Bull. Korean Math. Soc. **50** (2013), No. 3, pp. 705–717
http://dx.doi.org/10.4134/BKMS.2013.50.3.705

A CLASSIFICATION OF HALF LIGHTLIKE SUBMANIFOLDS OF A SEMI-RIEMANNIAN MANIFOLD WITH A SEMI-SYMMETRIC NON-METRIC CONNECTION

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Abstract. In this paper, we study the geometry of half lightlike submanifolds M of a semi-Riemannian manifold $\widetilde{M}$ with a semi-symmetric non-metric connection subject to the conditions; (1) the characteristic vector field of $\widetilde{M}$ is tangent to M, the screen distribution on M is totally umbilical in M and the co-screen distribution on M is conformal Killing, or (2) the screen distribution is integrable and the local lightlike second fundamental form of M is parallel.

1. Introduction

In the classical theory of spacetime, while the rest spaces of timelike curves are spacelike subspaces of the tangent spaces, the rest spaces of null curves are lightlike subspaces of the tangent spaces [13]. To investigate this, Hawking and Ellis introduced the notion of so-called screen spaces in Section 4.2 of their book [9]. As for any semi-Riemannian manifold there is a natural existence of lightlike subspaces, Duggal-Bejancu [6] published their work on the general theory of degenerate (lightlike) submanifolds to fill a gap in the study of submanifolds in 1996. Since then there has been very active study on lightlike geometry of submanifolds. The geometry of lightlike submanifolds is used in mathematical physics, in particular, in general relativity since lightlike submanifolds produce models of different types of horizons (event horizons, Cauchy's horizons, Kruskal's horizons). Lightlike hypersurfaces are also studied in the theory of electromagnetism [6].

Ageshe and Chafle [1] introduced the notion of a semi-symmetric non-metric connection on a Riemannian manifold. Although now we have lightlike version of a large variety of Riemannian submanifolds, a general notion of lightlike submanifolds of a semi-Riemannian manifold with a semi-symmetric non-metric

Received January 27, 2012.
2010 *Mathematics Subject Classification.* Primary 53C25, 53C40, 53C50.
Key words and phrases. half lightlike submanifold, semi-Riemannian manifold with semi-symmetric non-metric connection, totally umbilical screen distribution.

connection has not been introduced as yet. Only there are some limited papers on particular subcases recently studied by Yaşar et al. [14] and Jin [10].

Motivated by the notion of a semi-symmetric non-metric connection on a Riemannian manifold, the objective of this paper is to study the half lightlike version of above classical results. We focus on the geometry of half lightlike submanifolds M of a semi-Riemannian manifold $\widetilde{M}$ with a semi-symmetric non-metric connection subject to the conditions; (1) the characteristic vector field ζ of $\widetilde{M}$ is tangent to M, the screen distribution $S(TM)$ is totally umbilical in M the co-screen distribution $S(TM^{\perp})$ is conformal Killing, or (2) the screen distribution $S(TM)$ is integrable and the local lightlike second fundamental form B of M is parallel.

2. Semi-symmetric non-metric connection

Let $(\widetilde{M}, \widetilde{g})$ be a semi-Riemannian manifold. A connection $\widetilde{\nabla}$ on $\widetilde{M}$ is called a *semi-symmetric non-metric connection* [1] if $\widetilde{\nabla}$ and its torsion tensor $\widetilde{T}$ satisfy

$$(\widetilde{\nabla}_X \widetilde{g})(Y, Z) = -\pi(Y)\widetilde{g}(X, Z) - \pi(Z)\widetilde{g}(X, Y), \tag{2.1}$$

$$\widetilde{T}(X, Y) = \pi(Y)X - \pi(X)Y \tag{2.2}$$

for any vector fields X, Y and Z on $\widetilde{M}$, where π is a 1-form associated with a non-zero vector field ζ by $\pi(X) = \widetilde{g}(X, \zeta)$. We say that ζ is the *characteristic vector field* of $\widetilde{M}$. We shall assume ζ *to be unit spacelike* without loss of generality.

A submanifold (M, g) of a semi-Riemannian manifold $\widetilde{M}$ of codimension 2 is called a *half lightlike submanifold* if the radical distribution $Rad(TM) = TM \cap TM^{\perp}$ of M is a vector subbundle of the tangent bundle TM and the normal bundle $TM^{\perp}$ of rank 1. Then there exists complementary non-degenerate distributions $S(TM)$ and $S(TM^{\perp})$ of $Rad(TM)$ in TM and $TM^{\perp}$ respectively, called the *screen* and *co-screen distribution* on M, such that

$$TM = Rad(TM) \oplus_{orth} S(TM), \; TM^{\perp} = Rad(TM) \oplus_{orth} S(TM^{\perp}), \tag{2.3}$$

where the symbol $\oplus_{orth}$ denotes the orthogonal direct sum. We denote such a half lightlike submanifold by $M = (M, g, S(TM))$. Denote by $F(M)$ the algebra of smooth functions on M and by $\Gamma(E)$ the $F(M)$ module of smooth sections of a vector bundle E over M. Choose $L \in \Gamma(S(TM^{\perp}))$ as a unit vector field with $\widetilde{g}(L, L) = \epsilon = \pm 1$. Consider the orthogonal complementary distribution $S(TM)^{\perp}$ to $S(TM)$ in $T\widetilde{M}$. Certainly $Rad(TM)$ and $S(TM^{\perp})$ are subbundles of $S(TM)^{\perp}$. As $S(TM^{\perp})$ is non-degenerate, we have

$$S(TM)^{\perp} = S(TM^{\perp}) \oplus_{orth} S(TM^{\perp})^{\perp},$$

where $S(TM^{\perp})^{\perp}$ is the orthogonal complementary to $S(TM^{\perp})$ in $S(TM)^{\perp}$. For any null section ξ of $Rad(TM)$, there exists a uniquely defined lightlike vector bundle $ltr(TM)$ and a null vector field N of $ltr(TM)$ satisfying

$$\widetilde{g}(\xi, N) = 1, \; \widetilde{g}(N, N) = \widetilde{g}(N, X) = \widetilde{g}(N, L) = 0, \; \forall\, X \in \Gamma(S(TM)). \tag{2.4}$$

