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A CLASS OF UNICYCLIC GRAPHS DETERMINED BY THEIR LAPLACIAN SPECTRUM*

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Abstract. Let $G_{r,p}$ be a graph obtained from a path by adjoining a cycle C_r of length r to one end and the central vertex of a star S_p on p vertices to the other end. In this paper, it is proven that unicyclic graph $G_{r,p}$ with r even is determined by its Laplacian spectrum except for $n = p + 4$.

Key words. Adjacency spectrum, Laplacian spectrum, Cospectral graph, Unicyclic graph.

AMS subject classifications. 05C05, 05C50.

1. Introduction. Let G be a simple graph on n vertices and $A(G)$ be its adjacency matrix. Let $d_G(v)$ be the degree of vertex v in G, and $D(G)$ be the diagonal matrix with the degrees of the corresponding vertices of G on the diagonal and zero elsewhere. Matrix $Q(G) = D(G) - A(G)$ is called the Laplacian matrix of G. The eigenvalues of $A(G)$ (resp., $Q(G)$) and the spectrum (which consists of eigenvalues) of $A(G)$ (resp., $Q(G)$) are also called the adjacency (resp., Laplacian) eigenvalues of G and the adjacency (resp., Laplacian) spectrum of G. Since both matrices $A(G)$ and $Q(G)$ are real symmetric matrices, their eigenvalues are all real numbers. So we can assume that $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$ and $\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_n(G) = 0$ are the adjacency eigenvalues and the Laplacian eigenvalues of G, respectively.

Two graphs are adjacency (resp., Laplacian) cospectral if they have the same adjacency (resp., Laplacian) spectrum. Denote by $\phi(G) = \phi(G;\lambda) = \det(\lambda I - A(G))$ and $\chi(G;\mu) = \det(\mu I - Q(G))$ the characteristic polynomial of adjacency matrix and Laplacian matrix of G, respectively. A graph is said to be determined by the adjacency (resp., Laplacian) spectrum if there is no non-isomorphic graph with the same adjacency (resp., Laplacian) spectrum.

In general, the spectrum of a graph does not determine the graph and the question "Which graphs are determined by their spectrum?" ([3]) remains a difficult problem. For the background and some known results about this problem and related topics, we refer the readers to [4] and references therein. For the unicyclic graphs, Haemers

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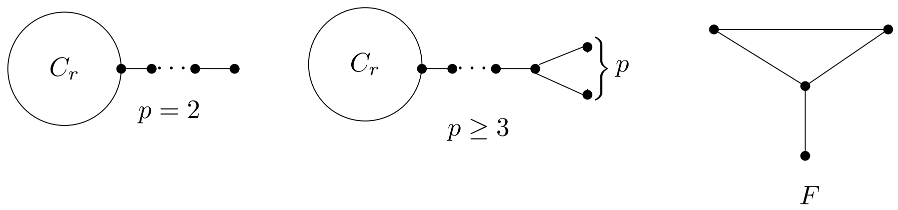

FIG. 1.1. *Graphs $G_{r,p}$ and F.*

et al. [5] showed that lollipop graphs H with p odd are determined by the adjacency spectrum. Boulet and Jouve proved in [1] that the remaining lollipop graphs are also determined by their adjacency spectrum. Haemers et al. showed that lollipop graphs are determined by their Laplacian spectrum as well. Let $U_{n,r}$ be the graph obtained by attaching $n-r$ pendent edges to a vertex of cycle C_r. Zhang et al. proved in [13] that $U_{n,r}$ is determined by its Laplacian spectrum. We shall prove a class of unicyclic graphs determined by their Laplacian spectra in this paper.

Let $G_{r,p}$ (see Fig. 1.1) be a graph obtained from a path by adjoining a cycle C_r of length r to one end and the central vertex of a star S_p on p vertices to the other end. For $p=2$, $G_{r,p}$ is a lollipop graph, which is determined by its adjacency spectrum and Laplacian spectrum respectively. Without loss of generality, we assume that $p \geq 3$ and n is the order of $G_{r,p}$. In this paper, we prove that $G_{r,p}$ with r even is determined by its Laplacian spectrum except for $n = p+4$, which extends the known families of unicyclic graphs determined by their Laplacian spectrum.

2. Preliminaries.

The following lemmas will be used in the next section.

LEMMA 2.1. ([3]) *For $n \times n$ matrices A and B, the following are equivalent:*

(i) A and B are cospectral;

(ii) A and B have the same characteristic polynomial;

(iii) $tr(A^i) = tr(B^i)$ for $i = 1, 2, \ldots, n$.

