 8.

$$\begin{aligned}&\left(\frac{(r_s,z_s)}{\lambda}\cdot\nabla_{(\tilde{r},\tilde{z})}\epsilon_2,\Lambda\Sigma\right)-\left(\frac{(r_s,z_s)}{\lambda}\cdot\nabla_{(\tilde{r},\tilde{z})}\epsilon_1,\Lambda\Theta\right)\\ &= \mathrm{Im}\left(\frac{(r_s,z_s)}{\lambda}\cdot\nabla_{(\tilde{r},\tilde{z})}\epsilon,\Lambda\tilde{Q}_b\right) \leq \frac{|(r_s,z_s)|}{\lambda}\mathcal{E}(t)^{1/2}.\end{aligned} \tag{11.33}$$

Term 9.

$$-\tilde{\gamma}_s[(\epsilon_1,\Lambda\Sigma)+(\epsilon_2,\Lambda\Theta)] \leq |\tilde{\gamma}_s|\,\mathcal{E}(t)^{1/2}. \tag{11.34}$$

Term 10.

$$(\mathrm{Re}\,\tilde{\Psi},\Lambda\Sigma)+(\mathrm{Im}\,\tilde{\Psi},\Lambda\Theta) = \mathrm{Re}(\tilde{\Psi}_b,\Lambda\tilde{Q}_b), \tag{11.35}$$

which is estimated by $\delta(\alpha^*)$, since $|\tilde{\Psi}_b,\Lambda\tilde{Q}_b| \leq e^{-C/|b|}$, see Lemma 4 in [28].
Term 11.

$$(R_1(\epsilon),\Lambda\Theta)-(R_2(\epsilon),\Lambda\Sigma) \leq \mathcal{E}(t), \tag{11.36}$$

estimating quadratic and cubic in ϵ terms similar to *Term 11* in (11.12).

Collecting the above estimates on terms (11.26)–(11.36), keeping only (11.26), we obtain

$$|b_s|\,\|\tilde{R}Q\|_{L^2}^2 \leq \mathcal{E}(t)^{1/2}\left(\left|\frac{\lambda_s}{\lambda}+b\right|+\frac{|(r_s,z_s)|}{\lambda}+|\tilde{\gamma}_s|+|b_s|\right)+\mathcal{E}(t)^{1/2}\Gamma_b^{\frac{1}{2}(1-C\eta)}+\mathcal{E}(t)\,. \tag{11.37}$$

11.4. **Computation of $\tilde{\gamma}_s$.** Multiply the equation for ϵ_1 (6.4) by $\Lambda^2\Theta$ and the equation for ϵ_2 (6.2) by $-\Lambda^2\Sigma$ and add.
Term 1.

$$b_s\left[(\partial_b\Sigma,\Lambda^2\Theta)-(\partial_b\Theta,\Lambda^2\Sigma)\right] = -b_s\,\mathrm{Im}(\partial_b\tilde{Q}_b,\Lambda^2\tilde{Q}_b) \leq \delta(\alpha^*) \tag{11.38}$$

by the properties of $\tilde{Q}_b$ (see the last estimate in Lemma 4 in [28]).
Term 2. Using ORTH 3 condition, we have

$$\begin{aligned}&(\partial_s\epsilon_1,\Lambda^2\Theta)-(\partial_s\epsilon_2,\Lambda^2\Sigma)\\ &\quad= \partial_s\left[(\epsilon_1,\Lambda^2\Theta)-(\epsilon_2,\Lambda^2\Sigma)\right]-b_s\left[(\epsilon_1,\Lambda^2\partial_b\Theta)-(\epsilon_2,\Lambda^2\partial_b\Sigma)\right]\\ &\quad= -b_s\,\mathrm{Im}(\epsilon,\Lambda^2\partial_b\tilde{Q}_b)\\ &\quad\leq |b_s|\,\mathcal{E}(t)^{1/2}\end{aligned} \tag{11.39}$$

by the estimate $|(\epsilon,P(R)\frac{d^m}{dR^m}\partial_b\tilde{Q}_b(R)| \leq \mathcal{E}(t)^{1/2}$, $0\leq m\leq 2$, from Lemma 4 in [28].

Term 3.

$$
\begin{aligned}
&(-M_-(\epsilon)+b\Lambda\epsilon_1,\Lambda^2\Theta)-(M_+(\epsilon)+b\Lambda\epsilon_2,\Lambda^2\Sigma)\\
&\quad= -(\epsilon_1,\Lambda^2\Sigma)-(\epsilon_2,\Lambda^2\Theta)\\
&\qquad+(\Delta_{(\tilde r,\tilde z)}\epsilon_1,\Lambda^2\Sigma)+(\Delta_{(\tilde r,\tilde z)}\epsilon_2,\Lambda^2\Theta)\\
&\qquad+\frac{\lambda}{\mu}(\partial_{\tilde r}\epsilon_1,\Lambda^2\Sigma)+\frac{\lambda}{\mu}(\partial_{\tilde r}\epsilon_2,\Lambda^2\Theta)\\
&\qquad+\Big((2\Theta^2+|\tilde Q_b|^2)\epsilon_2,\Lambda^2\Theta\Big)+\Big((2\Sigma^2+|\tilde Q_b|^2)\epsilon_1,\Lambda^2\Sigma\Big)\\
&\qquad+(2\Sigma\Theta\epsilon_2,\Lambda^2\Sigma)+(2\Sigma\Theta\epsilon_1,\Lambda^2\Theta)\\
&\qquad+b[(\Lambda\epsilon_2,\Lambda^2\Sigma)-(\Lambda\epsilon_1,\Lambda^2\Theta)]\\
&\quad=\mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}+\mathrm{V}+\mathrm{VI}.
\end{aligned}
\tag{11.40}
$$

For terms III and VI we use the smallness of λ and b, localization of $\tilde Q_b$ and Cauchy-Schwarz to estimate them by $\Gamma_b^{10}\,\mathcal{E}(t)^{1/2}$.

In terms I, II, IV and V we collect separately terms containing ϵ_1 and ϵ_2. Recall the closeness of $\tilde Q_b$ to Q and that Q is real, thus, for example, $\partial_r\Theta\approx 0$ as $b\to 0$ (recall QP 1), and hence, the terms containing $\partial_r\Theta$ will be on the order of $\delta(b)$ - these are the terms containing ϵ_2. The terms with ϵ_1 produce

$$(\epsilon_1,-\Lambda^2\Sigma+\Delta(\Lambda^2\tilde Q_b)+\Lambda^2\Sigma(|\tilde Q_b|^2+2\Sigma^2)+2\Sigma\Theta\Lambda^2\Theta)=(\epsilon_1,L_+(\Lambda^2 Q))+\delta(b),$$

where $L_+=-\Delta+1-3Q^2$, by property QP 1.

Term 4.

$$
\begin{aligned}
&\left(\frac{\lambda_s}{\lambda}+b\right)[(\Lambda\Sigma,\Lambda^2\Theta)-(\Lambda\Theta,\Lambda^2\Sigma)]\\
&=2\left(\frac{\lambda_s}{\lambda}+b\right)(\Lambda\Sigma,(\tilde r,\tilde z)\cdot\nabla(\Lambda\Theta))\\
&\le\left|\frac{\lambda_s}{\lambda}+b\right|\delta(b),
\end{aligned}
\tag{11.41}
$$

again by QP 1, closeness of $\tilde Q_b$ to Q (e.g., the terms such as $\partial_r\Theta\approx 0$ as $b\to 0$).

Term 5.

$$\left(\frac{1}{\lambda}(r_s,z_s)\cdot\nabla_{(\tilde r,\tilde z)}\Sigma,\Lambda^2\Theta\right)-\left(\frac{1}{\lambda}(r_s,z_s)\cdot\nabla_{(\tilde r,\tilde z)}\Theta,\Lambda^2\Sigma\right)\le\frac{|(r_s,z_s)|}{\lambda}\,\delta(b),\tag{11.42}$$

by QP 1 similar to *Term 4*.

Term 6.

$$\tilde\gamma_s[(\Sigma,\Lambda^2\Sigma)+(\Theta,\Lambda^2\Theta)]=\tilde\gamma_s(\|\Lambda Q\|_{L^2}^2+\delta(b)),\tag{11.43}$$

by integration by parts and closeness of $\tilde Q_b$ to Q.

Term 7.

$$\left(\frac{\lambda_s}{\lambda}+b\right)[(\Lambda\epsilon_1,\Lambda^2\Theta)-(\Lambda\epsilon_2,\Lambda^2\Sigma)] \le \left|\frac{\lambda_s}{\lambda}+b\right| \mathcal{E}(t)^{1/2}, \tag{11.44}$$

by Cauchy-Schwarz and properties of $\tilde{Q}_b$.
Term 8.

$$\left(\frac{(r_s,z_s)}{\lambda}\cdot\nabla_{(\tilde{r},\tilde{z})}\epsilon_1,\Lambda^2\Theta\right)-\left(\frac{(r_s,z_s)}{\lambda}\cdot\nabla_{(\tilde{r},\tilde{z})}\epsilon_2,\Lambda^2\Sigma\right) \le \frac{|(r_s,z_s)|}{\lambda}\mathcal{E}(t)^{1/2}, \tag{11.45}$$

similar to *Term 7.*
Term 9.

$$\tilde{\gamma}_s[(\epsilon_2,\Lambda^2\Theta)+(\epsilon_1,\Lambda^2\Sigma)] \le |\gamma_s|\,\mathcal{E}(t)^{1/2}. \tag{11.46}$$

Term 10.

$$(\operatorname{Im}\tilde{\Psi}_b,\Lambda^2\Theta)+(\operatorname{Re}\tilde{\Psi}_b,\Lambda^2\Sigma)=\operatorname{Re}(\tilde{\Psi}_b,\Lambda^2\tilde{Q}_b)\le\delta(\alpha^*). \tag{11.47}$$

Term 11.

$$-(R_1(\epsilon),\Lambda^2\Sigma)-(R_2(\epsilon),\Lambda^2\Theta)\le\mathcal{E}(t), \tag{11.48}$$

in the same fashion as *Term 11* in (11.12).

Collecting the above estimates on terms (11.38)–(11.48), keeping only (11.43) and the estimate for (11.40), we obtain

$$\begin{aligned}&\left|\tilde{\gamma}_s-\frac{1}{\|\Lambda Q\|_{L^2}^2}(\epsilon_1,L_+(\Lambda^2 Q))\right|\\ &\le \mathcal{E}(t)^{1/2}\left(|b_s|+\left|\frac{\lambda_s}{\lambda}+b\right|+\frac{|(r_s,z_s)|}{\lambda}+|\tilde{\gamma}_s|\right)+\mathcal{E}(t)^{1/2}\Gamma_b^{\frac{1}{2}(1-C\eta)}+\mathcal{E}(t)\,.\end{aligned} \tag{11.49}$$

We finish this section by observing that solving the system of equations (11.13)-(11.25)-(11.37)-(11.49) for parameters $(b_s,\frac{\lambda_s}{\lambda}+b,\frac{(r_s,z_s)}{\lambda},\tilde{\gamma}_s)$ gives (5.6) and (5.7).

12. Bootstrap Step 7. Deduction of BSO 4 from (5.6)

From BSI 3: $\mathcal{E}(t)\le\Gamma_{b(t)}^{3/4}$, we deduce from (5.6) the estimate

$$\left|\frac{\lambda_s}{\lambda}+b\right|+|b_s|\le\Gamma_{b(t)}^{1/2}.$$

Direct computation gives

$$\begin{aligned}\frac{d}{ds}(\lambda^2 e^{5\pi/b}) &= 2\lambda^2 e^{5\pi/b}\left(\frac{\lambda_s}{\lambda}-\frac{5\pi b_s}{2b^2}\right)\\ &= 2\lambda^2 e^{5\pi/b}\left(\frac{\lambda_s}{\lambda}+b-b-\frac{5\pi b_s}{2b^2}\right).\end{aligned} \tag{12.1}$$

Using that

$$\left|\frac{\lambda_s}{\lambda}+b\right|\le \Gamma_{b(t)}^{1/2}$$

and (from ZP 2)

$$\left|\frac{5\pi b_s}{2b^2}\right|\le \frac{\Gamma_{b(t)}^{1/2}}{b(t)^2}\le e^{-\pi/4b},$$

we get that (12.1) implies

$$\frac{d}{ds}(\lambda^2 e^{5\pi/b})\le -\lambda^2 b e^{5\pi/b}\le 0$$

and, integrating in s, we get

$$\lambda^2(t)e^{5\pi/b(t)}\le \lambda^2(0)e^{5\pi/b(0)}.$$

Since ZP 2: $e^{-5\pi/(4\,b(t))}\le \Gamma_{b(t)}\le e^{-3\pi/(4\,b(t))}$, we have

$$\frac{\lambda^2(t)|E_0|}{\Gamma_{b(t)}^4}\le \lambda^2(t)|E_0|e^{5\pi/b(t)}\le \lambda^2(0)|E_0|e^{5\pi/b(0)}\le \lambda^2(0)|E_0|\Gamma_{b(t)}^{-20/3}\le 1$$

by IDA 5, which gives BSO 4.

13. Bootstrap Step 8. The local virial inequality

Multiply the ϵ_1 equation (6.2) by $-\Lambda\Theta$, multiply the ϵ_2 equation by $\Lambda\Sigma$, and add. We analyze the resulting terms one at a time.

$$\mathrm{I}=b_s[(-\partial_b\Sigma,\Lambda\Theta)+(\partial_b\Theta,\Lambda\Sigma)].$$

Since, for b small, $\partial_b\tilde{Q}_b\sim \frac{i\tilde{R}^2}{4}Q$, we have $\partial_b\Sigma\approx 0$ and $\partial_b\Theta\approx\frac{R^2}{4}Q$, and thus, the above is essentially

$$\frac{1}{4}b_s(\tilde{R}^2Q,\Lambda Q)=-b_s\|\tilde{R}Q\|_{L^2}^2$$

by integration by parts.