We call N, $ltr(TM)$ and $tr(TM) = S(TM^{\perp}) \oplus_{orth} ltr(TM)$ the *lightlike transversal vector field, lightlike transversal vector bundle* and *transversal vector bundle* of M with respect to $S(TM)$, respectively [5]. Therefore $T\widetilde{M}$ is decomposed as

$$
\begin{aligned}
(2.5) \qquad T\widetilde{M} &= TM \oplus tr(TM) = \{Rad(TM) \oplus tr(TM)\} \oplus_{orth} S(TM) \\
&= \{Rad(TM) \oplus ltr(TM)\} \oplus_{orth} S(TM) \oplus_{orth} S(TM^{\perp}).
\end{aligned}
$$

From the decomposition (2.5), the vector field ζ is decomposed by

$$(2.6) \qquad \zeta = \omega + \mu\xi + \lambda N + \nu L,$$

where ω is a smooth vector field on $S(TM)$, and λ, μ and ν are smooth functions defined by $\lambda = \pi(\xi)$, $\mu = \pi(N)$ and $\nu = \epsilon\pi(L)$.

Let P be the projection morphism of TM on $S(TM)$. Then the local Gauss and Weingartan formulas of M and $S(TM)$ are given by

$$
\begin{aligned}
(2.7) \qquad \widetilde{\nabla}_X Y &= \nabla_X Y + B(X,Y)N + D(X,Y)L, \\
(2.8) \qquad \widetilde{\nabla}_X N &= -A_N X + \tau(X)N + \rho(X)L, \\
(2.9) \qquad \widetilde{\nabla}_X L &= -A_L X + \phi(X)N, \\
(2.10) \qquad \nabla_X PY &= \nabla^*_X PY + C(X,PY)\xi, \\
(2.11) \qquad \nabla_X \xi &= -A^*_\xi X - \sigma(X)\xi, \ \forall\, X,\, Y \in \Gamma(TM),
\end{aligned}
$$

where ∇ and ∇^* are induced linear connections on TM and $S(TM)$, respectively, B and D are called the *local lightlike* and *screen second fundamental forms* of M, respectively, C is called the *local second fundamental form* on $S(TM)$. A_N, A^*_ξ and A_L are linear operators on TM and τ, ρ, ϕ and σ are 1-forms on TM. Using (2.1), (2.2) and (2.7), we have

$$
\begin{aligned}
(2.12) \qquad (\nabla_X g)(Y,Z) &= B(X,Y)\eta(Z) + B(X,Z)\eta(Y) \\
&\quad -\pi(Y)g(X,Z) - \pi(Z)g(X,Y), \\
(2.13) \qquad T(X,Y) &= \pi(Y)X - \pi(X)Y, \ \forall\, X,\, Y,\, Z \in \Gamma(TM)
\end{aligned}
$$

and B and D are symmetric on TM, where T is the torsion tensor with respect to the induced connection ∇ and η is a 1-form on TM such that

$$(2.14) \qquad \eta(X) = \widetilde{g}(X,N), \ \forall\, X \in \Gamma(TM).$$

From the facts $B(X,Y) = \widetilde{g}(\widetilde{\nabla}_X Y, \xi)$ and $D(X,Y) = \epsilon\widetilde{g}(\widetilde{\nabla}_X Y, L)$, we know that B and D are independent of the choice of the screen distribution $S(TM)$. Using this equations and (2.1), we get

$$(2.15) \qquad B(X,\, \xi) = 0, \quad D(X,\, \xi) = -\epsilon\, \phi(X), \ \forall\, X \in \Gamma(TM).$$

The above second fundamental forms are related to their shape operators by

$$
\begin{aligned}
(2.16) \qquad & g(A^*_\xi X, Y) = B(X,Y) - \lambda g(X,Y), \qquad \widetilde{g}(A^*_\xi X, N) = 0, \\
(2.17) \qquad & g(A_L X, Y) = \epsilon\{D(X,Y) - \nu g(X,Y)\} + \phi(X)\eta(Y), \\
& \widetilde{g}(A_L X, N) = \epsilon\{\rho(X) - \nu\eta(X)\},
\end{aligned}
$$

$$\begin{aligned} g(A_N X, PY) &= C(X, PY) - \mu g(X, PY) - \pi(PY)\eta(X), \\ \widetilde{g}(A_N X, N) &= -\mu\eta(X), \qquad \sigma(X) = \tau(X) - \lambda\eta(X), \end{aligned} \tag{2.18}$$

for all $X, Y \in \Gamma(TM)$. From the equations of (2.16), we show that A^*_ξ is a $\Gamma(S(TM))$-valued self-adjoint shape operator related to B and satisfies

$$A^*_\xi \xi = 0. \tag{2.19}$$

A vector field X on $\widetilde{M}$ is called a *conformal Killing vector field* if $\widetilde{\mathcal{L}}_X \widetilde{g} = -2\delta\widetilde{g}$ for any smooth function δ, where $\widetilde{\mathcal{L}}$ denotes the Lie derivative of $\widetilde{M}$, i.e.,

$$(\widetilde{\mathcal{L}}_X \widetilde{g})(Y, Z) = X(\widetilde{g}(Y, Z)) - \widetilde{g}([X, Y], Z) - \widetilde{g}(Y, [X, Z]), \quad \forall\, Y, Z \in \Gamma(T\widetilde{M}).$$

In particular, if $\delta = 0$, then X is called a *Killing vector field* on $\widetilde{M}$. A distribution $\mathcal{G}$ on $\widetilde{M}$ is called a *conformal Killing* (or *Killing*) *distribution* on $\widetilde{M}$ if each vector field belonging to $\mathcal{G}$ is a conformal Killing (or Killing) vector field.