If A is the adjacency matrix of a graph, then $tr(A^i)$ gives the total number of closed walks of length i. So cospectral graphs have the same number of closed walks of each given length i. In particular, they have the same number of edges (taking $i=2$) and triangles (taking $i=3$).

LEMMA 2.2. ([2]) *Let G be a connected graph, and H a proper subgraph of G. Then $\lambda_1(H) < \lambda_1(G)$.*

LEMMA 2.3. ([2]) *Let G be the graph obtained from the disjoint union $H_1 \bigcup H_2$*

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by adding an edge v_1v_2 joining the v_1 of H_1 and v_2 of H_2, then $\phi(G) = \phi(H_1)\phi(H_2) - \phi(H_1 - v_1)\phi(H_2 - v_2)$, where $H_i - v_i$ denote the graph obtained from H_i by deleting the vertex v_i and the edges incident to v_i.

Hoffman and Smith defined an internal path [6] of a graph as a walk $v_0, v_1, \ldots, v_k$ $(k \geq 1)$ such that $v_1, \ldots, v_k$ are distinct (v_0, v_k need not be distinct), $d_{v_0} > 2, d_{v_k} > 2$ and $d_{v_i} = 2, 0 < i < k$.

LEMMA 2.4. ([6]) *Let G be a connected graph that is not isomorphic to W_n, where W_n is a graph obtained from the path P_{n-2} (indexed in natural order $1,\ 2, \ldots,\ n-2$) by adding two pendant edges at vertices 2 and $n-3$. Let G_{uv} be the graph obtained from G by subdividing the edge uv of G. If uv lies on an internal path of G, then $\lambda_1(G_{uv}) \leq \lambda_1(G)$.*

LEMMA 2.5. ([2]) *Let the eigenvalues of graphs G and $G - v$ be $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and $\lambda'_1 \geq \lambda'_2 \geq \cdots \geq \lambda'_{n-1}$, respectively. Then $\lambda_1 \geq \lambda'_1 \geq \lambda_2 \geq \lambda'_2 \geq \cdots \geq \lambda'_{n-1} \geq \lambda_n$.*

LEMMA 2.6. ([2]) *Let C_n, P_n be the cycle and path on n vertices respectively. Then*

$$\phi(C_n) = \prod_{j=1}^{n}(\lambda - 2\cos\frac{2\pi j}{n}) = \lambda\phi(P_{n-1}) - 2\phi(P_{n-2}) - 2;$$

$$\phi(P_n) = \prod_{j=1}^{n}(\lambda - 2\cos\frac{\pi j}{n}) = \lambda\phi(P_{n-1}) - \phi(P_{n-2}).$$

We write the Laplacian characteristic polynomial as $\chi(G;\mu) = q_0\mu^n + q_1\mu^{n-1} + \cdots + q_{n-1}\mu + q_n$.

LEMMA 2.7. ([3, 11]) *Let G be a graph with n vertices and m edges and $d = (d_1, \ldots, d_n)$ be its non-increasing degree sequence. Then*

$$q_0 = 1;\quad q_1 = -2m;\quad q_2 = 2m^2 - m - \frac{1}{2}\sum_{i=1}^{n} d_i^2;\quad q_{n-1} = (-1)^{n-1}nt(G);\quad q_n = 0;$$

where $t(G)$ is the number of spanning trees in G.

Part (i) and (ii) of the following are given in [10] and [9], respectively.

LEMMA 2.8. *Let G be a graph with $V(G) \neq \emptyset$ and $E(G) \neq \emptyset$.*

(i) Then $\Delta(G) + 1 \leq \mu_1 \leq \max\{\frac{d_u(d_u+m_u)+d_v(d_v+m_v)}{d_u+d_v}, uv \in E(G)\}$, where $\Delta(G)$ denotes the maximum vertex degree of G, u_1 is the largest Laplacian eigenvalue of G, d_vm_v means the sum of degrees of vertices adjacent to v in G.

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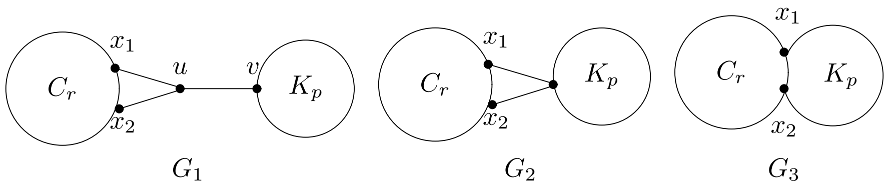

FIG. 3.1. *Graphs G_1, G_2 and G_3.*

(ii) If G is a connected graph with at least 2 vertices, then $\mu_1 = \Delta(G) + 1$ if and only if $|V(G)| = \Delta(G) + 1$.

