A CLASS OF SUBHARMONIC FUNCTIONS
AND INTEGRAL INEQUALITIES

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Let $U_\rho(a) = \{z \in C \mid |z - a| < \rho\}$, C the complex plane, and f be a holomorphic function on the closure of $U_\rho(a)$. Then f satisfies the inequality

$$(1) \qquad (\pi\rho^2)^{-1} \int_{U_\rho(a)} |f(z)|^2 dxdy \leqq \left((2\pi)^{-1} \int_0^{2\pi} |f(a + \rho e^{i\theta})| d\theta\right)^2 ,$$

where $z = x + iy$. This was proved by Carleman [4]. Beckenbach and Radó [2] introduced the functions of class PL; a real-valued function defined in a domain of R^2 is said to be of class PL, if $u \geqq 0$ and $\log u$ is subharmonic (cf. [12, 2.12]). It was proved in [3] that a continuous function u, $u \geqq 0$, is of class PL if and only if the inequality (1) holds for every disk $U_\rho(a)$ contained in the domain, and Radó raised a problem to deal with related inequalities in the case of higher dimensions (cf. [12, 3.27]). Our main purpose concerns this problem; it will be shown that, if u is continuous, $u \geqq 0$, and satisfies the inequality of the type (1) in R^n, then $\log u$ is subharmonic for all $n \geqq 2$, but the converse is not true for $n \geqq 3$.

Beckenbach [1] extended the Fejér-Riesz inequality from holomorphic functions to functions of class PL, and Yamashita [15] obtained various results. On the other hand, Hasumi and Mochizuki [9] and Mochizuki [10] extended the inequality to holomorphic functions of several complex variables. We shall note that this can be extended to the functions u such that $\log u$ are plurisubharmonic. A meaning of these inequalities lies in the fact that the growth conditions (3) and (5) imply the integrability of u over any hyperplanes $L \cap \varDelta$ and $L \cap B$, respectively (see §2). This was observed by Hardy for H^2 ([8, 5]), and then by Fejér and Riesz for H^1 ([6, §3]). Recently, another treatment of the Fejér-Riesz inequality based on a different viewpoint appeared in [11, 4.7, 5.5].

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1. Log. subharmonic functions and the Carleman inequality.

Let G be a domain in R^n, $n \geq 2$, and u be a real-valued function defined on G. We shall call u a *log. subharmonic function*, if $u \geq 0$ and $\log u$ is subharmonic. We do not exclude the case $u \equiv 0$. The log. subharmonic functions constitute a subclass of the strongly subharmonic functions introduced by Gårding and Hörmander [7]. Every log. subharmonic function u is subharmonic and u^p is log. subharmonic for $p > 0$. Let $B_\rho(a) = \{x \in R^n \mid \|x - a\| < \rho\}$ for $a \in R^n$, $\rho > 0$, where $\|x\|^2 = \sum_{j=1}^n x_j^2$ if $x = (x_1, \cdots, x_n) \in R^n$. We shall write $B_\rho(0) = B_\rho$ and $B_1 = B$, the unit ball. We denote by V_ρ the volume of B_ρ, and simply by V in place of V_1. The ordinary volume element in R^n and the element of the surface area in the unit sphere ∂B will be denoted by $d\omega$ and $d\tau$, respectively. The area of ∂B will be denoted by S. For $a \in G$, $\delta(a)$ will stand for the distance from the point a to the boundary of G.

We use the following notations, if necessary,

$$I(u) = \int_B u(x)d\omega(x) , \qquad J(u) = \int_{\partial B} u(x)d\tau(x) ,$$

and examine whether $V^{-1}I(u^p) \leq (S^{-1}J(u))^p$, $p > 0$, is valid or not for log. subharmonic functions u defined in a neighborhood of $\bar{B}$. Clearly, this is equivalent to the problem on the more general inequality $(V^{-1}I(u^p))^{1/p} \leq (S^{-1}J(u^q))^{1/q}$, $p, q > 0$. Note that u^p is bounded on $\bar{B}$. In R^n, we define

