Proceedings of the 1991 IEEE
International Conference on Robotics and Automation
Sacramento, California - April 1991

A Classification of 3R Regional Manipulator Singularities and Geometries

Joel W. Burdick
Dept. of Mechanical Engineering, California Institute of Technology
Pasadena, CA 91125

Abstract

This paper categorizes 3R manipulator singularities and geometries based on genericity. A novel recursive application of screw theory is used to generate singular configurations and provide a geometric interpretation of non-genericity. A generic manipulator classification scheme based on homotopy class is introduced. Non-generic geometries are interpreted as bifurcations of generic geometries with respect to kinematic parameter values. Some conjectures on the classes of manipulators which can change pose without passing through a singularity are also given.

1. Introduction

This paper investigates the singularities of $3R$ regional manipulators. A detailed knowledge of the number, geometry, and location of kinematic singularities is important for many problems in manipulator design, analysis, trajectory planning, and control. Manipulator singularities have been previously studied by many researchers, including [1,2,3,4]. Pai [5] has introduced the notion of *generic* manipulator singularities, which will be used in this work.

This paper introduces a novel recursive application of screw theory to develop the equations of singular configurations. The geometric insight of this approach guides the enumeration of non-generic geometries. A classification scheme for generic manipulator geometries, based on homotopy class, is also presented. Non-generic manipulator geometries divide the space of all possible geometries into disjoint sets of homotopicly similar generic geometries. Thus they serve as a classification scheme for all 3R geometries. Preliminary conjectures on the ability of manipulators which can or can not change pose without passing through a singular configuration are presented. Because of space limitations, interested readers should consult [6] for more complete details.

2. The Equations Describing 3R Manipulator Singular Configurations

The determinant of an arbitrary geometry 3R manipulator Jacobian is

$$\begin{aligned}\det(\mathbf{J}(\theta)) &= g(\overline{\theta}) \\ &= a_3\left[V_1(\theta_3)\cos\theta_2 + V_2(\theta_3)\sin\theta_2 + V_3(\theta_3)\right] \\ &= a_3\left[\frac{x_2^2(\rho_3-\rho_1)+2\rho_2(x_3)x_2+\rho_3+\rho_1}{(1+x_2^2)(1+x_3^2)^2}\right]\end{aligned} \quad (1)$$

where:

$$\begin{aligned}V_1(\theta_3) &= m_1 c^2\theta_3 + m_4 s\theta_3 c\theta_3 + m_6 c\theta_3 + m_7 s\theta_3 \\ V_2(\theta_3) &= m_2 c^2\theta_3 + m_3 s^2\theta_3 + m_5 s\theta_3 c\theta_3 + m_8 c\theta_3 \\ &\quad + m_9 s\theta_3 \\ V_3(\theta_3) &= m_{10} c\theta_3 s\theta_3 + m_{11} c\theta_3 + m_{12} s\theta_3 \\ \rho_1(x_3) &= (m_1-m_6)x_3^4 + 2(m_7-m_4)x_3^3 - 2m_1x_3^2 \\ &\quad + 2(m_7+m_4)x_3 + m_1 + m_6 \\ \rho_2(x_3) &= (m_2-m_8)x_3^4 + 2(m_9-m_5)x_3^3 + 2(2m_3 \\ &\quad - m_2)x_3^2 + 2(m_9+m_5)x_3 + m_2 + m_8 \\ \rho_3(x_3) &= -m_{11}x_3^4 + 2(m_{12}-m_{10})x_3^3 \\ &\quad + 2(m_{12}+m_{10})x_3 + m_{11}\end{aligned} \quad (2)$$

and:

$$\begin{aligned}&m_1 = a_3 d_2 s\alpha_1 s\alpha_2 \quad m_2 = a_1 a_3 c\alpha_1 s\alpha_2 \\ &m_3 = a_2 a_3 s\alpha_1 c\alpha_2 \quad m_4 = a_1 a_3 c\alpha_1 c\alpha_2 s\alpha_2 - a_2 a_3 s\alpha_1 \\ &m_5 = -a_3 d_2 s\alpha_1 c\alpha_2 s\alpha_2 \\ &m_6 = a_2 d_2 s\alpha_1 s\alpha_2 - a_1 d_3 c\alpha_1 s^2\alpha_2 \\ &m_7 = -a_2^2 s\alpha_1 \quad m_8 = d_2 d_3 s\alpha_1 s^2\alpha_2 + a_1 a_2 c\alpha_1 s\alpha_2 \\ &m_9 = -a_2 d_3 s\alpha_1 s\alpha_2 \quad m_{10} = -a_1 a_3 s\alpha_1 s^2\alpha_2 \\ &m_{11} = -a_1 d_3 s\alpha_1 c\alpha_2 s\alpha_2 \quad m_{12} = -a_1 a_2 s\alpha_1\end{aligned} \quad (3)$$

