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A CLASSIFICATION OF IMMERSED HYPERSURFACES IN SPHERES WITH PARALLEL BLASCHKE TENSORS

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Abstract. In this paper, we give a complete classification of all immersed hypersurfaces in the unit sphere with parallel Blaschke tensors. For this classification, two kinds of new examples are constructed.

1. Introduction. Let $S^n(r)$ be the n-dimensional standard sphere of radius r and $S^n = S^n(1)$ in the n-dimensional Euclidean space $\boldsymbol{R}^n$. Let $H^n(c)$ be the n-dimensional hyperbolic space of constant curvature $c < 0$ defined by

$$H^n(c) = \{y = (y_0, y_1) \in \boldsymbol{R}^{n+1}_1\,;\ \langle y, y\rangle_1 = 1/c,\ y_0 > 0\}\,,$$

where for any integer $n \geq 2$, $\boldsymbol{R}^n_1 \equiv \boldsymbol{R}_1 \times \boldsymbol{R}^{n-1}$ is the n-dimensional Lorentzian space with the standard Lorentzian inner product $\langle\cdot,\cdot\rangle_1$ given by

$$\langle y, y'\rangle_1 = -y_0y_0' + y_1\cdot y_1'\,, \quad y = (y_0, y_1),\ y' = (y_0', y_1') \in \boldsymbol{R}^n_1\,,$$

and '$\cdot$' denotes the standard Euclidean inner product on $\boldsymbol{R}^{n-1}$.

Denote by S^n_+ the hemisphere in S^n whose first coordinate is positive. Then there are two conformal diffeomorphisms $\sigma : \boldsymbol{R}^n \to S^n\backslash\{(-1, 0)\}$ and $\tau : H^n(-1) \to S^n_+$ defined as follows:

$$\sigma(u) = \left(\frac{1-|u|^2}{1+|u|^2}, \frac{2u}{1+|u|^2}\right), \quad u \in \boldsymbol{R}^n, \tag{1.1}$$

$$\tau(y) = \left(\frac{1}{y_0}, \frac{y_1}{y_0}\right), \quad y = (y_0, y_1) \in H^n \subset \boldsymbol{R}^{n+1}_1\,. \tag{1.2}$$

Now suppose that $x : M^m \to S^{m+p}$ is an immersed submanifold in S^{m+p} without umbilic points. We recall that there are four basic Möbius invariants of x given by Wang in [17], which are the Möbius metric g, the Möbius form Φ, the Blaschke tensor A and the Möbius second fundamental form B. Study of these invariants is closely related to Willmore hypersurfaces (in particular, Willmore surfaces) and other interesting topics in conformal differential geometry. In recent years, many interesting and important results have been obtained in related areas; see, for example, [1, 2, 5–7, 9–17] and references therein. Among these results, there are some interesting classification theorems of submanifolds with particular Möbius invariants, such as classification of surfaces with vanishing Möbius form [13], classification of

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Möbius isotropic submanifolds [15], and classification of hypersurfaces with Blaschke tensors that are linearly dependent on the Möbius metrics and the second Möbius fundamental forms [12, 16]. Also, Hu and Li [6] recently proved a classification theorem of all immersed hypersurfaces in S^{m+1} with parallel Möbius second fundamental forms, which can be stated as follows.

THEOREM 1.1 (Hu and Li [6]). *Let* $x : M^m \to S^{m+1}$, $m \geq 2$, *be an immersed umbilic-free hypersurface with parallel Möbius second fundamental form. Then* x *is locally Möbius equivalent to one of the following hypersurfaces.*

(1) *A standard torus* $S^{k_1}(r) \times S^{m-k_1}(\sqrt{1-r^2})$ *in* S^{m+1} *for some* $r > 0$ *and positive integer* k_1.

(2) *The image under the conformal diffeomorphism* σ *of a standard cylinder* $S^{k_1}(r) \times \boldsymbol{R}^{m-k_1}$ *in* $\boldsymbol{R}^{m+1}$ *for some* $r > 0$ *and positive integer* k_1.

(3) *The image under the conformal diffeomorphism* τ *of a standard cylinder* $S^{k_1}(r) \times H^{m-k_1}(-1/(1+r^2))$ *in* $H^{m+1}(-1)$ *for some* $r > 0$ *and positive integer* k_1.

(4) $CSS(p, q, r)$ *for some constants* p, q, r, *as indicated in Example* 3.1.

Thus, it is natural to study submanifolds in the unit sphere S^n with particular Blaschke tensors. It is easily seen that a submanifold in S^n with vanishing Blaschke tensor also has a vanishing Möbius form, and therefore is a special Möbius isotropic submanifold. By the argument of [15], we can conclude that each submanifold in S^n with vanishing Blaschke tensor is locally Möbius equivalent to the image under the conformal diffeomorphism $\sigma : \boldsymbol{R}^n \to S^n \backslash \{(-1, 0)\}$ of a minimal submanifold in the Euclidean space $\boldsymbol{R}^n$. On the other hand, by Theorem 1.1, it is interesting to find a classification of immersed submanifolds with parallel Blaschke tensors.

In this direction, the most important area is the study of hypersurfaces. In this paper, we give a Möbius classification of all immersed hypersurfaces in S^{m+1} with parallel Blaschke tensors. In pursuing this, we find two kinds of immersed hypersurfaces that have parallel Blaschke tensors but have non-parallel Möbius second fundamental forms (for details, see Examples 3.2 and 3.3). The main theorem of this paper is the following.

THEOREM 1.2. *Let* $x : M^m \to S^{m+1}$, $m \geq 2$, *be an immersed hypersurface without umbilics. If the Blaschke tensor* A *of* x *is parallel, then one of the following holds.*

(1) x *is Möbius isotropic and is therefore locally Möbius equivalent to*:

- (a) *a minimal immersed hypersurface in* S^{m+1} *with constant scalar curvature*; *or*
- (b) *the image under* σ *of a minimal immersed hypersurface in* $\boldsymbol{R}^{m+1}$ *with constant scalar curvature*; *or*
- (c) *the image under* τ *of a minimal immersed hypersurface in* $H^{m+1}(-1)$ *with constant scalar curvature.*

(2) x *is of parallel Möbius second fundamental form* B *and is therefore locally Möbius equivalent to*:

- (a) *a standard torus* $S^{k_1}(r) \times S^{m-k_1}(\sqrt{1-r^2})$ *in* S^{m+1} *for some* $r > 0$ *and positive integer* k_1; *or*

(b) *the image under* σ *of a standard cylinder* $S^{k_1}(r) \times \boldsymbol{R}^{m-k_1}$ *in* $\boldsymbol{R}^{m+1}$ *for some* $r > 0$ *and positive integer* k_1; *or*

(c) *the image under* τ *of a standard cylinder* $S^{k_1}(r) \times H^{m-k_1}(-1/(1+r^2))$ *in* $H^{m+1}(-1)$ *for some* $r > 0$ *and positive integer* k_1; *or*

(d) *CSS*(p, q, r) *for some constants* p, q, r.

(3) x *is non-isotropic with a non-parallel Möbius second fundamental form* B *and is locally Möbius equivalent to*:

(a) *one of the minimal hypersurfaces as indicated in Example* 3.2; *or*

(b) *one of the non-minimal hypersurfaces as indicated in Example* 3.3.

REMARK 1.3. It is directly verified that each of the immersed hypersurfaces without umbilics stated in the above theorem has a parallel Blaschke tensor.