LEMMA 2.9. ([7, 8]) *Let G be a graph with n vertices and $\overline{G}$ its complement, then $\mu_i(G) = n - \mu_{n-i}(\overline{G})$ for $1 \le i \le n-1$.*

LEMMA 2.10. ([12]) *Let F be the graph in Fig. 1.1, $N_G(F)$ the number of subgraphs F of a graph G, and $N_G(i)$ the number of closed walks of length i in G. Then $N_G(5) = 30N_G(K_3) + 10N_G(C_5) + 10N_G(F)$, where K_3 is the complete graph of order 3, C_5 is the circle of length 5.*

For a bipartite graph G with n vertices and m edges, the Laplacian matrix $Q(G) = D - A$ and signless Laplacian matrix $|Q(G)| = D + A$ are similar by a diagonal matrix with diagonal entries ± 1, hence they have the same spectrum. Let N be the vertex-edge incidence matrix of G and B the adjacency matrix of the line graph $L(G)$ of G. Since $|Q(G)| = NN^T$, $N^TN = 2I + B$, NN^T and N^TN have the same non-zero eigenvalues, for $\mu \neq 0$, μ is an eigenvalue of $|Q(G)|$ with multiplicity a if and only if $\mu - 2$ is an eigenvalue of B with multiplicity a, and the multiplicity of the eigenvalue -2 equals $m - n + 1$ ([3]). For a unicyclic connected bipartite graph G, $Q(G)$ has one eigenvalue 0, since $m = n$, the multiplicity of eigenvalue -2 of B is 1. Thus, we have the following lemma.

LEMMA 2.11. *Let G be a connected unicyclic bipartite graph with n vertices and $L(G)$ its line graph. Then $\mu_i(G) = \lambda_i(L(G)) + 2$ for $i = 1, 2, \ldots, n-1$, where $\lambda_i(L(G))$ is the i-th largest adjacency eigenvalue of $L(G)$.*

3. Main results. We need the following key lemmas to prove our results. Let K_p be a complete graph on p vertices, and G_i a graph depicted in Fig. 3.1, x_1x_2 an edge of G_i $(i = 1,\ 2,\ 3)$.

LEMMA 3.1. $\lambda_1(G_1) < \min\{\lambda_1(G_2), \lambda_1(G_3)\}$ *for $p > 3$.*

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Proof. By Lemma 2.3 and direct calculation, we obtain the characteristic polynomial of G_i $(i = 1, 2, 3)$:

$$\begin{aligned}
\phi(G_1) &= (\lambda+1)^{p-2}((\lambda(\lambda+1)(\lambda-p+1)-(\lambda-p+2))\phi(C_r)\\
&\quad -2(\lambda+1)(\lambda-p+1)(\phi(P_{r-1})+\phi(P_{r-2})+1)),\\
\phi(G_2) &= (\lambda+1)^{p-2}((\lambda(\lambda-p+2)-(p-1))\phi(C_r)-2(\lambda-p+2)(\phi(P_{r-1})\\
&\quad +\phi(P_{r-2})+1)),\\
\phi(G_3) &= (\lambda+1)^{p-3}((\lambda(\lambda-p+3)-2(p-2))\phi(P_{r-1})-2(\lambda+1)(\phi(P_{r-2})+1)).
\end{aligned}$$

Let

$$\begin{aligned}
\phi^*(G_1) &= (\lambda(\lambda+1)(\lambda-p+1)-(\lambda-p+2))\phi(C_r)-2(\lambda+1)(\lambda-p+1)(\phi(P_{r-1})\\
&\quad +\phi(P_{r-2})+1),\\
\phi^*(G_2) &= (\lambda(\lambda-p+2)-(p-1))\phi(C_r)-2(\lambda-p+2)(\phi(P_{r-1})+\phi(P_{r-2})+1),\\
\phi^*(G_3) &= (\lambda(\lambda-p+3)-2(p-2))\phi(P_{r-1})-2(\lambda+1)(\phi(P_{r-2})+1).
\end{aligned}$$

Obviously, $\lambda_1(G_i)$ is also the largest root of $\phi^*(G_i)$ $(i = 1, 2, 3)$. Since $\phi^*(G_1; p-1) = -\phi(C_r, p-1)$ and $p > 3$, $\phi^*(G_1; p-1) < 0$ by Lemma 2.6. By the intermediate value theorem, $\lambda_1(G_1) > p-1$. As G_1 is not regular, $\lambda_1(G_1) < \Delta(G_1)$, where $\Delta(G_1)$ is the maximum degree of G_1. Hence $\lambda_1(G_1) < p$. By Lemma 2.6, $\lambda\phi(P_{r-i}) = \phi(P_{r-i+1}) + \phi(P_{r-i-1})$, $i = 1, \ldots, r-1$.