where $s\theta_2 = \sin\theta_2$, $c\theta_2 = \cos\theta_2$, $s\alpha_1 = \sin\alpha_1$, etc. $x_2 = \tan(\theta_2/2)$; $x_3 = \tan(\theta_3/2)$. Singular configurations are the zero set of (1). The trivial case, $a_3 = 0$, in which the manipulator is always singular, will be neglected. Thus singular configuration arise if either:

$$x_2^2(\rho_3-\rho_2) + 2\rho_2(x_3)\ x_2 + \rho_1 + \rho_3 = 0 \quad (4)$$

or if (1) has "zeros at infinity:"

$$\rho_3(x_3) - \rho_1(x_3) = 0, \quad \text{and } \theta_2 = \pm\pi \quad (5.a)$$

$$m_6 - m_{11} - m_1 = 0, \quad \text{and } \theta_3 = \pm\pi \quad (5.b)$$

These singularities will be termed "singularities at infinity," and are considered in §5.3.

Definition: Let $\mathcal{C}$ denote the configuration space, which in this case is a 3-torus, T^3.

Definition: Let $\mathcal{G} = \{(\theta_1,\theta_2,\theta_3)|\ det(\mathbf{J}(\overline{\theta})) = 0\}$ be termed the ***critical point set*** of the forward kinematic function, $f(\overline{\theta})$. A connected and continuous subset of $\mathcal{G}$ is termed a *critical point surface*, or CPS.

Definition: Let $\mathcal{K}$ be the set of Denavit-Hartenberg kinematic parameters. A 3R manipulator has only 6 independent parameters. Since a_3 is assumed non-zero, we choose $a_3 = 1$ and rescale the remaining 6 independent parameters: a_1, a_2, d_2, d_3, α_1, and α_2.

$\mathcal{G}$ has co-dimension 1. Thus, the CPS form 2-dimensional surfaces which divide $\mathcal{C}$ into two (or more) disjoint regions. For example, Figure 1 depicts $\mathcal{G}$ for a $3R$ manipulator with kinematic parameters: $\alpha_1 = 60^0$, $\alpha_2 = -90^o$, $a_1 = 1.3$, $a_2 = 1.0$, $a_3 = 1.6$, $d_2 = 0.4$, $d_3 = -0.2$.

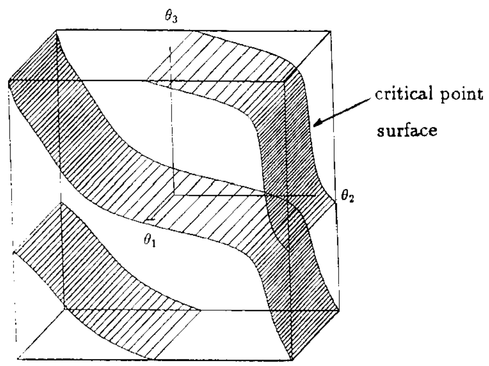

Figure 1: Loci of 3R Manipulator Singularities in $\mathcal{C}$

While $\mathcal{G}$ typically consists of two disjoint surfaces, $\mathcal{G}$ can consist of up to 8 CPS. For example, more additional CPS occur when:

$$V_1(x_3) = V_2(x_3) = V_3(x_3) = 0. \tag{6}$$

In this case, *all* values of θ_2 are zeros of (1), and condition (6) leads to extra 2-dimensional CPS.

Definition: The singularities which arise when (6) is satisfied are termed *extra branch* singularities.

Figure 2 shows the CPS for a 3R manipulator ($\alpha_1 = 60.0^o$, $\alpha_2 = -45.0^o$, $a_1 = 0.0$, $a_2 = 1.1$, $a_3 = 1.0$, $d_2 = 0.3$, $d_3 = 0.3$) which has two extra branch singularities satisfying (6).

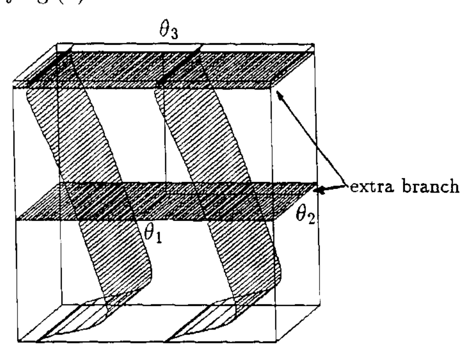

Figure 2: Extra Branch Singularities

3. Singularities: A Recursive Screw Approach

This section presents an alternate method for deriving the equations governing singularities.

Theorem [1,4,5]: Revolute-jointed regional manipulator singularities are characterized by a zero pitch screw which is reciprocal to all joint axis screws. This reciprocal screw axis intersects the end-effector and each joint axis (possibly at infinity).

This geometric relationship can be used to synthesize all possible singular configurations as follows.

1. Start with all potential reciprocal screw axes passing through the end-effector.
2. Constrain this set of axes to intersect joint axis 3.
3. Constrain these axes to intersect joint axis 2.
4. Constrain these axes to intersect joint axis 1.