2. Preliminaries. Let $x : M^m \to S^{m+p}$ be an immersed submanifold without umbilic points, and $n = m + p$. Denote by h the second fundamental form of x and $H = (1/m)\operatorname{tr} h$ the mean curvature vector field. Define

$$\rho = \left(\frac{m}{m-1}(|h|^2 - m|H|^2)\right)^{1/2}, \quad Y = \rho(1, x)\,. \tag{2.1}$$

Then $Y : M^m \to \boldsymbol{R}_1^{n+2}$ is an immersion of M^m into the Lorentzian space $\boldsymbol{R}_1^{n+2}$ and is called the canonical lift (or the Möbius position vector) of x. The function ρ given by (2.1) may be called the 'Möbius factor' of the immersion x. We define

$$C_+^{n+1} = \{Y = (Y_0, Y) \in \boldsymbol{R}_1 \times \boldsymbol{R}^{n+1}\,;\ \langle Y, Y\rangle_1 = 0,\ Y_0 > 0\}\,.$$

Let $O(n+1, 1)$ be the Lorentzian group of all elements in $GL(n+2; \boldsymbol{R})$ preserving the standard Lorentzian inner product $\langle\cdot,\cdot\rangle_1$ on $\boldsymbol{R}_1^{n+2}$, and $O^+(n+1,1)$ be a subgroup of $O(n+1,1)$ given by

$$O^+(n+1,1) = \{T \in O(n+1,1)\,;\ T(C_+^{n+1}) \subset C_+^{n+1}\}\,. \tag{2.2}$$

Then the following theorem is well known.

THEOREM 2.1 (Wang [17]). *Two submanifolds* $x, \tilde{x} : M^m \to S^{m+p}$ *with Möbius position vectors* $Y, \tilde{Y}$, *respectively, are Möbius equivalent if and only if there is a* $T \in O^+(n+1,1)$ *such that* $\tilde{Y} = T(Y)$.

By Theorem 2.1, the induced metric $g = Y^*\langle\cdot,\cdot\rangle_1 = \rho^2\, dx \cdot dx$ defined by Y on M^m from the Lorentzian product $\langle\cdot,\cdot\rangle_1$ is a Möbius invariant Riemannian metric (cf. [3, 4 17]), and is called the Möbius metric of x. Using the vector-valued function Y and the Laplacian Δ of the metric g, we can define another important vector-valued function $N : M^m \to \boldsymbol{R}_1^{n+2}$ by

$$N = -\frac{1}{m}\Delta Y - \frac{1}{2m^2}\langle \Delta Y, \Delta Y\rangle_1 Y\,. \tag{2.3}$$

Then it is verified that the Möbius position vector Y and the Möbius biposition vector N satisfy the following identities [17]:

$$\langle \Delta Y, Y\rangle_1 = -m\,, \quad \langle \Delta Y, dY\rangle_1 = 0\,, \quad \langle \Delta Y, \Delta Y\rangle_1 = 1 + m^2\kappa\,, \tag{2.4}$$

$$\langle Y, Y\rangle_1 = \langle N, N\rangle_1 = 0\,, \quad \langle Y, N\rangle_1 = 1\,, \tag{2.5}$$

where κ denotes the normalized scalar curvature of the Möbius metric g.

Let $V \to M^m$ be the vector subbundle of the trivial Lorentzian bundle $M^m \times \boldsymbol{R}^{n+2}_1$ defined to be the orthogonal complement of $\boldsymbol{R}Y \oplus \boldsymbol{R}N \oplus Y_*(TM^m)$ with respect to the Lorentzian product $\langle\cdot,\cdot\rangle_1$. Then V is called the Möbius normal bundle of the immersion x. Clearly, we have the following vector bundle decomposition:

$$M^m \times \boldsymbol{R}^{n+2}_1 = \boldsymbol{R}Y \oplus \boldsymbol{R}N \oplus Y_*(TM^m) \oplus V\,. \tag{2.6}$$

Now, let $T^\perp M^m$ be the normal bundle of the immersion $x : M^m \to S^{m+1}$. Then the mean curvature vector field H of x defines a bundle isomorphism $f : T^\perp M^m \to V$ by

$$f(e) = (H\cdot e, (H\cdot e)x + e) \quad \text{for any } e \in T^\perp M^m\,. \tag{2.7}$$

It is easily seen that f preserves the inner products as well as the connections on $T^\perp M^m$ and V (see [17]).

To simplify notation, we make the following conventions on the ranges of induces used frequently in this paper:

$$1 \le i, j, k, \cdots \le m\,, \quad m+1 \le \alpha, \beta, \gamma, \cdots \le n\,. \tag{2.8}$$

For a local orthonormal frame field $\{e_i\}$ for the induced metric $dx\cdot dx$ with dual $\{\theta^i\}$ and for an orthonormal normal frame field $\{e_\alpha\}$ of x, we set

$$E_i = \rho^{-1}e_i\,, \quad \omega^i = \rho\theta^i\,, \quad E_\alpha = f(e_\alpha)\,. \tag{2.9}$$

Then $\{E_i\}$ is a local orthonormal frame field for the Möbius metric g, $\{\omega^i\}$ is the dual of $\{E_i\}$, and $\{E_\alpha\}$ is a local orthonormal frame field of the Möbius normal bundle $V \to M$. By [17] and [15], the basic Möbius invariants Φ, A and B have the following local expressions:

$$\Phi = \sum \Phi^\alpha_i \omega^i E_\alpha\,, \quad A = \sum A_{ij}\omega^i\omega^j\,, \quad B = \sum B^\alpha_{ij}\omega^i\omega^j E_\alpha\,, \tag{2.10}$$

where

$$\Phi^\alpha_i = -\rho^{-2}\left(H^\alpha_{,i} + \sum (h^\alpha_{ij} - H^\alpha\delta_{ij})e_j(\log\rho)\right), \tag{2.11}$$

$$\begin{aligned} A_{ij} = &-\rho^{-2}\left(\mathrm{Hess}_{ij}(\log\rho) - e_i(\log\rho)e_j(\log\rho) - \sum H^\alpha h^\alpha_{ij}\right) \\ &- \frac{1}{2}\rho^{-2}(|d\log\rho|^2 - 1 + |H|^2)\delta_{ij}\,, \end{aligned} \tag{2.12}$$

$$B^\alpha_{ij} = \rho^{-1}(h^\alpha_{ij} - H^\alpha\delta_{ij})\,, \tag{2.13}$$

in which the subscript 'i' denotes the covariant derivative with respect to the induced metric $dx\cdot dx$ and in the direction e_i.

