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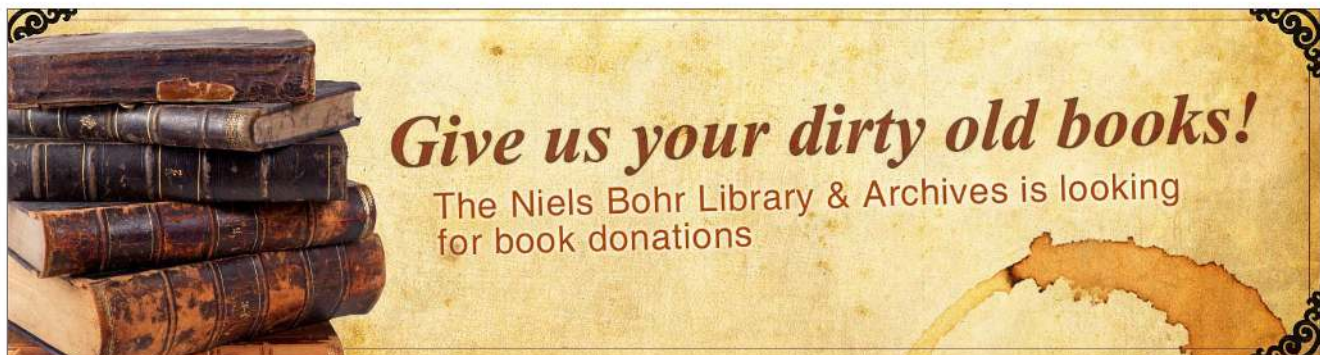
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A classification of near-horizon geometries of extremal vacuum black holes

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We consider the near-horizon geometries of extremal, rotating black hole solutions of the vacuum Einstein equations, including a negative cosmological constant, in four and five dimensions. We assume the existence of one rotational symmetry in four dimensions (4D), two commuting rotational symmetries in five dimensions (5D), and in both cases nontoroidal horizon topology. In 4D we determine the most general near-horizon geometry of such a black hole and prove it is the same as the near-horizon limit of the extremal Kerr-AdS₄ black hole. In 5D, without a cosmological constant, we determine all possible near-horizon geometries of such black holes. We prove that the only possibilities are one family with a topologically $S^1 \times S^2$ horizon and two distinct families with topologically S^3 horizons. The $S^1 \times S^2$ family contains the near-horizon limit of the boosted extremal Kerr string and the extremal vacuum black ring. The first topologically spherical case is identical to the near-horizon limit of two different black hole solutions: the extremal Myers–Perry black hole and the slowly rotating extremal Kaluza–Klein (KK) black hole. The second topologically spherical case contains the near-horizon limit of the fast rotating extremal KK black hole. Finally, in 5D with a negative cosmological constant, we reduce the problem to solving a sixth-order nonlinear ordinary differential equation of one function. This allows us to recover the near-horizon limit of the known, topologically S^3 , extremal rotating AdS₅ black hole. Further, we construct an approximate solution corresponding to the near-horizon geometry of a small, extremal AdS₅ black ring. © 2009 American Institute of Physics.
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I. INTRODUCTION

Asymptotically flat and anti-de Sitter (AdS) black hole solutions in four and five dimensions are of interest in the context of string theory and AdS/CFT, respectively, as they provide an effective description of the strong coupling dynamics in certain sectors of the dual conformal field theories. Focusing on supersymmetric states often allows one to evade the problem of performing computations at strong coupling, as such states tend to be protected. This provides the opportunity to reproduce the Hawking–Bekenstein entropy of the black hole in question from a microstate counting in the weakly coupled field theory.

In recent years, great progress has been made in the construction of supersymmetric black holes both in ungauged supergravity^{1–9} and gauged supergravity,^{10–16} largely due to systematic classification techniques available for BPS solutions.^{1,11,17} As is well known, supersymmetric black holes are necessarily extremal. Curiously, recent work on the attractor mechanism (see Ref. 18 for a comprehensive review) has revealed that, in fact, it may be extremality rather than

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supersymmetry which is responsible for the success of the entropy counting of black holes in flat space.^{19,20} In the case of extremal but nonsupersymmetric black holes, the attractor mechanism was established upon the assumption that the near-horizon geometry of an extremal black hole must have an $SO(2, 1)$ symmetry.²¹ This assertion was proven in four and five dimensions in Ref. 22, in a large class of theories, under the assumption that the black hole is axisymmetric in 4D and has two commuting rotational Killing vector fields in 5D (see also Ref. 23 for generalizations for $D > 5$).²⁴ Indeed, there has been recent success in counting the microstates of extremal, nonsupersymmetric black holes in four and five dimensions.^{25–29}

The classification problem of stationary black holes in higher dimensions is also of intrinsic interest.³⁰ From this point of view supersymmetry is merely a technical tool allowing one to study the classification problem in a more constrained setting. Similarly, extremality may also be used as a simplifying assumption. This is because any extremal black hole admits a near-horizon limit, a geometry in its own right which solves the same field equations.^{2,22} The advantage of this is that determining and thus classifying near-horizon geometries is a technically simpler problem: it becomes a $D-2$ dimensional problem of Riemannian geometry on a compact space (i.e., spatial sections of the horizon). Given a classification of near-horizon geometries in some theory, one can deduce certain information about what black hole solutions are allowed. In particular, it can allow one to rule out the existence of extremal black holes with a certain horizon topology. Furthermore, this analysis determines not only the possible horizon topologies but also determines their geometry explicitly. The one disadvantage of this method is that the existence of a near-horizon geometry does not guarantee the existence of an extremal black hole solution with that near-horizon geometry.

Previously, certain classifications of near-horizon geometries have been achieved in a variety of ungauged supergravities,^{2,31–33} where the combined use of supersymmetry and the near-horizon limit is particularly fruitful. The main success of this is it allowed the proof of a uniqueness theorem for asymptotically flat, topologically spherical, supersymmetric black holes in five dimensional ungauged supergravity: the only solution turns out to be BMPV.^{2,31} In the gauged case the near-horizon equations are more complicated and a classification of near-horizon geometries was achieved using an extra assumption: the black hole admits two commuting rotational symmetries.^{15,16} This ruled out the existence of supersymmetric AdS_5 black rings with these symmetries.

In this work, we consider the classification of near-horizon geometries in a setting without supersymmetry in four and five dimensions. For simplicity we will consider near-horizon geometries of extremal black hole solutions to Einstein's vacuum equations and allow for a negative cosmological constant. As a result, we can consider asymptotically flat [and Kaluza–Klein (KK) in 5D] and AdS black holes, respectively.³⁴ In the pure vacuum in 5D there are a number of known examples of extremal black holes and their associated near-horizon geometries: the extremal boosted Kerr string, the extremal Myers–Perry black hole,³⁵ the extremal black ring,³⁶ and two different extremal limits of the KK black hole^{37,38} (often termed “slow” and “fast” rotating).³⁹ In contrast, in the presence of a negative cosmological constant only one example is known: the extremal limit of the topologically spherical, rotating AdS_5 black hole found in Ref. 40. Indeed, an interesting open question concerns the existence of asymptotically AdS_5 black rings. No such solutions are currently known. Furthermore, the systematic solution generating techniques available for vacuum gravity,^{41,42} are not available in the presence of a cosmological constant. Thus it appears that a near-horizon analysis is one of the few systematic techniques available to obtain information on the existence of AdS_5 black rings (at least in the extremal sector).

We use the assumption of axisymmetry in 4D and two commuting rotational symmetries in 5D, which means the near-horizon geometry is cohomogeneity-1 in both cases; therefore everything reduces to ordinary differential equations (ODEs). Our analysis will employ both local and global considerations (i.e., compactness of spatial sections of the horizon). The global arguments allow one to avoid solving the differential equations generally, thus simplifying the problem. The main results of this paper may now be stated.

Theorem 1: Consider a four-dimensional nonstatic and axisymmetric near-horizon geometry,

with a compact horizon section of nontoroidal topology, satisfying $R_{\mu\nu} = \Lambda g_{\mu\nu}$ for $\Lambda \leq 0$. If $\Lambda = 0$ then it must be the near-horizon limit of the extremal Kerr black hole. If $\Lambda < 0$ it must be the near-horizon limit of the extremal Kerr-AdS₄ black hole.

Remarks:

- For $\Lambda = 0$ the same result has been proven in Ref. 43, and again in the context of isolated horizons in Ref. 44. Their analysis included a Maxwell field (in which case the result is the near-horizon geometry of extremal Kerr–Newman).⁴⁵
- Static near-horizon geometries of this form have been considered in Ref. 47. For $\Lambda = 0$ it was shown that the near-horizon geometry is a direct product of two-dimensional (2D) Minkowski space and a flat T^2 . However, in the context of black holes this may be excluded by the horizon topology theorems.^{48,49} For $\Lambda < 0$ it was shown that it is a direct product of AdS₂ and a compact Einstein space of negative curvature: this is incompatible with our assumption of axisymmetry.
- Topological censorship⁴⁹ implies that for asymptotically flat and globally AdS₄ black holes the horizon section cannot have toroidal topology. Thus our result implies that the near-horizon geometry of any asymptotically flat (or globally AdS₄), Ricci flat (or negative curvature Einstein space), stationary and axisymmetric extremal black hole is given by the near-horizon limit of Kerr (or Kerr-AdS₄).
- Note that for four dimensional *nonextremal* rotating black holes, axisymmetry has been proven to be a consequence of stationarity [even in AdS (Ref. 54)]. Therefore, it is reasonable to expect the same to occur for extremal black holes and thus their near-horizon limits.⁵⁰

Theorem 2: Consider a five-dimensional nonstatic near-horizon geometry, with a compact horizon section \mathcal{H} of nontoroidal topology, and a $U(1)^2$ isometry group with spacelike orbits, satisfying $R_{\mu\nu} = 0$. Then it must be contained in one of the following three families: a three parameter family with $\mathcal{H} = S^1 \times S^2$, given by Eq. (3); a two parameter family with $\mathcal{H} = S^3$ (case A) given by (4) and (5); a three parameter family with $\mathcal{H} = S^3$ (case B) given by (6). See the main results section II B for more details and explicit metrics.

Remarks:

- Static vacuum near-horizon geometries were considered in Ref. 47. It was shown that they must be the direct product of 2D Minkowski space and a flat compact three-dimensional space. However, in the context of black holes, these may be ruled out by the black hole horizon topology theorem.^{52,53}
- $S^1 \times S^2$ case: In a region of parameter space it is isometric to the near-horizon limit of extremal boosted Kerr string. Further, for a particular value of the boost parameter (i.e., such that the string is tensionless), it is isometric to the near-horizon limit of the asymptotically flat extremal vacuum black ring.²²
- S^3 case A: This is isometric to the near-horizon limit of two different black holes: extremal Myers–Perry (which must have two nonzero angular momenta J_i) and the slow rotating extremal KK black hole ($G_4 J < PQ$). In a special case (corresponding to $J_1 = \pm J_2$ and $J = 0$, respectively), the rotational symmetry group enhances to $SU(2) \times U(1)$ [or $SO(3) \times U(1)$] and the near-horizon geometry is a homogeneous space.
- S^3 case B: In a region of parameter space, it is isometric to the near-horizon limit of the fast rotating extremal KK black hole ($G_4 J > PQ$). This solution always has total rotational symmetry group $U(1)^2$ (i.e., it never gets enhanced as in case A).
- Any extremal vacuum black hole in 5D with $R \times U(1)^2$ isometry group and compact horizon sections of nontoroidal topology must have a near-horizon geometry contained in one of the three families above. Note that toroidal horizon topology is not allowed by the black hole topology theorems of Refs. 52 and 53.
- We should emphasize that our results do not rule out horizon sections with Lens space topology. In fact, under the assumptions of Theorem 2, all near-horizon geometries with $\mathcal{H} = L_{(p,q)}$ can be deduced from the S^3 cases by identifying points related by a particular (discrete) subgroup of the $U(1)^2$ isometry group.

- For a nonextremal rotating black hole in 5D, it has been proven that stationarity implies the existence of one rotational symmetry.^{54,57} Therefore one expects extremal black holes to also have one rotational symmetry. We have assumed two rotational symmetries, a property satisfied by all known black hole solutions in 5D, although there is no general argument for this.

We have not been able to determine all possible near-horizon geometries with two rotational symmetries and compact horizons in 5D with a negative cosmological constant. We have reduced the problem to solving one sixth-order ODE of one function. The only family of solutions to this ODE we know of corresponds to the two-parameter family of near-horizon geometries of the extremal rotating AdS₅ black holes of Ref. 40. If a vacuum extremal AdS black ring with two rotational symmetries does indeed exist, it must correspond to a solution to our ODE. An AdS black ring would possess a number of length scales: R_1 the radius of the S^1 of the horizon, R_2 the radius of the S^2 of the horizon, and ℓ the AdS length scale. A small AdS black ring would be one such that $R_1 \ll \ell$ and $R_2 \ll \ell$. In this regime the black ring would not “see” the effects of the AdS boundary conditions and one would expect it to be well approximated by an asymptotically flat black ring. Therefore, by perturbing about the solution corresponding to the near horizon of the asymptotically flat black ring, one should be able to construct a first-order correction (valid for small R_i/ℓ) representing the near-horizon of a small extremal AdS ring. We have performed this calculation and find that there exist regular perturbations which preserve the $S^1 \times S^2$ topology of the horizon. It is tempting to conclude that this provides some evidence for the existence of, at least a small, extremal vacuum black ring in AdS₅.

The organization of this paper is as follows. In Sec. II we present a self-contained summary of our main results. Section III provides a review of general features of near-horizon geometries with rotational symmetries, and we present the field equations to be analyzed. Section IV deals with the four dimensional case, including a negative cosmological constant. In Section V we consider five dimensional near-horizon geometries: first we examine the general case (including a negative cosmological constant), then turn to a classification of all solutions in the pure vacuum case, and finally we investigate the existence of solutions describing the near-horizon limit of an extremal black ring in AdS₅. Section VI concludes with a discussion of our results. The details of various technical results used throughout the paper are given in the appendices.

II. SUMMARY OF MAIN RESULTS

In this section we will state more explicitly the main results of this paper. This section is intended to be a self-contained summary without derivations; these are provided in the rest of the paper.