Let $({}^k\mathbf{P}, {}^k\overline{\mathbf{P}})$ denote the Plücker coordinates of possible reciprocal screw axes. k indicates that the Plücker coordinates are determined with respect to $\mathcal{F}_k$, the k^{th} link frame. Let $\mathcal{F}_{ee}$ be a frame parallel to $\mathcal{F}_3$ with origin coincident with the end-effector. ${}^{ee}\overline{\mathbf{P}} = 0$, since all reciprocal screws must intersect the end-effector. Let ${}^{ee}\mathbf{P} = [{}^{ee}P_1, {}^{ee}P_2, {}^{ee}P_3]^T$, where the $\{{}^{ee}P_j\}$ are as yet unknown direction cosines which must additionally satisfy ${}^{ee}P_1^2 + {}^{ee}P_2^2 + {}^{ee}P_3^2 = 1$.

Let $({}^{ee}\mathbf{Q}_3, {}^{ee}\overline{\mathbf{Q}}_3) = ([0,0,1]^T, [0, a_3, 0]^T)$ be the Plücker coordinates (in $\mathcal{F}_{ee}$) of joint axis 3. Each possible reciprocal screw axis will intersect joint axis 3 if:

$${}^{ee}\mathbf{Q}_3 \cdot {}^{ee}\overline{\mathbf{P}} + {}^{ee}\overline{\mathbf{Q}}_3 \cdot {}^{ee}\mathbf{P} = {}^{ee}\overline{\mathbf{Q}}_3 \cdot {}^{ee}\mathbf{P} = 0; \tag{7}$$

which results in ${}^{ee}P_2 = 0$. To intersect both axes 3 and 2, an additional constraint is required:

$${}^{ee}\mathbf{Q}_2 \cdot {}^{ee}\overline{\mathbf{P}} + {}^{ee}\overline{\mathbf{Q}}_2 \cdot {}^{ee}\mathbf{P} = {}^{ee}\overline{\mathbf{Q}}_2 \cdot {}^{ee}\mathbf{P} = 0; \tag{8}$$

where $({}^{ee}\mathbf{Q}_2, {}^{ee}\overline{\mathbf{Q}}_2)$ are the Plücker vector coordinates of joint axis 2, as expressed in $\mathcal{F}_{ee}$:

$$\begin{aligned} {}^{ee}\mathbf{Q}_2 &= [\, s\theta_3\ s\alpha_2 \quad c\theta_3\ s\alpha_2 \quad c\alpha_2\,]^T \\ {}^{ee}\overline{\mathbf{Q}}_2 &= \begin{bmatrix} d_3 s\alpha_2 c\theta_3 + a_2 c\alpha_2 s\theta_3 \\ -d_3 s\alpha_2 s\theta_3 + a_2 c\alpha_2 c\theta_3 + a_3 c\alpha_2 \\ -(a_2 + a_3 c\theta_3) s\alpha_2 \end{bmatrix}. \end{aligned} \tag{9}$$

Applying (8) and the norm constraint on ${}^{ee}\mathbf{P}$:

$${}^{ee}P_1 = \frac{b_1}{\sqrt{b_1^2 + b_2^2}}; \qquad {}^{ee}P_3 = \frac{b_2}{\sqrt{b_1^2 + b_2^2}} \tag{10}$$

where $b_1 = (a_2 + a_3 c\theta_3) s\alpha_2$ and $b_2 = d_3 s\alpha_2 c\theta_3 + a_2 c\alpha_2 s\theta_3$. Note that $|{}^{ee}\mathbf{P}| = 1$ is not absolutely required, and an unnormalized ${}^{ee}\mathbf{P}$, ${}^{ee}\mathbf{P}^*$ can be defined:

$${}^{ee}P_1^* = b_1; \qquad {}^{ee}P_3^* = b_2 \tag{11}$$

Definition: Let the line which intersects axes 2, 3, and the end-effector be termed the *Potential Reciprocal Screw* axis (PRS axis). The coordinates of this line are a function of θ_3 and the geometry of links 2 and 3. For given θ_3, there is a unique PRS axis, except for three degenerate cases (see §5).

To continue, transform the Plücker coordinates of the

PRS axis from $\mathcal{F}_{ee}$ to $\mathcal{F}_2$:

$$^2\mathbf{P} = \begin{bmatrix} {}^2P_1 \\ {}^2P_2 \\ {}^2P_3 \end{bmatrix} = \begin{bmatrix} {}^{ee}P_1 c_3 \\ {}^{ee}P_1 c\alpha_2 s\theta_3 - {}^{ee}P_3 s\alpha_2 \\ {}^{ee}P_1 s\alpha_2 s\theta_3 + {}^{ee}P_3 c\alpha_2 \end{bmatrix} \quad (12)$$

$$^2\overline{\mathbf{P}} = \begin{bmatrix} -{}^{ee}P_1 d_3 s\theta_3 + {}^{ee}P_3 a_3 s\theta_3 \\ {}^{ee}P_1(d_3 c\alpha_2 c\theta_3 - a_2 s\theta_3 s\alpha_2) - {}^{ee}P_3(a_2 + a_3 c\theta_3)c\alpha_2 \\ 0 \end{bmatrix} \quad (13)$$