$$c(p; n) = \sup \{ V^{-1}I(u^p)/(S^{-1}J(u))^p \} ,$$

where the supremum is taken over the log. subharmonic functions on neighborhoods of $\bar{B}$ such that $J(u) \neq 0$. Note that $c(p; n) \geq 1$ for every $p > 0$ and every n. Thus, $c(2; 2) = 1$ is the assertion of the Carleman inequality (1) extended by Beckenbach and Radó. Since $(V^{-1}I(u^p))^{1/p}$ is an increasing function of p, so is $c(p; n)$, hence we have $c(p; 2) = 1$, $0 < p \leq 2$. Clearly $V^{-1}I(u) \leq S^{-1}J(u)$ for any subharmonic function u, so $c(p; n) = 1$ for $0 < p \leq 1$. The following is our main result. It is proved in [3] in case $n = 2$ that u is log. subharmonic when the inequality (2) below is satisfied. The same method can be applied to the general case, as is done in the following, but one can see from this proof that the higher dimensional analogue of the Carleman inequality is a very strong condition for $n \geq 3$. It is the question posed by Radó whether the converse holds in general.

THEOREM 1. *Let G be a domain of R^n, $n \geq 2$, and u be a continuous*

function defined on G and $u \geq 0$. If u satisfies the following inequality (2) at every point $a \in G$ for any ρ, $0 < \rho < \delta(a)$, then u is log. subharmonic:

$$(2) \qquad V_\rho^{-1} \int_{B_\rho(a)} u(x)^2 d\omega(x) \leq \left(S^{-1} \int_{\partial B} u(a + \rho x) d\tau(x) \right)^2 .$$

The converse is not true for $n \geq 3$; that is, $c(p; n) > 1$, if $n \geq 3$ and $p \geq 2$. Moreover, if $n \geq 2$ and $p > (n + 1)(n - 1)^{-1}$, then $c(p; n) = \infty$.

PROOF. Note that, if u satisfies the inequality (2), then so does $u + \varepsilon$, $\varepsilon > 0$, and $\log(u + \varepsilon)$ tends to $\log u$ decreasingly as $\varepsilon \to 0$. Thus we can suppose $u(x) > 0$ on G. First, we prove the log. subharmonicity of u under the assumption that u is a C^2-function satisfying (2). Further, it is sufficient to show that the inequality (2) at $a = 0$ implies that $u(0)\Delta u(0) - \sum_{j=1}^n ((\partial u/\partial x_j)(0))^2 \geq 0$. We can see from the straightforward use of Pizzetti's formula ([5]) to the function u that

$$J_\rho := S^{-1} \int_{\partial B} u(\rho x) d\tau(x) = u(0) + \rho^2 (2n)^{-1} \Delta u(0) + o(\rho^2) .$$

Applying the same formula to the function u^2, we obtain

$$I_\rho := V_\rho^{-1} \int_{B_\rho} u(x)^2 d\omega(x) = n\rho^{-n} \int_0^\rho r^{n-1} dr \left(S^{-1} \int_{\partial B} u(rx)^2 d\tau(x) \right)$$

$$= u(0)^2 + \rho^2 (n + 2)^{-1} \left(\sum_j \left(\frac{\partial u}{\partial x_j}(0) \right)^2 + u(0)\Delta u(0) \right) + o(\rho^2) .$$

From the assumption that $J_\rho^2 - I_\rho \geq 0$, $0 < \rho < \delta(0)$, we get

$$u(0)\Delta u(0) - 2^{-1} n \sum_j \left(\frac{\partial u}{\partial x_j}(0) \right)^2 \geq 0 .$$

Now let $G_r = \{x \in G \mid \delta(x) > r\}$, $r > 0$. Define u_r by

$$u_r(x) = V_r^{-1} \int_{B_r(x)} u(t) d\omega(t) , \qquad x \in G_r .$$

Then u_r becomes a positive C^1-function on G_r and $\{u_r\}$ tends to u uniformly on compact subsets of G as $r \to 0$. If u is a C^1-function on G, then u_r is a C^2-function. Hence, for the proof, it is sufficient to verify that, if u satisfies the inequality (2), then so does each function u_r. This is done by using Minkowski's inequality as follows. Let $a \in G$. Take $r_0 > 0$ so that $a \in G_{r_0}$. Define u_r, $0 < r < r_0$, and take $\rho > 0$, sufficiently small. Then