REMARK 2.2. By definition, the Möbius form Φ and the Möbius second fundamental form B are V-valued differential forms. However, if we identify V with $T^{\perp}M^m$ via the canonical bundle isomorphism $f : T^{\perp}M^m \to V$, then Φ and B are identified, respectively, with $f^{-1}(\Phi)$, $f^{-1}(B)$. Therefore, (2.10) can be written as

$$f^{-1}(\Phi) = \rho \sum \Phi_i^{\alpha} \theta^i e_{\alpha}\,, \quad A = \rho^2 \sum A_{ij}\theta^i\theta^j\,, \quad f^{-1}(B) = \rho^2 \sum B_{ij}^{\alpha}\theta^i\theta^j e_{\alpha}\,.$$

Denote by D the Riemannian connection of the Möbius metric g. Then, with respect to the frame field $\{E_i\}$, the components R_{ijkl}, R_{ij} of the Riemannian curvature tensor and the Ricci tensor are defined, respectively, by

$$D_{E_k}D_{E_l}E_i - D_{E_l}D_{E_k}E_i - D_{[E_k,E_l]}E_i = \sum_j R_{ijkl}E_j\,, \quad R_{ij} = \sum_k R_{ikkj}\,.$$

Then by [17], we have

$$\text{(2.14)} \qquad \operatorname{tr} A = \frac{1}{2m}(1 + m^2\kappa)\,, \quad \operatorname{tr} B = \sum B_{ii}^{\alpha}E_{\alpha} = 0\,, \quad |B|^2 = \sum (B_{ij}^{\alpha})^2 = \frac{m-1}{m}\,.$$

$$\text{(2.15)} \qquad R_{ijkl} = \sum (B_{il}^{\alpha}B_{jk}^{\alpha} - B_{ik}^{\alpha}B_{jl}^{\alpha}) + A_{il}\delta_{jk} - A_{ik}\delta_{jl} + A_{jk}\delta_{il} - A_{jl}\delta_{ik}\,.$$

We should remark that (2.15) has the opposite sign to (2.31) in [17] due to the different defining equations of the Riemannian curvature tensor. Furthermore, if Φ_{ij}^{α}, A_{ijk}, B_{ijk}^{α} denote respectively the components with respect to the frame fields $\{E_i\}$ and $\{E_{\alpha}\}$ of the covariant derivatives of Φ, A, B, then the following Ricci identities hold [17]:

$$\text{(2.16)} \qquad \Phi_{ij}^{\alpha} - \Phi_{ji}^{\alpha} = \sum (B_{ik}^{\alpha}A_{kj} - B_{kj}^{\alpha}A_{ki})\,,$$

$$\text{(2.17)} \qquad A_{ijk} - A_{ikj} = \sum (B_{ik}^{\alpha}\Phi_j^{\alpha} - B_{ij}^{\alpha}\Phi_k^{\alpha})\,,$$

$$\text{(2.18)} \qquad B_{ijk}^{\alpha} - B_{ikj}^{\alpha} = \delta_{ij}\Phi_k^{\alpha} - \delta_{ik}\Phi_j^{\alpha}\,.$$

By taking a trace in (2.15) and (2.18), one obtains

$$\text{(2.19)} \qquad R_{ij} = -\sum B_{ik}^{\alpha}B_{kj}^{\alpha} + \delta_{ij}\operatorname{tr} A + (m-2)A_{ij}\,,$$

$$\text{(2.20)} \qquad (m-1)\Phi_i^{\alpha} = -\sum B_{ijj}^{\alpha}\,.$$

By (2.14), (2.19) and (2.20), if x is a hypersurface and $m \geq 3$, then the Möbius form Φ and the Blaschke tensor A are determined by the Möbius metric g and Möbius second fundamental form B. Thus, it is easily seen that the following theorem holds.

THEOREM 2.3 (Wang [17]). *Two hypersurfaces $x : M^m \to S^{m+1}$ and $\tilde{x} : \tilde{M}^m \to S^{m+1}$, $m \geq 3$, are Möbius equivalent if and only if there exists a diffeomorphism $\varphi : M^m \to \tilde{M}^m$ preserving the Möbius metric and the Möbius second fundamental forms.*

3. Examples. Before proving the main theorem, we present some concrete immersed hypersurfaces in S^{m+1} with parallel Blaschke tensors.

EXAMPLE 3.1 (Hu and Li [6]). Let $\boldsymbol{R}^+$ be the half line of positive real numbers. For any two natural numbers p, q satisfying $p + q < m$ and a real number $r \in (0, 1)$, consider the imbedded hypersurface $u : S^p(r) \times S^q(\sqrt{1-r^2}) \times \boldsymbol{R}^+ \times \boldsymbol{R}^{m-p-q-1} \to \boldsymbol{R}^{m+1}$ defined by $u = (tu', tu'', u''')$, where

$$u' \in S^p(r) \subset \boldsymbol{R}^{p+1}\,, \quad u'' \in S^q(\sqrt{1-r^2}) \subset \boldsymbol{R}^{q+1}\,, \quad t \in \boldsymbol{R}^+\,, \quad u''' \in \boldsymbol{R}^{m-p-q-1}.$$

Then $x = \sigma \circ u : S^p(r) \times S^q(\sqrt{1-r^2}) \times \boldsymbol{R}^+ \times \boldsymbol{R}^{m-p-q-1} \to S^{m+1}$ defines an immersed hypersurface in S^{m+1} without umbilics, which is denoted in [6] by $CSS(p, q, r)$. By a direct calculation, one easily finds that $CSS(p, q, r)$ has exactly three distinct Möbius principal curvatures and has parallel Möbius second fundamental form, see [6] for details. Then it follows from the arguments in the main theorem of [6] that $CCS(p, q, r)$ also has a parallel Blaschke tensor.

EXAMPLE 3.2. For any integers m and k_1 satisfying $m \geq 3$ and $2 \leq k_1 \leq m-1$, let $\tilde{y}_1 : M_1 \to S^{k_1+1}(r) \subset \boldsymbol{R}^{k_1+2}$ be an immersed minimal hypersurface without umbilics and with constant scalar curvature

$$\tilde{S}_1 = \frac{mk_1(k_1-1) - (m-1)r^2}{mr^2}\,, \tag{3.1}$$

and

$$\tilde{y} = (\tilde{y}_0, \tilde{y}_2) : H^{m-k_1}(-1/r^2) \to \boldsymbol{R}_1^{m-k_1+1} \tag{3.2}$$

be the canonical embedding. Set

$$\tilde{M}^m = M_1 \times H^{m-k_1}(-1/r^2)\,, \quad \tilde{Y} = (\tilde{y}_0, \tilde{y}_1, \tilde{y}_2)\,. \tag{3.3}$$

Then $\tilde{Y} : \tilde{M}^m \to \boldsymbol{R}_1^{m+3}$ is an immersion satisfying $\langle \tilde{Y}, \tilde{Y} \rangle_1 = 0$ and has the induced Riemannian metric

$$g = \langle d\tilde{Y}, d\tilde{Y} \rangle_1 = -d\tilde{y}_0^2 + d\tilde{y}_1^2 + d\tilde{y}_2^2\,.$$

Obviously,

$$(\tilde{M}^m, g) = (M_1, d\tilde{y}_1^2) \times (H^{m-k_1}(-1/r^2), \langle d\tilde{y}, d\tilde{y} \rangle_1) \tag{3.4}$$

as a Riemannian manifold. Define

$$\tilde{x}_1 = \frac{\tilde{y}_1}{\tilde{y}_0}\,, \quad \tilde{x}_2 = \frac{\tilde{y}_2}{\tilde{y}_0}\,, \quad \tilde{x} = (\tilde{x}_1, \tilde{x}_2)\,. \tag{3.5}$$

Then $\tilde{x}^2 = 1$ and $\tilde{x} : M^m \to S^{m+1}$ defines an immersed hypersurface without umbilics and has $\tilde{Y}$ as its Möbius position vector, see (3.7). Clearly, we have

$$d\tilde{x} = -\frac{d\tilde{y}_0}{\tilde{y}_0^2}(\tilde{y}_1, \tilde{y}_2) + \frac{1}{\tilde{y}_0} d(\tilde{y}_1, \tilde{y}_2)\,. \tag{3.6}$$