Theorem 2.1. *Let M be a half lightlike submanifold of a semi-Riemannian manifold $\widetilde{M}$ admitting a semi-symmetric non-metric connection. Then the following assertion are equivalent*:

(1) *B vanishes identically on M.*
(2) *A^*_ξ satisfies $A^*_\xi X = -\lambda PX$ for all $X \in \Gamma(TM)$.*
(3) *$Rad(TM)$ is a Killing distribution on M.*
(4) *∇ is a semi-symmetric non-metric connection on M.*

Proof. The equivalence of (1) and (2) follows from (2.16) and the fact that $S(TM)$ is non-degenerate. Next, if $B = 0$, from (2.12) and (2.13) we have

$$\begin{aligned} &(\nabla_X g)(Y, Z) = -\pi(Y)g(X, Z) - \pi(Z)g(X, Y), \\ &T(X, Y) = \pi(Y)X - \pi(X)Y, \quad \forall\, X, Y, Z \in \Gamma(TM). \end{aligned}$$

Thus ∇ is a semi-symmetric non-metric connection on M. Conversely if ∇ is a semi-symmetric non-metric connection on M, from (2.12) we have

$$B(X, Y)\eta(Z) + B(X, Z)\eta(Y) = 0, \quad \forall\, X, Y, Z \in \Gamma(TM).$$

Replacing Y by ξ to this result and using $(2.15)_1$, we have $B = 0$. Thus we obtain the equivalence of (1) and (4). Finally, from (2.12) and (2.13) we obtain

$$\begin{aligned} (\mathcal{L}_X g)(Y, Z) = g(\nabla_Y X, Z) &+ g(Y, \nabla_Z X) - 2\pi(X)g(Y, Z) \\ &+ B(X, Y)\eta(Z) + B(X, Z)\eta(Y), \quad \forall\, X, Y, Z \in \Gamma(TM). \end{aligned}$$

Taking $X = \xi$ to this and using (2.11) and the first equation of (2.16), we have

$$(\mathcal{L}_\xi g)(X, Y) = -B(X, Y), \quad \forall\, X, Y \in \Gamma(TM),$$

which proves the equivalence of (1) and (3). $\square$

Theorem 2.2 ([11]). *Let M be a half lightlike submanifold of a semi-Riemannian manifold $\widetilde{M}$ admitting a semi-symmetric non-metric connection. If $S(TM^{\perp})$ is conformal Killing, then there exists a smooth function δ such that*

$$D(X,Y) = \epsilon\delta\, g(X,Y), \quad \forall\, X,\, Y \in \Gamma(TM). \tag{2.20}$$

In particular, if $S(TM^{\perp})$ is a Killing distribution on $\widetilde{M}$, then $D = 0$.

3. Induced Ricci curvature tensors

Denote by $\widetilde{R}$, R and R^* the curvature tensors of the semi-symmetric non-metric connection $\widetilde{\nabla}$ on $\widetilde{M}$, the induced connection ∇ on M and the induced connection ∇^* on $S(TM)$, respectively. Using the Gauss-Weingarten equations for M and $S(TM)$, we obtain the Gauss-Codazzi equations for M and $S(TM)$:

$$\begin{aligned} &\widetilde{g}(\widetilde{R}(X,\, Y)Z,\, PW) \\ &\quad = g(R(X,\, Y)Z,\, PW) + B(X,\, Z)g(A_{N}Y,\, PW) - B(Y,\, Z)g(A_{N}X,\, PW) \\ &\qquad + D(X,\, Z)g(A_{L}Y,\, PW) - D(Y,\, Z)g(A_{L}X,\, PW), \end{aligned} \tag{3.1}$$

$$\begin{aligned} &\widetilde{g}(\widetilde{R}(X,\, Y)Z,\, \xi) \\ &\quad = (\nabla_X B)(Y,\, Z) - (\nabla_Y B)(X,\, Z) + B(Y,\, Z)\{\tau(X) - \pi(X)\} \\ &\qquad - B(X,\, Z)\{\tau(Y) - \pi(Y)\} + D(Y,\, Z)\phi(X) - D(X,\, Z)\phi(Y), \end{aligned} \tag{3.2}$$

$$\begin{aligned} &\widetilde{g}(\widetilde{R}(X,\, Y)Z,\, N) \\ &\quad = \widetilde{g}(R(X,\, Y)Z,\, N) + \mu\{B(Y,Z)\eta(X) - B(X,Z)\eta(Y)\} \\ &\qquad + \epsilon\{D(X,\, Z)[\rho(Y) - \nu\eta(Y)] - D(Y,\, Z)[\rho(X) - \nu\eta(X)]\}, \end{aligned} \tag{3.3}$$

$$\begin{aligned} &\epsilon\, \widetilde{g}(\widetilde{R}(X,\, Y)Z,\, L) \\ &\quad = (\nabla_X D)(Y,\, Z) - (\nabla_Y D)(X,\, Z) + B(Y,\, Z)\rho(X) - B(X,\, Z)\rho(Y) \\ &\qquad - D(Y,Z)\pi(X) + D(X,Z)\pi(Y), \end{aligned} \tag{3.4}$$