The PRS axis intersect axis 2 at a distance, l_2, from the origin of $\mathcal{F}_2$. Thus, $^2\overline{\mathbf{P}}$ must have the form:

$$^2\overline{\mathbf{P}} = \begin{bmatrix} -(l_2\ {}^2P_2) & (l_2\ {}^2P_1) & 0 \end{bmatrix}^T. \quad (14)$$

From (13) and (14), $l_2 = -a_2 \tan\theta_3 / \sin\alpha_2$.

In a singular configuration, the PRS axis must also intersect axis 1, whose Plücker are $({}^2\mathbf{Q}_1, {}^2\overline{\mathbf{Q}}_1)$:

$$^2\mathbf{Q}_1 = \begin{bmatrix} s\theta_2 s\alpha_1 \\ c\theta_2 s\alpha_1 \\ c\alpha_1 \end{bmatrix} \quad {}^2\overline{\mathbf{Q}}_1 = \begin{bmatrix} d_2 s\alpha_1 c\theta_2 + a_1 c\alpha_1 s\theta_2 \\ -d_2 s\alpha_1 s\theta_2 + a_1 c\alpha_1 c\theta_2 \\ -a_1 s\alpha_1 \end{bmatrix}. \quad (15)$$

For axis 1 and the PRS axis to intersect:

$$^2\overline{\mathbf{Q}}_1 \cdot {}^2\mathbf{P} + {}^2\mathbf{Q}_1 \cdot {}^2\overline{\mathbf{P}} = 0. \quad (16)$$

(16) can be expressed as a quadratic in $\tan(\frac{\theta_2}{2})$:

$$(B_3 - B_1)\tan^2(\frac{\theta_2}{2}) + 2B_2\tan(\frac{\theta_2}{2}) + B_3 + B_1 = 0 \quad (17)$$

where: $B_1 = (l_2 + d_2)s\alpha_1\ {}^2P_1 + {}^2P_2(a_1 c\alpha_1)$; $B_2 = {}^2P_1(a_1 c\alpha_1) - {}^2P_2(l_2 + d_2)s\alpha_1$; and $B_3 = -{}^2P_3(a_1 s\alpha_1)$. It can be verified that:

$$B_1 = \frac{V_1(\theta_3)}{\sqrt{b_1^2 + b_2^2}}; \quad B_2 = \frac{V_2(\theta_3)}{\sqrt{b_1^2 + b_2^2}}; \quad B_3 = \frac{V_3(\theta_3)}{\sqrt{b_1^2 + b_2^2}};$$

or if we use $^{ee}\mathbf{P}^*$ in place of $^{ee}\mathbf{P}$ while deriving equations (12), (13), and (16), then: $B_1^* = V_1(\theta_3); B_2^* = V_2(\theta_3); B_3^* = V_3(\theta_3)$. Consequently, the zero set of (17) is the same as the zero set of (1). Therefore:

> **Remark:** this geometric approach to generating singularities yields the same results as §2 *without an explicit construction of the Jacobian and its determinant.* In addition, the screw axis of the singular direction is determined with no additional cost.

This methodology can be recursively extended to redundant regional manipulators. Analogous methods have also been developed for spatial manipulators [7]. This approach has the following useful interpretation.

Geometric Interpretation. Given θ_3, (7) and (8) determine the Plücker vector coordinates of a unique (except 3 degenerate cases) PRS axis that intersects the end-effector and axes 2, 3. To intersect axis 1, thereby creating a singular configuration, axis 2 is rotated until the PRS axis intersects axis 1. The rotation of the PRS axis about axis 2 results in a double cone (with apex located at l_2 in $\mathcal{F}_2$) termed the *Potential Reciprocal Screw Cone*, or PRS cone. Axis 1 will nominally have zero, one, or two intersections with the PRS cone. An infinite number of intersections can occur if axis 1 intersects the cone apex or if the cone has special degenerate forms (see §5). Tangent intersection, degenerate PRS cones, and an infinite number of intersections are non-generic conditions.

4. Generic and Non-Generic 3R Manipulators

Pai [5] has introduced the concept of *generic* kinematic singularities. Pai has also investigated 3-jointed manipulator singularities though he did not attempt a complete classification. Only the necessary results from [5] are repeated here. A 3R manipulator will be non-generic if either one of the following two conditions hold.