Therefore, the induced metric $\tilde{g} = d\tilde{x} \cdot d\tilde{x}$ is related to g by

$$\begin{aligned} \tilde{g} &= \tilde{y}_0^{-4} d\tilde{y}_0^2(\tilde{y}_1^2 + \tilde{y}_2^2) + \tilde{y}_0^{-2}(d\tilde{y}_1^2 + d\tilde{y}_2^2) - 2\tilde{y}_0^{-3} d\tilde{y}_0(\tilde{y}_1\, d\tilde{y}_1 + \tilde{y}_2\, d\tilde{y}_2) \\ &= \tilde{y}_0^{-2} d\tilde{y}_0^2 + \tilde{y}^{-2}(d\tilde{y}_1^2 + d\tilde{y}_2^2) - 2\tilde{y}_0^{-2} d\tilde{y}_0^2 \\ &= \tilde{y}_0^{-2}(-d\tilde{y}_0^2 + d\tilde{y}_1^2 + d\tilde{y}_2^2) = \tilde{y}_0^{-2} g\,. \end{aligned} \tag{3.7}$$

If $\tilde{n}_1$ is the unit normal vector field of $\tilde{y}_1$ in $S^{k_1+1}(r) \subset \boldsymbol{R}^{k_1+2}$, then $\tilde{n} = (\tilde{n}_1, 0) \in \boldsymbol{R}^{m+2}$ is a unit normal vector field of $\tilde{x}$. Consequently, by (3.6), the second fundamental form $\tilde{h}$ of $\tilde{x}$ is related to the second fundamental form h of $\tilde{y}_1$ as

$$\tilde{h} = -d\tilde{n} \cdot d\tilde{x} = -\tilde{y}_0^{-1}(d\tilde{n}_1 \cdot d\tilde{y}_1) = \tilde{y}_0^{-1} h\,. \tag{3.8}$$

Let $\{E_i\,;\, 1 \le i \le k_1\}$ (resp. $\{E_i\,;\, k_1+1 \le i \le m\}$) be a local orthonormal frame field on $(M_1, d\tilde{y}_1^2)$ (resp. on $H^{m-k_1}(-1/r^2)$). Then $\{E_i\,;\, 1 \le i \le m\}$ gives a local orthonormal frame field on $(\tilde{M}^m, g)$. Put $e_i = \tilde{y}_0 E_i$, $i = 1, \dots, m$. Then $\{e_i\,;\, 1 \le i \le m\}$ yields a local orthonormal frame field on $(\tilde{M}^m, \tilde{g})$. Thus we obtain

$$\tilde{h}_{ij} = \begin{cases} \tilde{h}(e_i, e_j) = \tilde{y}_0^2 \tilde{h}(E_i, E_j) = \tilde{y}_0 h(E_i, E_j) = \tilde{y}_0 h_{ij}\,, & \text{when } 1 \le i, j \le k_1\,, \\ 0\,, & \text{when } i > k_1 \text{ or } j > k_1\,, \end{cases} \tag{3.9}$$

which implies, by the minimality of $\tilde{y}_1$, that the mean curvature $\tilde{H}$ of $\tilde{x}$ vanishes. Therefore, by definition, the Möbius factor $\tilde{\rho}$ of $\tilde{x}$ is determined by

$$\tilde{\rho}^2 = \frac{m}{m-1}\biggl(\sum_{i,j} \tilde{h}_{ij}^2 - m|\tilde{H}|^2\biggr) = \frac{m}{m-1}\tilde{y}_0^2 \sum_{i,j=1}^{k_1} h_{ij}^2 = \tilde{y}_0^2\,,$$

where we used the Gauss equation and (3.1). Hence, $\tilde{\rho} = \tilde{y}_0$ and thus $\tilde{Y}$ is the Möbius position of $\tilde{x}$. Consequently, the Möbius metric of $\tilde{x}$ is defined by $\langle d\tilde{Y}, d\tilde{Y}\rangle_1 = g$. Furthermore, the Möbius second fundamental form of $\tilde{x}$ is given by

$$\tilde{B} = \tilde{\rho}^{-1} \sum (\tilde{h}_{ij} - \tilde{H}\delta_{ij})\omega^i \omega^j = \sum_{i,j=1}^{k_1} h_{ij}\omega^i\omega^j\,, \tag{3.10}$$

where $\{\omega^i\}$ is the local coframe field on M^m dual to $\{E_i\}$.

On the other hand, by (3.4) and the Gauss equations of $\tilde{y}_1$ and $\tilde{y}$, one finds that the Ricci tensor of g is given as follows:

$$R_{ij} = \frac{k_1 - 1}{r^2}\delta_{ij} - \sum_{k=1}^{k_1} h_{ik}h_{kj}\,, \quad \text{if } 1 \le i,\ j \le k_1\,, \tag{3.11}$$

$$= -\frac{m - k_1 - 1}{r^2}\delta_{ij}\,, \quad \text{if } k_1 + 1 \le i,\ j \le m\,, \tag{3.12}$$

$$R_{ij} = 0\,, \quad \text{if } 1 \le i \le k_1,\ k_1 + 1 \le j \le m \quad \text{or} \quad k_1 + 1 \le i \le m,\ 1 \le j \le k_1\,, \tag{3.13}$$

which implies that the normalized scalar curvature of g is given by

$$\kappa = \frac{m(k_1(k_1-1) - (m-k_1)(m-k_1-1)) - (m-1)r^2}{m^2(m-1)r^2}\,. \tag{3.14}$$

Thus, we have

$$\frac{1}{2m}(1 + m^2\kappa) = \frac{k_1(k_1-1) - (m-k_1)(m-k_1-1)}{2(m-1)r^2}\,. \tag{3.15}$$

Since $m \geq 3$, it follows from (2.19) and (3.10)–(3.15) that the Blaschke tensor of $\tilde{x}$ is given by $A = \sum A_{ij}\omega^i\omega^j$, where

$$A_{ij} = \frac{1}{2r^2}\delta_{ij}\,, \quad \text{if } 1 \leq i, j \leq k_1\,, \tag{3.16}$$

$$A_{ij} = -\frac{1}{2r^2}\delta_{ij}\,, \quad \text{if } k_1 + 1 \leq i, j \leq m\,, \tag{3.17}$$

$$A_{ij} = 0, \quad \text{if } 1 \leq i \leq k_1,\ k_1 + 1 \leq j \leq m \quad \text{or} \quad k_1 + 1 \leq i \leq m,\ 1 \leq j \leq k_1\,. \tag{3.18}$$

Clearly, A is parallel and has two distinct eigenvalues

$$\lambda_1 = -\lambda_2 = \frac{1}{2r^2}. \tag{3.19}$$

EXAMPLE 3.3. For any integers m and k_1 satisfying $m \geq 3$ and $2 \leq k_1 \leq m - 1$, let $\tilde{y} = (\tilde{y}_0, \tilde{y}_1) : M_1 \to H^{k_1+1}\left(-1/r^2\right) \subset \boldsymbol{R}_1^{k_1+2}$ be an immersed minimal hypersurface without umbilics and with constant scalar curvature

$$\tilde{S}_1 = -\frac{mk_1(k_1 - 1) + (m - 1)r^2}{mr^2}\,, \tag{3.20}$$

and

$$\tilde{y}_2 : S^{m-k_1}(r) \to \boldsymbol{R}^{m-k_1+1} \tag{3.21}$$

be the canonical embedding. Set

$$\tilde{M}^m = M_1 \times S^{m-k_1}(r), \quad \tilde{Y} = (\tilde{y}_0, \tilde{y}_1, \tilde{y}_2)\,. \tag{3.22}$$