A. Vacuum near-horizon geometries in $D=4$ including a negative cosmological constant

Consider a 4D stationary, axisymmetric extremal black hole, with a compact horizon section of nontoroidal topology, satisfying $R_{\mu\nu} = \Lambda g_{\mu\nu}$ with $\Lambda \leq 0$. We have proven that its near-horizon limit must be given by

$$ds^2 = \Gamma(\sigma)[-C^2 r^2 dv^2 + 2dvdr] + \frac{\Gamma(\sigma)}{Q(\sigma)} d\sigma^2 + \frac{Q(\sigma)}{\Gamma(\sigma)} (dx + r dv)^2, \quad (1)$$

where

$$\Gamma = \beta^{-1} + \frac{\beta\sigma^2}{4}, \quad Q = -\frac{\beta\Lambda}{12}\sigma^4 - (C^2 + 2\Lambda\beta^{-1})\sigma^2 + 4\beta^{-3}(C^2\beta + \Lambda), \quad (2)$$

and $C > 0$, $\beta > 0$ are constants. Q must have four distinct real roots. The coordinate ranges are given by $\sigma_1 \leq \sigma \leq \sigma_2$, where σ_2 is the smallest positive root of Q and $\sigma_2 = -\sigma_1$ and $x \sim x + 2\pi k$, where $k = \Gamma(\sigma_2)/(C^2\sigma_2)$. This is actually a one-parameter family of solutions due to a scaling symmetry of the solution which allows one to set C^2 or β to any desired value. It has isometry

group $SO(2, 1) \times U(1)$ with the orbits given by circle bundles over AdS_2 and its cohomogeneity-1. The horizon is at $r=0$ and spatial sections of this are S^2 endowed with a cohomogeneity-1 metric. This near-horizon geometry is isometric to that of extremal Kerr ($\Lambda=0$) or extremal Kerr- AdS_4 ($\Lambda < 0$).

A consequence of the above result is that any stationary axisymmetric extremal black hole solution (with S^2 horizon sections) satisfying $R_{\mu\nu} = \Lambda g_{\mu\nu}$ for $\Lambda \leq 0$ must have a near-horizon geometry given by that of extremal Kerr ($\Lambda=0$) and Kerr- AdS_4 ($\Lambda < 0$).

B. Vacuum near-horizon geometries in $D=5$

Consider a 5D Ricci flat extremal black hole with $R \times U(1)^2$ symmetry (i.e., stationary plus two rotational symmetries) and assume spatial sections of the horizon are not toroidal. We have proven that its near-horizon geometry must be contained in one of three families.

$S^1 \times S^2$ horizon: The near-horizon geometry in this case can be written as

$$ds^2 = C^2 a^2 (1 + \sigma^2) [-C^2 r^2 dv^2 + 2dvdr] + \frac{a^2(1 + \sigma^2)}{1 - \sigma^2} d\sigma^2 + \frac{4a^2(1 - \sigma^2)}{(1 + \sigma^2)} (d\phi + \Omega dx^2 + C^2 r dv)^2 + \frac{1}{4C^4 a^2} (dx^2)^2, \quad (3)$$

where $-1 \leq \sigma \leq 1$, $\phi \sim \phi + 2\pi$, and $x^2 \sim x^2 + L$. The solution is parametrized by the constants (C, a, Ω, L) where $C, a, L > 0$, however, due to a scaling symmetry one of C, Ω, L may be set to any convenient value. It is therefore a three-parameter family of solutions. The isometry group of this geometry is $SO(2, 1) \times U(1)^2$. The orbits of $SO(2, 1)$ are circle bundles over AdS_2 and the geometry is cohomogeneity-1. The horizon is at $r=0$ and spatial sections of this are $S^1 \times S^2$ endowed with a cohomogeneity-1 metric. In fact, the $C^2 |\Omega| < 1/(4a^3)$ case is identical to the near-horizon limit of the boosted extremal Kerr string with boost parameter β and Kerr parameter a , see Ref. 22. This can be seen by defining $\tanh \beta = 4a^3 C^2 \Omega$ and setting $C^2 = 1/(2a^2 \cosh \beta)$ (which we are free to do due to the scaling symmetry mentioned). Further, if one chooses the boost, such that $\sinh^2 \beta = 1$, it is isometric to the near-horizon limit of the extremal vacuum black ring, see Ref. 22.

S^3 horizon (case A): The main assumption of our analysis is the existence of a $U(1)^2$ rotational symmetry. As is typical of rotating solutions in 5D, in this class there is a special case in which the rotational symmetry group enhances to $SU(2) \times U(1)$. It is convenient to write this special case in a separate coordinate system.

The more symmetric case can be written as

$$ds^2 = \Gamma [-C^2 r^2 dv^2 + 2dvdr] + \frac{2\Gamma}{C^2} (d\psi + \cos \theta d\phi + C^2 r dv)^2 + \frac{\Gamma}{C^2} (d\theta^2 + \sin^2 \theta d\phi^2), \quad (4)$$

where $0 \leq \psi \leq 4\pi$, $0 \leq \phi \leq 2\pi$, $0 \leq \theta \leq \pi$ are the usual Euler angles on S^3 . The solution is parametrized by the constants (C^2, Γ) ; however, due to a scaling symmetry it is a one-parameter family. This solution has an isometry group $SO(2, 1) \times SU(2) \times U(1)$. The orbits of $SO(2, 1)$ are circle bundles over AdS_2 and the geometry is a homogeneous space. The horizon is at $r=0$ and spatial sections of this are S^3 endowed with a homogeneous metric. It turns out that this case is isometric to both the near-horizon limit of the $J_1 = J_2$ extremal Myers–Perry black hole and the near-horizon limit of the $J=0$ extremal KK black hole.

The generic case is more complicated. It can be written as

$$ds^2 = \sigma [-C^2 r^2 dv^2 + 2dvdr] + \frac{\sigma d\sigma^2}{Q(\sigma)} + \left(C^2 \sigma - \frac{c_2}{\sigma} \right) \left(dx^1 + r dv + \frac{\sqrt{-c_2 c_1} dx^2}{C(C^2 \sigma^2 - c_2)} \right)^2 + \frac{Q(\sigma) (dx^2)^2}{(C^2 \sigma^2 - c_2)}, \quad (5)$$

where $Q(\sigma) = -C^2 \sigma^2 + c_1 \sigma + c_2$ and $\sigma_1 \leq \sigma \leq \sigma_2$, where σ_1, σ_2 are the roots of Q and $0 < \sigma_1 < \sigma_2$.

The parameters must satisfy $c_1 > 0$, $c_2 < 0$, and $c_1^2 + 4C^2c_2 > 0$. There is a scaling symmetry so it is really just a two-parameter family of metrics. The coordinates ϕ_i adapted to the $U(1)^2$ rotational symmetry are defined by $\partial/\partial\phi_i = -d_i((1/C\sigma_i)\sqrt{-c_2/c_1}(\partial/\partial x^1) - \partial/\partial x^2)$, where d_i are chosen so that the periods of ϕ_i are 2π . This near-horizon geometry has an isometry group $SO(2,1) \times U(1)^2$ whose generic orbits are T^2 bundles over AdS_2 and therefore it is cohomogeneity-1 [the orbits of $SO(2,1)$, in general, are line bundles over AdS_2]. Spatial sections of the horizon $r=0$ are given by S^3 endowed with a cohomogeneity-1 metric. It turns out this case is isometric to the near-horizon limit of two different black holes: the extremal $J_1 \neq J_2$ Myers–Perry and the extremal $0 < G_4J < PQ$ KK black hole.

In summary, this class of S^3 topology horizons is isometric to the near-horizon limit of either (i) extremal Myers–Perry or (ii) slowly rotating KK black hole.

S^3 horizon (case B): In this case the near-horizon geometry is of the form

$$ds^2 = (a_2\sigma^2 + a_0)[-C^2r^2dv^2 + 2dvdr] + \frac{(a_2\sigma^2 + a_0)d\sigma^2}{Q(\sigma)} + \frac{2P(\sigma)}{a_2\sigma^2 + a_0} \left[dx^1 + r dv - \frac{\kappa a_2 \sigma}{\alpha P(\sigma)} dx^2 \right]^2 + \frac{Q(\sigma)}{2P(\sigma)} (dx^2)^2, \quad (6)$$

where

$$Q = -C^2\sigma^2 + c_1\sigma + c_2, \quad P = \alpha\sigma^2 + \beta\sigma + \gamma, \quad (7)$$

with

$$\alpha = -a_2(C^2a_0 + a_2c_2), \quad \beta = 2c_1a_0a_2, \quad \gamma = a_0(C^2a_0 + a_2c_2), \quad (8)$$

and

$$\kappa \equiv \sqrt{\frac{(a_0C^2 - a_2c_2)[c_1^2a_0a_2 + (C^2a_0 + a_2c_2)^2]}{2}}. \quad (9)$$

The constants $(a_0, a_2, c_1, c_2, C^2)$ must satisfy $C^2a_0 - a_2c_2 > 0$, $c_1^2a_0a_2 + (C^2a_0 + a_2c_2)^2 > 0$, and $c_1^2 + 4C^2c_2 > 0$.⁵⁵ The latter condition ensures that Q has two distinct real roots $\sigma_1 < \sigma_2$ and the coordinate σ must belong to the interval $\sigma_1 \leq \sigma \leq \sigma_2$. This metric possesses two independent scaling symmetries and thus is really just a three-parameter family. It has an isometry group $SO(2,1) \times U(1)^2$ whose generic orbits are T^2 bundles over AdS_2 , and therefore it is cohomogeneity-1 [the orbits of $SO(2,1)$ are generically line bundles over AdS_2]. The horizon is at $r=0$ and spatial sections of this are S^3 endowed with a cohomogeneity-1 metric. Using one of the scaling symmetries, one can always set $c_1^2 + 4C^2c_2 = 4C^4$: then, the region of parameter space defined by $a_2 > 0$ and $4a_2C^{-2} + 2C^{-4}(C^2a_0 - a_2c_2) < [c_1^2a_0a_2 + (C^2a_0 + a_2c_2)^2]/(C^6a_2)$ can be shown to be identical to the near-horizon geometry of the fast rotating extremal KK black hole (i.e., $G_4J > PQ$).

C. Vacuum near-horizon geometries in $D=5$ with a negative cosmological constant

Consider a 5D near-horizon geometry with two commuting spacelike Killing vectors which satisfies $R_{\mu\nu} = \Lambda g_{\mu\nu}$. We have shown that the problem is equivalent to solving the two coupled ODEs,

$$\frac{d^2Q}{d\sigma^2} + 2C^2 + 6\Lambda\Gamma = 0, \quad \frac{d}{d\sigma} \left(\frac{Q^3}{\Gamma} \frac{d^3\Gamma}{d\sigma^3} \right) - 10\Lambda Q^2 \frac{d^2\Gamma}{d\sigma^2} = 0 \quad (10)$$

for the pair of functions $(\Gamma(\sigma), Q(\sigma))$, where $C > 0$ is a constant and $\Gamma > 0$. Observe that eliminating Γ gives a sixth-order nonlinear ODE. The near-horizon geometry is given in coordinates (v, r, σ, x^1, x^2) by

$$ds^2 = \Gamma[-C^2 r^2 dv^2 + 2dvdr] + \frac{\Gamma d\sigma^2}{Q} + \gamma_{11}(dx^1 + \omega(\sigma)dx^2)^2 + \frac{Q}{\Gamma\gamma_{11}}(dx^2)^2, \quad (11)$$

with

$$\gamma_{11} = \Gamma \frac{d}{d\sigma} \left(\frac{Q\dot{\Gamma}}{\Gamma} \right) + 2C^2\Gamma + 2\Lambda\Gamma^2 \quad (12)$$

and $\omega \equiv \gamma_{12}/\gamma_{11}$ is determined up to quadratures by either (80) and (82). Note that $\partial/\partial v$, $\partial/\partial x^1$, and $\partial/\partial x^2$ are all Killing so the metric depends on the single coordinate σ . The horizon is at $r=0$.

The most general polynomial solution to the pair of ODEs is

$$\Gamma = a_0 + a_1\sigma, \quad Q = -\Lambda a_1\sigma^3 - (C^2 + 3\Lambda a_0)\sigma^2 + c_1\sigma + c_2. \quad (13)$$

The resulting near-horizon geometry is a straightforward generalization of the Ricci flat near-horizon geometry with S^3 horizon (case A) in Sec. II B. It turns out this case (once compactness of the horizon is imposed) is exactly the near-horizon geometry of the most general known extremal rotating AdS₅ black hole,⁴⁰ which has horizon topology S^3 .

We have not been able to find all solutions to the pair of ODEs, which prevents us from providing a classification of near-horizon geometries in this case. It would be interesting to find a solution with horizon topology $S^1 \times S^2$, thus providing a candidate for the near-horizon geometry of an extremal AdS black ring. By linearizing the ODEs about the solution corresponding to the asymptotically flat black ring, we have constructed an approximate solution corresponding to the near-horizon limit of a “small” AdS black ring, see Sec. V D.

III. VACUUM NEAR-HORIZON EQUATIONS

Consider a stationary extremal black hole. In a neighborhood of the horizon we can introduce Gaussian null coordinates (v, r, x^a) , where $V = \partial/\partial v$ is a Killing field, the horizon is at $r=0$, and x^a are coordinates on a $D-2$ -dimensional spatial section of the horizon. We will refer to this $D-2$ -dimensional manifold as \mathcal{H} , which we assume is orientable, compact, and without a boundary. One can take the near-horizon limit of the metric by sending $v \rightarrow v/\epsilon$, $r \rightarrow \epsilon r$, and $\epsilon \rightarrow 0$, see Refs. 2 and 22. This gives

$$ds^2 = r^2 F(x) dv^2 + 2dvdr + 2rh_a(x)dvdx^a + \gamma_{ab}(x)dx^a dx^b, \quad (14)$$

where F, h_a, γ_{ab} are a function, one-form, and Riemannian metric on \mathcal{H} which we will refer to as the near-horizon data.