The next term is

$$\begin{aligned}\mathrm{II}&=-(\partial_s\epsilon_1,\Lambda\Theta)+(\partial_s\epsilon_2,\Lambda\Sigma)\\&=\partial_s[-(\epsilon_1,\Lambda\Theta)+(\epsilon_2,\Lambda\Sigma)]+b_s[-(\epsilon_1,\Lambda\partial_b\Theta)-(\epsilon_2,\Lambda\partial_b\Sigma)]\end{aligned}$$

which, by ORTH 4 is

$$=b_s[-(\epsilon_1,\Lambda\partial_b\Theta)-(\epsilon_2,\Lambda\partial_b\Sigma)]\le |b_s|\mathcal{E}(t)^{1/2}.$$

We continue, following the proof of [27, Lemma 8] or [28, Prop. 3]. Several terms come from $(M_-(\epsilon),\Lambda\Theta)+(M_+(\epsilon),\Lambda\Sigma)$ by taking adjoints using QP 5.

Terms requiring special attention in our case are:

(1) The local linear and quadratic in ϵ terms

$$-2(\epsilon_1,\Sigma)-2(\epsilon_2,\Theta)-3\int Q^2\epsilon_1^2-\int Q^2\epsilon_2^2.$$

These are related to the nonlocal quadratic-in-ϵ term $\int|\nabla_{\tilde r,\tilde z}\epsilon|^2\mu(\tilde r)\,d\tilde r\,d\tilde z$, but with favorable sign, via energy conservation (5.4). There remain quadratic terms

$$\int \tilde r Q^3\partial_{\tilde r}Q\epsilon_1^2+\int \tilde r Q^3\partial_{\tilde r}Q\epsilon_1^2$$

that are combined with $\int|\nabla_{\tilde r,\tilde z}\epsilon|^2\mu(\tilde r)\,d\tilde r\,d\tilde z$ to form $H(\epsilon,\epsilon)$, to which the spectral hypothesis can be applied. The $H(\epsilon,\epsilon)$ term is nonlocal, and the spectral hypothesis can only be applied locally, so this term is addressed with cutoffs as in Raphael [33].

(2) Terms in which $\frac{\lambda}{r}\partial_r\epsilon$ and $\frac{\lambda}{r}\partial_r\tilde Q_b$ are paired with $\tilde Q_b$. These are easily estimated due to the λ factor and the localization to the singular ring.

(3) $-(\epsilon_1,\mathrm{Re}(\Lambda\Psi))-(\epsilon_2,\mathrm{Im}(\Lambda\Psi))$. These terms are local and thus treated as in [27].

14. Bootstrap Step 9. Lower bound on $b(s)$

Using (5.8) (the local virial inequality), we obtain

$$\begin{aligned}\frac{d}{ds}(e^{\frac{3\pi}{4b}})&=-\frac{3\pi}{4b^2}e^{3\pi/4b}b_s\\&\le-\frac{3\pi}{4b^2}e^{3\pi/4b}(\delta_0\mathcal{E}(t)-\Gamma_b^{1-C\eta})\\&\le\frac{3\pi}{4b^2}e^{3\pi/4b}\Gamma_b^{1-C\eta}\\&\le 1\end{aligned}$$

by ZP 2 in the last step. We now integrate this to obtain

$$e^{3\pi/4b(s)}\le e^{3\pi/4b(s_0)}+s-s_0=s$$

by the definition of s_0 in (3.2). Thus, (5.9) follows. Now from the scaling law[3]

$$\begin{aligned}\left|\frac{\lambda_s}{\lambda}+b\right|\le\Gamma_b^{1/2}\le e^{-\frac{\pi}{4b(s)}}\le\frac{5}{9}\,b\\ \implies -\frac{14}{9}\,b\le\frac{\lambda_s}{\lambda}\le-\frac{4}{9}\,b\\ \implies -\frac{\lambda_s}{\lambda}\ge\frac{4}{9}\,b.\end{aligned}$$

[3]The constant $\frac{5}{9}$ is not very important, since $e^x/x\gg1$ for large x; it is chosen for convenience of calculations, e.g., in [33] it is chosen to be $\frac{1}{3}$.

Integrating this in s and inserting (5.9), we get

$$-\log\lambda(s) \geq -\log\lambda(s_0) + \int_{s_0}^{s} \frac{\pi}{3\log\sigma}\,d\sigma.$$

But

$$\int_{s_0}^{s}\frac{d\sigma}{\log\sigma} \geq \int_{s_0}^{s}\frac{\log\sigma - 1}{(\log\sigma)^2}\,d\sigma = \frac{s}{\log s} - \frac{s_0}{\log s_0},$$

and thus,

$$-\log\lambda(s) \geq -\log\lambda(s_0) + \frac{\pi}{3}\left(\frac{s}{\log s} - \frac{s_0}{\log s_0}\right).$$

By IDA 6: $0 < \lambda_0 < \exp(-\exp 8\pi/9b_0)$ and the definition of s_0: $s_0 = e^{3\pi/4b_0}$, we get $\lambda_0 = \lambda(s_0) \geq s_0^{32/37}$, and thus,

$$-\log\lambda(s) \geq -\frac{1}{2}\log\lambda(s_0) + \frac{\pi s}{3\log s} + \left(-\frac{1}{2}\log\lambda(s_0) - \frac{\pi}{3}\frac{s_0}{\log s_0}\right)$$

and since $s_0 \gg 1$ (or $b_0 \ll 1$), the term in parentheses is positive and we have

$$-\log\lambda(s) \geq -\frac{1}{2}\log\lambda(s_0) + \frac{\pi s}{3\log s}. \tag{14.1}$$

Exponentiating this equation gives (5.10). Now we show how (5.9) and (5.10) imply BSO 6. Recall (5.9) and (5.10) and that

$$\text{BSO 6} \iff \frac{\pi}{5} \leq b(t)\log|\log\lambda(t)|.$$

Subtract $\log s$ from both sides of (14.1) to get

$$\begin{aligned}-\log(s\lambda(s)) &= -\log s - \log\lambda(s)\\ &\geq -\log s - \frac{1}{2}\log\lambda(s_0) + \frac{\pi}{3}\frac{s}{\log s}.\end{aligned}$$

Since $\lambda(s_0) \ll 1$, we have $-\frac{1}{2}\log\lambda(s_0) \geq 0$, and hence,

$$-\log(s\lambda(s)) \geq \frac{\pi}{3}\frac{s}{\log s} - \log s \geq \frac{\pi}{3.1}\frac{s}{\log s}.$$

Taking the log, we obtain

$$\log(-\log(s\lambda(s))) \geq \log\frac{\pi}{3.1} + \log\left(\frac{s}{\log s}\right) \geq \frac{1}{2}\log s.$$

By (5.9),

$$\log(-\log(s\lambda(s))) \geq \frac{3\pi}{8b(s)}. \tag{14.2}$$

Since $s \gg 1$ and $s\lambda(s) \ll 1$, we have $-\log\lambda(s) \geq -\log(s\lambda(s))$, and BSO 6 follows.

15. Bootstrap Step 10. Control on the radius of concentration

The inequality (5.7) implies

$$\left|\frac{(r_s,z_s)}{\lambda}\right| \le \delta(\alpha^*)\mathcal{E}(t)^{1/2} + \Gamma_b^{1-C\eta} \le 1.$$

Thus,

$$\begin{aligned} |r(s)-r_0| &\le \int_{s_0}^{s} |r_s(\sigma)|\,d\sigma \\ &\le \int_{s_0}^{s} \lambda(\sigma)\,d\sigma \\ &\le \int_{2}^{+\infty} \sqrt{\lambda_0} e^{-\frac{\pi}{3}\frac{\sigma}{\log\sigma}}\,d\sigma \\ &\le \alpha^*, \end{aligned}$$

where we applied (5.10). It similarly follows that

$$|z(s)-z_0| \le \alpha^*\,.$$

16. Bootstrap Step 11. Momentum conservation implies BSO 5

Recall that the function $\psi(x,y,z)$ is an axially symmetric cutoff in r only, and is independent of z. Also recall the convention that the default (no subscript) notation is for xyz space, i.e., $\Delta = \partial_x^2+\partial_y^2+\partial_z^2$, $\nabla = (\partial_x,\partial_y,\partial_z)$, and $\int(\cdots)$ will mean $\int(\cdots)dxdydz$. Also note that (for axially symmetric u) we have $|\nabla u|^2 = |\nabla_{(r,z)}u|^2$. Pair the NLS equation with $\frac12\Delta\psi\bar u + \nabla\psi\cdot\nabla\bar u$, integrate and take the real part to obtain

$$\begin{aligned} \operatorname{Im}\int \partial_t u(\tfrac12\Delta\psi\bar u + \nabla\psi\cdot\nabla\bar u) & \\ = \ \operatorname{Re}\int \Delta u(\tfrac12\Delta\psi\bar u + \nabla\psi\cdot\nabla\bar u)& \\ + \operatorname{Re}\int |u|^2u(\tfrac12\Delta\psi\bar u + \nabla\psi\cdot\nabla\bar u)& \end{aligned}$$

that we write as I = II + III. We first note that by integration by parts, we have

$$\mathrm{I} = \frac12\partial_t\int \nabla\psi\cdot\nabla u\;\bar u.$$

Also, by integration by parts:

$$\mathrm{II.1} = -\frac14\int\Delta^2\psi\;|u|^2 + \frac12\int\Delta\psi\;|\nabla u|^2.$$

We convert the second term in II.1 to cylindrical coordinates to get

$$\mathrm{II.1} = -\frac14\int\Delta^2\psi\;|u|^2 + \frac12\int\partial_r^2\psi\;|\nabla u|^2 r drdz + \frac12\int\partial_r\psi\;|\nabla u|^2 drdz.$$

Since $\partial_z\psi = 0$, we calculate by switching to (r,z) coordinates and integrating by parts in this setting:

$$\text{II.2} = \frac{1}{2}\int \partial_r^2\psi(|\partial_r u|^2 - |\partial_z u|^2) r dr dz - \frac{1}{2}\int \partial_r\psi |\nabla_{(r,z)} u|^2 dr dz.$$

Adding these, we get

$$\text{II} = -\frac{1}{4}\int \Delta^2\psi \, |u|^2 + \int \partial_r^2\psi \, |\partial_r u|^2.$$

The two terms in III are easily manipulated (using integration by parts for the second one in the xyz variables):

$$\text{III} = -\frac{1}{4}\int \Delta\psi \, |u|^4.$$

Pulling it all together, we have

$$\frac{1}{2}\partial_t \ \text{Im} \int \nabla\psi \cdot \nabla u \ \bar{u} = -\frac{1}{4}\int \Delta^2\psi \, |u|^2 + \int \partial_r^2\psi \, |\partial_r u|^2 - \frac{1}{4}\int \Delta\psi \, |u|^4.$$

Thus, using the axial exterior Gagliardo-Nirenberg inequality (Lemma 2.1) to control the 4th power term, we obtain

$$\left|\partial_t \ \text{Im} \int \nabla\psi \cdot \nabla u \ \bar{u}\right| \leq \|\nabla u(t)\|_{H^1}^2 \leq c\frac{1}{\lambda^2(t)},$$

where in the last step we used the decomposition (6.1) and the estimate $\|\nabla\epsilon\|_{L^2} \leq 1$ furnished by Lemma 5.1. Since we explicitly chose the origin of the rescaled time as $s_0 = e^{3\pi/4b_0}$, we have

$$\int_0^t \frac{d\tau}{\lambda^2(\tau)} = \int_{s_0}^s d\sigma = s - s_0 \leq s.$$

We have, $\forall\, t \in [0, t_1)$, that

$$\lambda(t)\left|\text{Im}\int \nabla\psi \cdot \nabla u(t) \ \bar{u}(t)\right| \leq \lambda(t)\left|\text{Im}\int \nabla\psi \cdot \nabla u_0 \ \bar{u}_0\right| + c\lambda(t)s(t). \tag{16.1}$$

From (14.2) it follows that[4]

$$\forall\, s \in [s_0, s_1), \quad s\lambda(s) \leq \Gamma_{b(s)}^{10}. \tag{16.2}$$

Next, we have

$$\begin{aligned}\partial_s(\lambda e^{6\pi/b}) &= \lambda e^{6\pi/b}\left(\frac{\lambda_s}{\lambda} - \frac{6\pi b_s}{b^2}\right) \\ &= \lambda e^{6\pi/b}\left(\frac{\lambda_s}{\lambda} + b - \frac{6\pi b_s}{b^2}\right) - \lambda b e^{6\pi/b}.\end{aligned}$$

By (5.6), BSI 3, we have $|\lambda_s/\lambda + b| + |b_s| \leq \Gamma_{b(t)}^{3/4}$. Also, ZP 2 implies that $1/b \sim \ln(\Gamma_{b(t)}^{-1})$, and thus, $|6\pi b_s/b^2| \leq \Gamma_{b(t)}^{1/2}$. This shows that the first term is dominated by the second

[4]The power of Γ on the right side can be 5 or higher.

in the above display, i.e., that $\partial_s(\lambda e^{6\pi/b}) \leq -\frac{1}{2}\lambda b e^{6\pi/b} < 0$, and thus, for all $s > s_0$, we have $\lambda(s)e^{6\pi/b(s)} \leq \lambda_0 e^{6\pi/b_0}$. Using that $\Gamma_{b(t)}^{-5} \sim e^{5\pi/b(t)}$ (again a consequence of ZP 2), we obtain

$$\begin{aligned}\lambda(t)\Gamma_{b(t)}^{-5}&\left|\operatorname{Im}\int \nabla\psi\cdot\nabla u_0\,\bar{u}_0\right|\\ &\leq \lambda(t)e^{6\pi/b(t)}\left|\operatorname{Im}\int \nabla\psi\cdot\nabla u_0\,\bar{u}_0\right|\\ &\leq \lambda_0 e^{6\pi/b_0}\left|\operatorname{Im}\int \nabla\psi\cdot\nabla u_0\,\bar{u}_0\right|\\ &\leq e^{6\pi/b_0}\Gamma_{b_0}^{10}\end{aligned}$$

by IDA 5. Now since ZP 2 implies $\Gamma_{b_0}^{10} \sim e^{-10\pi/b_0}$, we have that the above is ≤ 1, or

$$\lambda(t)\left|\operatorname{Im}\int \nabla\psi\cdot\nabla u_0\,\bar{u}_0\right| \leq \Gamma_{b(t)}^{5}. \tag{16.3}$$

Inserting (16.3) and (16.2) into (16.1), we get

$$\lambda(t)\left|\operatorname{Im}\int \nabla\psi\cdot\nabla u(t)\,\bar{u}(t)\right| \leq \Gamma_{b(t)}^{5}.$$

17. Bootstrap Step 12. Refined virial inequality in the radiative regime

The goal of this section is to prove (5.11). This is the "radiative virial estimate" which will later be combined with (5.12) to obtain (5.13), which is a refinement (using the Lyapunov functional $\mathcal{J}$ in place of b^2) to the virial estimate (5.8). Once again, the spectral property is a key ingredient. The idea in (5.11), (5.12), (5.13) is that instead of modeling the solution as $\tilde{Q}_b+\epsilon$, where $\tilde{Q}_b$ is supported inside radius $\tilde{R} \leq \frac{2}{b}$, we model the solution as $\tilde{Q}_b + \tilde{\zeta}_b + \tilde{\epsilon}$, where $\tilde{\zeta}$ corrects the solution in the "radiative" region $\frac{2}{b} \leq \tilde{R} \ll e^{-\pi/b}$. There is still plenty of distance between the radiative region $\frac{2}{b} \leq \tilde{R} \ll e^{-\pi/b}$ and the radius $\frac{1}{\lambda}$, which corresponds to a unit-sized distance from the blow-up core in the original scale.