$$\begin{aligned} &\widetilde{g}(\widetilde{R}(X,\, Y)\xi,\, N) \\ &\quad = B(X, A_{N}Y) - B(Y, A_{N}X) - 2d\tau(X,Y) + \rho(X)\phi(Y) - \rho(Y)\phi(X) \\ &\quad = C(Y, A^*_{\xi}X) - C(X, A^*_{\xi}Y) - 2d\sigma(X,Y) + \rho(X)\phi(Y) - \rho(Y)\phi(X) \\ &\qquad + \nu\{\phi(X)\eta(Y) - \phi(Y)\eta(X)\}, \end{aligned} \tag{3.5}$$

$$\begin{aligned} &\widetilde{g}(R(X,\, Y)PZ,\, PW) \\ &\quad = g(R^*(X,\, Y)PZ,\, PW) + C(X,\, PZ)g(A^*_{\xi}Y,\, PW) \\ &\qquad - C(Y,\, PZ)g(A^*_{\xi}X,\, PW), \end{aligned} \tag{3.6}$$

$$\begin{aligned} &\widetilde{g}(R(X,Y)PZ,\, N) \\ &\quad = (\nabla_X C)(Y,\, PZ) - (\nabla_Y C)(X,\, PZ) + C(X,\, PZ)\{\sigma(Y) + \pi(Y)\} \\ &\qquad - C(Y,\, PZ)\{\sigma(X) + \pi(X)\}. \end{aligned} \tag{3.7}$$

Let $\widetilde{Ric}$ be the Ricci curvature tensor of $\widetilde{M}$ and let $R^{(0,2)}$ denote the induced Ricci type tensor on M given respectively by

$$\begin{aligned}\widetilde{Ric}(X,Y) &= trace\{Z \to \widetilde{R}(Z,X)Y\}, \quad \forall\, X,\, Y \in \Gamma(T\widetilde{M}),\\ R^{(0,2)}(X,Y) &= trace\{Z \to R(Z,X)Y\}\,, \quad \forall\, X,\, Y \in \Gamma(TM).\end{aligned}$$

Consider a quasi-orthonormal frame field $\{\xi; W_a\}$ on M, where $Rad(TM) = Span\{\xi\}$ and $S(TM) = Span\{W_a\}$ and let $E = \{\xi, N, L, W_a\}$ be the corresponding frame field on $\widetilde{M}$. Let $\dim\widetilde{M} = m+3$. Using this quasi-orthonormal frame field, we obtain

$$\begin{aligned}\widetilde{Ric}(X,Y) = \sum_{a=1}^{m} \epsilon_a\, \widetilde{g}(\widetilde{R}(W_a,X)Y,\, W_a) + \widetilde{g}(\widetilde{R}(\xi,X)Y,\, N)\\ + \widetilde{g}(\widetilde{R}(N,X)Y,\, \xi) + \epsilon\, \widetilde{g}(\widetilde{R}(L,X)Y,\, L),\ \forall\, X,\, Y \in \Gamma(T\widetilde{M}),\end{aligned} \tag{3.8}$$

$$R^{(0,2)}(X,Y) = \sum_{a=1}^{m} \epsilon_a g(R(W_a,X)Y,\, W_a) + \widetilde{g}(R(\xi,X)Y,\, N) \tag{3.9}$$

for all $X, Y \in \Gamma(TM)$, where ϵ_a denotes the causal character of W_a. Substituting (3.1) and (3.3) into (3.8) and using (2.16) and (2.17), we obtain

$$\begin{aligned}R^{(0,2)}(X,Y) = \widetilde{Ric}(X,Y) &+ B(X,Y)trA_{_N} + D(X,Y)trA_{_L}\\ &- g(A_N X, A^*_\xi Y) - \epsilon\, g(A_{_L}X, A_{_L}Y) - \lambda g(A_{_N}X,Y)\\ &- \nu g(A_{_L}X,Y) + \rho(X)\phi(Y) - \nu\phi(X)\eta(Y)\\ &- \widetilde{g}(\widetilde{R}(\xi,Y)X,\, N) - \epsilon\, \widetilde{g}(\widetilde{R}(L,X)Y,\, L).\end{aligned} \tag{3.10}$$

This shows that $R^{(0,2)}$ is not symmetric. The tensor field $R^{(0,2)}$ is called the *induced Ricci curvature tensor* [8, 10] of M, denoted by Ric, if it is symmetric. M is *Ricci flat* if its induced Ricci curvature tensor vanishes identically on M.

Theorem 3.1. *Let M be a half lightlike submanifold of a semi-Riemannian manifold $\widetilde{M}$ admitting a semi-symmetric non-metric connection. Then $R^{(0,2)}$ is symmetric if and only if the 1-form τ is closed, i.e., $d\tau = 0$, on TM.*

Proof. From the first equation of (2.17) and the fact D is symmetric, we have

$$g(A_{_L}X,\, Y) - g(X,\, A_{_L}Y) = \phi(X)\eta(Y) - \phi(Y)\eta(X),\ \forall\, X,\, Y \in \Gamma(TM). \tag{3.11}$$

Using $(3.5)_1$, (3.11) and the first Bianchi's identity, from (3.10) we obtain

$$\begin{aligned}&R^{(0,2)}(X,Y) - R^{(0,2)}(Y,X)\\ &= g(A^*_\xi X, A_{_N}Y) - g(A^*_\xi Y, A_{_N}X) + \lambda\{g(X, A_{_N}Y) - g(A_{_N}X,Y)\}\\ &\quad + \rho(X)\phi(Y) - \rho(Y)\phi(X) - \widetilde{g}(\widetilde{R}(X,Y)\xi, N)\\ &= 2d\tau(X,\, Y), \quad \forall\, X,\, Y \in \Gamma(TM).\end{aligned}$$