> **Non-Genericity Condition 1 (NG1):** The Jacobian matrix has rank 1 for some singular configuration.
>
> **Non-Genericity Condition 2 (NG2):** The following conditions are satisfied at critical points, $\overline{\theta}_c$:

$$\frac{\partial[\det(\mathbf{J}(\overline{\theta}_c))]}{\partial\theta_i} = 0 \qquad \text{for } i = 2, 3. \quad (18)$$

where $det(\mathbf{J}(\vec{\theta}_c)) = 0$. A singular configuration satisfying NG1 or NG2 will be termed a *non-generic singular configuration.* We start with condition **NG2**, as it can be shown [6] that **NG1** is a subset of **NG2**.

5. Non-Generic Rank 2 Singularities

Applying (18) to (1), a non-generic manipulator with rank 2 singularities must have a solution to the following three equations:

$$\begin{aligned} V_1(\theta_3)\cos\theta_2 + V_2(\theta_3)\sin\theta_2 + V_3(\theta_3) &= 0 \\ -V_1(\theta_3)\sin\theta_2 + V_2(\theta_3)\cos\theta_3 &= 0 \\ V_1'(\theta_3)\cos\theta_2 + V_2'(\theta_3)\sin\theta_2 + V_3'(\theta_3) &= 0 \end{aligned} \quad (19)$$

where $V_j'(\theta_3) = \partial V(\theta_3)/\partial\theta_3$:

$$\begin{aligned} V_1'(\theta_3) &= -2m_1 c\theta_3 s\theta_3 + m_4(c^2\theta_3 - s^2\theta_3) \\ &\quad - m_6 s\theta_3 + m_7 c\theta_3 \\ V_2'(\theta_3) &= 2(m_3 - m_2)s\theta_3 c\theta_3 + m_5(c^2\theta_3 - s^2\theta_3) \\ &\quad - m_8 s\theta_3 + m_9 c\theta_3 \\ V_3'(\theta_3) &= m_{10}(c^2\theta_3 - s^2\theta_3) - m_{11}s\theta_3 + m_{12}c\theta_3 \end{aligned} \quad (20)$$

These three equations have a simultaneous solution under the following conditions [5]:

Solution #1:

$$V_1(\theta_3) = V_2(\theta_3) = V_3(\theta_3) = 0 \quad (21.a)$$

$$(V_1'(\theta_3))^2 + (V_2'(\theta_3))^2 - (V_3'(\theta_3))^2 \geq 0 \qquad (21.b)$$

Solution #2:

$$V_3(\theta_3) \neq 0 \qquad (22.a)$$

$$V_1^2(\theta_3) + V_2^2(\theta_3) - V_3^2(\theta_3) = 0 \qquad (22.b)$$

$$V_1(\theta_3)V_1'(\theta_3) + V_2(\theta_3)V_2'(\theta_3) - V_3(\theta_3)V_3'(\theta_3) = 0 \qquad (22.c)$$

5.1. An Interpretation of Solution #1

Solution #1 implies that the PRS cone degenerately intersects axis 1 for all θ_2 values, thus leading to extra branch singularities, (6). The four possible degenerate intersections and their associated geometries are outlined below. It can also be shown that manipulators which satisfy (21.a) will also satisfy (21.b).

Solution #1.1: Joint axis 1 intersects the PRS cone apex. This occurs when ($a_1 = 0$) and the PRS axis intersects the concurrent point of axes 1 and 2. This occurs when:

$$\tan(\frac{\theta_3}{2}) = \left(-a_2 \pm \sqrt{a_2^2 + d_2^2 s^2\alpha_2}\right)/(d_2 s\alpha_2). \qquad (23)$$

Figure 2 is an example of this case.

Solution #1.2: When $\theta_3 = \pm\pi/2$, the PRS cone is a cylinder. If *axes 1 and 2 are parallel*, all generators of the PRS cylinder intersect axis 1 at infinity.

Solution #1.3: The PRS axis intersects axis 2 orthogonally–the PRS cone is a plane. If axis 1 lies in this plane, the PRS cone intersects axis 1 for all θ_2. The necessary conditions are: $\alpha_1 = \pm\pi/2$ and $d_2 = 0$. One of the following conditions must also hold:

- $\alpha_2 = \pm\pi/2$. Extra branches occur at $\theta_3 = 0, \pm\pi$.
- $d_3 = 0$. Extra branches occur at $\theta_3 = 0, \pm\pi$.
- $a_3^2 = a_2^2 + d_3^2 \tan^2\alpha_2$. There is only a single extra branch, which occurs at:

$$\theta_3 = \text{atan2}(\frac{d_3 \tan\alpha_2}{a_3}, \frac{-a_2}{a_3}). \qquad (24)$$

Solution #1.4: For some geometries the PRS axis is not unique. An infinite number of PRS axes intersect axes 2, 3, and the end-effector. One of these axes always intersects axis 1. This condition occurs when:

- **The last two axes are parallel.** When $\theta_3 = 0, \pm\pi$, axes 2, 3, and the end-effector lie in a plane. There are an infinite number of PRS axes in this plane.
- **The last two axes intersect.** When $\theta_3 = \pm\pi/2$, the last two axes and end-effector lie in a plane, leading to an infinite number of PRS axes.
- **The end-effector intersects joint axis 2.** The necessary kinematic condition is $a_3^2 = a_2^2 + d_3^2\tan^2\alpha_2$. There is only one extra branch singularity, occurring when θ_3 is given by (24). There can be two extra branches for the conditions: $\alpha_2 = \pm\pi/2$, $d_3 = 0$, and $a_3 \geq a_2$. The extra branches occurs when

$$\theta_3 = \pm\left(\pi/2 + \cos^{-1}(a_2/a_3)\right). \qquad (25)$$

Table 1 summarizes the rank 2 non-generic geometries satisfying Solution #1.