In this paper we will consider the problem of determining all vacuum near-horizon geometries allowing for a negative cosmological constant, i.e., metrics of the form (14) satisfying $R_{\mu\nu} = \Lambda g_{\mu\nu}$ with $\Lambda \leq 0$. It can be shown that these space-time Einstein equations for the metric (14) are equivalent to the following set of equations on \mathcal{H} :⁵⁶

$$R_{ab} = \frac{1}{2}h_a h_b - \nabla_{(a} h_{b)} + \Lambda \gamma_{ab}, \quad (15)$$

$$F = \frac{1}{2}h_a h^a - \frac{1}{2}\nabla_a h^a + \Lambda, \quad (16)$$

where R_{ab} and ∇_a are the Ricci tensor and metric connection of the horizon metric γ_{ab} . For later convenience it is worth noting that using (15) and (16), the contracted Bianchi identity for R_{ab} is equivalent to the following equation:

$$\nabla_a F = Fh_a + 2h^b \nabla_{[a} h_{b]} - \nabla^b \nabla_{[a} h_{b]}. \quad (17)$$

In this paper we will be concerned with solving Eqs. (15) and (16). Although we have not been able to solve them, in general, we will show how one can determine all solutions with a compact

\mathcal{H} under extra assumptions regarding rotational symmetries and horizon topology.

First, however, we will note a number of general implications of the above equations for $\Lambda = 0$. Observe that

$$\int_{\mathcal{H}} R = \int_{\mathcal{H}} F = \int_{\mathcal{H}} \frac{h^a h_a}{2} \geq 0 \quad (18)$$

with equality if and only if $h_a \equiv 0$. In the case $h_a \equiv 0$ it follows that $F \equiv 0$ and γ_{ab} is Ricci flat: the near-horizon geometry is then simply a direct product of $R^{1,1}$ and a Ricci flat metric on \mathcal{H} . In four and five dimensions this implies a flat metric on \mathcal{H} . In 4D this implies $\mathcal{H} = T^2$, whereas in 5D there are more possibilities (which include $\mathcal{H} = T^3$).⁵⁸ Also note that in 4D we see that the Euler number $\chi(\mathcal{H}) \geq 0$ and thus the only possible horizon topologies are S^2 and T^2 (the latter of which occurs only when the near-horizon geometry is a direct product of $R^{1,1}$ and flat T^2). Observe that for $r > 0$ the Killing vector $V = \partial/\partial v$ cannot be timelike everywhere, i.e., $F(x) < 0$ for all x is not allowed.

A. Cohomogeneity-1 near-horizon geometries

We will now restrict consideration to near-horizon geometries of stationary extremal black holes which are axisymmetric in $D=4$ and which admit two commuting rotational Killing vector fields in $D=5$. That is, black holes with an isometry group $R \times U(1)^{D-3}$ in $D=4,5$ whose generic orbits are $D-2$ dimensional. Denote the generators of the $U(1)^{D-3}$ isometry by m_i and introduce coordinates adapted to these so that $m_i = \partial/\partial \phi^i$ with $\phi^i \sim \phi^i + 2\pi$. The near-horizon geometry inherits the $U(1)^{D-3}$ isometry group, which implies that the full near-horizon geometry is cohomogeneity-1 (see Ref. 22). Therefore, the near-horizon data (F, h_a, γ_{ab}) defined on \mathcal{H} are all invariant under m_i .

The existence of the $U(1)^{D-3}$ isometry group restricts the horizon topology as follows. First note that the $U(1)^{D-3}$ isometry defines an effective group action on \mathcal{H} , although we do not assume it acts freely.⁵⁹ Thus spatial sections of the horizon \mathcal{H} are $D-2$ -dimensional closed (compact with no boundary) and orientable manifolds with a $U(1)^{D-3}$ effective action. For $D=4$ the only possible closed, oriented two manifolds admitting an effective $U(1)$ action are S^2 and T^2 . In $D=5$ the only possible closed oriented three manifolds which admit an effective $U(1)^2$ action are T^3 , $S^1 \times S^2$, and S^3 [as well as the Lens spaces $L(p, q)$ which occur as its quotients by discrete isometry subgroups], see, e.g., Ref. 60 and references therein. Note that only in the T^{D-2} case is the action free—in the other cases there are fixed points. In fact, there are global parts of our analysis which do not apply to the T^{D-2} case, and therefore we assume nontoroidal topology in $D=4,5$ henceforth. This does not represent much of a restriction though, as in view of the black hole topology theorems,^{52,49,53} one is mainly interested in S^2 topology in $D=4$ and $S^3, S^1 \times S^2$ in $D=5$. Let us now introduce some globally defined quantities which are central to our analysis.

Define the 1-form $\Sigma = -i_{m_1} \cdots i_{m_{D-3}} \epsilon_{D-2}$, where ϵ_{D-2} is the volume form associated with the metric γ_{ab} on \mathcal{H} . Since the m_i are Killing fields it follows that $d\Sigma = 0$. Therefore, if $H^1(\mathcal{H}) = 0$ there exists a globally defined function σ , such that $\Sigma = d\sigma$. For $\mathcal{H} = S^2, S^3$ (and Lens spaces) this is the case.⁶¹ For $\mathcal{H} = S^1 \times S^2$ we can argue that the 1-form Σ is a 1-form on the quotient space $\mathcal{H}/S^1 = S^2$ as follows. Since $\Sigma \cdot m_i = 0$, we simply need to show that some combination of the Killing fields m_i are a vector field on the S^1 . But this must be the case, as otherwise restricting to S^2 one would have a metric on S^2 with $U(1)^2$ isometry which is impossible. Hence as claimed, we have shown that even in the $S^1 \times S^2$ case the closed 1-form Σ must be globally exact (as it is also a closed 1-form on the S^2). To summarize, we have shown that for the cases of interest $\mathcal{H} = S^2, S^3, S^1 \times S^2$, one may define a globally defined function σ as above; notice that σ is invariant under the $U(1)^2$ isometries since $m_i \cdot d\sigma = 0$ and also that $d\sigma = 0$ at the fixed points of the rotational Killing fields.

In fact, the fixed points of the rotational Killing fields may be used to distinguish these topologies.⁶² For $S^2, S^1 \times S^2$ there is a unique Killing field which has fixed points, and further it only has two fixed points (the poles of the S^2). For S^3 again there are only two points where one

(or a combination) of the Killing fields vanishes, however, it is a different Killing field which vanishes at each of these points. From the definition of σ we deduce that $d\sigma=0$ only at fixed points of the m_i ; it follows that for the topologies under consideration $d\sigma$ vanishes only at these two points.

The 1-form h_a on \mathcal{H} , which is part of the near-horizon data, can be decomposed *globally* using the Hodge decomposition theorem as follows:

$$h = \beta + d\lambda, \quad (19)$$

where β is a coclosed 1-form⁶⁶ (i.e., $\nabla_a\beta^a=0$) and λ is a function on \mathcal{H} . Since $L_{m_i}h=0$ it follows that $L_{m_i}\beta=0$ and $L_{m_i}d\lambda=0$ separately. This follows from uniqueness of the Hodge decomposition together with the facts that $L_{m_i}\beta$ is coclosed (using the properties that β is coclosed and m_i are Killing fields) and also that $L_{m_i}d\lambda=d(m_i\cdot d\lambda)$ is exact. This also shows that $m_i\cdot d\lambda=c_i$, where c_i are constants. However, since λ is a function on \mathcal{H} it must be periodic in ϕ^i where $m_i=\partial/\partial\phi^i$ —this implies that $c_i=0$ and hence $m_i\cdot d\lambda=0$. We find it convenient to introduce the invariant function $\Gamma=e^{-\lambda}$ which satisfies $\Gamma>0$ on all \mathcal{H} .

We now introduce local coordinates (ρ, x^i) on a horizon section which are adapted to the $U(1)^{D-3}$ isometry,

$$\gamma_{ab}dx^a dx^b = d\rho^2 + \gamma_{ij}(\rho)dx^i dx^j, \quad (20)$$

where $\partial/\partial x^i$ are Killing fields and $i=1, \dots, D-3$. Observe that in $D=4$, it is necessarily the case that $m_1 \propto \partial/\partial x^1$, whereas in $D=5$ the $\partial/\partial x^i$ can be linear combinations of the m_i and thus need not have closed orbits. In these coordinates all scalar invariants only depend on ρ , so, for example, $\Gamma=\Gamma(\rho)$ and $\sigma=\sigma(\rho)$. Now consider the coclosed 1-form β above. The fact that it is invariant under the m_i means $\beta=\beta_\rho(\rho)d\rho+\beta_i(\rho)dx^i$. Then, $\nabla_a\beta^a=(1/\sqrt{\gamma})(d/d\rho)(\sqrt{\gamma}\beta^\rho)=0$, where $\gamma=\det \gamma_{ij}$, can be solved to get $\beta^\rho=c/\sqrt{\gamma}$, where c is a constant. This constant is related to an invariant as follows: consider the scalar $i_\beta \star(m_1 \wedge \dots \wedge m_{D-3})=cJ$, where \star is the Hodge dual with respect to the horizon metric γ_{ab} ⁶⁷ and J is the Jacobian of the coordinate transformation $\phi^i \rightarrow x^i$ (which is a nonzero constant). Therefore, if one of the rotational Killing fields on the horizon vanishes somewhere, we must have $c=0$ and hence $\beta_\rho \equiv 0$. This is indeed the case for all the topologies we are interested in (in fact, $c \neq 0$ only in the T^{D-2} case). Thus, exploiting the global representation for the one-form h derived above (19), in the coordinates (ρ, x^i) we have

$$h = \Gamma^{-1}k_i(\rho)dx^i - \frac{\Gamma'}{\Gamma}d\rho, \quad (21)$$

where, in general, we write $f'=df/d\rho$, and we have defined the functions $k_i(\rho)=\Gamma\beta_i$. Notice that the functions $k_i \equiv \Gamma h \cdot (\partial/\partial x^i)$ are, in fact, globally defined.

It is then convenient to introduce a new radial coordinate (for the full near-horizon geometry) by $r \rightarrow \Gamma(\rho)r$ and define the function $A \equiv \Gamma^2 F - k^i k_i$, where $k^i \equiv \gamma^{ij}k_j$. One of the main results found in Ref. 22 is that the vanishing of the ρi and ρv components of the Ricci tensor of the full near-horizon geometry in these new (v, r, ρ, i) coordinates implies that k^i are constants and that $A=A_0\Gamma$ for some constant A_0 . Then the near-horizon geometry written in the new r coordinate simplifies to

$$ds^2 = \Gamma(\rho)[A_0 r^2 dv^2 + 2dvdr] + d\rho^2 + \gamma_{ij}(\rho)(dx^i + k^i r dv)(dx^j + k^j r dv). \quad (22)$$

This form of the near-horizon geometry makes an $SO(2, 1)$ isometry group manifest.²² We will take this result as the starting point of our analysis and find all vacuum geometries of this form with compact \mathcal{H} . For completeness though, we will first show how this result is derived from the Einstein equations written in terms of the near-horizon data (F, h_a, γ_{ab}) defined purely on \mathcal{H} , i.e., Eqs. (15) and (16). First observe that the ρi component of (15) is $R_{\rho i} = -\frac{1}{2}\Gamma^{-1}\gamma_{ij}(k^j)'$ and since $R_{\rho i}=0$ for a metric of the form (20) this implies k^i are constants. Now, the ρ component of (17) is $F' = -(\Gamma'/\Gamma)F + \Gamma^{-1}k^i(\Gamma^{-1}k_i)'$, which when written in terms of the function A defined above is

equivalent to $A' + (\Gamma'/\Gamma)A + (k^i)'k_i = 0$. From this and the constancy of k^i , it follows that $A = A_0\Gamma$ for some constant A_0 and the result is established.⁶⁸

Note that we will be only interested in nonstatic⁶⁹ near-horizon geometries, as the static case has been analyzed previously⁴⁷ where it was found that the only solution is a direct product of $R^{1,1}$ and a flat compact space for $\Lambda = 0$ or a direct product of AdS_2 and a negative curvature compact Einstein space for $\Lambda < 0$. If all the constants $k^i = 0$ then the above near-horizon geometry is static.²² Therefore, we will assume that at least one of the constants $k^i \neq 0$ in this paper.⁷⁰

Let us now consider the near-horizon equation (16). Observe that $F = (A_0\Gamma + k^i k_i)/\Gamma^2$, and therefore (16) becomes

$$A_0 + \frac{k^i k_i}{2\Gamma} - \frac{1}{2}\nabla^2\Gamma = \Lambda\Gamma. \quad (23)$$

Integrating (23) over \mathcal{H} shows that

$$A_0 = \frac{1}{\text{vol}[\mathcal{H}]} \int_{\mathcal{H}} \left(-\frac{k^i k_i}{2\Gamma} + \Lambda\Gamma \right) \leq 0, \quad (24)$$

with equality if and only if $\Lambda = 0$ and $k^i = 0$. Therefore, for nonstatic near-horizon geometries $A_0 < 0$, and we will often set $A_0 = -C^2$ for some $C > 0$.

