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A CLASS OF STRONGLY COOPERATIVE SYSTEMS WITHOUT COMPACTNESS

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We consider a system of ordinary differential equations (ODE's)

$$\text{(1)} \qquad \dot{x} = F(x), \qquad x \in P \subset \mathbb{R}^n,\ x = (x_1, \ldots, x_n),\ F = (F_1, \ldots, F_n),$$

where P is open and $F : P \to \mathbb{R}^n$ is a C^1 vector field. System (1) is called *strongly cooperative* if at each $x \in P$, $(\partial F_i/\partial x_j)(x) > 0$ for $i \neq j$. For a survey the reader is referred to [S].

Strongly cooperative systems of ODE's have many interesting properties. To formulate them it is necessary to introduce some notation:

For $x, y \in P$ we write

$x \le y$ if $x_i \le y_i$ for all i,
$x < y$ if $x \le y$ and $x \neq y$,
$x \ll y$ if $x_i < y_i$ for all i.

Since F is C^1, it generates on P a local flow of class C^1, denoted by φ_t (this means that $\varphi_t(x)$ is the value taken on at time t by the solution to (1) passing through $x \in P$ at 0). A theorem due to Müller and Kamke states that for a strongly cooperative system defined on an open convex P the resulting local flow φ_t is strongly monotone, which means that $x < y$ implies $\varphi_t(x) \ll \varphi_t(y)$ for $t > 0$ as long as both exist. The reader interested in an abstract theory of strongly monotone (semi)flows is referred to [H2] and [ST].

A most important feature of strongly cooperative systems of ODE's is a strong tendency for their trajectories to converge to an equilibrium. More precisely, if for each $x \in P$ its forward semitrajectory has compact closure in P, then the set of points convergent to a stable equilibrium is open dense in P (cf. [H2] and [P]).

Results so far published on asymptotic behavior of (the majority of) forward semitrajectories for strongly cooperative systems have been formulated and proved under the assumption that (some of) the forward semitrajectories under consideration have compact closures. On the contrary, in the present note we require no compactness hypotheses (except, of course, local compactness of the ambient Euclidean space $\mathbb{R}^n$). Our Main Theorem is a generalization of the author's previous result, formulated without proof in [M2]; compare also [M1]. No knowledge of the latter papers is needed, however, for reading this note.

By a *first integral* for (1) we mean a C^1 function $H : P \to \mathbb{R}$ such that $\langle \operatorname{grad} H(x), F(x) \rangle = 0$ at each $x \in P$, where $\langle\ ,\ \rangle$ stands for the standard inner product in $\mathbb{R}^n$ and $\operatorname{grad} H$ is the vector with components $\partial H/\partial x_i$.

In the sequel, the letters x, y etc. represent points in P, whereas the letters u, v etc. represent tangent vectors. A vector is called *nonnegative* (resp. *positive*) if all its components are nonnegative (resp. positive). The set of nonnegative (resp. positive) vectors is denoted by C (resp. C°).

The remaining part of the note is devoted to the proof of the following

MAIN THEOREM. *Assume that* (1) *is a strongly cooperative system of ODE's admitting a first integral with positive gradient. Then any forward semitrajectory either leaves every compact contained in P or converges to an equilibrium.*

For $x \in P$ consider the system of nonautonomous linear ODE's:

$$\dot{v}_i = \sum_{j=1}^{n} (\partial F_i/\partial x_j)(\varphi_t(x)) \cdot v_j \,. \tag{2}$$

The systems (1)+(2) generate a local flow ϕ_t on the tangent bundle $P \times \mathbb{R}^n$. The flow ϕ_t can be written in the form

$$\phi_t(x, v) = (\varphi_t(x), \zeta(t, x)v) \,,$$

where $\zeta(t, x)$ is the corresponding transition operator for (2).