Table 1. Rank 2, Solution #1, Geometries

Geometric Constraint	Branch Locations	Number Branches
$a_1 = 0$	(23)	2
$\alpha_1 = 0$	$\theta_3 = \pm\pi/2$	2
$\alpha_1 = \pm\pi/2, d_2 = 0,$ $\alpha_2 = \pm\pi/2$	$\theta_3 = 0, \pm\pi$	2
$\alpha_1 = \pm\pi/2,$ $d_2 = 0, d_3 = 0$	$\theta_3 = 0, \pm\pi$	2
$a_3^2 = a_2^2 + d_3^2\tan^2\alpha_2$	(24)	1
$\alpha_2 = 0$	$\theta_3 = 0, \pm\pi$	2
$a_2 = 0$	$\theta_3 = \pm\pi/2$	2
$\alpha_2 = \pm\pi/2,$ $d_3 = 0, a_3 \geq a_2$	(25)	2

5.2. An Interpretation of Solution #2

In [6] it is shown that (22.a) simply implies that $a_1 \neq 0$, $\alpha_1 \neq 0$, and that the PRS cone does not assume a planar degenerate form. [6] also shows that (22.b) implies that axis 1 intersects the PRS cone tangentially, and (22.c) is a double root condition. Unfortunately, the Solution #2 geometries are not as easily characterized as the Solution #1 geometries. Assume that (22.a) is satisfied. Let $q(\theta_3)$ denote the expansion of (22.b). (22.b) can be put in the form:

$$q(x_3) = \sum_{j=0}^{8} q_j x_3^j/(1+x_3^2)^4 = \mu(x_3)/(1+x_3^2)^4 = 0. \qquad (26)$$

The $\{q_j\}$ are strictly functions of kinematic parameters, and their form can be found in [6]. (22.c) is the derivative of (22.b) and can be expressed as:

$$q'(x_3) = \sum_{j=0}^{8} q_j' x_3^j/(1+x_3^2)^4 = \mu'(x_3)/(1+x_3^2)^4 \qquad (27)$$

where the $\{q_j'\}$ can similarly be found in [6]. (26) and (27) will vanish simultaneously if:

$$q_8 = q_8' = 0; \quad \text{and} \quad \theta_3 = \pm\pi \qquad (28)$$

or if $\mu(x_3)$, $\mu'(x_3)$ have have a common root, which is signified by a zero resultant. Unfortunately, repeated attempts at finding a symbolic form for the resultant have failed, though numerical procedures can be used for verification.

The kinematic parameter constraints which lead to (28) follow from (3) and (26):

$$q_8 = (m_8 - m_2)^2 + (m_1 - m_6)^2 - m_{11}^2 = 0 \qquad (29)$$

$$q_8' = 2[(m_9 - m_5)(m_8 - m_2) + (m_7 - m_4)(m_1 - m_6) + m_{11}(m_{10} - m_{12})] = 0$$

Define the pseudo-variables and pseudo-constants: $x = (m_8 - m_2)$, $y = (m_1 - m_6)$, $z = m_{11}$, $a = (m_9 - m_5)$,

$b = (m_7 - m_4)$, $c = (m_{10} - m_{12})$. Equations (29) are of the form:

$$x^2 + y^2 - z^2 = 0; \qquad ax + by + cz = 0 \qquad (30)$$

which are respectively the equations for a cone and a plane. $x = y = z = 0$ is always a solution to (30). It can be shown that all geometries satisfying these conditions have already been enumerated. Additional solutions to (30) can also exist depending upon the values of a, b, and c. If $a^2 + b^2 - c^2 = 0$, the plane is tangent to the cone, and the intersection consists of a line (a continuum of kinematic parameter values). If $a^2 + b^2 - c^2 > 0$, the plane intersect the cone in two lines.

Solution #2 geometries do not have a simple "extra branch" CPS, as seen in Figure 3.