$$\left(V_\rho^{-1} \int_{B_\rho(a)} u_r(x)^2 d\omega(x) \right)^{1/2} \leq V_r^{-1} \int_{B_r} d\omega(t) \left(V_\rho^{-1} \int_{B_\rho(a)} u(x + t)^2 d\omega(x) \right)^{1/2}$$

$$\leq V_r^{-1} \int_{B_r} d\omega(t) \left(S^{-1} \int_{\partial B} u(a + t + \rho x) d\tau(x) \right) = S^{-1} \int_{\partial B} u_r(a + \rho x) d\tau(x) .$$

The second assertion can be seen from the function $u(x_1, \cdots, x_n) = (x_1 + 1)^2 + x_2^2$. Indeed, this is log. subharmonic and, recalling

$$\int_B x_1^{2k_1} \cdots x_n^{2k_n} d\omega(x) = \frac{\Gamma(k_1 + 1/2) \cdots \Gamma(k_n + 1/2)}{\Gamma(k_1 + \cdots + k_n + n/2 + 1)} ,$$

$$\int_{\partial B} x_1^{2k_1} \cdots x_n^{2k_n} d\tau(x) = \frac{2\Gamma(k_1 + 1/2) \cdots \Gamma(k_n + 1/2)}{\Gamma(k_1 + \cdots + k_n + n/2)} ,$$

for integers $k_1, \cdots, k_n \geq 0$, where Γ denotes the gamma function, we obtain $V^{-1}I(u^2) = 1 + (8n + 40)/(n + 2)(n + 4)$, $S^{-1}J(u) = 1 + 2/n$. It follows that $c(2; n) > 1$ for $n \geq 3$. In order to show the last assertion, we take the functions $u_k(x_1, \cdots, x_n) = e^{kx_1}$ in R^n, $n \geq 2$, $k = 1, 2, \cdots$. In what follows, we shall use the symbol C to denote various positive constants independent of k. We have

$$I(u_k^p) = \int_{-1}^{1} e^{pkx_1}dx_1 \int_{x_2^2 + \cdots + x_n^2 < 1 - x_1^2} dx_2 \cdots dx_n$$

$$= C \int_{-1}^{1} e^{pkx}(1 - x^2)^{(n-1)/2}dx > C \int_{0}^{1} e^{pkx}(1 - x)^{(n-1)/2}dx .$$

Using polar coordinates for ∂B, we have

$$J(u_k) = \int_{0}^{\pi} d\theta_1 \cdots \int_{0}^{\pi} d\theta_{n-2} \int_{0}^{2\pi} e^{k \cos \theta_1} \prod_{j=1}^{n-2} (\sin \theta_j)^{n-1-j} d\theta_{n-1}$$

$$= C \int_{-1}^{1} e^{kx}(1 - x^2)^{(n-3)/2}dx < C \int_{0}^{1} e^{kx}(1 - x)^{(n-3)/2}dx .$$

Now let $a > 0$ and $l > -2$. Then, by $a(1 - x) = t$, we can write

$$\int_{0}^{1} e^{ax}(1 - x)^{l/2}dx = e^{a}a^{-(l+2)/2}\Gamma((l + 2)/2)\Delta(a) ,$$

where $\Delta(a) \to 1$ as $a \to \infty$. Thus, we have

$$I(u_k^p)J(u_k)^{-p} > Ck^{-(n+1)/2}k^{p(n-1)/2}\Delta(k) ;$$

this gives the desired conclusion.

REMARK. The case of $n = 1$ may be helpful to see the influence of the dimension on the inequality. Let $u(x)$ be a positive function on an open interval (a, b). If $\log u(x)$ is a convex function, then we have

$$(\beta - \alpha)^{-1} \int_{\alpha}^{\beta} u(x)^2 dx \leqq (2^{-1}(u(\alpha) + u(\beta)))^2 ,$$

where $a < \alpha < \beta < b$; this is the 1-dimensional analogue of the inequality (2) and follows from the inequality $(s - t)^{-1}(e^s - e^t) < 2^{-1}(e^s + e^t)$, $s \neq t$. But the converse is not true, as is seen from the function $u(x) = x^2$, $x > 0$. *Thus, situations are different in three cases:* $n = 1$, $n = 2$, and $n \geq 3$.