Then $\tilde{Y} : M^m \to \boldsymbol{R}_1^{m+3}$ is an immersion satisfying $\langle \tilde{Y}, \tilde{Y}\rangle_1 = 0$ and has the induced Riemannian metric

$$g = \langle d\tilde{Y}, d\tilde{Y}\rangle_1 = -d\tilde{y}_0^2 + d\tilde{y}_1^2 + d\tilde{y}_2^2\,.$$

Obviously,

$$(\tilde{M}^m, g) = (M_1, \langle d\tilde{y}, d\tilde{y}\rangle_1) \times (S^{m-k_1}(r), d\tilde{y}_2^2) \tag{3.23}$$

as a Riemannian manifold. Define

$$\tilde{x}_1 = \frac{\tilde{y}_1}{\tilde{y}_0}\,, \quad \tilde{x}_2 = \frac{\tilde{y}_2}{\tilde{y}_0}\,, \quad \tilde{x} = (\tilde{x}_1, \tilde{x}_2)\,. \tag{3.24}$$

Then $\tilde{x}^2 = 1$ and $\tilde{x} : \tilde{M}^m \to S^{m+1}$ defines an immersed hypersurface without umbilics and has $\tilde{Y}$ as its Möbius position vector, see (3.26). Similarly, as in Example 3.2, we have

$$d\tilde{x} = -\frac{d\tilde{y}_0}{\tilde{y}_0^2}(\tilde{y}_1, \tilde{y}_2) + \frac{1}{\tilde{y}_0}(d\tilde{y}_1, d\tilde{y}_2)\,, \tag{3.25}$$

and the induced metric $\tilde{g} = d\tilde{x} \cdot d\tilde{x}$ is related to g by

$$\tilde{g} = \tilde{y}_0^{-2}(-d\tilde{y}_0^2 + d\tilde{y}_1^2 + d\tilde{y}_2^2) = \tilde{y}_0^{-2}g\,. \tag{3.26}$$

If $(\tilde{n}_0, \tilde{n}_1)$ is the unit normal vector field of $\tilde{y}$ in $H^{k_1+1}(-1/r^2) \subset \boldsymbol{R}_1^{m+2}$, then it is easy to see that

$$\tilde{n} = (\tilde{n}_1, 0) - \tilde{n}_0\tilde{x} \in \boldsymbol{R}^{m+2}$$

is a unit normal vector field of $\tilde{x}$. Hence, by (3.25), we have

$$\begin{aligned} d\tilde{n}\cdot d\tilde{x} &= (d\tilde{n}_1, 0)\cdot d\tilde{x} - \tilde{n}_0\, d\tilde{x}^2 \\ &= -(\tilde{y}_0^{-2}\, d\tilde{y}_0)d\tilde{n}_1\cdot\tilde{y}_1 + \tilde{y}_0^{-1}\, d\tilde{n}_1\cdot d\tilde{y}_1 - \tilde{n}_0\tilde{y}_0^{-2}(-d\tilde{y}_0^2 + d\tilde{y}_1^2 + d\tilde{y}_2^2) \\ &= \tilde{y}_0^{-1}(-d\tilde{n}_0\, d\tilde{y}_0 + d\tilde{n}_1\cdot d\tilde{y}_1) - n_0\tilde{y}_0^{-2} g\,, \end{aligned} \tag{3.27}$$

where the third equality comes from the fact that

$$-\tilde{y}_0^2 + \tilde{y}_2^2 = -1/r^2\,,\quad -\tilde{n}_0\tilde{y}_0 + \tilde{n}_1\cdot\tilde{y}_1 = -\tilde{n}_0\, d\tilde{y}_0 + \tilde{n}_1\cdot d\tilde{y}_1 = 0\,. \tag{3.28}$$

Consequently, the second fundamental form $\tilde{h}$ of $\tilde{x}$ is related to the second fundamental form h of $\tilde{y}$ and the metric g as

$$\tilde{h} = -d\tilde{n}\cdot d\tilde{x} = -\tilde{y}_0^{-1}\langle d(\tilde{n}_0, \tilde{n}_1), d\tilde{y}\rangle_1 + n_0\tilde{y}_0^{-2} g = y_0^{-1} h + n_0\tilde{y}_0^{-2} g\,. \tag{3.29}$$

Let $\{E_i\,;\, 1\le i\le k_1\}$ (resp. $\{E_i\,;\, k_1+1\le i\le m\}$) be a local orthonormal frame field on $(M_1, d\tilde{y}^2)$ (resp. on $S^{m-k_1}(r)$). Then $\{E_i\,;\, 1\le i\le m\}$ is a local orthonormal frame field on (M^m, g). Put $e_i = \tilde{y}_0 E_i$, $i = 1,\dots,m$. Then $\{e_i\,;\, 1\le i\le m\}$ is a local orthonormal frame field on $(M^m, \tilde{g})$. Thus

$$\tilde{h}_{ij} = \begin{cases} \tilde{h}(e_i, e_j) = \tilde{y}_0^2\tilde{h}(E_i, E_j) = \tilde{y}_0 h(E_i, E_j) + \tilde{n}_0 g(E_i, E_j) \\ \qquad\qquad = \tilde{y}_0 h_{ij} + \tilde{n}_0\delta_{ij}, \quad \text{when } 1\le i, j\le k_1, \\ \tilde{n}_0\delta_{ij}, \qquad\qquad\qquad\qquad \text{when } i > k_1 \text{ or } j > k_1\,, \end{cases} \tag{3.30}$$

which implies, by the minimality of $\tilde{y}_1$, that the mean curvature of $\tilde{x}$ is

$$\tilde{H} = \frac{1}{m}\sum \tilde{h}_{ii} = \frac{\tilde{y}_0}{m}\sum_{i=1}^{k_1} h_{ii} + \tilde{n}_0 = \tilde{n}_0\,. \tag{3.31}$$

Therefore, by definition, the Möbius factor $\tilde{\rho}$ of $\tilde{x}$ is determined by

$$\tilde{\rho}^2 = \frac{m}{m-1}\left(\sum_{i,j}\tilde{h}_{ij}^2 - m|\tilde{H}|^2\right) = \frac{m}{m-1}\tilde{y}_0^2\sum_{i,j} h_{ij}^2 = \tilde{y}_0^2\,,$$

where we used the Gauss equation and (3.20). Hence, $\tilde{\rho} = \tilde{y}_0$ and thus $\tilde{Y}$ is the Möbius position of $\tilde{x}$. Consequently, the Möbius metric of $\tilde{x}$ is defined by $\langle d\tilde{Y}, d\tilde{Y}\rangle_1 = g$. Furthermore, the Möbius second fundamental form of $\tilde{x}$ is given by

$$\tilde{B} = \tilde{\rho}^{-1}(\tilde{h}_{ij} - \tilde{H}\delta_{ij})\omega^i\omega^j = \sum_{i,j=1}^{k_1} h_{ij}\omega^i\omega^j\,, \tag{3.32}$$

where $\{\omega^i\}$ is the local coframe field on M^m dual to $\{E_i\}$.