$$\begin{aligned}
&\phi^*(G_1) - \lambda\phi^*(G_2)\\
&= (p-2-\lambda)\phi(C_r) + 2(p-1)(\phi(P_{r-1}) + \phi(P_{r-2}) + 1)\\
&= (p-2-\lambda)(\lambda\phi(P_{r-1}) - 2\phi(P_{r-2}) - 2) + 2(p-1)(\phi(P_{r-1}) + \phi(P_{r-2}) + 1)\\
&= (\lambda(p-2-\lambda) + 2(p-1))\phi(P_{r-1}) + 2(\lambda+1)(\phi(P_{r-2}) + 1)\\
&= (\lambda(p-2-\lambda) + 2(p-1))\phi(P_{r-1}) + 2(\phi(P_{r-1}) + \phi(P_{r-3})) + 2\phi(P_{r-2}) + 2(\lambda+1)\\
&= (\lambda(p-2-\lambda) + 2p)\phi(P_{r-1}) + 2(\phi(P_{r-2}) + \phi(P_{r-3})) + 2(\lambda+1).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\phi^*(G_1; \lambda_1(G_1)) - \lambda_1(G_1)\phi^*(G_2; \lambda_1(G_1))\\
&> (\lambda_1(G_1)(p-2-p) + 2p)\phi(P_{r-1}, \lambda_1(G_1)) + 2(\phi(P_{r-2}, \lambda_1(G_1))\\
&\quad +\phi(P_{r-3}, \lambda_1(G_1)) + 2(\lambda+1)\\
&> 0.
\end{aligned}$$

Since $p > \lambda_1(G_1) > p-1$, $\phi(P_{r-1}, \lambda_1(G_1)), \phi(P_{r-2}, \lambda_1(G_1), \phi(P_{r-3}, \lambda_1(G_1))$ are all positive for $p > 3$. Thus, $\phi^*(G_2; \lambda_1(G_1)) < 0$. By the intermediate value theorem the largest root of $\phi^*(G_2)$ exceeds $\lambda_1(G_1)$. So, $\lambda_1(G_1) < \lambda_1(G_2)$. Similarly, by Lemma

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2.6, we have

$$\begin{aligned}
&\phi^*(G_1) - \lambda^2(\lambda-2)\phi^*(G_3)\\
&= (2\lambda^4 - (2p-2)\lambda^3 - 2\lambda^2 p + (5p-8)\lambda + 2p - 2)\phi(P_{r-1})\\
&\quad +((2p-10)\lambda^3 + (6p-14)\lambda^2 + (4p-2)\lambda + 2)(\phi(P_{r-2}) + 1)\\
&= (2\lambda^4 - 2(p-1)\lambda^3 - 10\lambda^2 + (11p-22)\lambda + 8p - 14)\phi(P_{r-1}) + (6p-12)\phi(P_{r-2})\\
&\quad +(6p-12)\phi(P_{r-3}) + ((2p-10)\lambda + 6p - 14)\phi(P_{r-4}) + (2p-10)\lambda^3 + (6p-14)\lambda^2\\
&\quad +(4p-2)\lambda + 2.
\end{aligned}$$

For convenience, we set $\alpha = \lambda_1(G_1)$. Then

$$\begin{aligned}
&\phi^*(G_1;\alpha) - \alpha^2(\alpha-2)\phi^*(G_3;\alpha)\\
&= (2\alpha^4 - 2(p-1)\alpha^3 - 10\alpha^2 + (11p-22)\alpha + 8p - 14)\phi(P_{r-1},\alpha)\\
&\quad +(6p-12)\phi(P_{r-2},\alpha) + (6p-12)\phi(P_{r-3},\alpha) + ((2p-10)\alpha + 6p - 14)\phi(P_{r-4},\alpha)\\
&\quad +(2p-10)\alpha^3 + (6p-14)\alpha^2 + (4p-2)\alpha + 2.
\end{aligned}$$

Let

$$b = 2\alpha^4 - 2(p-1)\alpha^3 - 10\alpha^2 + (11p-22)\alpha + 8p - 14,$$

$$c = (2p-10)\alpha^3 + (6p-14)\alpha^2 + (4p-2)\alpha + 2.$$

Obviously, $c > 0$ for $p \geq 5$, and

$$\begin{aligned}
b &= (\alpha - p + 1)(\alpha-3)(2(\alpha-3)^2 + 18(\alpha-3) + 43) + \alpha^2 + 10\alpha - 13p + 7\\
&> (\alpha - p + 1)(\alpha-3)(2(\alpha-3)^2 + 18(\alpha-3) + 43) + (p-1)^2 - 3p - 3\\
&\quad +10(\alpha - p + 1)\\
&> 0
\end{aligned}$$