2. **Log. plurisubharmonic functions in C^n.** Hereafter, we restrict ourselves to the space of n complex variables, C^n. Let G be a domain in C^n and u be a real-valued function on G. We shall say that u is a *log. plurisubharmonic function*, if $u \geqq 0$ and $\log u$ is plurisubharmonic. In this case, u is plurisubharmonic and u^p is log. plurisubharmonic for $p > 0$. Let $F = (f_1, \cdots, f_m): G \to C^m$ be a holomorphic map. Then $\| F \| = (\sum_{j=1}^m |f_j|^2)^{1/2}$ is a log. plurisubharmonic function on G; this follows from the fact that, if u, v are functions of class PL in a domain of R^2, then so is $u + v$ ([12, 2.14]). Further, as is well known, log. plurisubharmonic functions frequently occur in complex analysis. We shall define a class of log. plurisubharmonic functions corresponding to the Hardy space H^p. Let $\varDelta$ be the unit polydisk of C^n and T denote the Bergman-Šilov boundary with Lebesgue measure $d\tau$. We define the class $LH^p(\varDelta)$, $0 < p < \infty$, as the totality of the log. plurisubharmonic functions u on $\varDelta$ satisfying the condition

$$(3) \qquad \sup \left\{ \int_T u(rz)^p d\tau(z) \,|\, 0 \leqq r < 1 \right\} < \infty \;.$$

This class was introduced by Yamashita [15] in case $n = 1$. Let $u \in LH^p(\varDelta)$. Then from the same argument as in [13, 3.3.3], we can see that

$$\int_{\partial U} u_z(r\alpha)^p d\alpha \leqq M < \infty \;, \qquad 0 \leqq r < 1 \;,$$

for almost every point $z \in T$, where ∂U denotes the unit circle and $d\alpha$ Lebesgue measure. The function $u_z(\lambda)^p$ is subharmonic in $|\lambda| < 1$, hence a theorem of Littlewood implies that $u_z(\lambda)$ has the radial limit for almost every $\alpha \in \partial U$ (cf. [14, Theorem IV. 34]). Thus, we can conclude that $u(z)$ has the radial limit $u^*(z)$ for almost all $z \in T$. Moreover, $u^* \in L^p(T)$, and

$$\int_T u(rz)^p d\tau(z) \to \int_T u^*(z)^p d\tau(z) \;, \qquad r \to 1 \;.$$

In the present situation, the Carleman inequality can easily be proved. We write $z_j = x_{2j-1} + ix_{2j}$, $1 \leqq j \leqq n$.

PROPOSITION 1. *Let $u \in LH^1(\varDelta)$. Then*

$$(4) \qquad \pi^{-n} \int_\varDelta u(z)^2 d\omega(z) \leqq \left((2\pi)^{-n} \int_T u^*(z) d\tau(z) \right)^2 \;.$$

PROOF. First, let u be a log. plurisubharmonic function defined in a neighborhood of $\bar{\varDelta}$. Write $z' = (z_2, \cdots, z_n) \in C^{n-1}$. $\varDelta'$, T' and $d\omega'$, $d\tau'$ are symbols with respect to the space C^{n-1}. Assume that the case $n - 1$ is valid. By means of Minkowski's inequality we have

$$\left(\pi^{-n}\int_\varDelta u(z)^2 d\omega(z)\right)^{1/2} = \left(\pi^{-1}\int_U dx_1 dx_2\left(\pi^{-(n-1)}\int_{\varDelta'} u(z_1,z')^2 d\omega'(z')\right)\right)^{1/2}$$

$$\leqq \left(\pi^{-1}\int_U dx_1 dx_2\left((2\pi)^{-(n-1)}\int_{T'} u(z_1,z')d\tau'(z')\right)^2\right)^{1/2}$$

$$\leqq (2\pi)^{-(n-1)}\int_{T'} d\tau'(z')\left(\pi^{-1}\int_U u(z_1,z')^2 dx_1 dx_2\right)^{1/2}$$