On the other hand, by (3.23) and the Gauss equations of $\tilde{y}_1$ and $\tilde{y}$, one finds that the Ricci tensor of g is given as follows:

$$R_{ij} = -\frac{k_1 - 1}{r^2}\delta_{ij} - \sum_{k=1}^{k_1} h_{ik}h_{kj}\,, \quad \text{if } 1 \le i,\ j \le k_1\,, \tag{3.33}$$

$$R_{ij} = \frac{m - k_1 - 1}{r^2}\delta_{ij}\,, \quad \text{if } k_1 + 1 \le i,\ j \le m\,, \tag{3.34}$$

$$R_{ij} = 0\,, \quad \text{if } 1 \le i \le k_1, k_1 + 1 \le j \le m \quad \text{or} \quad k_1 + 1 \le i \le m,\ 1 \le j \le k_1\,, \tag{3.35}$$

which implies that the normalized scalar curvature of g is given by

$$\kappa = \frac{m((m - k_1)(m - k_1 - 1) - k_1(k_1 - 1)) - (m - 1)r^2}{m^2(m - 1)r^2}\,. \tag{3.36}$$

Thus,

$$\frac{1}{2m}(1 + m^2\kappa) = \frac{(m - k_1)(m - k_1 - 1) - k_1(k_1 - 1)}{2(m - 1)r^2}\,. \tag{3.37}$$

Since $m \ge 3$, it follows from (2.19) and (3.32)–(3.37) that the Blaschke tensor of $\tilde{x}$ is given by $A = \sum A_{ij}\omega^i\omega^j$, where

$$A_{ij} = -\frac{1}{2r^2}\delta_{ij}\,, \quad \text{if } 1 \le i,\ j \le k_1, \tag{3.38}$$

$$A_{ij} = \frac{1}{2r^2}\delta_{ij}\,, \quad \text{if } k_1 + 1 \le i,\ j \le m\,, \tag{3.39}$$

$$A_{ij} = 0\,, \quad \text{if } 1 \le i \le k_1,\ k_1 + 1 \le j \le m \quad \text{or} \quad k_1 + 1 \le i \le m,\ 1 \le j \le k_1\,, \tag{3.40}$$

which, once again, implies that A is parallel and has two distinct eigenvalues

$$\lambda_1 = -\lambda_2 = -\frac{1}{2r^2}\,. \tag{3.41}$$

4. Proof of the main theorem. To make the argument simpler, we divide the proof into several lemmas. First, we recall a theorem proved by Liu et al. [15].

THEOREM 4.1 (Liu et al. [15]). *Any Möbius isotropic submanifold immersed in S^n must be locally Möbius equivalent to one of the following immersions.*

(1) *A minimal immersion $\tilde{x} : M \to S^n$ with constant scalar curvature.*

(2) *The image under σ of a minimal immersion with constant scalar curvature.*

(3) *The image under τ of a minimal immersion with constant scalar curvature.*

Now, let $x : M^m \to S^{m+1}$ be an immersed hypersurface without umbilics.

LEMMA 4.2. *If the Blaschke tensor A is parallel, then the Möbius form Φ vanishes identically*; *in particular, by* (2.18), *B_{ijk} is symmetric for all indices i, j, k.*

PROOF. For any given point $p \in M^m$, take an orthonormal frame field $\{E_i\}$ around p with respect to the Möbius metric g, such that the corresponding components B_{ij} of the

Möbius second fundamental form B are diagonalized at p, that is, $B_{ij}(p) = B_i\delta_{ij}$. Let $\{\omega^i\}$ be the dual of $\{E_i\}$, and write

$$A = \sum A_{ij}\omega^i\omega^j\,, \quad \Phi = \sum \Phi_i\omega^i\,.$$

Then it follows from (2.17) that

$$A_{ijk} - A_{ikj} = B_{ik}\Phi_j - B_{ij}\Phi_k\,. \tag{4.1}$$

Since A is parallel, $A_{ijk} = 0$ for any i, j, k. Thus, at the given point p, we have

$$B_i(\delta_{ik}\Phi_j(p) - \delta_{ij}\Phi_k(p)) = 0\,. \tag{4.2}$$

By (2.14), there are different indices i_1, i_2 such that $B_{i_1} \neq 0$ and $B_{i_2} \neq 0$. Then for any indices i, j, we have

$$\delta_{i_1j}\Phi_i(p) - \delta_{i_1i}\Phi_j(p) = 0\,, \quad \delta_{i_2j}\Phi_i(p) - \delta_{i_2i}\Phi_j(p) = 0\,. \tag{4.3}$$

If $i = i_1$, put $j = i_2$; if $i \neq i_1$, put $j = i_1$. Then it follows from (4.3) that $\Phi_i(p) = 0$. By the arbitrariness of i and p, we obtain that $\Phi \equiv 0$. □

LEMMA 4.3. *If A is parallel, then all eigenvalues of the Blaschke tensor A of x are constant on M^m.*

PROOF. Since A is parallel, there exists, around each point, a local orthonormal frame field $\{E_i\}$ such that

$$A_{ij} = A_i\delta_{ij}\,. \tag{4.4}$$

Substitute (4.4) into the equality

$$0 = \sum A_{ijk}\omega^k = dA_{ij} - A_{kj}\omega_i^k - A_{ik}\omega_j^k\,,$$

where $\{\omega^i\}$ is the dual of $\{E_i\}$ and ω_j^i are the connection forms of the Levi-Civita connection of g. Then it follows easily that

$$dA_i\delta_{ij} - (A_i - A_j)\omega_j^i = 0\,, \tag{4.5}$$

which implies that

$$dA_i = 0\,, \quad \omega_j^i = 0 \quad \text{in the case that } A_i \neq A_j\,. \tag{4.6}$$

□

LEMMA 4.4. *If* (4.4) *holds, then $B_{ij} = 0$ in the case that $A_i \neq A_j$.*

PROOF. It follows from (2.16) and Lemma 4.2 that $\sum B_{ik}A_{kj} - A_{ik}B_{kj} = \Phi_{ij} - \Phi_{ji} = 0$. Since (4.4) holds, we have $B_{ij}(A_j - A_i) = 0$. □

Now, let t be the number of the distinct eigenvalues of A, and $\lambda_1, \dots, \lambda_t$ denote the distinct eigenvalues of A. Fix a suitably chosen orthonormal frame field $\{E_i\}$ for which the matrix (A_{ij}) can be written as

$$(A_{ij}) = \mathrm{Diag}(\underbrace{\lambda_1, \dots, \lambda_1}_{k_1}, \underbrace{\lambda_2, \dots, \lambda_2}_{k_2}, \dots, \underbrace{\lambda_t, \dots, \lambda_t}_{k_t})\,, \tag{4.7}$$

or, equivalently,

$$A_1 = \cdots = A_{k_1} = \lambda_1, \ldots, A_{k_1+\cdots+k_{t-1}+1} = \cdots = A_m = \lambda_t \,. \tag{4.8}$$

LEMMA 4.5. *Suppose that $t \geq 3$. If, with respect to an orthonormal frame field $\{E_i\}$,* (4.7) *holds and at a point p, $B_{ij} = B_i \delta_{ij}$, then*

$$B_i = B_j \quad \textit{in the case that } A_i = A_j \,. \tag{4.9}$$

PROOF. By Lemma 4.3, for any i, j satisfying $A_i \neq A_j$, we have $\omega^i_j = 0$. Differentiating this equation, we obtain from (2.15) that

$$0 = R_{ijji} = B_{ii}B_{jj} - B_{ij}^2 + A_{ii} - A_{ij}\delta_{ij} + A_{jj} - A_{ij}\delta_{ij} \,.$$

Thus at p, it holds that

$$B_i B_j + A_i + A_j = 0 \,. \tag{4.10}$$

If there exist indices i, j such that $A_i = A_j$ but $B_i \neq B_j$, then for all k satisfying $A_k \neq A_i$, we have

$$B_i B_k + A_i + A_k = 0 \,, \quad B_j B_k + A_j + A_k = 0 \,. \tag{4.11}$$

It follows from (4.11) that $(B_i - B_j)B_k = 0$, which implies that $B_k = 0$. Thus, by (4.11), we obtain $A_k = -A_i = -A_j$. This implies that $t = 2$, contradicting the assumption of the lemma. □

COROLLARY 4.6. *If $t \geq 3$, then there exists an orthonormal frame field $\{E_i\}$ such that*

$$A_{ij} = A_i \delta_{ij} \,, \quad B_{ij} = B_i \delta_{ij} \,. \tag{4.12}$$

Furthermore, if (4.7) *holds, then*

$$(B_{ij}) = \mathrm{Diag}(\underbrace{\mu_1, \ldots, \mu_1}_{k_1}, \underbrace{\mu_2, \ldots, \mu_2}_{k_2}, \ldots, \underbrace{\mu_t, \ldots, \mu_t}_{k_t}) \,, \tag{4.13}$$

that is

$$B_1 = \cdots = B_{k_1} = \mu_1, \ldots, B_{k_1+\cdots+k_{t-1}+1} = \cdots = B_m = \mu_t \,, \tag{4.14}$$

where $\mu_1, \ldots, \mu_t$ are not necessarily different from each other.