Now let us turn to the equation for the Ricci tensor of the horizon (15). The nonzero components of the Ricci tensor of the horizon metric (20) are given by

$$R_{ij} = -\frac{1}{2}\gamma''_{ij} - \frac{\gamma'}{4\gamma}\gamma'_{ij} + \frac{1}{2}\gamma'_{ik}\gamma^{kl}\gamma'_{lj} = -\frac{1}{2}\nabla^2\gamma_{ij} + \frac{1}{2}\gamma'_{ik}\gamma^{kl}\gamma'_{lj}, \quad (25)$$

$$R_{\rho\rho} = -\frac{1}{2}(\log \gamma)'' - \frac{1}{4}\gamma^{ij}\gamma'_{jk}\gamma^{km}\gamma'_{ml}, \quad (26)$$

and note that for a function $f(\rho)$

$$\nabla^2 f \equiv f'' + \frac{\gamma'}{2\gamma}f', \quad (27)$$

where $\gamma \equiv \det \gamma_{ij}$. Evaluating the right hand side of (15) gives

$$R_{\rho\rho} = \frac{\Gamma''}{\Gamma} - \frac{1}{2}\frac{\Gamma'^2}{\Gamma^2} + \Lambda, \quad (28)$$

$$R_{ij} = \frac{1}{2}\Gamma^{-2}k_i k_j + \frac{1}{2}\gamma'_{ij}\frac{\Gamma'}{\Gamma} + \Lambda\gamma_{ij}. \quad (29)$$

Now, observe that (25) implies

$$R_{ij}\gamma^{jj} = -\frac{1}{2}(\log \gamma)'' - \frac{1}{4}(\log \gamma)'^2, \quad (30)$$

which using (29) implies

$$(\log \gamma)'' + \frac{\Gamma'}{\Gamma}(\log \gamma)' + \Gamma^{-2}k^2 + \frac{1}{2}(\log \gamma)'^2 + 2(D-3)\Lambda = 0. \quad (31)$$

By contracting (25) with $k^i k^j$ and using (29), one gets

$$(k^2)'' + \frac{\Gamma'}{\Gamma}(k^2)' - k_i' \gamma^{ij} k_j' + \frac{1}{2}(\log \gamma)'(k^2)' + 2\Lambda k^2 + \Gamma^{-2}(k^2)^2 = 0. \quad (32)$$

To integrate the above equations it proves useful to use the globally defined function σ introduced at the beginning of this section as a coordinate instead of ρ . Note that in ρ coordinates, it is given by $\sigma' = \sqrt{\det \gamma_{ij}}$. Observe that the volume form of \mathcal{H} is then given simply by $\epsilon_{D-3} = d\sigma \wedge dx^1 \wedge \cdots \wedge dx^{D-3}$ (choosing an orientation). Recall that σ is a globally defined function and $d\sigma$ is nonzero everywhere except where γ_{ij} degenerates. Indeed, σ cannot be constant as otherwise $\sigma' = 0$ and thus $\det \gamma_{ij} = 0$ everywhere. Therefore, it is legitimate to use σ as a coordinate everywhere except at these degeneration points (which occur at fixed points of m_i). Therefore, any expression we derive in this coordinate will be valid globally provided we can show that is also smooth where $d\sigma = 0$.

We will now derive some general results valid in both $D=4,5$. Substituting into Eq. (23) implies

$$\frac{k^i k_i}{2\Gamma^2} = \frac{\Gamma' \sigma''}{2\Gamma \sigma'} + \frac{C^2}{\Gamma} + \frac{\Gamma''}{2\Gamma} + \Lambda, \quad (33)$$

and Eq. (31) gives

$$\sigma''' + \frac{\Gamma' \sigma''}{\Gamma} + \sigma' \left(\frac{k^i k_i}{2\Gamma^2} + (D-3)\Lambda \right) = 0. \quad (34)$$

Eliminating $k^i k_i$ between these two equations leads to

$$\sigma''' + \frac{3\Gamma'}{2\Gamma} \sigma'' + \left(\frac{C^2}{\Gamma} + \frac{\Gamma''}{2\Gamma} + (D-2)\Lambda \right) \sigma' = 0. \quad (35)$$

This equation may actually be solved by noting the identity

$$\sigma''' + \frac{3\Gamma'}{2\Gamma} \sigma'' + \left(\frac{C^2}{\Gamma} + \frac{\Gamma''}{2\Gamma} + (D-2)\Lambda \right) \sigma' \equiv \sigma' \left[\frac{1}{2\Gamma} \frac{d^2 Q}{d\sigma^2} + \frac{C^2}{\Gamma} + (D-2)\Lambda \right], \quad (36)$$

where we have defined $Q(\sigma) \equiv \sigma'^2 \Gamma$. Therefore, we deduce that

$$\ddot{Q} + 2C^2 + 2(D-2)\Lambda\Gamma = 0, \quad (37)$$

where, in general, we denote $df/d\sigma = \dot{f}$. Observe that by working in the σ coordinate, the $d\rho^2$ part of the metric is given by

$$d\rho^2 = \frac{\Gamma}{Q} d\sigma^2. \quad (38)$$

Substituting $\sigma'^2 = Q/\Gamma$ back into (33) gives

$$k^i k_i = \Gamma \frac{d}{d\sigma} \left(\frac{Q\dot{\Gamma}}{\Gamma} \right) + 2C^2\Gamma + 2\Lambda\Gamma^2. \quad (39)$$

Since we are assuming the constants $k^i \neq 0$, we can always choose the coordinates x^i such that $k^i \partial / \partial x^i = \partial / \partial x^1$. This implies $k^i k_i = \gamma_{11}$, and therefore we have determined this component of the metric in terms of the functions Q and Γ . In $D=4$, together with (38), this determines the whole metric on \mathcal{H} in terms of the two functions Q and Γ .

Before closing this section let us derive a useful result based on global considerations. First notice that the norm of the one form $d\sigma$ is given by

$$(d\sigma)^2 = \frac{Q}{\Gamma}, \quad (40)$$

which implies $Q \geq 0$. Now since σ is a globally defined function on a closed manifold \mathcal{H} , it must have a distinct minimum (say σ_1) and maximum (say σ_2) so $\sigma_1 \leq \sigma \leq \sigma_2$ and $\sigma_1 < \sigma_2$ (note that σ cannot be a constant). Therefore $d\sigma$ must vanish at these two distinct points on \mathcal{H} , and as argued earlier there are, in fact, only two points where $d\sigma=0$ (they correspond to the fixed points of m_i). This implies that the function $Q \geq 0$ with equality if and only if $\sigma = \sigma_1$ or $\sigma = \sigma_2$. We deduce that any function we construct from σ -derivatives of globally defined functions can only fail to be defined at the points $\sigma = \sigma_1, \sigma_2$. Our subsequent analysis will be mostly local (integration of ODEs with respect to σ), although there are steps where we need to use the fact that certain functions are globally defined. In the appendices we introduce a (globally defined) vector field which allows us to prove that these functions are globally defined.

IV. Four Dimensions

In 4D the metric on the horizon is particularly simple,

$$\gamma_{ab} dx^a dx^b = d\rho^2 + \gamma(\rho) dx^2, \quad (41)$$

where we write $x^1 = x$ and note that $\gamma_{11} = \gamma$ in this case. Equation (39) therefore gives an expression for γ which, noting that $\gamma = \sigma'^2 = Q/\Gamma$, can be written as

$$Q = \dot{Q}\dot{\Gamma}\Gamma - \dot{\Gamma}^2 Q + Q\Gamma\ddot{\Gamma} + 2C^2\Gamma^2 + 2\Lambda\Gamma^3. \quad (42)$$

Now differentiate (42) with respect to σ . This gives an expression involving \ddot{Q} which can be eliminated using (37), leaving

$$\dot{Q} = Q\Gamma \frac{d^3\Gamma}{d\sigma^3} + \ddot{\Gamma}(2\dot{Q}\Gamma - Q\dot{\Gamma}) + 2C^2\Gamma\dot{\Gamma} + 2\Lambda\Gamma^2\dot{\Gamma}. \quad (43)$$

Now combine this with (42) in such a way to eliminate the C^2 and Λ terms to eventually get

$$Q \frac{d^3\Gamma}{d\sigma^3} + \left(\dot{Q} - \frac{\Gamma\dot{Q}}{\Gamma} \right) \left(2\ddot{\Gamma} - \frac{\dot{\Gamma}^2}{\Gamma} - \frac{1}{\Gamma} \right) = 0. \quad (44)$$

Now define

$$\mathcal{P} \equiv 2\ddot{\Gamma} - \frac{\dot{\Gamma}^2}{\Gamma} - \frac{1}{\Gamma} \quad (45)$$

and note the identity

$$2 \frac{d^3\Gamma}{d\sigma^3} \equiv \frac{\dot{\Gamma}\mathcal{P}}{\Gamma} + \dot{\mathcal{P}}. \quad (46)$$

Eliminate the third-order derivative terms between (44) and (46) to get

$$\dot{\mathcal{P}} = \left(\frac{\dot{\Gamma}}{\Gamma} - \frac{2\dot{Q}}{Q} \right) \mathcal{P}, \quad (47)$$

which integrates to

$$\frac{Q^2 \mathcal{P}}{\Gamma} = \alpha, \quad (48)$$

where α is some constant.

As discussed earlier, based on global analysis Q must vanish at two distinct points which from (48) would seem to say one must have $\alpha=0$. Indeed, in the appendices we prove that $Q^2 \mathcal{P}$ is a globally defined function which vanishes at the zeros of Q , and therefore one must have $\alpha=0$ for a compact \mathcal{H} . From (48) we see that therefore we must have $\mathcal{P}=0$, and this equation can be solved by noting the identity

$$\frac{\mathcal{P}\dot{\Gamma}}{\Gamma} \equiv \frac{d}{d\sigma} \left(\frac{\dot{\Gamma}^2 + 1}{\Gamma} \right), \quad (49)$$

which implies

$$\dot{\Gamma}^2 + 1 = \beta\Gamma, \quad (50)$$

where $\beta > 0$ is a constant. There are two solutions to this equation: either

$$\Gamma = \beta^{-1} + \frac{\beta(\sigma - \sigma_0)^2}{4}, \quad (51)$$

where σ_0 is a constant, or simply $\Gamma = \beta^{-1}$. This latter solution implies that Γ and Q are both constants—this is incompatible with having a compact \mathcal{H} , and therefore we discount it. Therefore Γ must be given by (51), and since by definition, σ is only defined up to an additive constant, without loss of generality we will set $\sigma_0=0$. We can now integrate easily for Q using (37) to find,

$$Q = -\frac{\beta\Lambda}{12}\sigma^4 - (C^2 + 2\Lambda\beta^{-1})\sigma^2 + c_1\sigma + c_2. \quad (52)$$

Now plugging back into Eq. (42) implies

$$c_2 = 4\beta^{-3}(C^2\beta + \Lambda). \quad (53)$$

The rest of the near-horizon equations are now satisfied without further constraint.

To summarize, so far we have shown that the near-horizon geometry is given by

$$ds^2 = \Gamma[-C^2 r^2 dv^2 + 2dvdr] + \frac{\Gamma}{Q} d\sigma^2 + \frac{Q}{\Gamma} (dx + r dv)^2, \quad (54)$$

where

$$\Gamma = \beta^{-1} + \frac{\beta\sigma^2}{4}, \quad Q = -\frac{\beta\Lambda}{12}\sigma^4 - (C^2 + 2\Lambda\beta^{-1})\sigma^2 + c_1\sigma + 4\beta^{-3}(C^2\beta + \Lambda), \quad (55)$$

and $C > 0$, $\beta > 0$, and c_1 are constants. Observe that the near-horizon geometry has the following scaling freedom:

$$C^2 \rightarrow KC^2, \quad \beta \rightarrow K^{-1}\beta, \quad c_1 \rightarrow K^2 c_1, \quad \sigma \rightarrow K\sigma, \quad x \rightarrow K^{-1}x, \quad v \rightarrow K^{-1}v \quad (56)$$

for constant $K > 0$, which allows one to fix one of the parameters (or a combination of them) to any desired value.

Although we have used some global information in our derivation, we need to complete the global analysis of this solution to determine the most general regular near-horizon geometry with compact horizon sections.

A. Global analysis

Consider the metric on \mathcal{H} ,

$$\gamma_{ab}dx^a dx^b = \frac{\Gamma}{Q}d\sigma^2 + \frac{Q}{\Gamma}dx^2. \quad (57)$$

As discussed earlier, compactness of \mathcal{H} requires $\sigma_1 \leq \sigma \leq \sigma_2$ and $Q \geq 0$ with equality occurring at σ_1, σ_2 only. It follows that $\dot{Q}(\sigma_1) > 0$ and $\dot{Q}(\sigma_2) < 0$. The Killing vector $\partial/\partial x$ must vanish at the endpoints. The horizon metric therefore is nondegenerate everywhere except at $\sigma = \sigma_1, \sigma_2$, where, in general, one has conical singularities. Simultaneous removal of the conical singularities at σ_1 and σ_2 is equivalent to

$$\frac{\dot{Q}(\sigma_1)}{\Gamma(\sigma_1)} = -\frac{\dot{Q}(\sigma_2)}{\Gamma(\sigma_2)}. \quad (58)$$

If this condition is satisfied we have a regular metric with $\partial/\partial x$ vanishing at the endpoints $\sigma = \sigma_1, \sigma_2$ and therefore \mathcal{H} has S^2 topology as expected.

Let us first consider $\Lambda = 0$ so $Q(\sigma) = -C^2\sigma^2 + c_1\sigma + c_2 = C^2(\sigma - \sigma_1)(\sigma_2 - \sigma)$. It follows that $\dot{Q}(\sigma_1) = -\dot{Q}(\sigma_2)$, and therefore using the condition for the absence of conical singularities (58), we have $\Gamma(\sigma_1) = \Gamma(\sigma_2)$. Since the roots must be distinct, using the form of Γ we see that $\sigma_1 = -\sigma_2 \neq 0$. This implies $c_1 = 0$ and from the expression for c_2 we get $\sigma_1 = -2\beta^{-1}$ so $Q = C^2(4\beta^{-2} - \sigma^2)$. Define a new coordinate $\phi = C^2x$, a parameter $a = 1/C\sqrt{\beta}$, and rescale $\sigma \rightarrow 2\sigma/\beta$. The horizon metric then becomes

$$\gamma_{ab}dx^a dx^b = a^2 \left(\frac{1 + \sigma^2}{1 - \sigma^2} \right) d\sigma^2 + 4a^2 \left(\frac{1 - \sigma^2}{1 + \sigma^2} \right) d\phi^2 \quad (59)$$

and regularity implies ϕ to be 2π periodic. This is an inhomogeneous metric on S^2 with $\partial/\partial\phi$ vanishing at $\sigma = \pm 1$. The full near-horizon geometry, upon rescaling $v \rightarrow \beta v/2$, is now given by

$$ds^2 = \frac{1 + \sigma^2}{2} \left[-\frac{r^2}{2a^2} dv^2 + 2dvdr \right] + a^2 \left(\frac{1 + \sigma^2}{1 - \sigma^2} \right) d\sigma^2 + 4a^2 \left(\frac{1 - \sigma^2}{1 + \sigma^2} \right) \left(d\phi + \frac{r}{2a^2} dv \right)^2. \quad (60)$$

This coincides exactly with the near-horizon geometry of extremal Kerr as given in Ref. 22 upon the change in variables $\sigma = \cos \theta$. This proves the following.

The only 4D Ricci flat axisymmetric near-horizon geometry with a nontoroidal horizon section is that of the extremal Kerr black hole.