PROPOSITION 1. *Let* (1) *be a strongly cooperative system of ODE's. Then for each $x \in P$, $t > 0$ such that $\varphi_t(x)$ exists, the inclusion $\zeta(t, x)(C \setminus \{0\}) \subset C^\circ$ holds.*

Proof. This is a particular case of Thm. 1.1(b) in [H1]. ∎

By definition, the level sets of the first integral H are invariant. The positivity of $\operatorname{grad} H$ implies that each level set of H is an $(n-1)$-dimensional C^1 submanifold of P.

For the rest of the proof of Main Theorem let a level set L of H be fixed. The tangent bundle of L is denoted by TL. We have

$$TL = \{(x, v) : \langle \operatorname{grad} H(x), v \rangle = 0,\ x \in L\} \,.$$

In particular, $\{(x, F(x)) :\ x \in L\} \subset TL$. A Finsler structure on TL is given by a continuous mapping

$$TL \ni (x, v) \mapsto |v|_x \in \mathbb{R}_+$$

such that for each $x \in L$ the mapping $|\ |_x$ is a norm on the subspace

$$S_x := \{v \in \mathbb{R}^n : \langle \operatorname{grad} H(x), v\rangle = 0\}.$$

Define $U_x := \{v \in \mathbb{R}^n : \langle \operatorname{grad} H(x), v\rangle = 1\}$. Since H is a first integral, we have

(3) $\quad U_{\varphi_t(x)} = \zeta(t,x)U_x \quad$ for $x \in L,\ t \in \mathbb{R}$ such that $\varphi_t(x)$ exists.

Further, set $A_x := C \cap U_x$, $A_x^\circ := C^\circ \cap U_x$. Proposition 1 and (3) yield

(4) $\quad \zeta(t,x)A_x \subset A^\circ_{\varphi_t(x)} \quad$ for $x \in L,\ t \in \mathbb{R}$ such that $\varphi_t(x)$ exists.

It is straightforward that A_x is the $(n-1)$-dimensional simplex whose vertex lying on the ith coordinate axis has coordinate equal to the reciprocal of $(\partial H/\partial x_i)(x)$. Finally, define

$$B_x := A_x - A_x := \{u \in \mathbb{R}^n : u = v - w,\ v, w \in A_x\}.$$

LEMMA 1. *For each $x \in L$ the set B_x constructed above has the following properties*:

(a) *B_x is a relative neighborhood of* 0 *in S_x*,
(b) *B_x is compact, convex and balanced*,
(c) *B_x is a (convex) polyhedron.*

Proof. (a) First, let $B_x \ni u = v - w,\ v, w \in A_x$. We have $\langle \operatorname{grad} H(x), u\rangle = \langle \operatorname{grad} H(x), v\rangle - \langle \operatorname{grad} H(x), w\rangle = 1 - 1 = 0$. This shows that $B_x \subset S_x$.

Further, fix some $v \in A_x^\circ$. We have

$$A_x - v \subset B_x.$$

The convex set $A_x - v$ has dimension $n-1$, so $B_x \subset S_x$ has the same dimension. Moreover, since $v \in A_x^\circ$ (= the relative interior of A_x in U_x), its translate $0 = v - v$ belongs to the relative interior of $A_x - v$ in $S_x = U_x - v$, hence to the relative interior of B_x in S_x.

(b) and (c). These are propositions in the theory of convex sets (see e.g. Theorems 8.1 and 8.6 in [L]). ∎

For any $x \in L$ define a norm on S_x as

(5) $$|u|_x := \inf\{\lambda \geq 0 :\ u \in \lambda B_x\}.$$

In order to show that the mapping $(x,u) \mapsto |u|_x$ is a Finsler structure it is necessary and sufficient to ensure that $|\ |_x$ and the Euclidean norm are equivalent uniformly in x in compact sets. This is established by the following