PROOF. Since A is parallel, we can find a local orthonormal frame field $\{E_i\}$, such that (4.7) or, equivalently, (4.8) holds. It suffices to show that, at any point, the component matrix (B_{ij}) of B with respect to $\{E_i\}$ is diagonal. Note that $k_1, \ldots, k_t$ are the multiplicities of the eigenvalues $\lambda_1, \ldots, \lambda_t$, respectively. By Lemma 4.4, we can write

$$(B_{ij}) = \mathrm{Diag}(B_{(1)}, \ldots, B_{(t)}) \,,$$

where $B_{(1)}, \ldots, B_{(t)}$ are square matrices of orders $k_1, \ldots, k_t$, respectively. For any point p, we can choose a suitable orthogonal matrix T of the form

$$T = \mathrm{Diag}(T_{(1)}, \ldots, T_{(t)}) \,,$$

with $T_{(1)}, \ldots, T_{(t)}$ being orthogonal matrices of orders $k_1, \ldots, k_t$, such that

$$T \cdot (B_{ij}(p)) \cdot T^{-1} = \mathrm{Diag}(B_1, \ldots, B_m)\,,$$

where $B_1, \ldots, B_m$ are the eigenvalues of tensor B at p. It then follows from Lemma 4.5 that

$$B_1 = \cdots = B_{k_1} := \mu_1, \ldots, B_{k_1+\cdots+k_{t-1}+1} = \cdots = B_m := \mu_t\,.$$

Hence, we have

$$\begin{aligned} T_{(1)} B_{(1)}(p) T_{(1)}^{-1} &= \mathrm{Diag}(\mu_1, \ldots, \mu_1)\,, \\ &\vdots \\ T_{(t)} B_{(t)}(p) T_{(t)}^{-1} &= \mathrm{Diag}(\mu_t, \ldots, \mu_t)\,. \end{aligned}$$

Therefore, we have

$$\begin{aligned} B_{(1)}(p) &= T_{(1)}^{-1} \cdot \mathrm{Diag}(\mu_1, \ldots, \mu_1) \cdot T_{(1)} = \mathrm{Diag}(\mu_1, \ldots, \mu_1)\,, \\ &\vdots \\ B_{(t)}(p) &= \mathrm{Diag}(\mu_t, \ldots, \mu_t)\,, \end{aligned}$$

that is,

$$(B_{ij}(p)) = \mathrm{Diag}(\mu_1, \ldots, \mu_1, \ldots, \mu_t, \ldots, \mu_t)\,. \qquad \square$$

LEMMA 4.7. *If $t \geq 3$, then all the Möbius principal curvatures $\mu_1, \ldots, \mu_t$ of x are constant, and hence x is Möbius isoparametric.*

PROOF. It suffices to show that μ_1 is constant. To this end, choose a frame field $\{E_i\}$ such that (4.7) and (4.13) hold. Note that, by (4.6), when $1 \leq i \leq k_1$ and $j > k_1$, we have

$$\sum B_{ijk}\omega^k = dB_{ij} - \sum B_{kj}\omega_i^k - \sum B_{ik}\omega_j^k = 0\,, \tag{4.15}$$

which implies that $B_{ijk} = 0$.

Then, by the symmetry of B_{ijk} (see Lemma 4.2), we see that $B_{ijk} = 0$, in the case that two indices in i, j, k are less than or equal to k_1 with the other index larger than k_1, or one index in i, j, k is less than or equal to k_1 with the other two indices larger than k_1. In particular, for any i, j satisfying $1 \leq i, j \leq k_1$,

$$\sum_{k=1}^{k_1} B_{ijk}\omega^k = dB_{ij} - \sum B_{kj}\omega_i^k - \sum B_{ik}\omega_j^k = dB_i\delta_{ij} - B_j\omega_i^j - B_i\omega_j^i\,.$$

Thus, by putting $j = i$, one obtains

$$\sum_{k=1}^{k_1} B_{iik}\omega^k = d\mu_1\,, \tag{4.16}$$

which implies that

$$E_k(\mu_1) = 0, \quad k_1 + 1 \leq k \leq m\,. \tag{4.17}$$

Similarly,

$$E_i(B_j) = 0, \quad 1 \le i \le k_1,\ k_1 + 1 \le j \le m\,. \tag{4.18}$$

On the other hand, it is easily seen from (4.10) that

$$\mu_1 B_j + \lambda_1 + A_j = 0\,, \quad k_1 + 1 \le j \le m\,, \tag{4.19}$$

hold identically. Differentiating (4.19) in the direction of E_k, $1 \le k \le k_1$, and using (4.18), we obtain

$$E_k(\mu_1) B_j = 0\,, \quad 1 \le k \le k_1,\ k_1 + 1 \le j \le m\,. \tag{4.20}$$

Since $t \ge 3$, one finds easily that there exists some index j such that $k_1 + 1 \le j \le m$ and $B_j \ne 0$. Therefore, $E_k(\mu_1) = 0$ for $1 \le k \le k_1$. This together with (4.17) implies that μ_1 is a constant. □

COROLLARY 4.8. *If $t \ge 3$, then $t = 3$ and B is parallel.*

PROOF. Indeed, the conclusion that B is parallel comes from (4.6), Corollary 4.6 and Lemma 4.7.

If $t > 3$, then there exist at least four indices i_1, i_2, i_3, i_4, such that $A_{i_1}, A_{i_2}, A_{i_3}, A_{i_4}$ are distinct each other. Then it follows from (4.10) that

$$B_{i_1} B_{i_2} + A_{i_1} + A_{i_2} = 0\,, \quad B_{i_3} B_{i_4} + A_{i_3} + A_{i_4} = 0\,,$$
$$B_{i_1} B_{i_3} + A_{i_1} + A_{i_3} = 0\,, \quad B_{i_2} B_{i_4} + A_{i_2} + A_{i_4} = 0\,.$$

Consequently, we obtain $(A_{i_1} - A_{i_4})(A_{i_2} - A_{i_3}) = 0$, a contradiction. □

LEMMA 4.9. *If $t \le 2$ and B is not parallel, then one of the following cases holds.*

(1) *$t = 1$ and x is Möbius isotropic.*

(2) *$t = 2$, $\lambda_1 + \lambda_2 = 0$ and $B_i = 0$ either for all $1 \le i \le k_1$ or for all $k_1 + 1 \le i \le m$.*

PROOF. It suffices to consider the case that $t = 2$. For any point $p \in M^m$, we can find an orthonormal frame field $\{E_i\}$ such that (4.12) holds at p.