It amounts to taking the inner product of the ϵ equation with suitable directions built on $\tilde{Q}_b + \tilde{\zeta}$. These computations are carried out in Lemma 6 of [24]. First, we proceed along the lines of Step 1–4 in the proof of Lemma 6 in [24]. Since the function $f_1(s)$ is supported in $B(0,2A)$, far inside the $\frac{1}{\lambda}$ width in $\tilde{R}$, all interaction estimates are two dimensional – the factor $\mu(\tilde{r})$ can be inserted up to possible correction terms with a λ coefficient, which can easily be estimated using (5.1) ($\lambda \leq \Gamma_b^{10}$). The result is (as at the beginning of Step 4 in the proof of Lemma 6 in [24], equation (4.18)), the bound

$$\begin{aligned}\partial_s f_1 \geq\ & H(\tilde{\epsilon},\tilde{\epsilon}) - \frac{1}{\|\Lambda Q\|_{L^2}^2}(\tilde{\epsilon}_1, L_+\Lambda^2 Q)(\tilde{\epsilon}_1,\Lambda Q) - C\lambda^2 E_0 \\ & + (\epsilon_1, \Lambda F_{\mathrm{re}}) + (\epsilon_2, \Lambda F_{\mathrm{im}}) - (\tilde{\zeta}_{\mathrm{re}}, \Lambda F_{\mathrm{re}}) - (\tilde{\zeta}_{\mathrm{im}}, \Lambda F_{\mathrm{im}}) \\ & - \delta(\alpha^*)(\tilde{\mathcal{E}}(s) + \lambda^2 |E_0|) - \Gamma_b^{1+z_0}.\end{aligned}$$

By the spectral property, we have

$$H(\tilde{\epsilon},\tilde{\epsilon}) - \frac{1}{\|\Lambda Q\|_{L^2}^2}(\tilde{\epsilon}_1, L_+\Lambda^2 Q)(\tilde{\epsilon}_1,\Lambda Q) \geq \tilde{\delta}_1 \tilde{\mathcal{E}} - C\lambda^2 E_0 - \delta_2 \Gamma_b.$$

We also have

$$|(\epsilon_1, \Lambda F_{\mathrm{re}}) + (\epsilon_2, \Lambda F_{\mathrm{im}})| \leq C\Gamma_b^{1/2}\left(\int_A^{2A} |\epsilon|^2\right)^{1/2} \leq \delta_2 \Gamma_b + \frac{1}{\delta_2}\int_A^{2A} |\epsilon|^2 .$$

A flux-type computation (see the (4.20) and its proof in [24])

$$-(\tilde{\zeta}_{\mathrm{re}}, \Lambda F_{\mathrm{re}}) - (\tilde{\zeta}_{\mathrm{im}}, \Lambda F_{\mathrm{im}}) \geq c\Gamma_b .$$

Combining these elements gives (5.11).

18. Bootstrap Step 13. L^2 dispersion at infinity in space

In this step, we prove (5.12). This gives us control of $\int_{A\leq \tilde{R}\leq 2A} |\epsilon|^2$ in terms of the s-derivative of the 3d L^2 norm of ϵ *outside* the radiative region. Since this involves estimating ϵ on the whole space, the factor $\mu(\tilde{r})$ is crucial, and we fully write out this step. With

$$\phi_4\left(\frac{\tilde{r}}{A}, \frac{\tilde{z}}{A}\right) = \phi_4\left(\frac{r-r(t)}{\lambda(t)A(t)}, \frac{z-z(t)}{\lambda(t)A(t)}\right).$$

we write

$$\begin{aligned}&4\pi \frac{d}{dt}\int \phi_4 |u|^2\, r dr dz \\ &\quad = -4\pi \int \nabla\phi_4 \cdot (\partial_t \alpha, \partial_t \beta)|u|^2\, r dr dz + \frac{1}{4\pi\lambda A}\,\mathrm{Im}\int (\nabla\phi_4\cdot\nabla u)\bar{u}\, r dr dz,\end{aligned}$$

where $\alpha(t) = \frac{r-r(t)}{A(t)\lambda(t)}$ and $\beta = \frac{z-z(t)}{A(t)\lambda(t)}$. Using that $\frac{ds}{dt} = \frac{1}{\lambda^2}$, we compute

$$(\alpha_t, \beta_t) = -\frac{1}{\lambda^2}\left(\frac{\lambda_s}{\lambda} + \frac{A_s}{A}\right)\left(\frac{r-r(s)}{\lambda}, \frac{z-z(s)}{\lambda}\right) - \frac{1}{\lambda^2}\left(\frac{r_s}{\lambda}, \frac{z_s}{\lambda}\right).$$

We have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{ds}\int \phi_4\left(\frac{\tilde{R}}{A}\right)|\epsilon|^2\mu(\tilde{r})\,d\tilde{r}\,d\tilde{z} \\
&\quad = \frac{1}{A}\operatorname{Im}\int (\nabla\phi_4\cdot\nabla\epsilon)\,\bar{\epsilon}\,\mu(\tilde{r})\,d\tilde{r}\,d\tilde{z} + \frac{b}{2}\int \frac{\Lambda^2}{A}\cdot\nabla\phi_4\left(\frac{(\tilde{r},\tilde{z})}{A}\right)|\epsilon|^2\,\mu(\tilde{r})\,d\tilde{r}\,d\tilde{z} \\
&\quad\quad - \frac{1}{2A}\int\left(\frac{\lambda_s}{\lambda}+b+\frac{A_s}{A}\right)(\tilde{r},\tilde{z})\cdot\nabla\phi_4\Big(\frac{\tilde{R}}{A}\Big)|\epsilon|^2\mu(\tilde{r})\,d\tilde{r}\,dz \\
&\quad\quad - \frac{1}{2A}\int \frac{(r_s,z_s)}{\lambda}\cdot\nabla\phi_4\Big(\frac{\tilde{R}}{A}\Big)|\epsilon|^2\mu(\tilde{r})\,d\tilde{r}\,dz.
\end{aligned}$$

Using that $A_s/A = -b_s/b^2$ and (5.6), we obtain that

$$|\mathrm{III}| \leq \frac{b}{40}\int |\nabla_{\tilde{r},\tilde{z}}\phi_4(\tilde{R}/A)|\epsilon|^2\mu(\tilde{r})d\tilde{r}\,d\tilde{z}.$$

Using ((5.7) $\implies |(r_s,z_s)|/\lambda \leq 1$), we obtain

$$|\mathrm{IV}| \leq \lambda\|\tilde{u}\|_{L^2_{xyz}} \leq \Gamma_b^2.$$

Term I is estimated via Cauchy-Schwarz and basic interpolation

$$|\mathrm{I}| \leq \frac{b}{40}\int |\nabla\phi_4(\tilde{R}/A)||\epsilon|^2\mu(\tilde{r})d\tilde{r}d\tilde{z} + \Gamma_b^{a/2}\int |\nabla\epsilon|^2\mu(\tilde{r})\,d\tilde{r}\,dz\,.$$

For Term II, we use that

$$\frac{(\tilde{r},\tilde{z})}{A}\cdot\nabla\phi_4\left(\frac{(\tilde{r},\tilde{z})}{A}\right) \geq \begin{cases}\frac{1}{4} & \text{for } A\leq|(\tilde{r},\tilde{z})|\leq 2A\\ 0 & \text{otherwise}\end{cases}$$

and the fact that $\mu(\tilde{r})\sim 1$ for $|(\tilde{r},\tilde{z})|\sim A$ to obtain

$$\mathrm{II} \geq \frac{b}{30}\int_{A\leq|(\tilde{r},\tilde{z})|\leq 2A}|\epsilon|^2 d\tilde{r}\,d\tilde{z}\,,$$

finishing the proof of (5.12).

19. Bootstrap Step 14. Lyapunov functional in H^1

In this section, we exhibit the Lyapunov function $\mathcal{J}\sim b^2$ and prove the upper bound on $\partial_s\mathcal{J}$ given in (5.13). This is obtained by combining (5.11) (virial identity for dynamics in the soliton core) and (5.12) (dispersion relation in the radiative regime) and the L^2 conservation (a global quantity that links the two).

Let

$$\begin{aligned}\mathcal{J}(s) = & \int |\tilde{Q}_b|^2 - \int Q^2 + 2(\epsilon_1, \Sigma) + 2(\epsilon_2, \Theta) \\ & + \frac{1}{r(s)} \int (1 - \phi_4 \left(\frac{(\tilde{r}, \tilde{z})}{A} \right) |\epsilon|^2 \mu(\tilde{r}) \, d\tilde{r} \, d\tilde{z} \\ & - \frac{\delta_1}{800} \left(b \tilde{f}_1(b) - \int_0^b \tilde{f}_1(v) dv + b(\epsilon_2, \Lambda \tilde{\zeta}_{\rm re}) - b(\epsilon_1, \Lambda \tilde{\zeta}_{\rm im}) \right),\end{aligned}$$

where

$$\tilde{f}_1(b) = \frac{b}{4} \| y \tilde{Q}_b \|_{L^2}^2 + \frac{1}{2} \operatorname{Im} \int (\tilde{r}, \tilde{z}) \cdot \nabla \tilde{\zeta} \, \bar{\tilde{\zeta}} \, .$$

The argument follows the proof of Prop. 4 in [24]. Multiply (5.11) by $\delta_1 b/800$ and sum with (5.12) to obtain

(19.1)

$$\begin{aligned}&\partial_s \left(\frac{1}{r(s)} \int \phi_4 \left(\frac{(\tilde{r}, \tilde{z})}{A} \right) |\epsilon|^2 \mu(\tilde{r}) \, d\tilde{r} \, d\tilde{z} \right) + \frac{\delta_1 b}{800} \partial_s f_1 \\ & \quad \geq \frac{\delta_1^2 b}{800} \tilde{\mathcal{E}} + \frac{b}{800} \int_{A \leq \tilde{R} \leq 2A} |\epsilon|^2 d\tilde{r} \, d\tilde{z} + \frac{c\delta_1 b}{1000} \Gamma_b - \frac{c}{b^2} \lambda^2 E_0 - \Gamma_b^{a/2} \int |\nabla_{(\tilde{r}, \tilde{z})} \epsilon|^2 \mu(\tilde{r}) \, d\tilde{r} \, d\tilde{z}.\end{aligned}$$

The last term on the right side is estimated as

$$\begin{aligned}\Gamma_b^{a/2} \int |\nabla_{(\tilde{r}, \tilde{z})} \epsilon|^2 \mu(\tilde{r}) \, d\tilde{r} \, d\tilde{z} & \leq \Gamma_b^{a/2} \left(\Gamma_b^{1 - C\eta} + \int |\nabla_{(\tilde{r}, \tilde{z})} \tilde{\epsilon}|^2 \mu(\tilde{r}) d\tilde{r} \, d\tilde{z} \right) \\ & \leq \Gamma_b^{1 + a/4} + \Gamma_b^{a/2} \int |\nabla_{(\tilde{r}, \tilde{z})} \tilde{\epsilon}|^2 \mu(\tilde{r}) \, d\tilde{r} \, d\tilde{z}.\end{aligned}$$

Using that

$$f_1(s) = \tilde{f}_1(s) + (\epsilon_2, \Lambda \tilde{\zeta}_{\rm re}) - (\epsilon_1, \Lambda \tilde{\zeta}_{\rm im}) \, ,$$

we also rewrite the second term on the left side as

$$\begin{aligned}b \partial_s f_1 & = \partial_s (b f_1) - b_s f_1 \\ & = \partial_s (b f_1) - b_s \tilde{f}_1 - b_s [(\epsilon_2, \Lambda \tilde{\zeta}_{\rm re}) - (\epsilon_1, \Lambda \zeta_{\rm im})] \\ & = \partial_s \left(b \tilde{f}_1(b) - \int_0^b \tilde{f}_1(v) \, dv + b(\epsilon_2, \Lambda \tilde{\zeta}_{\rm re}) - b(\epsilon_1, \Lambda \zeta_{\rm im}) \right) \\ & \quad - b_s [(\epsilon_2, \Lambda \tilde{\zeta}_{\rm re}) - (\epsilon_1, \Lambda \zeta_{\rm im})].\end{aligned}$$