Now consider the $\Lambda < 0$ case and set $\Lambda = -3g^2$. We have argued that Q must have distinct roots $\sigma_1 < \sigma_2$ and be positive in the interval in between these roots. Therefore, since Q is a quartic with a positive σ^4 coefficient, it must have four real roots and further they must be all distinct (for compactness), such that $\sigma_0 < \sigma_1 < \sigma_2 < \sigma_3$. Therefore

$$Q = \frac{\beta g^2}{4} (\sigma - \sigma_0)(\sigma - \sigma_1)(\sigma - \sigma_2)(\sigma - \sigma_3), \quad (61)$$

and due to the absence of a cubic term in Q we must have

$$\sigma_1 + \sigma_2 + \sigma_3 + \sigma_0 = 0. \quad (62)$$

The condition for the absence of conical singularities (58) becomes

$$\frac{(\sigma_2 - \sigma_0)(\sigma_3 - \sigma_2)}{(\sigma_1 - \sigma_0)(\sigma_3 - \sigma_1)} = \frac{\Gamma(\sigma_2)}{\Gamma(\sigma_1)}. \quad (63)$$

Now, we prove that this implies $\Gamma(\sigma_1) = \Gamma(\sigma_2)$. To do so, first assume $\Gamma(\sigma_2) > \Gamma(\sigma_1)$. This implies that the left hand side of (63) is greater than 1 and in turn this implies, using (62), that $\sigma_2 + \sigma_1 < 0$ and thus $\sigma_2^2 < \sigma_1^2$. It follows that $\Gamma(\sigma_2) < \Gamma(\sigma_1)$ in contradiction to our assumption. Similarly assuming $\Gamma(\sigma_2) < \Gamma(\sigma_1)$ implies $\sigma_1 + \sigma_2 > 0$ and hence $\sigma_2^2 > \sigma_1^2$ providing another contradiction. We conclude that $\Gamma(\sigma_1) = \Gamma(\sigma_2)$ and hence $\sigma_2 = -\sigma_1 \neq 0$. From (62) it follows that $\sigma_3 = -\sigma_0$ and therefore Q is an even function of σ , i.e., $c_1 = 0$.

Now we will show that the⁷¹ $c_1 = 0$ solution is the near-horizon limit of Kerr-AdS₄. Comparing coefficients of Q gives

$$\sigma_2^2 + \sigma_3^2 = \frac{4C^2}{g^2\beta} - \frac{24}{\beta^2}, \quad (64)$$

$$\sigma_2^2\sigma_3^2 = \frac{16C^2}{g^2\beta^3} - \frac{48}{\beta^4}. \quad (65)$$

These two equations are equivalent to

$$(\beta\sigma_2)^2(\beta\sigma_3)^2 - 4(\beta\sigma_2)^2 - 4(\beta\sigma_3)^2 = 48, \quad (66)$$

$$(\beta\sigma_2)^2(\beta\sigma_3)^2 - 2(\beta\sigma_2)^2 - 2(\beta\sigma_3)^2 = \frac{8C^2\beta}{g^2}. \quad (67)$$

Now define two positive constants a , r_+ by

$$a \equiv \frac{\sigma_2}{g\sigma_3}, \quad r_+ \equiv \frac{2}{g\beta\sigma_3} \quad (68)$$

and so it follows $ag < 1$. Note that the parameters a and r_+ are actually invariant under the scale transformation (56). Using these definitions to eliminate σ_2 , σ_3 from (66) implies $g^2r_+^2 < 1$ and

$$a^2 = \frac{r_+^2(1 + 3g^2r_+^2)}{1 - g^2r_+^2}. \quad (69)$$

Next, eliminate σ_2 , σ_3 in (67) and then use the expression for a (69) to get

$$\beta C^2 = \frac{1 + 6g^2r_+^2 - 3g^4r_+^4}{r_+^2(1 - g^2r_+^2)} = \frac{1 + a^2g^2 + 6g^2r_+^2}{r_+^2}. \quad (70)$$

Next use the scale invariance (56) of the near-horizon geometry to set

$$C^2 = \frac{1 + a^2g^2 + 6g^2r_+^2}{\Xi(r_+^2 + a^2)} \quad (71)$$

where we define $\Xi \equiv 1 - a^2g^2$. Using this choice of C^2 (70) implies

$$\beta = \frac{\Xi(r_+^2 + a^2)}{r_+^2}. \quad (72)$$

Plugging this into the definition of r_+ gives

$$\sigma_3 = \frac{2r_+}{g\Xi(r_+^2 + a^2)}, \quad (73)$$

and then from the definition of a , it follows that

$$\sigma_2 = \frac{2r_+a}{\Xi(r_+^2 + a^2)}. \quad (74)$$

Finally change coordinates from (σ, x) to (θ, ϕ) defined by

$$\phi = \frac{2ar_+x}{(r_+^2 + a^2)^2}, \quad \cos \theta = \frac{\sigma}{\sigma_2} \quad (75)$$

so $0 \leq \theta \leq \pi$ provides a unique parametrization of the interval. This gives

$$Q = \frac{4r_+^2 a^2 \sin^2 \theta \Delta_\theta}{\Xi^3(r_+^2 + a^2)^3}, \quad \Gamma = \frac{\rho_+^2}{\Xi(r_+^2 + a^2)}, \quad (76)$$

where $\Delta_\theta = 1 - a^2 g^2 \cos^2 \theta$ and $\rho_+^2 = r_+^2 + a^2 \cos^2 \theta$. It follows that

$$\frac{\Gamma d\sigma^2}{Q} + \frac{Q}{\Gamma} dx^2 = \frac{\rho_+^2 d\theta^2}{\Delta_\theta} + \frac{\sin^2 \theta \Delta_\theta (r_+^2 + a^2)^2}{\rho_+^2 \Xi^2} d\phi^2, \quad (77)$$

and it is easy to see that the absence of conical singularities implies $\phi \sim \phi + 2\pi$. Inspecting the appendices we see that this is exactly the horizon geometry of Kerr-AdS₄ and the rest of the near-horizon data Γ, k^ϕ also agrees. Therefore we have proved the following.

The only 4D axisymmetric near-horizon geometry with a nontoroidal horizon section which satisfies $R_{\mu\nu} = \Lambda g_{\mu\nu}$, with $\Lambda < 0$, is the near-horizon limit of Kerr-AdS₄.

This completes the proof of Theorem 1 stated in Sec. I.

V. Five Dimensions

A. Near-horizon equations

In 5D it is useful to rewrite the horizon metric as

$$\gamma_{ab} dx^a dx^b = d\rho^2 + \gamma_{11}(\rho)(dx^1 + \omega(\rho)dx^2)^2 + \frac{\gamma(\rho)}{\gamma_{11}(\rho)}(dx^2)^2, \quad (78)$$

where we define $\omega(\rho) \equiv \gamma_{12}/\gamma_{11}$ and recall $\gamma = \det \gamma_{ij}$. We have already determined $k^i k_i = \gamma_{11}$ in terms of Γ, Q (39). Since we also know $\gamma = \sigma'^2 = Q/\Gamma$ we need to determine only one other component of γ_{ij} , say γ_{12} or equivalently ω .

Consider (32), which since we have chosen $k = \partial/\partial x^1$ is equivalent to the R_{11} equation. To simplify this equation we will need the identity

$$k_i' \gamma^{ij} k_j' \equiv \frac{(\gamma_{11}')^2}{\gamma_{11}} + \frac{\gamma_{11}^3}{\gamma} \left[\left(\frac{\gamma_{12}}{\gamma_{11}} \right)' \right]^2, \quad (79)$$

substitute for $\gamma = \sigma'^2 = Q/\Gamma$, convert all ρ derivatives to σ derivatives, and note the fact $\sigma'' = (d/d\sigma)(Q/2\Gamma)$. The result is that (32) becomes

$$\gamma_{11}^2 \dot{\omega}^2 = \frac{1}{\Gamma} \frac{d}{d\sigma} \left(\frac{Q \dot{\gamma}_{11}}{\gamma_{11}} \right) + 2\Lambda + \frac{\gamma_{11}}{\Gamma^2}. \quad (80)$$

Now consider the $\rho\rho$ component of (15) which is given by equating (26) and (28). To evaluate (26) it proves useful to note the identity

$$\begin{aligned} \gamma^{jj} \gamma'_{jk} \gamma^{km} \gamma'_{ml} &\equiv \frac{(\gamma'_{11})^2}{\gamma_{11}^2} + \left(\frac{\gamma'_{11}}{\gamma_{11}} - \frac{\gamma'}{\gamma} \right)^2 + \frac{2\gamma_{11}^2}{\gamma} \left[\left(\frac{\gamma_{12}}{\gamma_{11}} \right)' \right]^2 \\ &= \frac{Q(\dot{\gamma}_{11})^2}{\Gamma \gamma_{11}^2} + \frac{Q}{\Gamma} \left(\frac{\dot{\gamma}_{11}}{\gamma_{11}} - \frac{\Gamma}{Q} \frac{d}{d\sigma} \left(\frac{Q}{\Gamma} \right) \right)^2 + 2\gamma_{11}^2 \dot{\omega}^2, \end{aligned} \quad (81)$$

wherein the second line we have converted to σ derivatives. The other term in (26) is given by $\log \gamma'$, which using $\gamma = \sigma'^2$ contains a σ''' , and we eliminate this using (35). After some calculation the $\rho\rho$ equation simplifies to

$$\gamma_{11}^2 \dot{\omega}^2 = \frac{2C^2}{\Gamma} + 4\Lambda - \frac{Q\ddot{\Gamma}}{\Gamma^2} + \frac{\dot{Q}\dot{\Gamma}}{\Gamma^2} - \frac{Q\dot{\gamma}_{11}^2}{\Gamma \gamma_{11}^2} + \frac{\dot{\gamma}_{11}}{\gamma_{11}} \frac{d}{d\sigma} \left(\frac{Q}{\Gamma} \right). \quad (82)$$

Equating (80) and (82), using (39) to write the γ_{11}/Γ^2 term in (80), leads to

$$\frac{d}{d\sigma} (\Gamma \dot{\gamma}_{11}) + \left(2\ddot{\Gamma} - \frac{\dot{\Gamma}^2}{\Gamma} \right) \gamma_{11} = 0. \quad (83)$$

Differentiating (39) with respect to σ gives

$$\dot{\gamma}_{11} = Q \frac{d^3 \Gamma}{d\sigma^3} + \left(2\ddot{\Gamma} - \frac{\dot{\Gamma}^2}{\Gamma} \right) \left(\dot{Q} - \frac{Q\dot{\Gamma}}{\Gamma} \right) - 2\Lambda \Gamma \dot{\Gamma}, \quad (84)$$

where (37) has been used to eliminate \ddot{Q} . Substituting (84) and (39) into (83), again using (37) to eliminate \ddot{Q} , leads to the remarkably simple equation,

$$Q \frac{d^4 \Gamma}{d\sigma^4} + \left(3\dot{Q} - \frac{\dot{\Gamma}Q}{\Gamma} \right) \frac{d^3 \Gamma}{d\sigma^3} - 10\Lambda \Gamma \ddot{\Gamma} = 0, \quad (85)$$

which can be written more compactly as

$$\frac{d}{d\sigma} \left(\frac{Q^3 d^3 \Gamma}{\Gamma d\sigma^3} \right) - 10\Lambda Q^2 \ddot{\Gamma} = 0. \quad (86)$$

We must now examine the remaining components of the near-horizon equations, i.e., the $x^1 x^2$ components of (15). One can check that the 12 component of (25) is

$$\begin{aligned} R_{12} &= -\frac{\gamma''_{11} \omega}{2} - \gamma'_{11} \omega' - \frac{\gamma_{11} \omega''}{2} + \frac{\gamma_{11} \gamma' \omega'}{4\gamma} + \frac{\gamma_{11}^3 \omega \omega'^2}{2\gamma} + \frac{\gamma_{11}^2 \omega}{2\gamma_{11}} - \frac{\gamma'_{11} \gamma' \omega}{4\gamma} \\ &= -\frac{Q\ddot{\gamma}_{11} \omega}{2\Gamma} - \frac{d}{d\sigma} \left(\frac{Q}{\Gamma} \right) \frac{\dot{\gamma}_{11} \omega}{2} + \frac{\gamma_{11}^3 \omega \dot{\omega}^2}{2} - \frac{Q\dot{\gamma}_{11} \dot{\omega}}{\Gamma} - \frac{Q\gamma_{11} \ddot{\omega}}{2\Gamma} + \frac{Q\dot{\gamma}_{11}^2 \omega}{2\Gamma \gamma_{11}}, \end{aligned} \quad (87)$$

and (29) requires that

$$R_{12} = \frac{\gamma_{11}^2 \omega}{2\Gamma^2} + \frac{Q\dot{\Gamma}}{2\Gamma^2} [\dot{\omega} \gamma_{11} + \omega \dot{\gamma}_{11}] + \Lambda \gamma_{11} \omega. \quad (88)$$

Eliminating the $\dot{\omega}^2$ term in (87) using (80) leads to many cancellations and the $x^1 x^2$ component of (15) becomes simply

$$\gamma_{11} \dot{\omega} + 2\dot{\gamma}_{11} \omega + \frac{\dot{\Gamma}}{\Gamma} \gamma_{11} \dot{\omega} = 0, \quad (89)$$

which integrates to

$$\dot{\omega} = \frac{k}{\gamma_{11}^2 \Gamma}, \quad (90)$$

where k is a constant. In fact, (90) is automatically satisfied as a consequence of the other components of the near-horizon equations. Indeed, using (82) and (39) one can check that $(d/d\sigma)(\gamma_{11}^4 \Gamma^2 \dot{\omega}^2) = 0$ as a consequence of (37) and (86).

In fact, the above equations exhibit certain scaling symmetries which translate to scaling symmetries of the full near-horizon geometry. It is important to keep track of these when it comes to counting the parameters of a solution. The two ODEs (37) and (86) possess the following two symmetries:

$$\mathcal{S}_1: Q \rightarrow K^3 Q, \quad \Gamma \rightarrow K\Gamma, \quad C^2 \rightarrow KC^2, \quad \sigma \rightarrow K\sigma, \quad (91)$$

$$\mathcal{S}_2: Q \rightarrow L^2 Q, \quad \sigma \rightarrow L\sigma \quad (92)$$

for constant $K > 0$ and constant L (of either sign). It follows that

$$\mathcal{S}_1: \gamma_{11} \rightarrow K^2 \gamma_{11}, \quad x^1 \rightarrow K^{-1} x^1, \quad v \rightarrow K^{-1} v, \quad (93)$$

$$\mathcal{S}_2: \gamma_{12} \rightarrow L \gamma_{12}, \quad x^2 \rightarrow L^{-1} x^2 \quad (94)$$

provide scaling symmetries of the full near-horizon geometry. Observe that these scalings can be combined, e.g., $\mathcal{S}_2^{-1} \mathcal{S}_1$ (with $K=L$) generates the near-horizon symmetry $Q \rightarrow KQ$, $\Gamma \rightarrow K\Gamma$, $C^2 \rightarrow KC^2$, $x^1 \rightarrow K^{-1} x^1$, $x^2 \rightarrow Kx^2$, $v \rightarrow K^{-1} v$.