By (4.6), we see that

$$\omega^i_j = 0\,, \quad 1 \le i \le k_1\,,\ \ k_1 + 1 \le j \le m\,, \tag{4.21}$$

hold identically. Taking exterior differentiation of (4.21) and making use of (2.15), we find that, at p

$$B_i B_j + A_i + A_j = 0\,, \quad 1 \le i \le k_1\,,\ \ k_1 + 1 \le j \le m\,. \tag{4.22}$$

If there exist one pair of indices i_0, j_0 satisfying $1 \le i_0 \le k_1$, $k_1 + 1 \le j_0 \le m$ such that $B_{i_0} \ne 0$ and $B_{j_0} \ne 0$, then for each index i satisfying $1 \le i \le k_1$, we obtain

$$B_{i_0} B_{j_0} + A_{i_0} + A_{j_0} = 0\,, \quad B_i B_{j_0} + A_i + A_{j_0} = 0\,,$$

from which it follows that $(B_i - B_{i_0}) B_{j_0} = 0$ or, equivalently,

$$B_i = B_{i_0}\,, \quad 1 \le i \le k_1\,. \tag{4.23}$$

Similarly, we obtain

$$B_j = B_{j_0}\,, \quad k_1 \le j \le m\,. \tag{4.24}$$

Consequently, (4.13) also holds in the case that $t = 2$. Now, an argument similar to that in the proof of Lemma 4.7 shows that the principal curvatures B_i are all constant. Therefore, B is parallel by (4.21), contradicting the assumption. Thus, either $B_i = 0$ for all indices i satisfying $1 \le i \le k_1$ or $B_j = 0$ for all indices j satisfying $k_1 \le j \le m$. In both cases we have, by (4.22), $\lambda_1 + \lambda_2 = 0$. □

PROOF OF THEOREM 1.2. By Theorems 1.1 and 4.1, it suffices to consider the case that x neither is Möbius isotropic nor has parallel Möbius second fundamental form. Hence, from the lemmas proved in this section, we can suppose without loss of generality that

$$t = 2\,, \quad \lambda_1 = -\lambda_2 = \lambda \ne 0\,, \quad B_{k_1+1} = \cdots = B_m = 0\,. \tag{4.25}$$

Since $\sum B_i = 0$ and $\sum B_i^2 = (m-1)/m$, one sees easily that $m \ge 3$. Since A is parallel, the tangent bundle TM^m of M^m has a decomposition $TM^m = V_1 \oplus V_2$, where V_1 and V_2 are the eigenspaces of A corresponding to the eigenvalues $\lambda_1 = \lambda$ and $\lambda_2 = -\lambda$, respectively.

Let $\{E_i\,;\, 1 \le i \le k_1\}$ and $\{E_j\,;\, k_1 \le j \le m\}$ be orthonormal frame fields for subbundles V_1 and V_2, respectively. Then $\{E_i\,;\, 1 \le i \le m\}$ is an orthonormal frame field on M^m with respect to the Möbius metric g. Then (4.21) implies that both V_1 and V_2 are integrable, and thus the Riemannian manifold (M^m, g) can be locally decomposed into a direct product of two Riemannian manifolds (M_1, g_1) and (M_2, g_2), that is, as a Riemannian manifold, locally

$$(M^m, g) = (M_1, g_1) \times (M_2, g_2)\,. \tag{4.26}$$

It follows from (2.15), (4.7), (4.25) and (4.26) that the Riemannian curvature tensors of (M_1, g_1) and (M_2, g_2) have the following components, respectively,

$$R_{ijkl} = 2\lambda(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}) + (B_{il}B_{jk} - B_{ik}B_{jl})\,, \quad 1 \le i, j, k, l \le k_1\,, \tag{4.27}$$

$$R_{ijkl} = -2\lambda(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl})\,, \quad k_1 + 1 \le i, j, k, l \le m\,. \tag{4.28}$$

Thus (M_2, g_2) is of constant sectional curvature -2λ.

Next we consider the following cases separately.

Case (1): $\lambda > 0$. In this case, set $r = (2\lambda)^{-1/2}$. Then (M_2, g_2) can be locally identified with $H^{m-k_1}(-1/r^2)$. Let $\tilde{y} = (\tilde{y}_0, \tilde{y}_2) : H^{m-k_1}(-1/r^2) \to \boldsymbol{R}_1^{m-k_1+1}$ be the canonical embedding.

Since $h = \sum_{i,j=1}^{k_1} B_{ij}\omega^i\omega^j$ is a Codazzi tensor on (M_1, g_1), it follows from (4.27) that there exists a minimal immersed hypersurface

$$\tilde{y}_1 : (M_1, g_1) \to S^{k_1+1}(r) \subset \boldsymbol{R}^{k_1+2}\,, \quad 2 \le k_1 \le m-1\,, \tag{4.29}$$

which has h as its second fundamental form. Clearly, $\tilde{y}_1$ is umbilic free and has constant scalar curvature

$$S_1 = \frac{mk_1(k_1-1) - (m-1)r^2}{mr^2}\,,$$

and M^m can be locally identified with $\tilde{M}^m = (M_1, g_1) \times H^{m-k_1}(-1/r^2)$.

Define $\tilde{x}_1 = \tilde{y}_1/\tilde{y}_0$, $\tilde{x}_2 = \tilde{y}_2/\tilde{y}_0$ and $\tilde{x} = (\tilde{x}_1, \tilde{x}_2)$. Then, by the discussion in Example 3.2, $\tilde{x} : \tilde{M}^m \to S^{m+1}$ yields an immersed hypersurface with the given g and B as its Möbius metric and Möbius second fundamental form, respectively. Therefore, by Theorem 2.3, x is Möbius equivalent to $\tilde{x}$.

Case (2): $\lambda < 0$. In this case, set $r = (-2\lambda)^{-1/2}$, then (M_2, g_2) can be locally identified with $S^{m-k_1}(r)$. Let $\tilde{y}_2 : S^{m-k_1}(r) \to \boldsymbol{R}^{m-k_1+1}$ be the canonical embedding.

Since $h = \sum_{i,j=1}^{k_1} B_{ij}\omega^i\omega^j$ is a Codazzi tensor on (M_1, g_1), it follows from (4.27) that there exists a minimal immersed hypersurface

$$\tilde{y} = (\tilde{y}_0, \tilde{y}_1) : (M_1, g_1) \to H^{k_1+1}(-1/r^2) \subset \boldsymbol{R}_1^{k_1+2}, \quad 2 \le k_1 \le m-1, \tag{4.30}$$

which has h as its second fundamental form. Clearly, $\tilde{y}$ is umbilic free and has constant scalar curvature

$$S_1 = -\frac{mk_1(k_1-1) - (m-1)r^2}{mr^2},$$

and M^m can be locally identified with $\tilde{M}^m = (M_1, g_1) \times H^{m-k_1}(-1/r^2)$.

Define $\tilde{x}_1 = \tilde{y}_1/\tilde{y}_0$, $\tilde{x}_2 = \tilde{y}_2/\tilde{y}_0$ and $\tilde{x} = (\tilde{x}_1, \tilde{x}_2)$. Then, by the discussion in Example 3.3, $\tilde{x} : \tilde{M}^m \to S^{m+1}$ defines an immersed hypersurface with the given g and B as its Möbius metric and Möbius second fundamental form, respectively. Therefore, by Theorem 2.3, x is Möbius equivalent to $\tilde{x}$. $\square$

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