Formula (19.1) becomes, using the bound on $|b_s|$ in (5.6),

$$\begin{aligned}
&\partial_s\Bigg(\frac{1}{r(s)}\int \phi_4(\frac{\tilde{R}}{A})|\epsilon|^2\mu(\tilde{r})\,d\tilde{r}\,d\tilde{z} \\
&\quad + \frac{\delta_1}{800}\left[b\tilde{f}_1(b) - \int_0^b \tilde{f}_1(v)dv + b(\epsilon_2,\Lambda\tilde{\zeta}_{\mathrm{re}}) - b(\epsilon_1,\Lambda\tilde{\zeta}_{\mathrm{im}})\right]\Bigg) \\
&\qquad \geq \frac{\delta_1^2 b}{800}\left(\tilde{\mathcal{E}}(s) + \int_A^{2A}|\epsilon|^2\right) + \frac{c\delta_1 b}{1000}\Gamma_b.
\end{aligned} \tag{19.2}$$

The conservation of L^2 norm, written in terms of ϵ and $\tilde{Q}_b$ is

$$\int|\epsilon|^2\mu(\tilde{r})d\tilde{r}d\tilde{z} + \int|\tilde{Q}_b|^2\mu(\tilde{r})\,d\tilde{r}d\tilde{z} + 2\,\mathrm{Re}\int\epsilon\overline{\tilde{Q}_b}\mu(\tilde{r})d\tilde{r}d\tilde{z} = \int|u_0|^2,$$

which we rewrite as

$$\begin{aligned}
&\int|\epsilon|^2\mu(\tilde{r})\,d\tilde{r}d\tilde{z} + r(t)\left(|\tilde{Q}_b|^2 - \int|Q|^2 + 2\,\mathrm{Re}(\epsilon,\overline{\tilde{Q}_b})\right) \\
&\quad = \int|u_0|^2 - r(t)\int Q^2 - 2\lambda\,\mathrm{Re}(\epsilon,\tilde{r}\tilde{Q}_b).
\end{aligned} \tag{19.3}$$

Then write

$$\int\phi_4(\tilde{R}/A)|\epsilon|^2\mu(\tilde{r})d\tilde{r}d\tilde{z} = \int|\epsilon|^2\mu(\tilde{r})d\tilde{r}d\tilde{z} - \int(1-\phi_4(\tilde{R}/A))|\epsilon|^2\mu(\tilde{r})\,d\tilde{r}d\tilde{z}$$

and obtain by taking ∂_s of (19.3),

$$\begin{aligned}
&\partial_s\left(\frac{1}{r(s)}\int\phi_4(\tilde{R}/A)|\epsilon|^2\mu(\tilde{r})\,d\tilde{r}d\tilde{z}\right) \\
&\quad = \;-\partial_s\left(\int|\tilde{Q}_b|^2 - \int Q^2 + 2(\epsilon_1,\Sigma) - 2(\epsilon_2,\Theta)\right) \\
&\qquad + \partial_s\left(\frac{1}{r(s)}\int(1-\phi_4(\tilde{R}/A))|\epsilon|^2\mu(\tilde{r})\,d\tilde{r}d\tilde{z}\right) \\
&\qquad - 2\partial_s\left(\frac{\lambda}{r(s)}\,\mathrm{Re}(\epsilon,\tilde{r}\overline{\tilde{Q}_b})\right) - \frac{r_s}{r^2}\int|u_0|^2.
\end{aligned}$$

The last two terms are easily estimate with (5.6) and (5.7). Inserting into (19.2), we obtain (5.13).

We next show the two estimates (5.14) and (5.15) for $\mathcal{J}$.

We first show (5.14). We have $\|\tilde{Q}_b\|_{L^2}^2 - \|Q\|_{L^2}^2 = (d_0+o(1))b^2$, and now we estimate the rest of the terms in the definition of $\mathcal{J}$. By Cauchy-Schwarz,

$$|(\epsilon_1,\Sigma)+(\epsilon_2,\Theta)| \lesssim \int|\epsilon|^2 e^{-\tilde{R}}d\tilde{r}d\tilde{z} \leq \Gamma_b^{1/2} \ll b\,.$$

Next, we have

$$\int (1-\phi_4(\tilde{R}/A))|\epsilon|^2\mu(\tilde{r})d\tilde{r}d\tilde{z} \leq \int_{\substack{|r-r(t)|\lesssim \lambda A\\ |z-z(t)|\lesssim \lambda A}} |\tilde{u}|^2 dxdydz.$$

The set $\{(x,y,z)\,|\,|r-r(t)|\lesssim \lambda A\,,|z-z(t)|\lesssim \lambda A\}$ has a volume $\sim (\lambda A)^2$ in $\mathbb{R}^3$. Therefore, by Hölder, the above is bounded by

$$(\lambda A)\left(\int_{\substack{|r-r(t)|\lesssim \lambda A\\ |z-z(t)|\lesssim \lambda A}} |\tilde{u}|^4 dxdydz\right)^{1/2}.$$

Then in the L^4, we widen the spatial restriction to $r\geq \frac{1}{2}$ and apply the axial Gagliardo-Nirenberg (Lemma 2.1) and the definition of A to obtain

$$\lambda A\|\tilde{u}\|_{L^2}\|\nabla\tilde{u}\|_{L^2} \lesssim A\left(\int_{\tilde{r},\tilde{z}} |\nabla_{(\tilde{r},\tilde{z})}\epsilon|^2\,\mu(\tilde{r})\,d\tilde{r}\,d\tilde{z}\right)^{1/2} \ll b\,.$$

We claim that $|\tilde{f}_1(b)|\lesssim b$. Indeed, we first note that $\frac{1}{4}b\|\tilde{R}\tilde{Q}_b\|_{L^2}^2\lesssim b$. Also, recalling that $\tilde{\zeta}=\phi_3(\tilde{R}/A)\zeta$, we have

$$\begin{aligned}\left|\operatorname{Im}\int \tilde{R}\,\partial_{\tilde{R}}\tilde{\zeta}\,\tilde{\zeta}d\tilde{r}d\tilde{z}\right| &\leq \int_{\tilde{R}\lesssim A}\tilde{R}\,|\partial_{\tilde{R}}\zeta|\,|\zeta|\,d\tilde{r}d\tilde{z}+\frac{1}{A}\int_{\tilde{R}\lesssim A}\tilde{R}\,|\zeta|^2 d\tilde{r}\,d\tilde{z}\\ &\leq \|\tilde{R}\,\zeta\|_{L^\infty}A\|\nabla\zeta\|_{L^2}+\frac{1}{A}\|\tilde{R}\,\zeta\|_{L^\infty}^2\int \tilde{R}^{-1}\,d\tilde{r}\,d\tilde{z}\\ &\leq \Gamma_b A(\Gamma_b^{1-C\eta})^{1/2}+\Gamma_b \ll b,\end{aligned}$$

which establishes that $|\tilde{f}_1(b)|\lesssim b$. It is similarly straightforward to show that $\left|\int_0^b \tilde{f}_1(v)\,dv\right|\lesssim b$. Finally, the local terms $(\epsilon_2,\Lambda\tilde{\zeta}_{\mathrm{re}})$ and $(\epsilon_1,\Lambda\tilde{\zeta}_{\mathrm{im}})$ are estimated by Cauchy-Schwarz and the ZP 1–2 properties, and shown to be $\ll b$. This concludes the proof of (5.14).

For the proof of (5.15), see the proof of Prop. 5 in [24], which uses the conservation of energy.

20. Bootstrap Step 15. Control on $\mathcal{E}(t)$ and upper bound on $b(s)$

Next, we prove (5.16). Using that (5.14) implies $\left|\sqrt{\frac{d_0}{\mathcal{J}}}-b^{-1}\right|\ll 1$, we have

$$\frac{1}{b^2}\exp\left(\frac{5\pi}{4}\sqrt{\frac{d_0}{\mathcal{J}}}\right)\gtrsim \frac{1}{b^2}\exp\left(\frac{7\pi}{6}b^{-1}\right)\gtrsim \Gamma_b^{-1}.$$

Thus, by (5.13) (note that $-\mathcal{J}_s\geq 0$)

$$\partial_s\exp\left(\frac{5\pi}{4}\sqrt{\frac{d_0}{\mathcal{J}}}\right)\sim \mathcal{J}^{-3/2}(-\mathcal{J}_s)\exp\left(\frac{5\pi}{4}\sqrt{\frac{d_0}{\mathcal{J}}}\right)\gtrsim -\frac{\mathcal{J}_s}{b\Gamma_b}\geq 1.$$

Integrating from s_0 to s, we obtain

$$\exp\left(\frac{5\pi}{4}\sqrt{\frac{d_0}{\mathcal{J}(s)}}\right) - \exp\left(\frac{5\pi}{4}\sqrt{\frac{d_0}{\mathcal{J}(s_0)}}\right) \geq s - s_0.$$

Since $\exp\left(\frac{5\pi}{4}\sqrt{\frac{d_0}{\mathcal{J}(s_0)}}\right) \geq e^{\pi/b_0} > s_0$, we conclude that

$$\exp\left(\frac{5\pi}{4}\sqrt{\frac{d_0}{\mathcal{J}(s)}}\right) \geq s$$

and (5.16) follows.

Next, we obtain (5.17). Divide (5.13) by $J^{1/2}$ and use (5.14) ($\mathcal{J} \sim b^2$) to obtain

$$\Gamma_{b(t)} + \tilde{\mathcal{E}}(t) \leq -CJ^{-1/2}\mathcal{J}_s.$$

Integrate from s_0 to s to obtain

$$\int_{s_0}^{s} (\Gamma_{b(\sigma)} + \tilde{\mathcal{E}}(\sigma))\, d\sigma \lesssim \mathcal{J}^{1/2}(s_0) - \mathcal{J}^{1/2}(s) \lesssim b(s_0) \lesssim \alpha^*,$$

where in the last step we applied IDA 2.

Finally, we prove BSO 3. Let $s \in [s_0, s_1)$ (recall that we are carrying out the bootstrap argument on $0 \leq t < t_1$, and $s_1 = s(t_1)$). If $b_s(s) \leq 0$, then BSO 3 at s follows immediately from (5.8) at s (the local virial identity, $b_s \geq \delta_0\mathcal{E}(s) - \Gamma_{b(s)}^{1-C\eta}$).

If $b_s(s) > 0$, let $s_2 \in [s_0, s)$ be the smallest time such that for all $\sigma \in (s_2, s)$, we have $b_s(\sigma) > 0$. Then $b(s_2) \leq b(s)$ and either $s_2 = s_0$ or $b_s(s_2) = 0$. In either case, we claim

$$\mathcal{E}(s_2) \leq \Gamma_{b(s_2)}^{6/7}. \tag{20.1}$$

If $s_2 = s_0$, then (20.1) is just IDA 4. If $b_s(s_2) = 0$, then (20.1) just follows from the local virial identity (5.8) at s_2. From the second of the two estimates in (5.15) and also (20.1), we have

$$\mathcal{J}(s_2) - f_2(b(s_2)) \leq \Gamma_{b(s_2)}^{5/6}.$$

By the first of the two estimates in (5.15), the fact that $\mathcal{J}$ is (nonstrictly) decreasing, and the above estimate, we have

$$\begin{aligned} f_2(b(s)) + \frac{1}{C}\mathcal{E}(s) &\leq \mathcal{J}(s) + \Gamma_{b(s)}^{5/6} \\ &\leq \mathcal{J}(s_2) + \Gamma_{b(s)}^{5/6} \\ &\leq f_2(b(s_2)) + \Gamma_{b(s_2)}^{5/6} + \Gamma_{b(s)}^{5/6}. \end{aligned}$$

Combining this estimate with

$$0 < \left.\frac{df_2}{db^2}\right|_{b^2=0} < +\infty \;\text{ and }\; b(s_2) \leq b(s) \quad \Longrightarrow \quad f_2(b(s_2)) \leq f_2(b(s))$$

and

$$b(s_2) \le b(s) \implies \Gamma_{b(s_2)} \le \Gamma_{b(s)}$$

yields

$$f_2(b(s)) + \frac{1}{C}\mathcal{E}(s) \le f_2(b(s)) + 2\Gamma_{b(s)}^{5/6}.$$

Canceling $f_2(b(s))$ yields BSO 3.

21. Bootstrap Step 16. $H^{1/2}$ interior smallness.

21.1. **Preliminaries.** We consider two fractional derivative operators in our analysis: D^s_{rz} and D^s_{xyz}. The operator D^s_{rz} acts directly on a function $f(r',z')$ defined in $\mathbb{R}^2$ and returns a function of $(r,z)\in\mathbb{R}^2$:

$$D^s_{rz} = \mathcal{F}_2^{-1}(\rho^2+\zeta^2)^{s/2}\mathcal{F}_2,$$

where $\mathcal{F}_2$ is the Fourier transform on $\mathbb{R}^2$ and we adopt the practice of writing the dual variables of (r,z) as (ρ,ζ). The operator D^s_{xyz} acts on a function $f(x',y',z')$ defined on $\mathbb{R}^3$ and returns a function of (x,y,z) on $\mathbb{R}^3$

$$D^s_{xyz} = \mathcal{F}_3^{-1}(\xi^2+\eta^2+\zeta^2)^{s/2}\mathcal{F}_3,$$

where we adopt the practice of writing the dual variables of (x,y,z) as (ξ,η,ζ). Now if $f(x,y,z)$ is an axially symmetric function on $\mathbb{R}^3$, then $D^s_{xyz}f(x,y,z)$ is axially symmetric and can thus be viewed as a function of $(r,z)\in\mathbb{R}^+\times\mathbb{R}$ and, when extended to an even function in r, it can be viewed as a function on $\mathbb{R}^2$. In this case, we can write

$$D^s_{xyz} = \mathcal{H}^{-1}(\rho^2+\zeta^2)^{s/2}\mathcal{H},$$

where $\mathcal{H}$ is a zero-order Hankel transform[5] (p. 341 of Bracewell [2]) in r and a Fourier transform in z:

$$\mathcal{H}f(\rho,\zeta) = \int_{z=-\infty}^{+\infty}\int_{r=0}^{\infty} J_0(r\rho)e^{-iz\zeta}f(r,z)r\,dr\,dz$$

and the zero-order Bessel function on $\mathbb{R}$ is:

$$J_0(\omega) = \frac{1}{2\pi}\int_{\theta=-\pi}^{\pi} e^{i\omega\sin\theta}d\theta$$

(which extends $J_0(\omega)$ as an even function in ω, and thus $\mathcal{H}f(\rho,\zeta)$ is even in ρ.)[6] We note that

$$\mathcal{H}^{-1}F(r,z) = \int_{\zeta=-\infty}^{+\infty}\int_{\rho=0}^{+\infty} J_0(r\rho)e^{iz\zeta}F(\rho,\zeta)\rho\,d\rho\,d\zeta.$$

[5]We will not bother with constant factors (such as $(2\pi)^{-1}$) in the definitions of the Fourier and Hankel transforms, as they are irrelevant in the analysis.