Summary: We have shown that the functions $\Gamma(\sigma)$ and $Q(\sigma)$ satisfy the coupled ODEs (37) and (86). Further, given a solution to these ODEs ($\Gamma(\sigma), Q(\sigma)$), a near-horizon geometry satisfying the vacuum Einstein equations $R_{\mu\nu} = \Lambda g_{\mu\nu}$ can be constructed as follows. Firstly γ_{11} is determined from (39); next $\omega = \gamma_{12}/\gamma_{11}$ can be got up to quadratures from either (80) or (82); finally note (38) gives $\gamma_{\sigma\sigma}$. This determines the horizon metric (78) in the coordinates (σ, x^1, x^2) . Recalling that we chose a gauge where $k^i = \delta_1^i$, one can write down the full near-horizon geometry from (22).

B. A class of near-horizon geometries with S^3 horizons

Observe that one set of solutions to (86) is given by

$$\Gamma = a_1 \sigma + a_0, \quad (95)$$

where a_1, a_0 are constants. Then, (37) implies

$$Q = -\Lambda a_1 \sigma^3 - (C^2 + 3\Lambda a_0) \sigma^2 + c_1 \sigma + c_2, \quad (96)$$

where c_1, c_2 are integration constants. The analysis naturally splits into two, depending on whether a_1 vanishes or not.⁷²

1. Homogeneous horizon

First, suppose $a_1 = 0$ and so Γ is a constant. Then, the equation for $k^i k_i$ (39) gives

$$\gamma_{11} = 2C^2 \Gamma + 2\Gamma^2 \Lambda, \quad (97)$$

which is a constant and thus $C^2 + \Lambda \Gamma > 0$. Equation (80) gives

$$\dot{\omega}^2 = \frac{(C^2 + 2\Lambda \Gamma)}{2\Gamma^3 (C^2 + \Lambda \Gamma)^2}, \quad (98)$$

which is also a constant and implies $C^2 + 2\Lambda \Gamma \geq 0$. Therefore

$$\omega = \pm \left(\frac{(C^2 + 2\Lambda\Gamma)}{2\Gamma^3(C^2 + \Lambda\Gamma)^2} \right)^{1/2} \sigma + c_3, \quad (99)$$

where c_3 is an integration constant. We may set $c_3=0$ using the coordinate freedom of the $x^1 \rightarrow x^1 + \text{const } x^2$ which we will now assume we have done. Note that $Q = -(C^2 + 3\Lambda\Gamma)\sigma^2 + c_1\sigma + c_2$, and since σ is only defined up to an additive constant, without loss of generality we may translate σ in order to set $c_1=0$. This implies $Q = c_2 - (C^2 + 3\Lambda\Gamma)\sigma^2$. Recall that in order to have a compact horizon one needs $\sigma_1 \leq \sigma \leq \sigma_2$ with $Q \geq 0$ in this interval and vanishing only at the endpoints. It is easy to see this implies $C^2 + 3\Lambda\Gamma > 0$ (which is automatic when $\Lambda=0$). It now follows that $c_2 > 0$ and $\sigma_2 = -\sigma_1 = \sqrt{c_2(C^2 + 3\Lambda\Gamma)^{-1}}$. We now define new coordinates (θ, ψ, ϕ) as follows:

$$\cos \theta = \frac{\sigma}{\sigma_2}, \quad \phi = \pm x^2 \sqrt{\frac{c_2(C^2 + 3\Lambda\Gamma)}{2\Gamma^3(C^2 + \Lambda\Gamma)}}, \quad \psi = x^1(C^2 + 3\Lambda\Gamma) \sqrt{\frac{C^2 + \Lambda\Gamma}{C^2 + 2\Lambda\Gamma}} \quad (100)$$

so that $0 \leq \theta \leq \pi$ parametrizes the interval $\sigma_1 \leq \sigma \leq \sigma_2$ uniquely and $Q = c_2 \sin^2 \theta$. The near-horizon data are then given by

$$\gamma_{ab} dx^a dx^b = \frac{2\Gamma(C^2 + 2\Lambda\Gamma)}{(C^2 + 3\Lambda\Gamma)^2} (d\psi + \cos \theta d\phi)^2 + \frac{\Gamma}{C^2 + 3\Lambda\Gamma} (d\theta^2 + \sin^2 \theta d\phi^2), \quad (101)$$

$$k^\psi = (C^2 + 3\Lambda\Gamma) \sqrt{\frac{C^2 + \Lambda\Gamma}{C^2 + 2\Lambda\Gamma}}, \quad (102)$$

with Γ a constant. It is clear that regularity of the metric on \mathcal{H} implies the usual restrictions $0 \leq \psi \leq 4\pi$ and $0 \leq \phi \leq 2\pi$ resulting in a homogeneous metric on S^3 written in Euler angles. This near-horizon geometry has the scaling symmetry,

$$C^2 \rightarrow KC^2, \quad \Gamma \rightarrow K\Gamma, \quad v \rightarrow K^{-1}v, \quad (103)$$

where $K > 0$ is a constant. This allows one to fix one (or a combination) of the parameters (C^2, Γ) of the above solution, and therefore it is a one-parameter family. In fact, as we show in the appendices that it is isometric to the near-horizon limit of the extremal self-dual rotating AdS₅ black hole⁴⁰ (i.e., with $J_1 = J_2$). In the case $\Lambda=0$ it turns out (as we also show in the Appendices) that it is also isometric to the near-horizon limit of the $J=0$ extremal KK black hole.³⁷

2. Inhomogeneous horizon

Now, suppose $a_1 \neq 0$. We are free to perform a translation in σ to set $a_0=0$, which without loss of generality we will do. The equation for $k^i k_i$ (39) gives

$$\gamma_{11} = a_1 \left(C^2 \sigma - \frac{c_2}{\sigma} \right). \quad (104)$$

We can now solve for ω using (82). After some calculation, Eq. (82) gives

$$\dot{\omega}^2 = \frac{4\sigma^2 c_2 (\Lambda a_1 c_2 - c_1 C^2)}{a_1^3 (C^2 \sigma^2 - c_2)^4}, \quad (105)$$

and therefore the parameters must satisfy the inequality

$$c_2 (\Lambda a_1 c_2 - c_1 C^2) \geq 0. \quad (106)$$

Integrating one gets

$$\omega = \pm \frac{\sqrt{a_1^{-3}c_2(\Lambda a_1c_2 - c_1C^2)}}{C^2(C^2\sigma^2 - c_2)} + c_3, \quad (107)$$

where c_3 is a constant. Collecting the above results the horizon metric is

$$\gamma_{ab}dx^a dx^b = \frac{a_1\sigma d\sigma^2}{Q(\sigma)} + a_1\left(C^2\sigma - \frac{c_2}{\sigma}\right)\left(dx^1 + \frac{\sqrt{a_1^{-3}c_2(\Lambda a_1c_2 - c_1C^2)}}{C^2(C^2\sigma^2 - c_2)}dx^2\right)^2 + \frac{Q(\sigma)(dx^2)^2}{a_1^2(C^2\sigma^2 - c_2)}, \quad (108)$$

where by shifting $x^1 \rightarrow x^1 + \text{const } x^2$, we have eliminated the constant c_3 , used the freedom $x^2 \rightarrow \pm x^2$ to arrange $\omega > 0$, and

$$Q = -\Lambda a_1\sigma^3 - C^2\sigma^2 + c_1\sigma + c_2. \quad (109)$$

This near-horizon metric has two independent scaling symmetries (corresponding to \mathcal{S}_1 and \mathcal{S}_2),

$$C^2 \rightarrow KC^2, \quad c_1 \rightarrow K^2c_1, \quad c_2 \rightarrow K^3c_2, \quad \sigma \rightarrow K\sigma, \quad x^1 \rightarrow K^{-1}x^1, \quad v \rightarrow K^{-1}v, \quad (110)$$

where $K > 0$ is constant, and

$$a_1 \rightarrow L^{-1}a_1, \quad c_1 \rightarrow Lc_1, \quad c_2 \rightarrow L^2c_2, \quad \sigma \rightarrow L\sigma, \quad x^2 \rightarrow L^{-1}x^2, \quad (111)$$

where L is constant (which can be either sign). These allow one to fix two (or two combinations) of the parameters (C^2, a_1, c_1, c_2) and thus this solution is a two-parameter family.

3. Global analysis of inhomogeneous horizon

We now turn to a global analysis of the $a_1 \neq 0$ solution just derived. First we will use the second scaling symmetry (111) to fix $a_1 = 1$ and thus $\Gamma = \sigma$. Since $\Gamma > 0$ we see that $\sigma > 0$. Now, observe that since $\gamma_{11} \geq 0$ (with equality only possible at isolated points), we must have $\sigma_1^2 \geq c_2C^{-2}$. In fact, it is easy to show that the case⁷³ $\sigma_1^2 = c_2C^{-2}$ (so $c_2 > 0$) is incompatible with $\dot{Q}(\sigma_1) > 0$ and $\sigma_1 < \sigma_2$. Therefore, we must have $\sigma_1^2 > c_2C^{-2}$, which implies we have $\gamma_{11} > 0$ everywhere, and therefore the 2-metric γ_{ij} degenerates only at the zeros of $Q(\sigma)$. From the form of the metric on the horizon it follows that the Killing vectors,

$$m_i = d_i \left(\frac{\partial}{\partial x^2} - \omega(\sigma_i) \frac{\partial}{\partial x^1} \right) \quad (112)$$

for constants d_i and $i=1,2$ vanish at the degeneration points $\sigma = \sigma_i$. Further, since $\omega(\sigma_1) \neq \omega(\sigma_2)$ it follows that $m_1 \neq m_2$. Regularity of the metric on the horizon requires the orbits of m_i to close in such a way there are no conical singularities at the points where they vanish. We choose the constants d_i such that in terms of adapted coordinates defined by $m_i = \partial/\partial\phi_i$, the periodicity of the orbits is given by $\phi_i \sim \phi_i + 2\pi$. The coordinate transformation between (x^1, x^2) and (ϕ_1, ϕ_2) is given by

$$x^1 = -[\omega(\sigma_1)d_1\phi_1 + \omega(\sigma_2)d_2\phi_2], \quad x^2 = d_1\phi_1 + d_2\phi_2. \quad (113)$$

To ensure the absence of the conical singularities at $\sigma = \sigma_1$ and $\sigma = \sigma_2$, one must take

$$d_i^2 = \frac{4\sigma_i(C^2\sigma_i^2 - c_2)}{\dot{Q}(\sigma_i)^2}, \quad (114)$$

which therefore determines the d_i up to a sign. The solution is now globally regular, with m_1 vanishing at $\sigma = \sigma_1$ and m_2 vanishing at $\sigma = \sigma_2$. Hence the horizon \mathcal{H} has S^3 topology (or that of a Lens space).

Now we will show that this near-horizon geometry is, in fact, isometric to the near-horizon limit of known black holes. In the $\Lambda = 0$ case we will show that it is isometric to the near-horizon

limits of two different known extremal black holes: the Myers–Perry ($J_1 \neq J_2$) and the slowly rotating KK black hole ($0 < G_4 J < PQ$). In the $\Lambda < 0$ case we will show that it is isometric to the near-horizon limit of the known extremal rotating AdS₅ black hole⁴⁰ ($J_1 \neq J_2$). We provide the near-horizon limits of all these black holes in the appendices.

$\Lambda=0$ case: In this case some of the above formulas simplify. In particular, using $Q(\sigma_i)=0$ one gets $C^2\sigma_i^2 - c_2 = c_1\sigma_i$. Therefore, since above we argued that $C^2\sigma_i^2 - c_2 > 0$, it follows that $c_1 > 0$. Then we see that (106) implies $c_2 \leq 0$. Further, the fact that Q must have two positive roots requires $c_2 < 0$ and $c_1^2 + 4C^2c_2 > 0$. Using these results one gets

$$d_i^2 = \frac{4c_1\sigma_i^2}{c_1^2 + 4C^2c_2}, \quad \omega(\sigma_i) = \frac{\sqrt{-c_2c_1}}{c_1C\sigma_i}. \quad (115)$$

In fact, from the results of Ref. 22, it is straightforward to show that this near-horizon geometry is isometric to the near-horizon limit of the 5D extremal Myers–Perry solution. To see this, first using the scaling freedom (110) to set $C^2 = c_1$ (this can be done as C^2 and c_1 transform differently) and hence $c_1 + 4c_2 > 0$. Next define two positive constants $a > b > 0$ by

$$a \equiv \frac{1}{\sqrt{c_1}} + \frac{\sqrt{c_1 + 4c_2}}{c_1}, \quad b \equiv \frac{1}{\sqrt{c_1}} - \frac{\sqrt{c_1 + 4c_2}}{c_1}, \quad (116)$$

from which it follows that

$$C^2 = c_1 = \frac{4}{(a+b)^2}, \quad c_2 = -\frac{4ab}{(a+b)^4}, \quad \sigma_1 = \frac{b}{a+b}, \quad \sigma_2 = \frac{a}{a+b}. \quad (117)$$

The coordinate change defined by

$$\cos^2 \theta = \frac{\sigma - \sigma_1}{\sigma_2 - \sigma_1}, \quad x^1 = \frac{\sqrt{ab}(a+b)^2}{2(a-b)}(\psi - \phi), \quad x^2 = \frac{(a+b)}{(a-b)}(b\psi - a\phi), \quad (118)$$

where $0 \leq \theta \leq \pi/2$ and $\psi = \phi_1$ and $\phi = \phi_2$, shows that our near-horizon geometry is identical to that of extremal Myers–Perry as given in the appendices in (θ, ψ, ϕ) coordinates and (a, b) parameters (which is also the same form as in Ref. 22).