LEMMA 2. *Let $|\ |_x$ be the family of norms defined by* (5) *and let $\|\ \|$ be the Euclidean norm. Then for any compact $K \subset L$ we can find positive constants $d \leq D$ such that for every $x \in K$, $u \in S_x$*

$$d|u|_x \leq \|u\| \leq D|u|_x.$$

P r o o f. For $x \in K$ we define d_x to be the Euclidean distance between 0 and the relative boundary of B_x in S_x, and D_x to be the maximum Euclidean norm of vertices of B_x. Evidently for $u \in S_x$, $d_x|u|_x \le \|u\| \le D_x|u|_x$. From the continuous dependence of vertices of the polyhedra B_x on x it follows that the assignments $x \mapsto d_x$ and $x \mapsto D_x$ are continuous. Now it suffices to take $d := \inf\{d_x : x \in K\}$, $D := \sup\{D_x : x \in K\}$. ■

PROPOSITION 2. *Assume that* (1) *is a strongly cooperative system of ODE's admitting a first integral with positive gradient. Then for each* $x \in P$, *each* $t > 0$ *such that* $\varphi_t(x)$ *exists*, *and each* $u \in S_x \setminus \{0\}$ *we have*

$$|\zeta(t,x)u|_{\varphi_t(x)} < |u|_x \,.$$

P r o o f. Assume $|u|_x = 1$. This means that u belongs to the relative boundary of B_x in S_x. By (4)

$$\zeta(t,x)B_x = \zeta(t,x)A_x - \zeta(t,x)A_x \subset A^{\circ}_{\varphi_t(x)} - A^{\circ}_{\varphi_t(x)} \,.$$

The last set lies in the relative interior of $B_{\varphi_t(x)}$ in $S_{\varphi_t(x)}$, so $|\zeta(t,x)u|_{\varphi_t(x)} < 1$. ■

Now, our Main Theorem will be a consequence of the following abstract result, which can be regarded as a form of the Invariance Principle.

PROPOSITION 3. *Let* F *be a* C^1 *vector field on a manifold* L, *generating a local flow* φ_t *with derivative* ϕ_t. *Assume that there is a Finsler structure* $|\ |$ *on the tangent bundle* TL *of* L *such that for any nonzero* $v \in TL$ *one has* $|\phi_t v| < |v|$ *for all* $t > 0$ *such that* $\phi_t v$ *exists. Then any forward semitrajectory of* φ_t *either leaves every compact set or converges to an* (*exponentially asymptotically stable*) *equilibrium.*

P r o o f. The assertion of the proposition is equivalent to saying that for any $x \in L$ its ω-limit set $\omega(x)$ is either empty or a singleton $\{y\}$ such that y is exponentially asymptotically stable.

First, suppose that some $\omega(x)$ contains a point y which is not an equilibrium. Choose $T > 0$ such that $\varphi_T(y) \ne y$, and two neighborhoods, M of y and N of $\varphi_T(y)$, such that $\sup\{|F(z)| : z \in N\} < \inf\{|F(z)| : z \in M\}$. Since $y, \varphi_T(y) \in \omega(x)$, there are t_1, $t_2 > t_1 + T$ such that $\varphi_{t_1}(x), \varphi_{t_2}(x) \in M$, $\varphi_{t_1+T}(x), \varphi_{t_2+T}(x) \in N$. By the choice of M and N we have $|F(\varphi_{t_2}(x))| > |F(\varphi_{t_1+T}(x))|$. But from the assumptions of the proposition it follows that

$$|F(\varphi_{t_2}(x))| = |\phi_{t_2-t_1+T}F(\varphi_{t_1+T}(x))| < |F(\varphi_{t_1+T}(x))| \,.$$

We have thus proved that any nonempty ω-limit set consists entirely of equilibria. Now, let $y \in \omega(x)$ be an equilibrium, and let v be a nonzero vector tangent at y. We have $|\phi_1 v| < |v|$, so the spectral radius of ϕ_1, considered a linear operator from the tangent space at y into itself, is less than 1. Therefore y is exponentially asymptotically stable, hence $\omega(x) = \{y\}$. ■

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