From which, we show that $R^{(0,2)}$ is symmetric if and only if $d\tau = 0$. □

A semi-Riemannian manifold $\widetilde{M}$ of constant curvature c is called a *semi-Riemannian space form* and denote it by $\widetilde{M}(c)$. In this case, the curvature tensor $\widetilde{R}$ of $\widetilde{M}$ is given by

$$\widetilde{R}(X,Y)Z = c\{\widetilde{g}(Y,Z)X - \widetilde{g}(X,Z)Y\}, \quad \forall\, X,\, Y,\, Z \in \Gamma(T\widetilde{M}). \tag{3.12}$$

In case of a semi-Riemannian space form $\widetilde{M}(c)$, we have

$$\begin{aligned} R^{(0,\,2)}(X,Y) = mcg(X,Y) &+ B(X,Y)trA_{_N} + D(X,Y)trA_{_L} \\ &- g(A_{_N}X, A^*_\xi Y) - \epsilon\, g(A_{_L}X, A_{_L}Y) - \lambda g(A_{_N}X,Y) \\ &- \nu g(A_{_L}X,Y) + \rho(X)\phi(Y) - \nu\phi(X)\eta(Y). \end{aligned} \tag{3.13}$$

4. Tangential characteristic vector field

In this section, we may assume that ζ is tangent to M. Then we show that $\lambda = g(\zeta,\, \xi) = 0$, $\nu = \epsilon\, \widetilde{g}(\zeta,\, L) = 0$ and $\tau = \sigma$ by $(2.18)_3$.

Proposition 4.1. *Let M be a half lightlike submanifold of a semi-Riemannian manifold $\widetilde{M}$ admitting a semi-symmetric non-metric connection. If ζ is tangent to M, then there exists a screen distribution $S(TM)$ which contains ζ.*

Proof. If ζ belongs to $Rad(TM)$, then $\zeta = \mu\xi$ and $\mu \neq 0$. It follows that

$$1 = \widetilde{g}(\zeta,\, \zeta) = \mu^2\widetilde{g}(\xi,\, \xi) = 0.$$

It is a contradiction. Thus ζ does not belong to $Rad(TM)$. Due to $(2.3)_1$, this result enables one to choose a screen distribution $S(TM)$ which contains ζ. We call such a $S(TM)$ the *natural screen distribution* of M. □

Note 1. Although $S(TM)$ is not unique, it is canonically isomorphic to the factor vector bundle $S(TM)^\sharp = TM/Rad(TM)$ considered by Kupeli [12]. Thus all $S(TM)$ are mutually isomorphic. For this reason, we consider only half lightlike submanifolds M of a semi-Riemannian manifold $\widetilde{M}$ admitting a semi-symmetric non-metric connection with a natural screen distribution.

Definition. We say that $S(TM)$ is *totally umbilical* [6] (in M) if, on any coordinate neighborhood $\mathcal{U} \subset M$, there is a smooth function γ such that

$$C(X, PY) = \gamma\, g(X,Y), \quad \forall\, X,\, Y \in \Gamma(TM). \tag{4.1}$$

In case $\gamma = 0$ on $\mathcal{U}$, we say that $S(TM)$ is *totally geodesic* (in M).

Theorem 4.2. *Let M be a half lightlike submanifold of a semi-Riemannian manifold $\widetilde{M}$ admitting a semi-symmetric non-metric connection. If ζ is tangent to M and $S(TM)$ is totally umbilical in M, then $S(TM)$ is totally geodesic.*

Proof. Applying $\widetilde{\nabla}_X$ to $\widetilde{g}(A_{_N}Y, N) = 0$ and using (2.1), (2.6), (2.7), we have

$$\widetilde{g}(\nabla_X(A_{_N}Y), N) = \pi(A_{_N}Y)\eta(X) + g(A_{_N}X, A_{_N}Y),\ \forall\, X,\, Y \in \Gamma(TM).$$

Substituting this equation into the last term of the following equation

$$0 = \widetilde{g}(\widetilde{R}(X,Y)N,\, N) = -g(\nabla_X(A_{_N}Y), N) + g(\nabla_Y(A_{_N}X),\, N)$$

due to $(2.17)_2$, $(2.18)_2$ and the fact $\mu = \nu = 0$, we have

$$\pi(A_N X)\eta(Y) = \pi(A_N Y)\eta(X), \quad \forall\, X, Y \in \Gamma(TM).$$

Replacing Y by ξ to this equation and using (2.14), we have

$$\pi(A_N X) = \pi(A_N \xi)\eta(X), \quad \forall\, X \in \Gamma(TM).$$

As $S(TM)$ is totally umbilical in M, using $(2.18)_1$ and (4.1) we have

$$\pi(A_N X) = g(A_N X,\, \zeta) = \gamma\pi(X) - \eta(X),\ \pi(A_N \xi) = g(A_N \xi,\, \zeta) = -1.$$

From this results, we have $\gamma\pi(X) = 0$ for any $X \in \Gamma(TM)$. Replacing X by ζ to this and using $g(\zeta, \zeta) = 1$, we get $\gamma = 0$. Thus $S(TM)$ is totally geodesic. $\square$

Theorem 4.3. *Let M be a half lightlike submanifold of a semi-Riemannian manifold $\widetilde{M}$ admitting a semi-symmetric non-metric connection. If $S(TM^\perp)$ is a conformal Killing distribution on $\widetilde{M}$, then $S(TM^\perp)$ is a Killing distribution.*