[6]Note that if we wanted to write the domain of r-integration in the definition of $\mathcal{H}$ as $(-\infty,+\infty)$, assuming $f(r,z)$ is even in r, we would need to put $|r|$ in the integrand in place of r.

If a function $h(x,y,z)$ is axially symmetric and the support of h is contained in $0<\delta\le r\le \delta^{-1}$, then we will say that h is *shell supported.* We note that for a shell supported function h, $\|h\|_{L^p_{rz}}\sim \|h\|_{L^p_{xyz}}$. Unfortunately, if h is shell-supported, then fractional derivatives $D^\alpha_{rz}h$ are no longer shell-supported. To deal with this, and handle the conversion from D^s_{rz} to D^s_{xyz}, we use the following standard microlocal fact, which is a consequence of the pseudodifferential calculus (see Stein [35], Chapter VI, §2, Theorem 1 on p. 234 and §3, Theorem 2 on p. 237; see also Evans-Zworski [8]).

Lemma 21.1 (disjoint smoothing)**.** *Suppose that* $\operatorname{dist}(\operatorname{supp}\psi_1,\operatorname{supp}\psi_2)>0$ *and* $s,\alpha\ge 0$. *Then for* $h(r,z)$ *we have*

$$\|\psi_1\, D^s_{rz}\, \psi_2\, h\|_{H^\alpha_{rz}}\lesssim_{s,\alpha}\|\psi_2 h\|_{L^2_{rz}}\,,$$

and for $h(x,y,z)$ *we have*

$$\|\psi_1\, D^s_{xyz}\, \psi_2\, h\|_{H^\alpha_{xyz}}\lesssim_{s,\alpha}\|\psi_2 h\|_{L^2_{xyz}}\,.$$

Next, we address the matter of converting from D^s_{rz} to D^s_{xyz}. We need the composition (Bracewell [1])

$$\mathcal{F}_2\mathcal{H}^{-1}=\mathcal{A},$$

where $\mathcal{A}$ is the Abel transform (see p. 351 of Bracewell [2])

$$\mathcal{A}f(\rho,\zeta)=\int_\rho^{+\infty}\frac{\rho' f(\rho',\zeta)}{(\rho'-\rho)^{1/2}(\rho'+\rho)^{1/2}}\,d\rho'$$

(which is the identity in ζ).

Lemma 21.2 (fractional derivative conversion)**.** *For any* $0\le s\le 2$*, the composition*

$$r^{1/2}D^s_{rz}D^{-s}_{xyz}r^{-1/2}$$

is bounded as an operator $L^2_{rz}\to L^2_{rz}$.

We remark that scaling requires that the weights in any such estimate be r^α on the left and $r^{-\alpha}$ on the right for some $\alpha\in\mathbb{R}$. We only need $\alpha=\frac12$ in our analysis and have not explored the possible validity of other values of α.

Proof. Note that the composition under consideration is

$$U=r^{1/2}\mathcal{F}^{-1}(\rho^2+\zeta^2)^{s/2}\mathcal{A}((\rho')^2+\zeta^2)^{-s/2}\mathcal{H}^{-1}r^{-1/2}\,.$$

Note that $(r+i0)^{1/2}\mathcal{F}^{-1}=\mathcal{F}^{-1}\partial_\rho I^{1/2}_\rho$where I^α is the fractional integral operator $I^\alpha f(\rho)=\int_{\rho'=\rho}^{+\infty}(\rho'-\rho)^{\alpha-1}f(\rho')\,d\rho'$. This, together with the fact that $\mathcal{F}^{-1}$ is an $L^2_{\rho\zeta}\to L^2_{rz}$ unitary map, and the fact that $\rho^{1/2}\mathcal{H}r^{-1/2}$ is an $L^2_{rz}\to L^2_{\rho\zeta}$ unitary map, we reduce to proving the $L^2_{\rho\zeta}\to L^2_{\rho\zeta}$ boundedness of

$$U=\partial_\rho I^{1/2}_\rho((\rho')^2+\zeta^2)^{s/2}\mathcal{A}((\rho'')^2+\zeta^2)^{-s/2}(\rho'')^{-1/2}\,.$$

We obtain $Uf = \partial_\rho I_\rho^{1/2} h$, where

$$h(\rho',\zeta) = \int_{\rho''=\rho'}^{\rho''=+\infty} \frac{(\rho'')^{1/2}((\rho')^2+\zeta^2)^{s/2} f(\rho'',\zeta)}{(\rho''-\rho')^{1/2}(\rho''+\rho')^{1/2}((\rho'')^2+\zeta^2)^{s/2}}\, d\rho''.$$

By Fubini's theorem, we obtain

$$Uf(\rho,\zeta) = \partial_\rho \int_{\rho''=\rho}^{\rho''=+\infty} K(\rho,\rho'') f(\rho'',\zeta)\, d\rho'',$$

where

$$K(\rho,\rho'') = \int_{\rho'=\rho}^{\rho'=\rho''} \frac{1}{(\rho'-\rho)^{1/2}(\rho''-\rho')^{1/2}} \left(\frac{(\rho')^2+\zeta^2}{(\rho'')^2+\zeta^2}\right)^{s/2} \left(\frac{\rho''}{\rho''+\rho'}\right)^{1/2} d\rho'.$$

By changing the variables twice, first $\rho' \mapsto \rho' + \rho$ and then $\rho' = \sigma(\rho'' - \rho)$, we have

$$K(\rho,\rho'') = \int_{\sigma=0}^{1} \frac{1}{\sigma^{1/2}(1-\sigma)^{1/2}} \left(\frac{(\sigma\rho''+(1-\sigma)\rho)^2+\zeta^2}{(\rho'')^2+\zeta^2}\right)^{s/2} \left(\frac{\rho''}{(1+\sigma)\rho''+(1-\sigma)\rho}\right)^{1/2} d\sigma.$$

Thus,

$$Uf(\rho,\zeta) = -K(\rho,\rho) f(\rho,\zeta) + \int_{\rho''=\rho}^{+\infty} \partial_\rho K(\rho,\rho'') f(\rho'',\zeta) d\rho''.$$

We have $K(\rho,\rho) = 2^{-1/2}\int_{\sigma=0}^1 \sigma^{-1/2}(1-\sigma)^{-1/2} d\sigma < \infty$. For the second term, we change variable $\rho'' = \rho\mu$ to obtain

$$Uf(\rho,\zeta) = -cf(\rho,\zeta) + \int_{\mu=1}^{+\infty} \rho(\partial_\rho K)(\rho,\rho\mu) f(\rho\mu,\zeta)\, d\mu\,. \tag{21.1}$$

But we have $\rho\partial_\rho K(\rho,\rho\mu) = I(\mu,\zeta/\rho)$, where

$$I(\mu,\lambda) = \int_{\sigma=0}^{1} \frac{1}{\sigma^{1/2}(1-\sigma)^{1/2}} \left(\frac{(\sigma\mu+(1-\sigma))^2+\lambda^2}{\mu^2+\lambda^2}\right)^{\frac{s}{2}} \left(\frac{\mu}{(1+\sigma)\mu+1-\sigma}\right)^{\frac12} J(\mu,\lambda)\, d\sigma$$

and

$$J(\mu,\lambda) = s\frac{(\sigma\mu+(1-\sigma))(1-\sigma)}{(\sigma\mu+(1-\sigma))^2+\lambda^2} + \frac12 \frac{1-\sigma}{(1+\sigma)\mu+(1-\sigma)} = J_1(\mu,\lambda) + J_2(\mu,\lambda).$$

Denote by I_k the result of substituting J_k into the expression for I. Since $|J_2(\mu,\lambda)| \le (\mu+1)^{-1}$, we have

$$|I_2(\mu,\lambda)| \lesssim \frac{1}{\mu}.$$

If $s \le 2$, we have

$$\begin{aligned} |I_1(\mu,\lambda)| &\le \int_{\sigma=0}^1 \sigma^{-1/2}(1-\sigma)^{1/2} \frac{s(\sigma\mu+(1-\sigma))}{(\mu^2+\lambda^2)^{\frac{s}{2}}((\sigma\mu+(1-\sigma))^2+\lambda^2)^{1-\frac{s}{2}}}\, d\sigma \\ &\le \frac{s}{\mu^s}\int_{\sigma=0}^1 \frac{(1-\sigma)^{1/2}}{\sigma^{1/2}(\sigma\mu+(1-\sigma))^{1-s}}\, d\sigma \end{aligned}$$

By separately considering the regions $0 \le \sigma \le \frac{1}{2}$ and $\frac{1}{2} \le \sigma \le 1$, we obtain

$$\begin{aligned} |I_1(\mu,\lambda)| &\lesssim \frac{1}{\mu} + \int_{\sigma=0}^{1/2} \frac{d\sigma}{\sigma^{1/2}(\sigma\mu+1)^{1-s}} \\ &\lesssim \frac{1}{\mu} + \int_{\eta=0}^{\eta=\mu} \frac{d\eta}{\eta^{1/2}(\eta+1)^{1-s}} \\ &\lesssim \begin{cases} \frac{1}{\mu} & \text{if } s > \frac{1}{2} \\ \frac{\log(\mu+1)}{\mu} & \text{if } s = \frac{1}{2} \\ \frac{s}{\mu^{s+\frac{1}{2}}} & \text{if } 0 < s < \frac{1}{2}. \end{cases} \end{aligned}$$

Applying the L^2_ρ norm to (21.1) and using Minkowski's integral inequality, we bound by

$$\|Uf(\cdot,\zeta)\|_{L^2_\rho} \lesssim \|f(\cdot,\zeta)\|_{L^2_\rho} + \left(\int_{\mu=1}^{+\infty} \max(\mu^{-1}, s\mu^{-\frac{1}{2}-s})\mu^{-\frac{1}{2}}\, d\mu \right) \|f(\cdot,\zeta)\|_{L^2_\rho}.$$

Following through with the L^2_ζ norm, we complete the proof. □

The following were established by Strichartz [37], Keel-Tao [17] and Kenig-Ponce-Vega [19]. We say that (q,p) is 2d admissible if $\frac{2}{q} + \frac{2}{p} = 1$ and $2 \le p < \infty$.

Lemma 21.3 (2d Strichartz estimates and local smoothing)**.** *Suppose that $h(r,z,t)$ satisfies*

$$i\partial_t h + \Delta_{rz} h = g.$$

Then for 2d admissible pairs (q_1,p_1) and (q_2,p_2),

$$\|h\|_{L_t^{q_1} L_{rz}^{p_1}} \lesssim \|h_0\|_{L^2_{rz}} + \begin{cases} \|g\|_{L_t^{q_2'} L_{rz}^{p_2'}} \\ \|\langle D_{rz}\rangle^{-1/2} g\|_{L^2_t L^2_{rz}} & \textit{if } \operatorname{supp} g \textit{ compact}, \end{cases}$$

where $h_0(r,z) = h_0(r,z,0)$ is the initial data. In the case of the last bound $\operatorname{supp} g$ *should be contained in a fixed compact set for all t.*

We say that (q,p) is 3d admissible if $\frac{2}{q} + \frac{3}{p} = \frac{3}{2}$ and $2 \le p < 6$. [7]

Lemma 21.4 (3d Strichartz estimates and local smoothing)**.** *Suppose that $f(x,y,z,t)$ solves*

$$i\partial_t f + \Delta_{xyz} f = g.$$

Then for 3d admissible pairs (q_1,p_1) and (q_2,p_2), we have

$$\|f\|_{L_t^{q_1} L_{xyz}^{p_1}} \lesssim \|f_0\|_{L^2_{xyz}} + \begin{cases} \|g\|_{L_t^{q_2'} L_{xyz}^{p_2'}} \\ \|\langle D_{xyz}\rangle^{-1/2} g\|_{L^2_t L^2_{xyz}} & \textit{if } \operatorname{supp} g \textit{ compact}, \end{cases}$$

[7]We thank Fabrice Planchon for pointing out that, although the endpoint case $(q_1,p_1) = (2,6)$ is available for the local smoothing estimate, it does not follow from the Christ-Kiselev lemma. We will avoid the use of the endpoint in our analysis.

where $f_0(x,y,z) = f(x,y,z,0)$ *is the initial data. In the case of the last bound* $\operatorname{supp} g$ *should be contained in a fixed compact set for all* t.

The local smoothing estimates in the above lemmas are established in the homogeneous case by [19] Theorem 4.1 on p. 54. The inhomogeneous version as stated above then follows from the Christ-Kiselev [3] lemma.

Finally, we record some Gagliardo-Nirenberg embeddings for shell-supported functions.