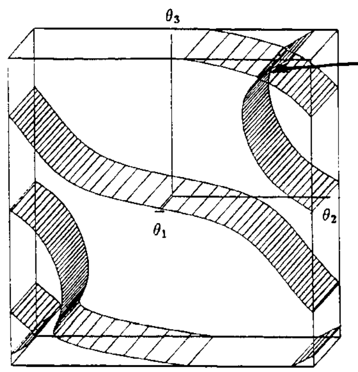

Figure 3: Example of Solution #2 CPS

5.3. The singularities at infinity

Singularities at infinity occur when:

$$\rho_3(x_3) - \rho_1(x_3) = 0; \qquad \text{and } \theta_2 = \pm\pi \qquad (5.a)$$

$$m_6 - m_{11} - m_1 = 0; \qquad \text{and } \theta_3 = \pm\pi \qquad (5.b)$$

It can be shown [3R] that (5.a) leads to only one additional geometry previously not found:

$$a_1 = a_2; \qquad d_2 = 0; \qquad \sin(\alpha_2 - \alpha_1) = 0. \qquad (31)$$

Similarly, (5.b) leads to one additional non-generic geometry:

$$\tan\alpha_1 = \frac{a_1 d_3 \sin\alpha_2}{(a_2 - a_3)d_2 + a_1 d_3 \cos\alpha_2} \qquad (32)$$

with extra branches at $\theta_3 = \pm\pi$.

6. Rank 1 Singularities

[6] shows that all 3R manipulators with rank 1 singularities are a subset of the rank 2 non-generic manipulators. The rank 1 geometries are listed in [6].

7. The Generic Critical Point Homotopy Classes

A means to categorize generic geometries is desired. A scheme based on the number and *homotopy class* of the generic CPS is proposed. Two maps, $f_0: X \to Y$ and $f_1: X \to Y$ are *homotopic* if there exists a smooth map, $F(x,t): X \times I \to y$ such that $F(x,0) = f_o(x)$ and $F(x,1) = f_1(x)$. All maps of a given class which are homotopic belong to the same homotopy class. Since (1) is independent of θ_1, the interesting features of $\mathcal{G}$ can be studied by projection onto the θ_2-θ_3 subspace, which will be denoted as $\mathcal{C}_2$. For generic manipulators, the projected CPS are 1-dimensional manifolds that can be characterized by their fundamental group.

Definition: Let Ω, denote the set of all closed loops in $\mathcal{C}_2$. All homotopic loops in $\mathcal{C}_2$ form a *path component* in the loop space. The fundamental group of $\mathcal{C}_2$ is the group of loop equivalence classes: $\pi_1(\mathcal{C}_2) = \Omega/\sim$ (loop homotopies).

For $\mathcal{C}_2$, $\pi_1(\mathcal{C}_2) = \mathbf{Z} \times \mathbf{Z}$, where $\mathbf{Z}$ is the set of integers. Thus, generic CPS can be labeled by a pair of integers, (I_2, I_3), which characterize how many times a CPS "wraps around" a generator of $\mathcal{C}_2$. Thus, generic manipulator geometries can be classified by the number and homotopy class of their CPS. Examples are given in [6].

8. Non-Generic Singularities as Bifurcations

NG2 can also be interpreted as a bifurcation condition on $\mathcal{G}$. The classical one-dimensional bifurcations (transcritical, pitchfork, and saddle) can be used to categorize the geometry of non-generic CPS in $\mathcal{C}_2$. $g(\theta_2, \theta_3)$ can be interpreted as a quartic in x_3 with coefficients which are a function of the "parameter" θ_2. (18) implies that: (1) the roots of the $g(\theta_2, \theta_3) = 0$ bifurcate at non-generic singular configurations; (2) $g(\theta_2, \theta_3) = 0$ has non-unique tangents, or intersecting CPS, at a non-generic singularity (Figure 2,3).

Alternatively, $g(\theta_2, \theta_3, \mathcal{K}) = det(\mathrm{J}(\overline{\theta}, \mathcal{K}))$ can be considered to be a function of two state variables, θ_2 and θ_3, and six kinematic parameters. In this interpretation, the non-generic singularities are bifurcations of $\mathcal{G}$ with respect to variations in $\mathcal{K}$.

> **Remark:** The set of non-generic manipulators is codimension 1 in $\mathcal{K}$, and thus divides $\mathcal{K}$ into disjoint regions. In each region, all generic geometries have the same homotopy class. Thus, the enumeration of non-generic geometries can serve as a classification scheme for all 3R manipulators.

As kinematic parameters are perturbed, one generic geometry homotopy class bifurcates into another homotopy class by passing through a non-generic (or unstable) geometry.

9. Discussion and Applications

This paper enumerated all non-generic 3R manipulators and introduced a classification scheme for generic manipulators. The enumeration was guided by a novel construction of the singularity set. Further, non-generic manipulators were interpreted as bifurcations of the generic geometry classes. Non-generic manipulators divide the space of all 3R manipulators into disjoint sets, and thus serve as the basis for a 3R manipulator classification scheme.