Now we will show how our near-horizon geometry is also isometric to the near-horizon geometry of the slowly rotating extremal KK black hole. Define the following positive parameters:

$$p \equiv \frac{1}{C^2} \sqrt{c_1 \left(1 - \frac{c_2}{C^2}\right)}, \quad q^2 \equiv \frac{c_1}{c_2^2} \left(1 - \frac{c_2}{C^2}\right), \quad \eta^2 \equiv 1 + \frac{4C^2c_2}{c_1^2}, \quad (119)$$

so $\eta < 1$. It follows that

$$C^2 = \frac{2(p+q)}{(pq)^{3/2}(1-\eta^2)^{1/2}}, \quad c_1 = \frac{2C^2}{\sqrt{1-\eta^2}} \sqrt{\frac{p}{q}}, \quad c_2 = -\frac{C^2p}{q}, \quad (120)$$

and

$$\sigma_1 = \sqrt{\frac{p}{q(1-\eta^2)}}(1-\eta), \quad \sigma_2 = \sqrt{\frac{p}{q(1-\eta^2)}}(1+\eta). \quad (121)$$

Writing the near-horizon geometry in coordinates (θ, y, ϕ) defined by

$$\cos \theta = \frac{2\sigma - \sigma_1 - \sigma_2}{\sigma_2 - \sigma_1}, \quad x^1 = -\frac{\sqrt{1-\eta^2}}{C^2\eta}\phi, \quad x^2 = \frac{2}{C^2q}\sqrt{\frac{(p+q)}{p(1-\eta^2)}}\left(\frac{\phi}{\eta} + \sqrt{\frac{p+q}{p^3}}y\right), \quad (122)$$

where $0 \leq \theta \leq \pi$, shows that it is identical to the near-horizon limit of the slowly rotating extremal KK black hole given in the appendices in (θ, y, ϕ) coordinates and (p, q, η) parameters.

$\Lambda < 0$ case: Set $\Lambda = -4g^2$. It is convenient to work with the roots $\sigma_1, \sigma_2, \sigma_3$ of Q as parameters as well as the original parameters C^2, c_1, c_2 . These are related by

$$C^2 = 4g^2(\sigma_1 + \sigma_2 + \sigma_3), \quad c_1 = 4g^2(\sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_3), \quad c_2 = -4g^2\sigma_1\sigma_2\sigma_3, \quad (123)$$

so $Q = 4g^2(\sigma - \sigma_1)(\sigma - \sigma_2)(\sigma - \sigma_3)$ where $\sigma_3 > \sigma_2$. Define the quantity $W = (\sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_3)/\sigma_1\sigma_2$ which is invariant under the scaling freedom (110). Use the scaling freedom (110) to set $\sigma_3/\sigma_1\sigma_2 = W$; this can be done as the left hand side transforms homogeneously and the right hand side is invariant. This implies $\sigma_1 + \sigma_2 < 1$ and

$$\sigma_3 = \frac{\sigma_1\sigma_2}{1 - \sigma_1 - \sigma_2}. \quad (124)$$

Now define the positive constants a, b, r_+ by

$$\frac{1}{1 + g^2r_+^2} = \sigma_1 + \sigma_2, \quad \frac{r_+^2}{r_+^2 + a^2} = \sigma_1, \quad \frac{r_+^2}{r_+^2 + b^2} = \sigma_2, \quad (125)$$

so $a > b$ (as $\sigma_1 < \sigma_2$). This implies that

$$\sigma_3 = \frac{r_+^2(1 + g^2r_+^2)}{g^2(r_+^2 + a^2)(r_+^2 + b^2)} \quad (126)$$

and

$$C^2 = \frac{4r_+^2(1 + a^2g^2 + b^2g^2 + 3g^2r_+^2)}{(r_+^2 + a^2)(r_+^2 + b^2)}. \quad (127)$$

Now define a new variable θ by

$$\cos^2 \theta = \frac{\sigma - \sigma_1}{\sigma_2 - \sigma_1}, \quad (128)$$

so $0 \leq \theta \leq \pi/2$ uniquely parametrizes the interval $\sigma_1 \leq \sigma \leq \sigma_2$. This implies

$$\Gamma = \sigma = \frac{r_+^2\rho_+^2}{(r_+^2 + a^2)(r_+^2 + b^2)}, \quad Q = \frac{4r_+^6(a^2 - b^2)^2 \sin^2 \theta \cos^2 \theta \Delta_\theta}{(r_+^2 + a^2)^3 (r_+^2 + b^2)^3}, \quad (129)$$

where we have defined

$$\rho_+^2 = r_+^2 + a^2 \cos^2 \theta + b^2 \sin^2 \theta, \quad \Delta_\theta = 1 - a^2g^2 \cos^2 \theta - b^2g^2 \sin^2 \theta. \quad (130)$$

It follows that

$$\frac{\Gamma d\sigma^2}{Q} = \frac{\rho_+^2 d\theta^2}{\Delta_\theta}, \quad (131)$$

which proves that the $\sigma\sigma$ component of our horizon metric coincides with the $\theta\theta$ component of the known extremal rotating AdS₅ black hole of Ref. 40 (see the appendices). It remains to check

the $x^i x^j$ components of the horizon metric. To do this we need the constants $d_i, \omega(\sigma_i)$ appearing in the coordinate transformation (113) which work out to be

$$d_1 = -\frac{r_+^2 + b^2}{\Xi_b(a^2 - b^2)} \sqrt{(1 + b^2 g^2 + 2g^2 r_+^2)(2r_+^2 + a^2 + b^2)}, \quad (132)$$

$$d_2 = \frac{r_+^2 + a^2}{\Xi_a(a^2 - b^2)} \sqrt{(1 + a^2 g^2 + 2g^2 r_+^2)(2r_+^2 + a^2 + b^2)}, \quad (133)$$

$$\omega(\sigma_1) = \sqrt{\frac{(r_+^2 + b^2)(r_+^2 + a^2)^3(1 + a^2 g^2 + 2g^2 r_+^2)(1 + g^2 r_+^2)}{4r_+^4(1 + b^2 g^2 + 2g^2 r_+^2)(2r_+^2 + a^2 + b^2)(1 + a^2 g^2 + b^2 g^2 + 3g^2 r_+^2)^2}}, \quad (134)$$

$$\omega(\sigma_2) = \sqrt{\frac{(r_+^2 + a^2)(r_+^2 + b^2)^3(1 + b^2 g^2 + 2g^2 r_+^2)(1 + g^2 r_+^2)}{4r_+^4(1 + a^2 g^2 + 2g^2 r_+^2)(2r_+^2 + a^2 + b^2)(1 + a^2 g^2 + b^2 g^2 + 3g^2 r_+^2)^2}}, \quad (135)$$

where we have defined $\Xi_a = 1 - g^2 a^2$, $\Xi_b = 1 - g^2 b^2$, and without loss of generality we have chosen a particular sign for each of the d_i (note $d_1 < 0$ and $d_2 > 0$). Using the transformation (113) one can now compute the $\phi_i \phi_j$ components of the horizon metric. We have checked that $\gamma_{\phi_i \phi_j}$ is identical to the $a, b = \psi, \phi$ components of the horizon metric of the rotating AdS₅ black hole solutions of Ref. 40 (see the appendices) upon identifying $\phi_1 = \psi$ and $\phi_2 = \phi$. Therefore, we have verified that the horizon metric of our solutions coincides exactly with that of the known extremal rotating AdS₅ black hole. Finally, let us turn to the remaining near-horizon data, the vector $k^i \partial_i = \partial / \partial x^1$. Using the coordinate change (113),

$$\frac{\partial}{\partial x^1} = \frac{1}{d_1[\omega(\sigma_2) - \omega(\sigma_1)]} \frac{\partial}{\partial \phi_1} + \frac{1}{d_1[\omega(\sigma_1) - \omega(\sigma_2)]} \frac{\partial}{\partial \phi_2} \quad (136)$$

$$= \frac{2br_+}{\Xi_b(r_+^2 + b^2)^2} \frac{\partial}{\partial \phi_1} + \frac{2ar_+}{\Xi_a(r_+^2 + a^2)^2} \frac{\partial}{\partial \phi_2}, \quad (137)$$

where the first equality follows from the coordinate change (113) and the second upon using our expressions for $d_i, \omega(\sigma_i)$. Therefore, the k^i agree with those of the extremal rotating AdS₅ black hole upon the same identification $\phi_1 = \psi$ and $\phi_2 = \phi$. Therefore, to summarize, we have proved that $\gamma_{ab}, k^i, C^2, \Gamma$ all coincide with those of the most general known extremal rotating AdS₅ black hole⁴⁰ (as given in the appendices), thus proving equivalence of the near-horizon geometries.

C. All Ricci flat solutions with compact horizons

In the $\Lambda=0$ case we can actually determine all possible near-horizon geometries with compact horizons as we will now show. Equation (86) integrates to

$$Q^3 \frac{d^3 \Gamma}{d\sigma^3} = \alpha \Gamma, \quad (138)$$

where α is a constant. In the appendices we prove that the left hand side is a globally defined function which vanishes at the zeros of Q . Therefore evaluating at one of the zeros of Q implies that $\alpha=0$. It follows that

$$\frac{d^3 \Gamma}{d\sigma^3} = 0, \quad (139)$$

and therefore

$$\Gamma = a_2\sigma^2 + a_1\sigma + a_0, \quad (140)$$

where a_i are integration constants. Also, Eq. (37) determines Q ,

$$Q = -C^2\sigma^2 + c_1\sigma + c_2, \quad (141)$$

where c_1, c_2 are constants. The analysis now splits into two cases: either $a_2=0$ or $a_2 \neq 0$. We have already analyzed the former case in Sec. IV where it was shown that the resulting near-horizon geometry is identical to the near-horizon limit of extremal Myers–Perry, or equivalently the near-horizon limit of the slowly rotating extremal KK black hole.

We now analyze the $a_2 \neq 0$ case. Since σ is only defined up to an additive constant, we can always shift σ to set $a_1=0$, and thus without loss of generality we take

$$\Gamma = a_2\sigma^2 + a_0. \quad (142)$$

Substituting into the equation for $k^i k_i$ (39) gives

$$\gamma_{11} = \frac{2P(\sigma)}{\Gamma}, \quad (143)$$

where we have defined

$$P(\sigma) \equiv \alpha\sigma^2 + \beta\sigma + \gamma \quad (144)$$

and

$$\alpha = -C^2 a_0 a_2 - c_2 a_2^2, \quad \beta = 2a_0 a_2 c_1, \quad \gamma = C^2 a_0^2 + a_2 a_0 c_2, \quad (145)$$

which satisfy $\gamma a_2 + \alpha a_0 = 0$ and the discriminant of the quadratic P is

$$D \equiv \beta^2 - 4\alpha\gamma = 4a_0 a_2 [c_1^2 a_0 a_2 + (C^2 a_0 + a_2 c_2)^2]. \quad (146)$$

Now, plugging into (80) gives

$$\dot{\omega}^2 = \frac{(a_0 C^2 - a_2 c_2)[c_1^2 a_0 a_2 + (C^2 a_0 + a_2 c_2)^2] \Gamma^2}{2P(\sigma)^4}. \quad (147)$$

Notice that this implies that the constants satisfy

$$(a_0 C^2 - a_2 c_2)[c_1^2 a_0 a_2 + (C^2 a_0 + a_2 c_2)^2] \geq 0. \quad (148)$$

The analysis thus splits into a number of subcases. In the appendices we show that $[c_1^2 a_0 a_2 + (C^2 a_0 + a_2 c_2)^2] = 0$ does not lead to a compact horizon and therefore we exclude this. It follows that there are two possibilities: (i) $a_0 C^2 - a_2 c_2 = 0$ or (ii) $a_0 C^2 - a_2 c_2 \neq 0$.

1. Inhomogeneous $S^1 \times S^2$ horizon

We now consider case (i) and eliminate a_0 using $a_0 = a_2 c_2 C^{-2}$. Observe that (147) implies ω is a constant. Also note that in this case the quadratic $P(\sigma) \propto Q(\sigma)$; in particular,

$$\gamma_{11} = \frac{4c_2 a_2^2 Q(\sigma)}{C^2 \Gamma}. \quad (149)$$

The horizon metric reads

$$\gamma_{ab}dx^a dx^b = \frac{a_2(c_2 C^{-2} + \sigma^2)}{Q(\sigma)} d\sigma^2 + \frac{4c_2 a_2}{C^2(c_2 C^{-2} + \sigma^2)} Q(\sigma) (dx^1 + \omega dx^2)^2 + \frac{C^2}{4c_2 a_2^2} (dx^2)^2, \quad (150)$$

with $\Gamma = a_2(c_2 C^{-2} + \sigma^2)$. This metric is nondegenerate everywhere except at the end points $\sigma = \sigma_1$ and $\sigma = \sigma_2$ where $Q=0$. At these points $\partial/\partial x^1$ vanishes and the metric has conical singularities, in general. The simultaneous removal of these conical singularities leads to a regular metric on $S^1 \times S^2$. The condition for this is easily shown to be

$$\frac{\dot{Q}(\sigma_1)}{\Gamma(\sigma_1)} = -\frac{\dot{Q}(\sigma_2)}{\Gamma(\sigma_2)}, \quad (151)$$

which noting $\dot{Q}(\sigma_i) = \mp C^2(\sigma_1 - \sigma_2)$ implies $\Gamma(\sigma_1) = \Gamma(\sigma_2)$. It follows that $\sigma_2 = -\sigma_1$, and hence $c_1 = 0$ and $c_2 > 0$. Since $\Gamma > 0$, now it follows that $a_2 > 0$. Now, rescaling $\sigma \rightarrow \sqrt{c_2} C^{-1} \sigma$ and $x^2 \rightarrow C c_2^{-1/2} x^2$ and defining a new coordinate and parameter by

$$\phi = C^2 x^1, \quad a \equiv \frac{\sqrt{a_2 c_2}}{C^2}, \quad (152)$$

one finds

$$\gamma_{ab}dx^a dx^b = \frac{a^2(1 + \sigma^2)}{1 - \sigma^2} d\sigma^2 + \frac{4a^2(1 - \sigma^2)}{(1 + \sigma^2)} (d\phi + \Omega dx^2)^2 + \frac{1}{4C^4 a^4} (dx^2)^2, \quad (153)$$

where $\Gamma = C^2 a^2 (1 + \sigma^2)$ and we have defined a new constant $\Omega \equiv \omega C^3 c_2^{-1/2}$. The Killing vector $k = \partial/\partial x^1 = C^2 \partial/\partial \phi$ vanishes at $\sigma = \pm 1$; the absence of conical singularities at these points implies $\phi \sim \phi + 2\pi$, and therefore $\partial/\partial \phi$ generates a rotational symmetry. Finally, we use the shift freedom $\phi \rightarrow \phi + \text{const } x^2$ in order to ensure $\partial/\partial x^2$ corresponds to the other rotational symmetry generator, so $x^2 \sim x^2 + L$. We have thus derived a near-horizon geometry whose horizon topology is $S^1 \times S^2$. It is parametrized by (a, C, Ω, L) , although there is a scaling symmetry,

$$C^2 \rightarrow KC^2, \quad \Omega \rightarrow K^{-1}\Omega, \quad L \rightarrow KL, \quad x^2 \rightarrow Kx^2, \quad (154)$$

which allows one to fix a combination of (C, Ω, L) (note that a is invariant) and hence it is a three-parameter family.