for $p \geq 6$. If $p = 5$, then $5 > \alpha > 4$, $c = 16\alpha^2 + 18\alpha + 2 > 0$. Using

$$5\phi(P_{r-i},\alpha) > \alpha\phi(P_{r-i},\alpha) = \phi(P_{r-i+1},\alpha) + \phi(P_{r-i-1},\alpha),$$

we have

$$\begin{aligned}
&\phi^*(G_1;\alpha) - \alpha^2(\alpha-2)\phi^*(G_3;\alpha)\\
&= ((\alpha-4)(\alpha-3)(2(\alpha-3)^2 + 18(\alpha-3) + 43) + \alpha^2 + 10\alpha - 58)\phi(P_{r-1},\alpha)\\
&\quad +18\phi(P_{r-2},\alpha) + 18\phi(P_{r-3},\alpha) + 16\phi(P_{r-4},\alpha) + c\\
&> ((\alpha-4)(\alpha-3)(2(\alpha-3)^2 + 18(\alpha-3) + 43) + \alpha^2 + 10\alpha - 54)\phi(P_{r-1},\alpha)\\
&\quad +2\phi(P_{r-2},\alpha) + \phi(P_{r-3},\alpha) + 20\phi(P_{r-4},\alpha) + c.
\end{aligned}$$

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Since $\alpha^2+10\alpha-54=(\alpha-4)(\alpha+14)+2>0$, $-\alpha^2(\alpha-2)\phi^*(G_3;\alpha)>0$. This implies that $\phi^*(G_3;\alpha)<0$.

Similarly, for $p=4$, $4>\alpha>3$, $c=-2\alpha^3+10\alpha^2+8\alpha+2=-2\alpha^2(\alpha-5)+8\alpha+2>0$. Then

$$\begin{aligned}
&\phi^*(G_1;\alpha)-\alpha^2(\alpha-2)\phi^*(G_3;\alpha)\\
>&((\alpha-3)^2(2(\alpha-3)^2+18(\alpha-3)+43)+\alpha^2+10\alpha-39)\phi(P_{r-1},\alpha)\\
&+2\phi(P_{r-2},\alpha)+5\phi(P_{r-3},\alpha)+12\phi(P_{r-4},\alpha)+c\\
=&((\alpha-3)^2(2(\alpha-3)^2+18(\alpha-3)+43)+(\alpha-3)(\alpha+13))\phi(P_{r-1},\alpha)\\
&+2\phi(P_{r-2},\alpha)+5\phi(P_{r-3},\alpha)+12\phi(P_{r-4},\alpha)+c>0,
\end{aligned}$$

which implies that $\phi^*(G_3;\alpha)<0$. Hence, by the intermediate value theorem, the largest root of $\phi^*(G_3)$ exceeds $\lambda_1(G_1)$. Thus, $\lambda_1(G_1)<\lambda_1(G_3)$. ☐

LEMMA 3.2. *Let graphs G and $G_{r,p}$ be Laplacian cospectral. Then G is a connected unicyclic graph with circle length r and the same degree sequence with $G_{r,p}$.*

Proof. By Lemma 2.8(i), the largest eigenvalue of $G_{r,p}$ satisfies $p+1\le\mu_1<p+2$. Suppose that graph G is Laplacian cospectral to $G_{r,p}$. By Lemma 2.8, the largest vertex degree of G is at most p. By Lemma 2.7, G and $G_{r,p}$ have the same number of vertices, edges, spanning trees. So G is a connected unicyclic graph with n vertices. Since $G_{r,p}$ has r spanning trees, the length of cycle in G is also r. Assume that G has n_i vertices of degree i, for $i=1,\dots,p$. By Lemma 2.7, we have

$$\sum_{i=1}^{p}n_i=n,\sum_{i=1}^{p}in_i=2n,\sum_{i=1}^{p}i^2n_i=p^2+3^2+2^2(n-p-1)+p-1. \tag{3.1}$$

This gives

$$\sum_{i=3}^{p}(i-1)(i-2)n_i=p^2-3p+4. \tag{3.2}$$

By Lemma 2.11, $L(G)$ and $L(G_{r,p})$ are adjacency cospectral, so they have the same number of triangles. This gives

$$\sum_{i=3}^{p}\binom{i}{3}n_i=\binom{p}{3}+1. \tag{3.3}$$

Obviously, $n_p\le 1$ for $p>3$. We assert that $n_p=1,\ n_3=1$. Assume that $n_p=0$. Combining equations (3.2) and (3.3), we have

$$\begin{aligned}
p(p-1)(p-2)+6=\sum_{i=3}^{p}(i(i-1)(i-2))n_i&\le(p-1)(\sum_{i=3}^{p-1}(i-1)(i-2)n_i)\\
&=(p-1)(p^2-3p+4).
\end{aligned}$$