$$\leqq (2\pi)^{-(n-1)}\int_{T'} d\tau'(z')\left((2\pi)^{-1}\int_0^{2\pi} u(e^{i\theta},z')d\theta\right)$$

$$= (2\pi)^{-n}\int_T u(z)d\tau(z) .$$

Now let $u \in LH^1(\varDelta)$ and let $u_r(z) = u(rz)$, $0 < r < 1$. Then each u_r satisfies the inequality (4). Let $\tilde{u}_r(z) = r^{-n}u(z)$ for $z \in \varDelta_r$ and $\tilde{u}_r(z) = 0$ for $z \notin \varDelta_r$, where $\varDelta_r = \{z \in C^n \mid |z_j| < r,\ 1 \leqq j \leqq n\}$. From the fact that $\tilde{u}_r(z) \to u(z)$, $r \to 1$, for $z \in \varDelta$ and Fatou's lemma, we can see that the inequality (4) holds for the function u. This completes the proof.

We can define $LH^p(B)$, $0 < p < \infty$, for the unit ball B of C^n as the class of log. plurisubharmonic functions u on B such that

$$(5) \qquad \sup\left\{\int_{\partial B} u(rz)^p d\tau(z)\mid 0 \leqq r < 1\right\} < \infty .$$

By the same method as in the case of $LH^p(\varDelta)$, we can see that every $u \in LH^p(B)$ has the radial limit $u^*(z)$ for almost all $z \in \partial B$ and $u^* \in L^p(\partial B)$. On the other hand, there is a boundary function $v(z)$ of u for almost every $z \in \partial B$ in the sense of [7] such that $J(|u_r^p - v^p|) \to 0$, $r \to 1$. Thus $v = u^*$ a.e., and $J(u_r^p) \to J(u^{*p})$, $r \to 1$.

For functions in $LH^p(B)$ and $LH^p(\varDelta)$, we have inequalities of the Fejér-Riesz type. Since u^p is log. plurisubharmonic with u, it is sufficient to state the results for functions in LH^1.

THEOREM 2.   *There exists a constant K, $K \leqq 1$, such that every function u in $LH^1(B)$ satisfies the following inequality for any hyperplane L in R^{2n} with Lebesgue measure $d\sigma$:*

$$(6) \qquad \int_{L\cap B} u(z)d\sigma(z) \leqq K\int_{\partial B} u^*(z)d\tau(z) .$$

If L passes through the origin, then $K \leqq 1/2$.

PROOF.   If u is log. plurisubharmonic on a neighborhood of $\bar{B}$, then the inequality (6) is derived by the same method as in [9] and [10] based on the inequality extended by Beckenbach and Yamashita for $n = 1$. Let $u \in LH^1(B)$ and let $\tilde{u}_r(z) = r^{-(2n-1)}u(z)$ for $z \in B_r$ and $\tilde{u}_r(z) = 0$ for $z \notin B_r$. Assume that the hyperplane L is defined by the equation $x_{2n} = a$,

$0 \leqq a < 1$, and $L \cap B \neq \varnothing$. Take r such that $a < r < 1$ and define a hyperplane L' by $x_{2n} = r^{-1}a$. A parametrization $\Phi \colon G \to L' \cap B$ is defined by $\Phi \colon x_1 = \cos \theta_1$, $x_j = \sin \theta_1 \cdots \sin \theta_{j-1} \cos \theta_j$, $2 \leqq j \leqq 2n - 1$, $x_{2n} = r^{-1}a$. Further, $\phi(x_1, \cdots, x_{2n}) = (rx_1, \cdots, rx_{2n})$ gives the parametrization $\phi \circ \Phi \colon G \to L \cap B_r$. Writing $(\theta_1, \cdots, \theta_{2n-1}) = \theta$ and $d\theta_1 \cdots d\theta_{2n-1} = d\theta$, we have

$$\int_{L \cap B} \widetilde{u}_r(z) d\sigma(z) = r^{-(2n-1)} \int_G u((\phi \circ \Phi)(\theta)) r^{2n-1} \prod_{j=1}^{2n-1} (\sin \theta_j)^{2n-j} d\theta$$

$$= \int_{L' \cap B} u_r(z) d\sigma(z) \leqq K \int_{\partial B} u_r(z) d\tau(z) .$$

If L is a hyperplane defined by a general equation, a suitable unitary transformation U can be used to derive the above inequality, since $\widetilde{u}_r \circ U = (u \circ U)\widetilde{}_r$ and $u_r \circ U = (u \circ U)_r$. Fatou's lemma proves the inequality (6).