We briefly mention how these results might be applied in the future. Smith [8] has investigated the conditions under which the 3R manipulator 4^{th} order inverse kinematic solution degenerates for all end-effector locations. Smith termed such robots "degenerate" or "solvable." It is noted that:

Remark: *All solvable 3R manipulators are a subset of non-generic 3R manipulators.*

This observation is only an empirical one, and the fundamental connection underlying non-genericity and solvability still must be established.

Can a manipulator change pose without going through a singularity? It was once thought [9] that all manipulators must pass through a singular configuration when changing pose. In [4] a 3R manipulator which could change pose without passing through a singular configuration was given. Figure 4 shows the CPS of this manipulator ($\alpha_1 = 10^o$, $\alpha_2 = 75^o$, $a_1 = 3.5$, $a_2 = 2.0$, $a_3 = 1.75$, $d_2 = 1.0$, $d_3 = 0.5$). The four inverse kinematic solutions which place the end-effector at $(3.5, 0.0, 0.85)$ are superimposed. There are two inverse kinematic solutions in each connected singularity free region (c-sheet [6]), thus these poses can be connected with a trajectory which does not intersect a CPS. It has been numerically verified that the trajectories in Figure 4 are non-singular pose changing motions.

While the relationship between pose-flipping and singularity is still a subject of research, the author would like to make the following conjectures:

Conjecture: Non-generic manipulators with two extra-branch singularities *must* pass through a singular configuration while changing pose.

Conjecture: Not all generic manipulators can change pose without passing through a singularity.

Conjecture: Generic manipulators which can change pose with out becoming singular can not do so in all parts of their workspace.

The reasoning behind these conjectures, and their relation to the work of Smith can be found in [6].

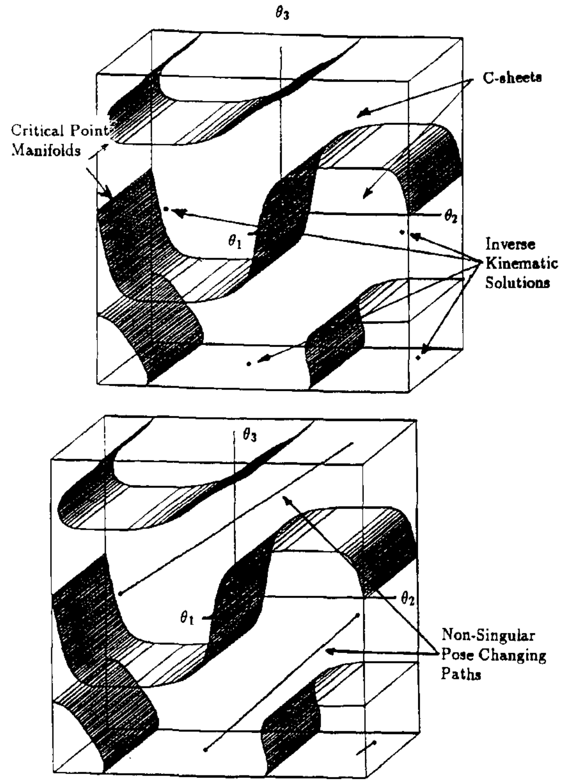

Figure 4: Non-Singular Pose Flipping Manipulator

10. References

[1] K. Sugimoto, J. Duffy, and K. H. Hunt, "Special Configurations of Spatial Mechanisms and Robot Arms," *Mechanism and Machine Theory*, vol. 17, no. 2, 1982, pp. 119-132.

[2] K.J. Waldron, Shih-Liang Wang, and S.J. Bolin, "A Study of the Jacobian Matrix of Serial Manipulators," *J. Mechanisms, Trans., and Automation in Design*, vol. 107, June 1985, pp. 230-238.

[3] Shih-Liang Wang and Kenneth J. Waldron, "A Study of the Singular Configurations of Serial Manipulators," ASME Paper 86-DET-61.

[4] J.W. Burdick, "Kinematic Analysis and Design of Redundant Robot Manipulators," Stanford Computer Science Report no. STAN-CS-88-1207.

[5] D.K. Pai and M.C. Leu, "Generic Singularities of Robot Manipulators," *Proc. IEEE Int. Conf. Robotics and Aut.*, 1989, Scottsdale, AZ, pp. 738-744.

[6] J.W. Burdick, "A Classification of 3R Regional Manipulator Singularities and Geometries," *Report No.* RMS-89-02, Dept. of Mech. Eng., Caltech, Sept., 1990.

[7] J.W. Burdick, "A Recursive Factorization of Manipulator Singularities." *Report No.* RMS-90-02, Dept. of Mech. Eng., Caltech, February, 1991.

[8] D.R. Smith, "Design of Solvable Manipulators," Ph.D. Thesis, Dept. of Mech. Eng., Georgia Inst. of Technology, May 1990.

[9] P. Borrel and A. Liegeois, "A Study of Manipulator Inverse Kinematic Solutions with Applications to Trajectory Planning and Workspace Determination, *Proc. IEEE Int. Conf. Robotics and Aut.*, 1986, pp. 1180-1185.