Lemma 21.5 (embeddings). *Suppose that* $q(r,z) = q(x,y,z)$ *is axially symmetric and shell-supported. Then*

$$\|q\|_{L_t^4 L_{xyz}^4} \lesssim \|q\|_{L_t^\infty L_{xyz}^2}^{1/2} \|\nabla q\|_{L_t^2 L_{xyz}^2}^{1/2}, \tag{21.2}$$

$$\|D_{xyz}^{1/2} |q|^2\|_{L_t^{4/3} L_{xyz}^2} \lesssim \|q\|_{L_t^\infty L_{xyz}^2}^{1/2} \|\nabla q\|_{L_t^2 L_{xyz}^2}^{3/2}. \tag{21.3}$$

Proof. First we prove (21.2). We have the 2d Gagliardo-Nirenberg estimate

$$\|q\|_{L_{rz}^4} \lesssim \|q\|_{L_{rz}^2}^{1/2} \|\nabla_{rz} q\|_{L_{rz}^2}^{1/2}.$$

Since q and $\nabla_{rz} q$ are shell supported and $|\nabla_{rz} q| \le |\nabla_{xyz} q|$, we obtain

$$\|q\|_{L_{xyz}^4} \lesssim \|q\|_{L_{xyz}^2}^{1/2} \|\nabla_{xyz} q\|_{L_{xyz}^2}^{1/2}.$$

Integrating in time, we obtain (21.2). Next we prove (21.3). We begin by noting

$$\begin{aligned} \|D_{xyz}^{1/2} |q|^2\|_{L_{xyz}^2}^2 &= |\langle D_{xyz}^{1/2} |q|^2, D_{xyz}^{1/2} |q|^2 \rangle_{xyz}| \\ &= \langle |q|^2, D |q|^2 \rangle_{xyz} \\ &= \langle |q|^2, R \nabla |q|^2 \rangle_{xyz} \\ &= \langle |q|^2, R(\operatorname{Re} q \nabla \bar{q}) \rangle_{xyz}, \end{aligned}$$

where R is the vector Riesz transform. Using the boundedness of the Riesz transform on $L^{3/2}$, we obtain

$$\begin{aligned} \|D_{xyz}^{1/2} |q|^2\|_{L_{xyz}^2}^2 &\lesssim \|q\|_{L_{xyz}^6}^2 \|R(\operatorname{Re} q \nabla \bar{q})\|_{L_{xyz}^{3/2}} \\ &\lesssim \|q\|_{L_{xyz}^6}^3 \|\nabla q\|_{L_{xyz}^2}. \end{aligned}$$

By the 2d Gagliardo-Nirenberg estimate $\|q\|_{L_{rz}^6} \le \|q\|_{L_{rz}^2}^{1/3} \|\nabla_{rz} q\|_{L_{rz}^2}^{2/3}$, the fact that q and $\nabla_{rz} q$ are shell-supported, and $|\nabla_{rz} q| \le |\nabla_{xyz} q|$, we obtain

$$\|D_{xyz}^{1/2} |q|^2\|_{L_{xyz}^2} \lesssim \|q\|_{L_{xyz}^2}^{1/2} \|\nabla_{xyz} q\|_{L_{xyz}^2}^{3/2}.$$

Integrating in t, we obtain (21.3). □

21.2. **Outline and notation.** Effectively, we would like to restrict to outside the singular circle and reduce matters to the local theory, which is set in $H^{1/2}$.

Let

$$\tilde{\chi}_1(R) = \begin{cases} 0 & \text{for } R \le \frac{1}{4} \\ 1 & \text{for } R \ge \frac{3}{4} \end{cases}$$

and set $\chi_1(r,z) = \tilde{\chi}_1(((r-1)^2+z^2)^{1/2})$. This is a cutoff to *outside* the singular circle, and we call the support of χ_1 the *external region.* Let

$$\tilde{\chi}_2(R) = \begin{cases} 1 & \text{for } \frac{1}{4} \le R \le \frac{3}{4} \\ 0 & \text{for } R \le \frac{1}{8} \text{ and } R \ge \frac{7}{8} \end{cases},$$

$$\tilde{\chi}_3(R) = \begin{cases} 1 & \text{for } \frac{1}{8} \le R \le \frac{7}{8} \\ 0 & \text{for } R \le \frac{1}{16} \text{ and } R \ge \frac{15}{16} \end{cases}$$

and let $\chi_j(r,z) = \tilde{\chi}_j(((r-1)^2+z^2)^{1/2})$ for $j = 1,2$. Note that $\chi_2(r,z)$ is 1 on the support of $\chi_1(1-\chi_1)$ and $\chi_3(r,z)$ is 1 on the support of $\chi_2(r,z)$. We will call the support of χ_2 the *tight singular periphery* and the support of χ_3 the *wide singular periphery.* Let $v = \chi_1 u$, $w = \chi_2 u$ and $q = \chi_3 u$.

Our goal is to obtain the estimate BSO 8, i.e.,

$$\|D_{xyz}^{1/2} v\|_{L_t^\infty L_{xyz}^2} \le (\alpha^*)^{3/8}. \tag{21.4}$$

This will be a multistep process – estimates on q (Step A) obtained from BSI 3 will imply estimates on w (Step B), which will imply estimates on v (Step C).

21.3. **Step A. Estimates on q.** Recall $q = \chi_3 u$, i.e., u restricted to the wide singular periphery. Control on q is inherited from the local virial arguments in the previous sections. Specifically, mass conservation, (5.17), and BSI 3 imply (by rescaling) that

$$\|q\|_{L_t^\infty L_{xyz}^2} \le \|u_0\|_{L_{xyz}^2}, \qquad \|\nabla q\|_{L_t^2 L_{xyz}^2} \lesssim (\alpha^*)^{10}. \tag{21.5}$$

We also obtain an upper bound on t_1. Indeed, from (5.10),

$$t_1 = \int_{s_0}^{s_1} \lambda^2(s)\, ds \lesssim C\lambda_0 \int_2^{+\infty} e^{-(\pi/3)(s/\log s)} \le \alpha^*. \tag{21.6}$$

It will be understood that t is always confined to $[0,t_1)$ below (when writing L_t^∞, etc.)

21.4. **Step B. Estimates on w.** Recall $w = \chi_2 u$, i.e., u restricted to the tight singular periphery. We use the control on u in the larger wide singular periphery obtained above in Step A to provide control in a stronger norm on the smaller tight singular periphery. Since w is supported away from $r = 0$ and away from $r = \infty$, it effectively solves a 2d NLS equation. This effective 2d equation can be analyzed using the (locally-in-time) stronger 2d Strichartz estimates as well as the 2d Gagliardo-Nirenberg estimates.

Specifically, the goal of this subsection is to obtain, using (21.5), the following bounds on w:

$$\|\langle D_{xyz}\rangle^{1/2} w\|_{L_t^q L_{xyz}^p} \lesssim (\alpha_*)^{1/2} \tag{21.7}$$

for all 2d admissible (q,p).

The function $D_{xyz}^{1/2} w = D_{xyz}^{1/2}\chi_2 u$ is still axially symmetric but unfortunately (due the nonlocality of $D_{xyz}^{1/2}$) no longer shell supported. By Lemma 21.1, we have that $(1-\chi_3)\, D_{xyz}^{1/2}\,\chi_2 u$ is infinitely smoothing, and thus,

$$\|(1-\chi_3)\, D_{xyz}^{1/2}\, w\|_{L_t^q L_{xyz}^p} \lesssim \|w\|_{L_t^\infty L_{xyz}^2}.$$

Hence, to establish (21.7), it suffices to prove, for $p = \chi_3\, D_{xyz}^{1/2}\, w$, the estimates

$$\|p\|_{L_t^q L_{xyz}^p} \lesssim \|p_0\|_{L_{xyz}^2} + (\alpha^*)^{1/2} \tag{21.8}$$

and

$$\|w\|_{L_t^\infty L_{xyz}^2} \lesssim \|w_0\|_{L_{xyz}^2} + (\alpha^*)^{1/2}\,. \tag{21.9}$$

The equation (21.7) will then follow from IDA 7 and IDA 8.

We begin with (21.8). Note that p is axially symmetric and shell supported. Moreover, it solves an equation of the form

$$i\partial_t p + \Delta_{rz} p = \sum_j G_j,$$

where the inhomogeneities can be put into the forms

$$\begin{aligned} G_1 &= \chi_3\, D_{xyz}^{1/2}\, \chi_2\, (i\partial_t + \Delta_{rz}) u \\ G_2 &= \psi_3\, D_{xyz}^{1/2}\, \psi_2\, u \\ G_3 &= \nabla_{rz}\, \psi_3\, D_{xyz}^{1/2}\, \psi_2\, u\,. \end{aligned}$$

For each type of term G_j, the functions ψ_2 and ψ_3 are axial cutoff functions with support in $\frac14 \le ((r-1)^2+z^2)^{1/2} \le \frac34$ and $\frac18 \le ((r-1)^2+z^2)^{1/2} \le \frac78$, respectively. For term G_3, we substitute the equation $(i\partial_t + \Delta_{rz})u = -r^{-1}\partial_r u - |u|^2 u$ and then reexpress the term involving $r^{-1}\partial_r u$ to be of the type G_2 and G_3. At this point, the inhomogeneities take the form

$$\begin{aligned} G_1 &= \chi_3\, D_{xyz}^{1/2}\, \chi_2\, |u|^2 u \\ G_2 &= \psi_3\, D_{xyz}^{1/2}\, \psi_2\, u \\ G_3 &= \nabla_{rz}\, \psi_3\, D_{xyz}^{1/2}\, \psi_2\, u. \end{aligned}$$

By Lemma 21.3,

$$\|p\|_{L_t^q L_{rz}^p} \lesssim \|p_0\|_{L_{rz}^2} + \|G_1\|_{L_t^{4/3} L_{rz}^{4/3}} + \|G_2\|_{L_t^1 L_{rz}^2} + \|\langle D_{rz}\rangle^{-1/2} G_3\|_{L_t^2 L_{rz}^2}, \tag{21.10}$$

since G_3 has compact support in r and z.

Since G_1 is shell supported, the rz norm converts to an xyz norm. The fractional Leibniz rule and the identity $\chi_2 u|u|^2 = w|q|^2$ allow us to bound as follows:

$$\|G_1\|_{L_t^{4/3}L_{xyz}^{4/3}} \leq \|D_{xyz}^{1/2}w\|_{L_t^4L_{xyz}^4}\|q\|_{L_t^4L_{xyz}^4}^2 + \|w\|_{L_t^\infty L_{xyz}^4}\|D_{xyz}^{1/2}|q|^2\|_{L_t^{4/3}L_{xyz}^2}. \tag{21.11}$$

Since w is shell-supported, we can apply the 2d Sobolev embedding

$$\begin{aligned}\|w\|_{L_t^\infty L_{xyz}^4} &\sim \|w\|_{L_t^\infty L_{rz}^4}\\ &\lesssim \|D_{rz}^{1/2}w\|_{L_t^\infty L_{rz}^2}\\ &\lesssim \|\chi_3 D_{rz}^{1/2}w\|_{L_t^\infty L_{rz}^2} + \|(1-\chi_3)D_{rz}^{1/2}w\|_{L_t^\infty L_{rz}^2}.\end{aligned} \tag{21.12}$$

We note that we have

$$\begin{aligned}\chi_3 D_{rz}^{1/2}w &= \chi_3 D_{rz}^{1/2}D_{xyz}^{-1/2}D_{xyz}^{1/2}w\\ &= (\chi_3 r^{-1/2})(r^{1/2}D_{rz}^{1/2}D_{xyz}^{-1/2}r^{-1/2})(r^{1/2}D_{xyz}^{1/2}w).\end{aligned}$$

By Lemma 21.2 and the support properties of χ_3,

$$\begin{aligned}\|\chi_3 D_{rz}^{1/2}w\|_{L_t^\infty L_{rz}^2} &\lesssim \|r^{1/2}D_{xyz}^{1/2}w\|_{L_t^\infty L_{rz}^2} = \|D_{xyz}^{1/2}w\|_{L_t^\infty L_{xyz}^2}\\ &\lesssim \|p\|_{L_t^\infty L_{xyz}^2} + \|(1-\chi_3)D_{xyz}^{1/2}w\|_{L_t^\infty L_{xyz}^2}.\end{aligned}$$

Applying Lemma 21.1 to the second of these terms,

$$\|\chi_3 D_{rz}^{1/2}w\|_{L_t^\infty L_{rz}^2} \lesssim \|p\|_{L_t^\infty L_{xyz}^2} + \|w\|_{L_t^\infty L_{xyz}^2}. \tag{21.13}$$

On the other hand, by Lemma 21.1,

$$\|(1-\chi_3)D_{rz}^{1/2}w\|_{L_t^\infty L_{rz}^2} \lesssim \|w\|_{L_t^\infty L_{rz}^2}. \tag{21.14}$$

Combining (21.12), (21.13), and (21.14),

$$\|w\|_{L_t^\infty L_{xyz}^4} \lesssim \|p\|_{L_t^\infty L_{xyz}^2} + \|w\|_{L_t^\infty L_{xyz}^2}. \tag{21.15}$$

Returning to (21.11), writing $D_{xyz}^{1/2}w = \chi_3 D_{xyz}^{1/2}w + (1-\chi_3)D_{xyz}^{1/2}w$, we have

$$\|D_{xyz}^{1/2}w\|_{L_t^4L_{xyz}^4} \lesssim \|p\|_{L_t^4L_{xyz}^4} + \|(1-\chi_3)D_{xyz}^{1/2}\chi_2 u\|_{L_t^\infty L_{xyz}^4}.$$

Applying Lemma 21.1 to the second of these terms, we obtain

$$\|D_{xyz}^{1/2}w\|_{L_t^4L_{xyz}^4} \lesssim \|p\|_{L_t^4L_{xyz}^4} + \|w\|_{L_t^\infty L_{xyz}^2}. \tag{21.16}$$

Combining (21.11), (21.15), (21.16), (21.2), (21.3), and (21.5), we obtain

$$\|G_1\|_{L_t^{4/3}L_{xyz}^{4/3}} \lesssim (\alpha^*)^5(\|p\|_{L_t^4L_{xyz}^4} + \|p\|_{L_t^\infty L_{rz}^2} + \|w\|_{L_t^\infty L_{xyz}^2}). \tag{21.17}$$

Since G_2 is shell-supported, we convert the rz integration to xyz integration and appeal to (21.5) to obtain

$$\|G_2\|_{L_t^1L_{rz}^2} \lesssim t_1\|q\|_{L_t^\infty L_{xyz}^2} + t_1^{1/2}\|\nabla_{xyz}q\|_{L_t^2L_{xyz}^2} \lesssim (\alpha^*)^{1/2}. \tag{21.18}$$

Finally, for G_3 we take ψ_4 to be a smooth function that is 1 on the support of ψ_2 and ψ_3 but 0 near $r = 0$. Then decompose