Proof. As $S(TM^\perp)$ is conformal Killing, by $(2.15)_2$ and (2.20), we have $\phi = 0$. From the Weingarten equations $\widetilde{R}(X,Y)N$ and $\widetilde{R}(X,Y)L$ for M, we obtain

$$\begin{aligned}
&\widetilde{g}(\widetilde{R}(X,Y)N,\, L)\\
&= \epsilon\{D(Y, A_N X) - D(X, A_N Y) + 2d\rho(X,Y) + \rho(X)\tau(Y) - \rho(Y)\tau(X)\}\\
&= \widetilde{g}(\nabla_X(A_L Y) - \nabla_Y(A_L X) - A_L[X,Y] + \phi(Y)A_N X - \phi(X)A_N Y,\, N).
\end{aligned}$$

Using this, $(2.17)_2$ and $(2.18)_2$, we show that

$$\begin{aligned}
(4.2)\qquad &\epsilon\{D(Y, A_N X) - D(X, A_N Y) + 2d\rho(X,Y) + \rho(X)\tau(Y) - \rho(Y)\tau(X)\}\\
&= \widetilde{g}(\nabla_X(A_L Y) - \nabla_Y(A_L X),\, N) - \epsilon\rho([X,Y]).
\end{aligned}$$

Applying $\widetilde{\nabla}_X$ to $\widetilde{g}(A_L Y, N) = \epsilon\rho(Y)$ and using (2.1), (2.7), (2.8) and $(2.17)_2$, for all $X,\, Y \in \Gamma(TM)$, we have

$$\widetilde{g}(\nabla_X(A_L Y), N) = \epsilon X(\rho(Y)) + \pi(A_L Y)\eta(X) + g(A_L Y, A_N X) - \epsilon\tau(X)\rho(Y).$$

Substituting this into (4.2) and using $(2.17)_1$ and $(2.18)_2$, we have

$$\pi(A_L X)\eta(Y) = \pi(A_L Y)\eta(X), \quad \forall\, X, Y \in \Gamma(TM).$$

Replacing Y by ξ to this equation, we have

$$\pi(A_L X) = \pi(A_L \xi)\eta(X), \quad \forall\, X \in \Gamma(TM).$$

Taking $X = \xi$ and $Y = \zeta$ to $(2.17)_1$, we get $\pi(A_L \xi) = -\phi(\zeta) = 0$. Replacing Y by ζ to $(2.17)_1$ and using the above result, we have

$$D(X, \zeta) = \pi(A_L X) = 0, \quad \forall\, X \in \Gamma(TM).$$

Taking $X = Y = \zeta$ to (2.20), we get $\delta = 0$ and $S(TM^\perp)$ is Killing. $\square$

Theorem 4.4. *Let M be a half lightlike submanifold of a semi-Riemannian space form $\widetilde{M}(c)$ admitting a semi-symmetric non-metric connection such that the characteristic vector field ζ of $\widetilde{M}$ is tangent to M. If $S(TM)$ is totally umbilical in M and $S(TM^\perp)$ is conformal Killing, then M is Ricci flat.*

Proof. As $S(TM^{\perp})$ is conformal Killing, we get $D = 0$ by Theorem 4.3. Also as $S(TM)$ is totally umbilical, $C = 0$ by Theorem 4.2. Thus (3.7) becomes

$$\widetilde{g}(R(X,Y)PZ,\, N) = 0, \quad \forall\, X,\, Y,\, Z \in \Gamma(TM).$$

Substituting this and (3.12) into (3.3) with $\mu = \nu = 0$ and using $D = 0$, we get

$$c\,\{g(Y, PZ)\eta(X) - g(X, PZ)\eta(Y)\} = 0, \quad \forall\, X,\, Y,\, Z \in \Gamma(TM).$$

Taking $X = PZ = \zeta$ and $Y = \xi$ to this result, we have $c = 0$.

Using (2.17), (2.18) and the fact $D = C = \mu = \nu = 0$, we have

$$A_{N} X = -\eta(X)\zeta, \quad A_{L} X = \epsilon\rho(X)\xi, \quad \forall\, X \in \Gamma(TM). \tag{4.3}$$

From (3.5) and the facts $\lambda = \nu = 0$ and $\tau = \sigma$, we obtain

$$B(X, A_{N} Y) - B(Y, A_{N} X) = C(Y, A^{*}_{\xi} X) - C(X, A^{*}_{\xi} Y), \quad \forall\, X, Y \in \Gamma(TM).$$

Using $(2.16)_1$, $(2.18)_1$ and the fact A^{*}_{ξ} is self-adjoint, we have

$$\pi(A^{*}_{\xi} X)\eta(Y) = \pi(A^{*}_{\xi} Y)\eta(X), \ \forall\, X, Y \in \Gamma(TM).$$

Replacing Y by ξ to this equation and using $(2.16)_1$ and (2.19), we have

$$B(X, \zeta) = \pi(A^{*}_{\xi} X) = 0, \quad \forall\, X \in \Gamma(TM). \tag{4.4}$$

Substituting (4.3) into (3.13) and using (4.4), we have

$$R^{(0,\,2)}(X, Y) \;=\; B(X, Y)\, tr A_{N}, \quad \forall\, X,\, Y \in \Gamma(TM). \tag{4.5}$$

Thus we show that $R^{(0,\,2)}$ is symmetric. Using (4.3), we have

$$tr A_{N} = \sum_{a=1}^{m} \epsilon_a g(A_{N} W_a,\, W_a) + \widetilde{g}(A_{N}\xi,\, N) = 0 + 0 = 0.$$

Substituting this result into (4.5), we have $R^{(0,\,2)} = 0$. Thus M is Ricci flat. □

Theorem 4.5. *Let M be a half lightlike submanifold of a semi-Riemannian space form $\widetilde{M}(c)$ admitting a semi-symmetric non-metric connection. If ζ is tangent to M, $S(TM)$ is totally umbilical in M and $S(TM^{\perp})$ is conformal Killing, then the following are equivalent*:

(1) *M is flat, i.e., the curvature tensor R of M satisfies $R = 0$.*
(2) *The local lightlike second fundamental form B of M satisfies $B = 0$.*
(3) *The connection ∇ of M is a semi-symmetric non-metric connection.*

Proof. Using (2.12), (3.1)∼(3.4), (4.3) and the fact $c = D = 0$, we show that

$$\begin{aligned} R(X, Y)Z = {} & \widetilde{R}(X, Y)Z - B(X, Z)A_{N} Y + B(Y, Z)A_{N} X \\ & = \{B(X, Z)\eta(Y) - B(Y, Z)\eta(X)\}\,\zeta, \ \forall\, X,\, Y,\, Z \in \Gamma(TM), \end{aligned}$$

which implies the equivalence of (1) and (2). Next, the equivalence of (2) and (3) follows from Theorem 1.1. Thus we have our assertions. □

5. Integrable screen distributions

In general, the screen distribution $S(TM)$ is not necessarily integrable. The following result gives equivalent conditions for the integrability of $S(TM)$:

Theorem 5.1. *Let M be a half lightlike submanifold of a semi-Riemannian manifold $\widetilde{M}$ admitting a semi-symmetric non-metric connection. Then the following are equivalent*:

(1) *The screen distribution $S(TM)$ is an integrable distribution.*
(2) *C is symmetric, i.e., $C(X,Y)=C(Y,X)$ for all $X,\,Y\in\Gamma(S(TM))$.*
(3) *The shape operator A_{N} is a self-adjoint with respect to g, i.e.,*

$$g(A_{N}X,Y)=g(X,A_{N}Y),\quad \forall\, X,\,Y\in\Gamma(S(TM)).$$

Proof. First, note that a vector field X on M belongs to $S(TM)$ if and only if we have $\eta(X)=0$. Next, by using (2.10) and (2.13), we have

$$C(X,Y)-C(Y,X)=\eta([X,Y]),\quad \forall\, X,\,Y\in\Gamma(S(TM)),$$

which implies the equivalence of (1) and (2). Finally, the equivalence of (2) and (3) follows from the first equation of (2.18). $\square$

Note 2. In case $S(TM)$ is integrable, M is locally a product manifold $\mathcal{C}\times M^*$ where $\mathcal{C}$ is a null curve and M^* is a leaf of $S(TM)$ [6, 8].

Theorem 5.2. *Let M be a half lightlike submanifold of a Lorentzian manifold $\widetilde{M}$ admitting a semi-symmetric non-metric connection. If $S(TM)$ is integrable and the lightlike second fundamental form B of M is parallel, then M is locally a product manifold $\mathcal{C}\times M_o\times M_\lambda$, where $\mathcal{C}$ is a null curve, and M_o and M_λ are leaves of some integrable distributions of M.*

Proof. Under the hypotheses, $S(TM)$ is a Riemannian vector bundle and M is locally a product $\mathcal{C}\times M^*$ where $\mathcal{C}$ is a null curve and M^* is a leaf of $S(TM)$. Applying ∇_X to $B(Y,\xi)=0$ and using (2.11), $(2.15)_1$ and $(2.16)_1$, we have

$$g(A^*_\xi X, A^*_\xi Y)=\lambda g(A^*_\xi X,Y),\quad \forall\, X,\,Y\in\Gamma(TM). \tag{5.1}$$

Since ξ is an eigenvector field of A^*_ξ corresponding to the eigenvalue 0 due to (2.19) and A^*_ξ is an $S(TM)$-valued real self-adjoint operator, A^*_ξ have m real orthonormal eigenvector fields and is diagonalizable. Consider a frame field of eigenvectors $\{\xi, E_1,\,\ldots,\,E_m\}$ of A^*_ξ such that $\{E_1,\,\ldots,\,E_m\}$ is an orthonormal frame field of $S(TM)$ and $A^*_\xi E_i=\lambda_i E_i$ for each i. Put $X=Y=E_i$ in (5.1), each eigenvalue λ_i is a solution of the equation

$$x^2-\lambda\, x=0. \tag{5.2}$$

(5.2) has at most two distinct solutions 0 and λ. Assume that there exists $p\in\{0,\,1,\,\ldots,\,m\}$ such that $\lambda_1=\cdots=\lambda_p=0$ and $\lambda_{p+1}=\cdots=\lambda_m=\lambda$, by renumbering if necessary.

In case $p = 0$ or $p = m$: As $S(TM)$ is integrable, we show that $M = \mathcal{C} \times M^* \cong \mathcal{C} \times M^* \times \{x\}$ for any $x \in M$. In this case either $M_o = M^*$ and $M_\lambda = \{x\}$ or $M_\lambda = M^*$ and $M_o = \{x\}$. Thus this theorem is true.

In case $0 < p < m$: Consider the distributions D_o, D_λ, D_o^s and D_λ^s on M;

$$D_o = \{X \in \Gamma(TM) \mid A_\xi^* X = 0 \text{ and } PX \neq 0\}, \qquad D_o^s = PD_o,$$
$$D_\lambda = \{U \in \Gamma(TM) \mid A_\xi^* U = \lambda PU \text{ and } PU \neq 0\}, \quad D_\lambda^s = PD_\lambda.$$

Clearly we show that $D_o \cap D_\lambda = \{0\}$ and $D_o^s \cap D_\lambda^s = \{0\}$ as $\lambda \neq 0$.