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This gives $p^2 - 5p + 10 \leq 0$, which is a contradiction. It is easy to obtain $n_3 = 1$, and $n_i = 0$, $i = 4, \ldots, p-1$ from equation (3.3). By equation (3.1), we easily get that $n_2 = n - p - 1$, $n_1 = p - 1$. For $p = 3$, by equation (3.1), we have

$$n_1 + n_2 + n_3 = n; n_1 + 2n_2 + 3n_3 = 2n; n_1 + 4n_2 + 9n_3 = 4 + 4n.$$

Solving these equations gives that $n_1 = 2$, $n_2 = n - 4$, $n_3 = 2$, which is the same degree sequence with $G_{r,3}$. □

LEMMA 3.3. *If r is even, $n > p + r$, $p > 3$, then $G_{r,p}$ is determined by its Laplacian spectrum.*

Proof. Assume that G and $G_{r,p}$ are Laplacian cospectral. By Lemma 3.2, G is a connected unicyclic graph with circle length r and has the same degree sequence as $G_{r,p}$. Since r is even, G and $G_{r,p}$ are bipartite graphs. By Lemma 2.11, their line graphs are adjacency cospectral. Since G and $G_{r,p}$ have the same degree sequence, the line graph $L(G)$ is a connected graph with n vertices and contains a subgraph G_i ($i = 1, 2, 3$) or a subgraph obtained by subdividing edge uv of G_1 several times. For $n = p + r + 1$, the line graph of $G_{r,p}$ is G_1. By Lemma 3.1, $L(G) \cong G_1$. For $n > p + r + 1$, by Lemma 2.4, $\lambda_1(L(G_{r,p})) \leq \lambda_1(G_1)$. Since $L(G)$ and $L(G_{r,p})$ are adjacency cospectral, neither G_2 nor G_3 is a subgraph of $L(G)$ by Lemma 3.1. Since $n > p+r+1$, G_1 is not a subgraph of $L(G)$. Thus, $L(G)$ contains a subgraph obtained by subdividing edge uv of G_1 several times. By Lemmas 2.4 and 2.2, $L(G) \cong L(G_{r,p})$.□

For $n > p + r, p = 3$, we also have the following.

LEMMA 3.4. *$G_{r,3}$ is determined by its Laplacian spectrum for $n > 3 + r$.*

Proof. Let G and $G_{r,3}$ be Laplacian cospectral. By Lemma 3.2, G is a unicyclic graph with circle length r and has the same degree sequence as $G_{r,3}$. Then G is either G_4 or G_5 depicted in Fig. 3.2. Let a be the length of path from vertex u to v, b the length of path from u' to v', c the length of path from z to w and d the length of path from z' to w' in Fig. 3.2. Note that x is not necessarily adjacent to y in G_5, $L(G_{r,3})$ is G_6 with $a = b = 0$.

By Lemmas 2.1 and 2.11, $L(G)$ and $L(G_{r,3})$ are adjacency cospectral , so they have the same number of closed walks of length i for each i. Consider the closed walks of length 5. Since the line graphs of $G_{r,3}$ and G have the same number of triangles and C_5's, we only need to enumerate $N(F)$ in G_i ($i = 6, 7$) by Lemma 2.10. Clearly, $N_{L(G_{r,3})}(F) = 4$.

If there is a path with length no less than 1 between two triangles, then

$$N_{G_6}(F) = \begin{cases} 6, & a \neq 0, b \neq 0; \\ 5, & \text{either } a \text{ or } b \text{ is } 0. \end{cases}$$

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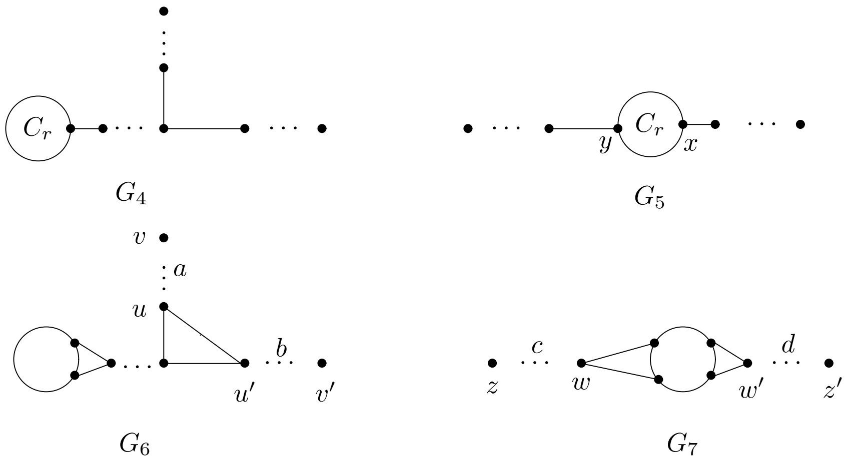