(21.19)
$$\begin{aligned}\|\langle D_{rz}\rangle^{-1/2}G_3\|_{L^2_tL^2_{rz}} &\lesssim \|\langle D_{rz}\rangle^{1/2}\,\psi_3\,D^{1/2}_{xyz}\,\psi_2\,u\|_{L^2_tL^2_{rz}}\\ &\lesssim \|\psi_4 D^{1/2}_{rz}\psi_3 D^{1/2}_{xyz}\psi_2 q\|_{L^2_tL^2_{rz}} + \|(1-\psi_4)D^{1/2}_{rz}\psi_3 D^{1/2}_{xyz}\psi_2 q\|_{L^2_tL^2_{rz}}\\ &\quad + \|\psi_3 D^{1/2}_{xyz}\psi_2 q\|_{L^2_tL^2_{rz}}.\end{aligned}$$

By Lemma 21.1

$$\begin{aligned}\|(1-\psi_4)D^{1/2}_{rz}\psi_3 D^{1/2}_{xyz}\psi_2 q\|_{L^2_tL^2_{rz}} &\lesssim \|\psi_3 D^{1/2}_{xyz}\psi_2 q\|_{L^2_tL^2_{rz}} \sim \|\psi_3 D^{1/2}_{xyz}\psi_2 q\|_{L^2_tL^2_{xyz}}\\ &\lesssim t_1^{1/2}\|q\|_{L^\infty_tL^2_{xyz}} + \|\nabla q\|_{L^2_tL^2_{xyz}}\\ &\lesssim (\alpha^*)^{1/2}.\end{aligned} \tag{21.20}$$

Note that

$$\psi_4 D^{1/2}_{rz}\psi_3 D^{1/2}_{xyz}\psi_2 q = (\psi_4 r^{-1/2})(r^{1/2}D^{1/2}_{rz}D^{-1/2}_{xyz}r^{-1/2})(r^{1/2}D^{1/2}_{xyz}\psi_3 D^{1/2}_{xyz}\psi_2 q),$$

and thus,

$$\|\psi_4 D^{1/2}_{rz}\psi_3 D^{1/2}_{xyz}\psi_2 q\|_{L^2_tL^2_{rz}} \lesssim \|r^{1/2}D^{1/2}_{xyz}\psi_3 D^{1/2}_{xyz}\psi_2 q\|_{L^2_tL^2_{rz}} = \|D^{1/2}_{xyz}\psi_3 D^{1/2}_{xyz}\psi_2 q\|_{L^2_tL^2_{xyz}}.$$

By the fractional Leibniz rule

$$\|\psi_4 D^{1/2}_{rz}\psi_3 D^{1/2}_{xyz}\psi_2 q\|_{L^2_tL^2_{rz}} \lesssim t_1^{1/2}\|q\|_{L^2_tL^2_{xyz}} + \|\nabla q\|_{L^2_tL^2_{xyz}} \lesssim (\alpha^*)^{1/2}. \tag{21.21}$$

Now

(21.22)
$$\|\psi_3 D^{1/2}_{xyz}\psi_2 q\|_{L^2_tL^2_{rz}} \sim \|\psi_3 D^{1/2}_{xyz}\psi_2 q\|_{L^2_tL^2_{xyz}} \lesssim t_1^{1/2}\|q\|_{L^\infty_tL^2_{xyz}} + \|\nabla q\|_{L^2_tL^2_{xyz}} \lesssim (\alpha^*)^{1/2}.$$

Collecting (21.19), (21.20), (21.21), (21.22),

$$\|\langle D_{rz}\rangle^{-1/2}G_3\|_{L^2_tL^2_{rz}} \lesssim (\alpha^*)^{1/2}. \tag{21.23}$$

Combine (21.10), (21.17), (21.18), and (21.23) to obtain (with conversion of some xyz-norms on p and w back to rz norms)

$$\|p\|_{L^q_tL^p_{rz}} \lesssim (\alpha^*)^{1/2}(\|p\|_{L^4_tL^4_{rz}} + \|p\|_{L^\infty_tL^2_{rz}} + \|w\|_{L^\infty_tL^2_{rz}}) + (\alpha^*)^{1/2}\,,$$

from which it follows that (21.8) holds once (21.9) is available.

Next we prove (21.9). Note that w is axially symmetric and shell supported. Moreover, it solves an equation of the form

$$i\partial_t w + \Delta_{rz} w = \sum_j G_j,$$

where the inhomogeneities can be put into the forms

$$\begin{aligned} G_1 &= \chi_2\,(i\partial_t + \Delta_{rz})u \\ G_2 &= \psi_2\, u \\ G_3 &= \nabla_{rz}\psi_2\, u\,. \end{aligned}$$

For each type of term G_j, the function ψ_2 is an axial cutoff function with support in $\frac{1}{4} \leq ((r-1)^2+z^2)^{1/2} \leq \frac{3}{4}$. For term G_1, we substitute the equation $(i\partial_t + \Delta_{rz})u = -r^{-1}\partial_r u - |u|^2 u$ and then reexpress the term involving $r^{-1}\partial_r u$ to be of the type G_2 and G_3. At this point, the inhomogeneities take the form

$$\begin{aligned} G_1 &= \chi_2\,|u|^2 u \\ G_2 &= \psi_2\, u \\ G_3 &= \nabla_{rz}\,\psi_2\, u\,. \end{aligned}$$

By Lemma 21.3,

$$\|w\|_{L_t^q L_{xyz}^p} \lesssim \|w_0\|_{L^2_{rz}} + \|G_1\|_{L_t^{4/3}L_{rz}^{4/3}} + \|G_2\|_{L_t^1 L_{rz}^2} + \|G_3\|_{L_t^1 L_{rz}^2}. \tag{21.24}$$

For G_1, we use that q is shell-supported and apply (21.2) to obtain

$$\|G_1\|_{L_t^{4/3}L_{xyz}^{4/3}} \lesssim \|q\|^3_{L_t^4 L_{xyz}^4} \lesssim \|q\|^{3/2}_{L^2_{xyz}} \|\nabla q\|^{3/2}_{L_t^2 L_{xyz}^2} \lesssim (\alpha^*)^5.$$

For the remaining inhomogeneities, we argue as follows:

$$\|G_2\|_{L_t^1 L_{rz}^2} + \|G_3\|_{L_t^1 L_{rz}^2} \lesssim t_1 \|q\|_{L_t^\infty L_{xyz}^2} + t_1^{1/2} \|\nabla q\|_{L_t^2 L_{xyz}^2} \lesssim (\alpha^*)^{1/2}.$$

Collecting, we obtain (21.9).

21.5. **Step C. Control on v.** We use the control on u in the tight singular periphery to obtain control on u in the external region. In this region, u solves a genuinely 3d NLS equation, and we thus need to work with the 3d Strichartz estimates, which are, locally-in-time, weaker than the 2d Strichartz estimates. Also, we must only use the 3d Gagliardo-Nirenberg estimates.

Specifically, we use (21.5) and (21.7) to prove

$$\|D^{1/2}_{xyz} v\|_{L_t^q L_{xyz}^p} \lesssim \|D^{1/2}_{xyz} v_0\|_{L^2_{xyz}} + (\alpha^*)^{1/2}. \tag{21.25}$$

Via IDA 8, we obtain (21.4).

The equation for v is

$$i\partial_t v + \Delta_{xyz} v + |v|^2 v = \sum F_j,$$

where

$$\begin{aligned} F_1 &= \psi |u|^2 u \\ F_2 &= \psi\, u \\ F_3 &= \psi\, \nabla\, u, \end{aligned}$$

where in each term F_j, the function ψ is supported in the *intersection* of the supports of χ_1 and $1-\chi_1$. Recall that χ_2 and χ_3 are 1 on this set. By Lemma 21.4,

$$\|D_{xyz}^{1/2}v\|_{L_t^qL_{xyz}^p} \lesssim \|D_{xyz}^{1/2}v_0\|_{L_{xyz}^2} + \|D_{xyz}^{1/2}\,|v|^2v\|_{L_t^{10/7}L_{xyz}^{10/7}} + \|D_{xyz}^{1/2}F_1\|_{L_t^{8/5}L_{xyz}^{4/3}} + \|D_{xyz}^{1/2}F_2\|_{L_t^1L_{xyz}^2} + \|F_3\|_{L_t^2L_{xyz}^2}.$$

Note that for F_3, we have used the local smoothing property, and the fact that F_3 is compactly supported.

For F_1, we use that $F_1 = \psi w|w|^2$, the fractional Leibniz rule, and estimate as

$$\|D_{xyz}^{1/2}F_1\|_{L_t^{8/5}L_{xyz}^{4/3}} \lesssim \|D_{xyz}^{1/2}w\|_{L_t^4L_{xyz}^4}\|w\|_{L_t^\infty L_{xyz}^4}^2 \lesssim (\alpha^*)^{3/2}$$

by (21.15) and (21.7). For F_2, we estimate as

$$\|D_{xyz}^{1/2}F_2\|_{L_t^1L_{xyz}^2} \lesssim t_1\|\langle D_{xyz}\rangle^{1/2}w\|_{L_t^\infty L_{xyz}^2} \lesssim (\alpha^*)^{3/2}$$

by (21.7). For F_3, we estimate as

$$\begin{aligned}\|F_3\|_{L_t^2L_{xyz}^2} &\lesssim \|q\|_{L_t^2L_{xyz}^2} + \|\nabla q\|_{L_t^2L_{xyz}^2}\\ &\lesssim t_1^{1/2}\|q\|_{L_t^\infty L_{xyz}^2} + \|\nabla q\|_{L_t^2L_{xyz}^2}\\ &\lesssim (\alpha^*)^{1/2}.\end{aligned}$$

We also estimate, using the fractional Leibniz rule, and the 3d Sobolev embedding,

$$\begin{aligned}\|D_{xyz}^{1/2}(|v|^2v)\|_{L_t^{10/7}L_{xyz}^{10/7}} &\le \|D_{xyz}^{1/2}v\|_{L_t^{10/3}L_{xyz}^{10/3}}\|v\|_{L_t^5L_{xyz}^5}^2\\ &\lesssim \|D_{xyz}^{1/2}v\|_{L_t^{10/3}L_{xyz}^{10/3}}\|D_{xyz}^{1/2}v\|_{L_t^5L_{xyz}^{30/11}}^2.\end{aligned}$$

Collecting, we have

$$\|D_{xyz}^{1/2}v\|_{L_t^qL_{xyz}^p} \lesssim \|D_{xyz}^{1/2}v_0\|_{L_{xyz}^2} + \|D_{xyz}^{1/2}v\|_{L_t^{10/3}L_{xyz}^{10/3}}\|D_{xyz}^{1/2}v\|_{L_t^5L_{xyz}^{30/11}}^2 + (\alpha^*)^{1/2},$$

which yields (21.25).

This completes the proof of BSO 1–8.

22. Finite time blow-up and log-log speed

In this section, we deduce the finite-time blow-up and log-log speed. Specifically we deduce (22.6), (22.7), and (22.8) below. The convergence (1.7) claimed in Theorem 1.1 is a refined version of (22.6) that can be obtained by following the techniques of [24, Prop. 6].

From the above bootstrap argument, we conclude that $t_1 = T$, and hence, by (21.6), $T \le \alpha^* < \infty$. From the local theory in H^1, we have that $\|u(t)\|_{H^1} \nearrow +\infty$ as $t \nearrow T$. By BSO 3 and the fact that $\|\nabla u(t)\|_{L^2} \le \lambda^{-1}(\|Q_b\|_{L^2} + \mathcal{E}(t))$, we conclude that

$$\lambda(t) \to 0 \quad \text{as} \quad t \to T. \tag{22.1}$$

Also, since $|\lambda_s/\lambda| \leq 1$ on $[s_0, s_1)$ (see BSO 2 and (5.6), recall s_1 corresponds to $t_1 = T$), integrating from s_0 to s, we obtain

$$|\log \lambda(s)| \leq s - s_0 + |\log \lambda(s_0)|\,.$$

By (22.1) $\lambda(s) \to 0$ as $s \to s_1$, thus, the above line implies $s_1 = +\infty$.

We now claim that for t close enough to T, λ satisfies the differential inequality

$$\frac{1}{C} \leq -(\lambda^2 \log|\log \lambda|)_t \leq C \tag{22.2}$$

for some universal constant $C > 0$. To establish (22.2), we first show that

$$\frac{1}{C} \leq b \log|\log \lambda| \leq C. \tag{22.3}$$

To prove (22.3), observe that the lower bound is equivalent to BSO 6. For the upper bound, first note that (5.9) and (5.16) give

$$b(s) \sim \frac{1}{\log s}\,. \tag{22.4}$$

Moreover, (5.6) gives

$$-\frac{\lambda_s}{\lambda} \sim b. \tag{22.5}$$

Integrating (22.5) and using (22.4), we obtain

$$\begin{aligned} -\log \lambda(s) + \log \lambda(s_0) &\sim \int_{s_0}^{s} b(\sigma)\, d\sigma \\ &\sim \int_{s_0}^{s} \frac{d\sigma}{\log \sigma} \\ &\geq \int_{s_0}^{s} \frac{\log \sigma - 1}{(\log \sigma)^2}\, d\sigma \\ &= \frac{s}{\log s} - \frac{s_0}{\log s_0}. \end{aligned}$$

Recall that $s_0 = e^{3\pi/4b_0} \gg 1$ and $0 < \lambda \ll 1$ by BSI 6. These imply $\log \lambda(s_0) < 0$, $\log \lambda(s) < 0$, $s_0/\log s_0 > 0$. Consequently,

$$|\log \lambda(s)| \leq \frac{s}{\log s} + |\log \lambda(s_0)|\,.$$

Taking the log of both sides, and taking $s \geq s_0 \gg 1$ such that $s/\log s \gg |\log \lambda(s_0)|$, gives

$$\log|\log \lambda(s)| \leq (\log s - \log\log s) \sim \log s \sim \frac{1}{b(s)}\,,$$

which gives the upper bound in (22.3). Now we proceed to justify (22.2) using (22.3). Since $\lambda \ll 1$, we have

$$\begin{aligned} \frac{d}{dt}(\lambda^2 \log|\log\lambda|) &= \lambda\lambda_t(\log|\log\lambda|)\left(2+\frac{\lambda}{|\log\lambda|\,(\log|\log\lambda|)^2}\right) \\ &\sim \lambda\lambda_t(\log|\log\lambda|) \\ &= \frac{\lambda_s}{\lambda}(\log|\log\lambda|). \end{aligned}$$

By (22.3) and (22.5), we obtain (22.2). Integrating (22.2) from t up to the blow-up time T (where $\lambda(T)=0$), we obtain

$$\lambda^2(t)\log|\log\lambda(t)| \sim (T-t).$$

This implies

$$\lambda(t) \sim \left(\frac{T-t}{\log|\log(T-t)|}\right)^{1/2}. \tag{22.6}$$

From the relationship between $\lambda(t)$ and t given by (22.6), the relationship between $b(s)$ and s given by (22.4), and the relationship between b and λ given by (22.3), we obtain the relationship between s and t as

$$|\log(T-t)|^{\alpha_1} \lesssim s \lesssim |\log(T-t)|^{\alpha_2} \quad \text{for some } \alpha_1<\alpha_2 \tag{22.7}$$

and the relationship between b and t as

$$b \sim \frac{1}{\log|\log(T-t)|}. \tag{22.8}$$

The proof of the exact convergence in (22.6) and (22.8) follows [24, Prop. 6]. This proves (1.7), (1.8) and (1.9).