For any $X \in \Gamma(D_o)$ and $U \in \Gamma(D_\lambda)$, we get $A_\xi^* PX = A_\xi^* X = 0$ and $A_\xi^* PU = A_\xi^* U = \lambda PU$. This imply $PX \in \Gamma(D_o)$ and $PU \in \Gamma(D_\lambda)$. Thus P maps $\Gamma(D_o)$ onto $\Gamma(D_o^s)$ and $\Gamma(D_\lambda)$ onto $\Gamma(D_\lambda^s)$. Since PX and PU are eigenvector fields of the real self-adjoint operator A_ξ^* corresponding to the different eigenvalues 0 and λ, respectively, we have $g(PX, PU) = 0$. From the facts $g(X,U) = g(PX,PU) = 0$ and $B(X,U) = g(A_\xi^* X,U) + \lambda g(X,U) = \lambda g(X,U) = 0$, we show that $D_o \perp_g D_\lambda$ and $D_o \perp_B D_\lambda$, respectively.

Since $\{E_i\}_{1 \leq i \leq p}$ and $\{E_a\}_{p+1 \leq a \leq m}$ are vector fields of D_o^s and D_λ^s, respectively, and D_o^s and D_λ^s are mutually orthogonal vector subbundles of $S(TM)$, D_o^s and D_λ^s are non-degenerate distributions of rank p and rank $(m-p)$, respectively. Thus $S(TM) = D_o^s \oplus_{orth} D_\lambda^s$.

From (5.1), we show that $A_\xi^*(A_\xi^* - \lambda P) = (A_\xi^* - \lambda P)A_\xi^* = 0$. Let $Y \in Im\, A_\xi^*$, then there exists $X \in \Gamma(TM)$ such that $Y = A_\xi^* X$. Then we have $(A_\xi^* - \lambda P)Y = 0$ and $Y \in \Gamma(D_\lambda)$. Thus $Im\, A_\xi^* \subset \Gamma(D_\lambda)$. Since the morphism A_ξ^* maps $\Gamma(TM)$ onto $\Gamma(S(TM))$, we have $Im\, A_\xi^* \subset \Gamma(D_\lambda^s)$. By duality, we also have $Im(A_\xi^* - \lambda P) \subset \Gamma(D_o^s)$.

For any $X, Y \in \Gamma(D_o)$ and $U, V \in \Gamma(D_\lambda)$, applying ∇_X to $B(U,V) = 2\lambda g(U,V)$ and ∇_U to $B(X,Y) = \lambda g(X,Y)$ and then, using (2.12), $(2.16)_1$ and the facts $\nabla B = 0$ and $D_o \perp_g D_\lambda$; $D_o \perp_B D_\lambda$, we have $(X\lambda)g(U,V) = 0$ and $(U\lambda)g(X,Y) = 0$, i.e., $X\lambda = 0$ and $U\lambda = 0$. This imply $X\lambda = 0$ for all $X \in \Gamma(D_o \oplus_{orth} D_\lambda)$. Thus λ is a constant on $S(TM)$.

For any $X, Y, Z \in \Gamma(D_o^s)$, applying ∇_Z to $B(X,Y) = \lambda g(X,Y)$ and using (2.12), $(2.16)_1$ and $\nabla B = 0$ and λ is constant on $S(TM)$, we get $\nabla_X g = 0$, i.e.,

$$\pi(X)g(Y,Z) + \pi(Y)g(X,Z) = 0. \tag{5.3}$$

Using this and the fact D_o^s is non-degenerate, we have

$$\pi(X)Y = -\pi(Y)X. \tag{5.4}$$

Taking the skew-symmetric part of (5.3) for X and Z, we get $\pi(X)g(Y,Z) = \pi(Z)g(X,Y)$, from which we have

$$\pi(X)Y = \pi(Y)X. \tag{5.5}$$

From (5.4) and (5.5), we have $\pi(X) = 0$ for all $X \in \Gamma(D_o^s)$. By duality, we have $\pi(U) = 0$ for all $U \in \Gamma(D_\lambda^s)$. Thus $\pi = 0$ and $\nabla g = 0$ on $S(TM)$.

For any $X, Y \in \Gamma(D_o^s)$ and $U, V \in \Gamma(D_\lambda^s)$, applying ∇_X to $B(Y,U) = 0$ and ∇_V to $B(Y,U) = 0$ and using (2.12), $(2.16)_1$, $\nabla B = 0$ and $\nabla g = 0$ on $S(TM)$, we have

$$g(A^*_\xi \nabla_X Y, U) = 0, \quad g((A^*_\xi - \lambda P)\nabla_V U, Y) = 0, \tag{5.6}$$

$$g(\nabla_X Y, U) = 0, \qquad g(\nabla_V U, Y) = 0. \tag{5.7}$$

From (5.6), since D^s_λ is non-degenerate and $Im\, A^*_\xi \subset \Gamma(D^s_\lambda)$, we have $A^*_\xi \nabla_X Y = 0$. Thus $\nabla_X Y \in \Gamma(D_o)$. By duality, we have $\nabla_V U \in \Gamma(D_\lambda)$. Thus we have

$$g(\nabla^*_X Y, U) = 0, \qquad g(\nabla^*_V U, Y) = 0,$$

due to (5.7). This results imply that $\nabla^*_X Y \in \Gamma(D^s_o)$ for all $X, Y \in \Gamma(D^s_o)$ and $\nabla^*_V U \in \Gamma(D^s_\lambda)$ for all $U, V \in \Gamma(D^s_\lambda)$. Thus D^s_o and D^s_λ are integrable and auto-parallel distributions on $S(TM)$.

Since the leaf M^* of $S(TM)$ is a Riemannian manifold and $S(TM) = D^s_o \oplus_{orth} D^s_\lambda$, where D^s_o and D^s_λ are auto-parallel distributions with respect to the induced connection ∇^* on $S(TM)$, by the decomposition theorem of de Rham [4], we have $M^* = M_o \times M_\lambda$, where M_o and M_λ are leaves of D^s_o and D^s_λ respectively. Thus we have our theorem. □

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