FIG. 3.2. *Graphs G_4, G_5 and the corresponding line graphs G_6, G_7, respectively.*

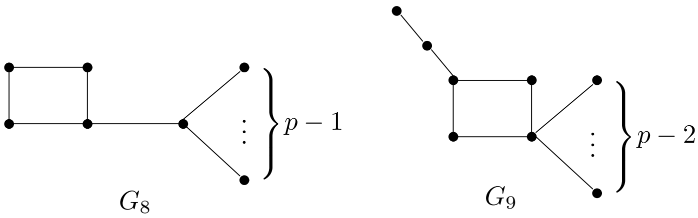

FIG. 3.3. *A family of non-isomorphic but Laplacian cospectral graphs.*

If two triangles share a common vertex, then

$$N_{G_6}(F) = \begin{cases} 8, & a \neq 0, b \neq 0; \\ 7, & \text{either } a \text{ or } b \text{ is } 0 \ . \end{cases}$$

If $c = 0$ (resp., $d = 0$), then $d \neq 0$ (resp., $c \neq 0$) for $n > 3 + r$.

$$N_{G_7}(F) = \begin{cases} 5, & \text{either } c \text{ or } d \text{ is } 0, \ x \text{ is not adjacent to } y, \\ 7, & \text{either } c \text{ or } d \text{ is } 0, \ x \text{ is adjacent to } y, \\ 6, & c \neq 0, d \neq 0, \ x \text{ is not adjacent to } y, \\ 8, & c \neq 0, d \neq 0, \ x \text{ is adjacent to } y. \end{cases}$$

Thus, the number of closed walks of length 5 in $L(G_{r,3})$ is different to G_i $(i = 6, 7)$ if $G_i \ncong L(G_{r,3})$. Hence G is isomorphic to $G_{r,3}$ for $n > 3 + r$. □

Let $n = p + r$. We determine a family of non-isomorphic Laplacian cospectral graphs for $r = 4$, see Fig. 3.3. Since the line graph of G_8 is isomorphic to G_2 in Fig.

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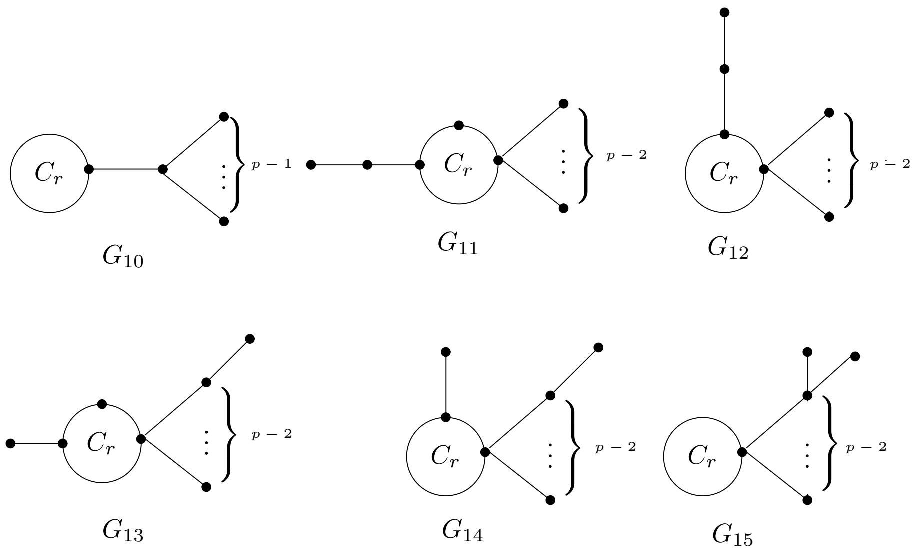

FIG. 3.4. *Graphs* G_j *(*$j = 10, \ldots, 15$*).*

3.1, it is easy to check that the line graphs of G_8 and G_9 have the same adjacency characteristic polynomial: $\lambda(\lambda+1)^{p-2}(\lambda+2)(\lambda^4-p\lambda^3+(p-5)\lambda^2+4(p-1)\lambda+4-2p)$.