THEOREM 3. *Let* $u \in LH^1(\Delta)$. *Then* u *satisfies the inequality for any hyperplane* L:

$$(7) \qquad \int_{L \cap \Delta} u(z) d\sigma(z) \leqq 2^{-(n-1)} n^{1/2} \int_T u^*(z) d\tau(z) .$$

PROOF. Let $u \in LH^1(\Delta)$, and $\widetilde{u}_r(z) = r^{-(2n-1)} u(z)$ for $z \in \Delta_r$, $= 0$ for $z \notin \Delta_r$. Suppose L is defined by the equation $x_{2n} = \sum_{j=1}^{2n-1} a_j x_j + a$ and define a hyperplane L' by $x_{2n} = r^{-1}(\sum a_j x_j + a)$. $L' \cap \Delta$ is parametrized by $\Phi \colon x_j = x_j$, $1 \leqq j \leqq 2n - 1$, $x_{2n} = $ the defining equation of L', and the measure $d\sigma'$ on $L' \cap \Delta$ is given by $d\sigma' = c_r dx_1 \cdots dx_{2n-1}$, where $c_r = r^{-1}(\sum_{j=1}^{2n-1} a_j^2 + r^2)^{1/2}$. On the other hand, $L \cap \Delta_r$ is parametrized by $\phi \circ \Phi$ with the measure $d\sigma = r^{2n-1} c_r dx_1 \cdots dx_{2n-1}$ on it. The inequality (7) follows from Fatou's lemma.

REFERENCES

[1] E. F. BECKENBACH, On a theorem of Fejér and Riesz, J. London Math. Soc. 13 (1938), 82–86.

[2] E. F. BECKENBACH AND T. RADÓ, Subharmonic functions and minimal surfaces, Trans. Amer. Math. Soc. 35 (1933), 648–661.

[3] E. F. BECKENBACH AND T. RADÓ, Subharmonic functions and surfaces of negative curvature, ibid. 35 (1933), 662–674.

[4] T. CARLEMAN, Zur Theorie der Minimalflächen, Math. Z. 9 (1921), 154–160.

[5] N. DU PLESSIS, An introduction to potential thery, Oliver & Boyd, Edinburgh, 1970.

[6] L. FEJÉR AND F. RIESZ, Über einige funktionentheoretische Ungleichungen, Math. Z. 11 (1921), 305–314.

[7] L. GÅRDING AND L. HÖRMANDER, Strongly subharmonic functions, Math. Scand. 15 (1964), 93–96. Correction, ibid. 18 (1966), 183.

[8] G. H. HARDY, A theorem concerning Taylor's series, Quart. J. Math. 44 (1913), 147–160.

[9] M. HASUMI AND N. MOCHIZUKI, Fejér-Riesz inequality for holomorphic functions of several complex variables, Tôhoku Math. J. 33 (1981), 493-501.

[10] N. MOCHIZUKI, Inequalities of Fejér-Riesz type for holomorphic functions on certain product domains, ibid. 34 (1982), 367-372.

[11] A. NAGEL, W. RUDIN, AND J. H. SHAPIRO, Tangential boundary behavior of function in Dirichlet-type spaces, Ann. of Math. 116 (1982), 331-360.

[12] T. RADÓ, Subharmonic functions, Ergebnisse der Mathematik und ihrer Grenzgebiete 5, Springer-Verlag, Berlin-Heidelberg-New York, 1937.

[13] W. RUDIN, Function theory in polydiscs, Benjamin, New York, Amsterdam, 1969.

[14] M. TSUJI, Potential theory in modern function theory, Maruzen, Tokyo, 1959.

[15] S. YAMASHITA, The hyperbolic M. Riesz theorem, Acta Sci. Math. 43 (1981), 141-146.

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