23. Convergence of the location of the singular circle

In this section, we just note that parameters $r(t)$ and $z(t)$ converge. Indeed, by (5.7), we have $|r_s|/\lambda \le 1$ and $|z_s|/\lambda \le 1$. Hence, for $0<t_1<t_2<T$, we have

$$|r(t_2)-r(t_1)| \le \int_{t_1}^{t_2} |r_t|\,dt = \int_{t_1}^{t_2} \frac{1}{\lambda}\left|\frac{r_s}{\lambda}\right|\,dt \le \int_{t_1}^{t_2}\frac{dt}{\lambda}.$$

By the estimates obtained on $\lambda(t)$ above, this implies that $r(t)$ is Cauchy as $t\to T$. Similarly, we obtain that $z(t)$ converges as $t\to T$. BSO 1 (which we proved in Bootstrap Step 10) then implies that $|r(T)-1| \le (\alpha^*)^{2/3}$. This proves (1.5) and (1.6).

24. L^2 convergence of the remainder

In this section, we will establish the existence of $u^* \in L^2(\mathbb{R}^3)$ such that $\tilde{u} \to u^*$ in $L^2(\mathbb{R}^3)$. This will prove (1.3) in Theorem 1.1.

First, we observe that there exists $u^* \in L^2_{\text{loc}}(\mathbb{R}^3\backslash\{0\})$ such that for each $R > 0$,

$$\lim_{t\nearrow T} \int_{|r(t)-r(T)|^2+|z(t)-z(T)|^2 \geq R^2} |\tilde{u}(t) - u^*|^2 \, dxdydz = 0\,. \tag{24.1}$$

This is essentially a consequence of the fact that, as in Bootstrap Step 16, we can carry out the local well-posedness theory in $H^{1/2}$ in this external region. This local theory gives a continuity of the flow (in this external region) right up to time T.

Having established that (24.1) holds, we note that it follows from (24.1) that for each $R > 0$,

$$\int_{|r-r(T)|^2+|z-z(T)|^2 \geq R^2} |u^*|^2 dxdydz \leq 2\|Q\|^2_{L^2_{rz}}\,,$$

and thus, in fact $u^* \in L^2(\mathbb{R}^3)$. We next upgrade (24.1) and show that in fact

$$\tilde{u}(t) \to u^* \text{ in } L^2(\mathbb{R}^3) \tag{24.2}$$

(not just in $L^2_{\text{loc}}(\mathbb{R}^3\backslash\{0\})$). The following estimate is, in the 3d case, a straightforward consequence of the Hardy inequality. But in 2d, it is a little more subtle and requires a logarithmic correction:

Lemma 24.1. *Suppose that $\phi(x, y)$ is a function on $\mathbb{R}^2$. Then for any $D \gg 1$, we have (with $r = \sqrt{x^2 + y^2}$),*

$$\|\phi\|^2_{L^2_{xy}(r\leq D)} \lesssim D^2(\|\phi\|^2_{L^2_{xy}(r\leq 1)} + (\log D)\|\nabla\phi\|^2_{L^2_{xy}(r\leq D)})\,.$$

Proof. We follow [24, Apx. C] for the reader's convenience. Represent ϕ in polar variables as $\phi(r, \theta)$. Let $\hat{\phi}(r, n)$ denote the Fourier series representation in the θ-variable, i.e.,

$$\hat{\phi}(r, n) = \int e^{-in\theta}\phi(r, \theta)\, d\theta. \tag{24.3}$$

The first step is to obtain (24.4), the bound for nonzero frequencies. By the Plancherel theorem,

$$\begin{aligned}\int_{\theta=0}^{2\pi} |\phi(r,\theta) - \hat{\phi}(r,0)|^2 \, d\theta \\ &= \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} |\hat{\phi}(r,n)|^2 \\ &= \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} n^{-2} |\widehat{\partial_\theta \phi}(r,n)|^2 \\ &\leq \sum_{n\in\mathbb{Z}} |\widehat{\partial_\theta \phi}(r,n)|^2 \\ &\leq \int_{\theta=0}^{2\pi} |\partial_\theta \phi(r,\theta)|^2 \, d\theta \, .\end{aligned}$$

Integrating in rdr from $r = 0$ to $r = D$, we obtain the desired bound for nonzero angular frequencies:

$$\|\phi(r,\theta) - \hat{\phi}(r,0)\|_{L^2_{xy}(r\leq D)} \lesssim \|\nabla \phi\|_{L^2_{xy}(r\leq D)} \, . \tag{24.4}$$

The next step is to obtain (24.7), the bound for zero angular frequencies. By the mean-value theorem for integrals, there exists $r_0 \in [\frac{1}{2}, 1]$ such that

$$\hat{\phi}(r_0, 0) = 2 \int_{r=1/2}^{1} \hat{\phi}(r,0) dr \, .$$

By Cauchy-Schwarz

$$|\hat{\phi}(r_0,0)| \leq \frac{2}{\sqrt{2}} \left(\int_{1/2}^{1} |\hat{\phi}(r,0)|^2 dr \right)^{1/2} \, .$$

By the definition of $\hat{\phi}(r_0, 0)$ from (24.3) with $n = 0$ and Cauchy-Schwarz,

$$|\hat{\phi}(r,0)| \lesssim \left(\int |\phi(r,\theta)|^2 d\theta \right)^{1/2} \, .$$

Combining we obtain

$$|\hat{\phi}(r_0,0)| \lesssim \left(\int_{r=1/2}^{1} \int_{\theta=0}^{2\pi} |\phi(r,\theta)|^2 r dr \, d\theta \right)^{1/2} \lesssim \|\phi\|_{L^2_{xy}(r\leq 1)} \, . \tag{24.5}$$

By the fundamental theorem of calculus,

$$\hat{\phi}(r,0) - \hat{\phi}(r_0,0) = \int_{s=r_0}^{s=r} \partial_s \hat{\phi}(s,0) \, ds.$$

Inserting the definition of $\hat{\phi}(s,0)$,

$$\hat{\phi}(r,0)-\hat{\phi}(r_0,0)=\int_{s=r_0}^{s=r}\int_{\theta=0}^{2\pi}\partial_s\phi(s,\theta)d\theta ds.$$

By Cauchy-Schwarz,

(24.6)

$$\begin{aligned}|\hat{\phi}(r,0)-\hat{\phi}(r_0,0)| &\lesssim \left(\int_{s=r_0}^{s=D}\int_{\theta=0}^{2\pi}|\partial_s\phi(s,\theta)|^2 sd\theta ds\right)^{1/2}\left(\int_{s=r_0}^{s=D}\int_{\theta=0}^{2\pi}s^{-1}dsd\theta\right)^{1/2}\\ &\lesssim (\log D)^{1/2}\|\nabla\phi\|_{L^2_{xy}(r\le D)}.\end{aligned}$$

Combining (24.5) and (24.6) gives

$$|\hat{\phi}(r,0)|\lesssim \|\phi\|_{L^2_{xy}(r\le 1)}+(\log D)^{1/2}\|\nabla\phi\|_{L^2_{xy}(r\le D)}\,.$$

Integrating against $rdrd\theta$,

$$\|\hat{\phi}(r,0)\|^2_{L^2_{xy}(r\le D)}\lesssim D^2\left(\|\phi\|^2_{L^2_{xy}(r\le 1)}+(\log D)\|\nabla\phi\|^2_{L^2_{xy}(r\le D)}\right)\,. \tag{24.7}$$

Combining (24.4) and (24.7), we obtain the claimed bound. □

Now we return to proving (24.2). It follows easily from (24.1) that $\tilde{u}(t)\to u^*$ weakly in $L^2(\mathbb{R}^3)$. Thus, to establish (24.2), it suffices to establish

$$\int |u^*|^2=\lim_{t\nearrow T}\int|\tilde{u}(t)|^2. \tag{24.8}$$

Let $R(t)=e^{a/b(t)}\lambda(t)$ for some constant $0<a\ll 1$, so that $R(t)\searrow 0$ as $t\nearrow T$ just a little slower that $\lambda(t)$. We first claim that

$$\int_{|r-r(t)|^2+|z-z(t)|^2\le R(t)^2}|\tilde{u}(t)|^2dxdydz\to 0. \tag{24.9}$$

We now prove (24.9). By rescaling and applying Lemma 24.1,

$$\begin{aligned}&\int_{|r-r(t)|^2+|z-z(t)|^2\le R(t)^2}|\tilde{u}(t)|^2dxdydz\\ &\quad=\int_{\tilde{r}^2+\tilde{z}^2\lesssim e^{2a/b(t)}}|\epsilon(\tilde{r},\tilde{z})|^2\mu(\tilde{r})d\tilde{r}d\tilde{z}\\ &\quad\lesssim\int_{\tilde{r}^2+\tilde{z}^2\le e^{2a/b(t)}}|\epsilon(\tilde{r},\tilde{z})|^2d\tilde{r}d\tilde{z}\\ &\quad\lesssim e^{2a/b(t)}\left(\int_{\tilde{r}^2+\tilde{z}^2\le 1}|\epsilon(\tilde{r},\tilde{z})|^2d\tilde{r}d\tilde{z}+\frac{1}{b}\int_{\tilde{r}^2+\tilde{z}^2\le e^{2a/b(t)}}|\nabla\epsilon(\tilde{r},\tilde{z})|^2d\tilde{r}d\tilde{z}\right)\\ &\quad\lesssim\frac{e^{2a/b(t)}}{b}\mathcal{E}(t)\to 0.\end{aligned}$$

Let $\phi(r,z)=1$ when $r^2+z^2\geq 1$ but $\phi(r,z)=0$ for $r^2+z^2\leq \frac{1}{2}$. It remains to prove that

$$\lim_{t\nearrow T}\int \phi\left(\frac{r-r(t)}{R(t)},\frac{z-z(t)}{R(t)}\right)|\tilde{u}(t)|^2dxdydz=\int |u^*|^2dxdydz.$$

To prove this, it suffices to show that

$$\left|\int \phi\left(\frac{r-r(t)}{R(t)},\frac{z-z(t)}{R(t)}\right)|u(t)|^2-\int \phi\left(\frac{r-r(T)}{R(t)},\frac{z-z(T)}{R(t)}\right)|u^*|^2\right|\underset{t\nearrow T}{\to} 0. \tag{24.10}$$

(Note that here we have replaced $\tilde{u}$ by u; this is acceptable, since the contribution from Q_b is negligible in this region of space.) To prove this, let, for t fixed,

$$v(\tau)=\int \phi\left(\frac{r-r(\tau)}{R(t)},\frac{z-z(\tau)}{R(t)}\right)|u(\tau)|^2dxdydz.$$

Then observe

$$\partial_\tau v=-\frac{(r_\tau,z_\tau)}{R(t)}\cdot\int\nabla\phi(\cdot,\cdot)|u|^2\,dxdydz+\frac{2}{R(t)}\operatorname{Im}\int\nabla\phi(\cdot,\cdot)\cdot\nabla u\,\bar{u}\,dxdydz.$$

By (5.7), $|(r_\tau,z_\tau)|\leq\lambda^{-1}$, and by BSO 3, $\|\nabla u\|_{L^2}\sim\lambda^{-1}(1+O(\Gamma^{4/5}))\sim\lambda^{-1}$. Hence,

$$|\partial_\tau v(\tau)|\lesssim\frac{1}{e^{a/b(t)}\lambda(t)}\cdot\frac{1}{\lambda(\tau)}\,.$$

Note that by (22.6),

$$\begin{aligned}\int_t^T\frac{d\tau}{\lambda(\tau)}&\lesssim\int_t^T\left(\frac{\log|\log(T-\tau)|}{T-\tau}\right)^{1/2}d\tau\\&\sim\int_t^T(T-\tau)^{-1/2}(\log|\log(T-\tau)|)^{1/2}\left(1+\frac{1}{\log|\log(T-\tau)|\ \log(T-\tau)}\right)d\tau\\&=2(T-t)^{1/2}(\log|\log(T-t)|)^{1/2}.\end{aligned}$$

The rate of $b(t)$ in (22.8) implies that $e^{a/b(t)}\sim|\log(T-t)|^C$ for some $C>0$. Hence,

$$|v(T)-v(t)|\lesssim\frac{1}{e^{a/b(t)}}\cdot\frac{1}{\lambda(t)}\int_t^T\frac{d\tau}{\lambda(\tau)}\lesssim\frac{\log|\log(T-t)|}{|\log(T-t)|^C}.$$

But by (24.1),

$$v(T)=\lim_{\tau\nearrow T}v(\tau)=\int\phi\left(\frac{r-r(T)}{R(t)},\frac{z-z(T)}{R(t)}\right)|u^*|^2$$

yielding (24.10), and hence, (24.2) is established. This finishes the proof of Theorem 1.1.

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Brown University

Arizona State University