For $n = p + r, r \neq 4$, we have:

LEMMA 3.5. *$G_{r,p}$ is also determined by its Laplacian spectrum if $n = p + r$, $r \neq 4$.*

Proof. Let graphs G and $G_{r,p}$ be Laplacian cospectral. By Lemma 3.2, G is a connected unicycic graph with the same degree sequence as $G_{r,p}$. Then G is just one of these graphs depicted in Fig. 3.4, here G_{10} is $G_{r,p}$ for $n = p + r$.

By Lemma 2.11, their line graphs have the same adjacency spectrum, thus the closed walks of length i in these line graphs are the same by Lemma 2.1. The line graph of G_j ($j = 10, \ldots, 15$) is depicted in Fig. 3.5, here x is adjacent to y in G_k ($k = 16, \ldots, 21$).

Consider the closed walks of length 5 in G_k ($k = 16, \ldots, 21$). By Lemma 2.10, since there are the same number of triangles and C_5's respectively in these graphs, we only need to enumerate the number of subgraphs F in G_k. It is easy to get the

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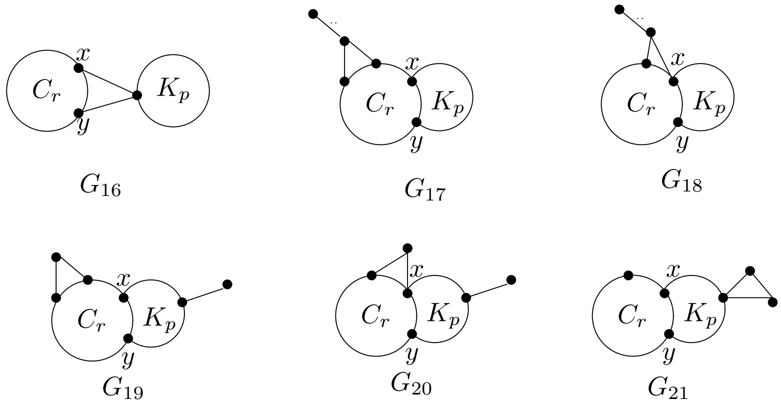

FIG. 3.5. *Graph G_k, the corresponding line graphs of G_j, $j = 10, \ldots, 15$.*

following:

$$N_{G_{16}}(F) = p + 1 + 2\binom{p-1}{2} + N_{K_p}(F); N_{G_{17}}(F) = 3 + 2\binom{p-1}{2} + N_{K_p}(F);$$

$$N_{G_{18}}(F) = p + 1 + 3\binom{p-1}{2} + N_{K_p}(F); N_{G_{19}}(F) = 2 + 3\binom{p-1}{2} + N_{K_p}(F);$$

$$N_{G_{20}}(F) = p + 4\binom{p-1}{2} + N_{K_p}(F); N_{G_{21}}(F) = p - 1 + 4\binom{p-1}{2} + N_{K_p}(F);$$

Obviously, $N_{G_k}(F) \neq N_{G_{16}}(F)$ $(k = 17, \ldots, 21)$ except for $N_{G_{19}}(F)$ for $p = 4$. For $p = 4$, by Lemmas 2.5 and 2.2, we have $\lambda_2(G_{16}) \leq 2$ and $\lambda_2(G_{19}) > 2$. So if G is not isomorphic to $G_{r,p}$, then their line graphs are not adjacency cospectral. Hence, G is isomorphic to $G_{r,p}$ for $r \neq 4$ and $n = p + r$. ☐

From Lemmas 3.3, 3.4 and 3.5, we obtain our main result.

THEOREM 3.6. *Unicyclic graph $G_{r,p}$ with r even is determined by its Laplacian spectrum except for $n = p + 4$.*

By Lemma 2.9, the complement of $G_{r,p}$ $(n \neq p+4)$ with r even is also determined by its Laplacian spectrum.

For r odd, a family of non-isomorphic but Laplacian cospectral graphs is given in Fig. 3.6.

If r is odd, since $G_{r,p}$ is not a bipartite graph, $u_i(G_{r,p}) \neq \lambda_i(L(G_{r,p})) + 2$ for $i = 1, \ldots, n$ in general, and hence we cannot use line graph to characterize the spectrum

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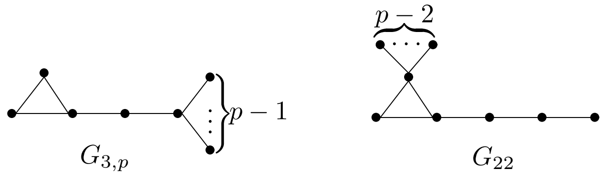

FIG. 3.6. *Graphs $G_{3,p}$ and its Laplacian cospectral graph.*

of $G_{r,p}$. The methods used here are invalid if r is odd. Some new techniques are needed to prove whether $G_{r,p}$ with r odd is determined by its Laplacian spectrum.

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