In fact, in a particular region of the parameter space, the above near-horizon geometry is isometric to that of the extremal boosted Kerr string. This region is given by $C^2|\Omega| < 1/(4a^3)$ (which is invariant under the scaling symmetry above). In this region define a boost parameter β (invariant under the scaling symmetry) by $\tanh \beta \equiv 4a^3 C^2 \Omega$. Then use the scaling freedom to set $C^2 = 1/(2a^2 \cosh \beta)$, and thus one can solve for $\Omega = (\sinh \beta)/(2a)$. Changing coordinates to $\sigma = \cos \theta$, with $0 \leq \theta \leq \pi$, we see that this near-horizon geometry is identical to that of the extremal boosted Kerr string as given in Ref. 22. Note that the special case $\sinh^2 \beta = 1$ corresponds to the near-horizon geometry of the asymptotically flat extremal vacuum black ring³⁶ as first observed in Ref. 22. It is curious that the boosted Kerr string ‘‘misses’’ the region of parameter space given by $C^2|\Omega| \geq 1/(4a^3)$.

2. Inhomogeneous S^3 horizon

We now analyze case (ii), i.e., $a_0 \neq a_2 c_2 C^{-2}$. It proves convenient to split the analysis into two cases depending on whether $\alpha = 0$ or not. First consider $\alpha \neq 0$. Integrating (147) gives

$$\omega = \pm \left[-\frac{\kappa a_2 \sigma}{\alpha P(\sigma)} + c_3 \right], \quad (155)$$

where for convenience we have defined a constant $\kappa > 0$ by

$$\kappa \equiv \sqrt{\frac{(a_0 C^2 - a_2 c_2)[c_1^2 a_0 a_2 + (C^2 a_0 + a_2 c_2)^2]}{2}}, \tag{156}$$

and c_3 is an integration constant. The remaining equations are satisfied without further constraint.

We will use the shift freedom $x^1 \rightarrow x^1 + \text{const } x^2$ to set $c_3 = 0$ and $x^2 \rightarrow \pm x^2$ to pick a sign for ω . The horizon metric is

$$\gamma_{ab} dx^a dx^b = \frac{\Gamma d\sigma^2}{Q(\sigma)} + \frac{2P(\sigma)}{\Gamma} \left[dx^1 - \frac{\kappa a_2 \sigma}{\alpha P(\sigma)} dx^2 \right]^2 + \frac{Q(\sigma)}{2P(\sigma)} (dx^2)^2. \tag{157}$$

The following identity is easily verified,

$$P(\sigma) \equiv (C^2 a_0 - a_2 c_2) \Gamma(\sigma) + 2a_0 a_2 Q(\sigma), \tag{158}$$

which implies $P(\sigma_i) = (C^2 a_0 - a_2 c_2) \Gamma(\sigma_i)$. For a positive definite metric we must have $P(\sigma_i) \geq 0$, which implies $a_0 > a_2 c_2 C^{-2}$ and thus $P(\sigma_i) > 0$. Observe that from (148) it follows that $[c_1^2 a_0 a_2 + (C^2 a_1 + a_2 c_2)^2] > 0$. There are now two cases to consider: either the discriminant $D \geq 0$ or $D < 0$. Using (146) we see that $D \geq 0$ is then equivalent to $a_0 a_2 \geq 0$ and $D < 0$ is equivalent to $a_0 a_2 < 0$. Therefore, in the case $D \geq 0$, Eq. (158) implies $P(\sigma) > 0$ for $\sigma_1 \leq \sigma \leq \sigma_2$. On the other hand, if $D < 0$, in which case P has no real roots, then it must be the case that $P(\sigma) > 0$ for all σ (so $\alpha > 0$). Therefore, we see that in both cases $P > 0$ for $\sigma_1 \leq \sigma \leq \sigma_2$, and therefore the metric on the horizon is nondegenerate everywhere except at the endpoints σ_1, σ_2 where Q vanishes. The Killing vectors,

$$m_i = d_i \left(\frac{\partial}{\partial x^2} - \omega(\sigma_i) \frac{\partial}{\partial x^1} \right), \tag{159}$$

for constant d_i vanish at the endpoints $\sigma = \sigma_i$, where the metric has conical singularities, in general. Using $Q(\sigma_i) = 0$ it can be shown that $\omega(\sigma_1) \neq \omega(\sigma_2)$ and therefore $m_1 \neq m_2$. Thus, removing the conical singularities (which corresponds to a particular choice of d_i) gives a metric which S^3 topology. The values of d_i work out to be

$$d_i^2 = \frac{8P(\sigma_i)\Gamma(\sigma_i)}{\dot{Q}(\sigma_i)^2} = \frac{8(C^2 a_0 - a_2 c_2)\Gamma(\sigma_i)^2}{C^4(\sigma_1 - \sigma_2)^2}. \tag{160}$$

Now let us consider the $\alpha = 0$ case. In the appendices we show that this arises as a limit of the $\alpha \neq 0$ case. In fact, in the appendix we give expressions valid for $\beta \neq 0$ which maybe be viewed as complementary to the $\alpha \neq 0$ case, since one cannot have both $\alpha = \beta = 0$ (as then $P \equiv 0$).

The near-horizon metric has the following scaling symmetries (corresponding to $S_2^{-1} S_1$ and S_2):

$$\begin{aligned} C^2 &\rightarrow KC^2, & a_0 &\rightarrow Ka_0, & a_2 &\rightarrow Ka_2, & c_1 &\rightarrow Kc_2, & c_2 &\rightarrow Kc_2 \\ x^1 &\rightarrow K^{-1}x^1, & x^2 &\rightarrow Kx^2, & v &\rightarrow K^{-1}v, \end{aligned} \tag{161}$$

where $K > 0$, and

$$a_2 \rightarrow L^{-2}a_2, \quad c_1 \rightarrow Lc_1, \quad c_2 \rightarrow L^2c_2, \quad \sigma \rightarrow L\sigma, \quad x^2 \rightarrow L^{-1}x^2, \tag{162}$$

where L is a constant (of either sign). These may be used to fix two (or two combinations) of the parameters $(a_0, a_2, c_1, c_2, C^2)$. Therefore, this is a three-parameter family of solutions.

We will now show that in a particular region of parameter space the $a_2 > 0$ solution is isometric to the near-horizon geometry of the fast rotating extremal KK black hole (i.e., $G_4 J > PQ$). Observe that $X \equiv (c_1^2 + 4C^2 c_2) / 4C^4$ is invariant under the first symmetry (161) and scales as X

$\rightarrow L^2X$ under the second symmetry (162). Therefore, use the second symmetry to set $X=1$. Note that since the condition $X=1$ is invariant under the first symmetry, we are still free to use (161). Define a positive constants p, Z by

$$p^2 \equiv \frac{c_1^2 a_0 a_2 + (C^2 a_0 + a_2 c_2)^2}{C^6 a_2}, \quad Z \equiv \frac{4a_2}{C^2} + \frac{2}{C^4}(C^2 a_0 - a_2 c_2). \quad (163)$$

Note that p and Z are invariant under (161). There are now two possibilities: either $Z/p^2 < 1$ or $Z/p^2 \geq 1$. The former region of parameter space gives the fast KK black hole as we now show. Use the first symmetry (161) to set

$$C^2 a_0 - a_2 c_2 = \frac{C^2}{2a_2} \left(1 - \frac{Z}{p^2}\right), \quad (164)$$

which is possible as the left hand side transforms homogeneously (i.e., as K^2), but the right hand side is invariant, and also for our solution $C^2 a_0 - a_2 c_2 > 0$. We also define positive constants a, q by

$$a^2 \equiv \frac{a_2}{C^2}, \quad q \equiv \frac{1}{pC^2 a_2}, \quad (165)$$

which can be inverted to give

$$C^2 = \frac{1}{a\sqrt{pq}}, \quad a_2 = \frac{a}{\sqrt{pq}}. \quad (166)$$

Now, using (164) it follows that

$$C^2 a_0 - a_2 c_2 = \frac{p^2 - 4a^2}{2a^2 p(p+q)}, \quad (167)$$

and thus $p^2 - 4a^2 > 0$. Note that using $X=1$, (163) can be written as $p^2 \equiv 4a_0/C^2 + (1/C^6 a_2)(C^2 a_0 - a_2 c_2)^2$; this, together with (167), can then be used to solve for a_0 to give

$$a_0 = \frac{1}{a\sqrt{pq}} \left(\frac{p^2}{4} - \frac{q^2(p^2 - 4a^2)^2}{16a^2(p+q)^2} \right). \quad (168)$$

Then (167) can be used to solve for c_2 giving

$$c_2 = \frac{1}{a\sqrt{pq}} - \frac{p^2(p^2 - 4a^2)(q^2 - 4a^2)}{16a^5 \sqrt{pq}(p+q)^2}. \quad (169)$$

Finally use $X=1$ to solve for c_1^2 ,

$$c_1^2 = \frac{p(p^2 - 4a^2)(q^2 - 4a^2)}{4a^6 q(p+q)^2}, \quad (170)$$

which implies $q^2 \geq 4a^2$. Thus c_1 is determined up to a sign. To fix the sign recall that when we used the second symmetry to set $X=1$ we did not specify the sign of L ; therefore, we can use this sign freedom to ensure $c_1 > 0$. Using the scaling symmetries, we have therefore shown how to go between the two sets of parameters $(C^2, a_0, a_2, c_1, c_2)$ and p, q, a in the region defined by $a_2 > 0$ and $Z < p^2$.

Now, define a new coordinate by

$$\cos \theta = \frac{2\sigma - \sigma_1 - \sigma_2}{\sigma_2 - \sigma_1} = \sigma - \frac{c_1}{2C^2}. \quad (171)$$

so $0 \leq \theta \leq \pi$ uniquely parametrizes the interval $\sigma_1 \leq \sigma \leq \sigma_2$. This implies

$$Q(\sigma) = C^2 \sin^2 \theta, \quad \Gamma = C^2 H_p, \quad (172)$$

where H_p is defined in (D13) from which it follows

$$\frac{\Gamma d\sigma^2}{Q(\sigma)} = H_p d\theta^2. \quad (173)$$

This proves that the $\sigma\sigma$ component of our near-horizon geometry written in the θ coordinate introduced agrees with the $\theta\theta$ component of the near-horizon limit of the fast rotating extremal KK black hole as given in the appendices. In order to verify the rest of the horizon metric we need to evaluate the constants d_i and $e_i \equiv -d_i \omega(\sigma_i)$ appearing in the coordinate transformation defined by (159) and $m_i = \partial/\partial\phi_i$. One finds⁷⁴

$$d_i = \epsilon_i \sqrt{\frac{q(p^2 - 4a^2)}{p + q}} \Gamma(\sigma_i), \quad e_i = -\frac{\epsilon_i a \sigma_i}{\alpha q}, \quad (174)$$

where

$$\begin{aligned} \Gamma(\sigma_1) &= \frac{P}{2a\sqrt{pq(p+q)}} [(pq + 4a^2) - \sqrt{(p^2 - 4a^2)(q^2 - 4a^2)}], \\ \Gamma(\sigma_2) &= \frac{P}{2a\sqrt{pq(p+q)}} [(pq + 4a^2) + \sqrt{(p^2 - 4a^2)(q^2 - 4a^2)}], \end{aligned} \quad (175)$$

and we will chose the signs by $\epsilon_1 = +1$ and $\epsilon_2 = -1$. In fact, the KK black hole is usually written in ‘‘Euler-type’’ coordinates which are related to the ϕ_i by $\phi = \phi_1 + \phi_2$ and $y = 2P(\phi_2 - \phi_1)$, where $P \equiv \sqrt{p(p^2 - 4a^2)}/4(p+q)$. It is thus convenient to change coordinates directly from x^i to (y, ϕ) which is performed by

$$x^1 = \frac{1}{2}(e_1 + e_2)\phi + \frac{1}{4P}(e_2 - e_1)y, \quad x^2 = \frac{1}{2}(d_1 + d_2)\phi + (d_2 - d_1)\frac{y}{4P}. \quad (176)$$

We have checked that for our near-horizon geometry γ_{ij} written in (y, ϕ) coordinates coincides with the (y, ϕ) components of the horizon metric of the fast rotating KK black hole given in the appendices. Furthermore, using the coordinate transformation above, one can calculate k^y and k^ϕ (recall $k^1 = 1$, $k^2 = 0$) which also coincide with those of the fast rotating KK black hole given in the appendices. This ends the proof of the equivalence of our $a_2 > 0$ near-horizon geometry in the region of parameter space defined by $Z < p^2$ [where Z and p are defined as in (163)] and the near-horizon limit of the fast rotating KK black hole.

D. Near-horizon geometry of a ‘‘small’’ extremal AdS₅ black ring?

We have not been able to solve the near-horizon equations, in general, in $D=5$ with $\Lambda < 0$. Earlier we showed that to do this one needs to solve the two coupled ODEs (37) and (86) for Q and Γ . We first note that when $\Lambda \neq 0$ it is possible to eliminate Γ from (86) and (37), resulting in a 6th order ODE for Q ,

$$\frac{d}{d\sigma} \left(\frac{Q^3}{\ddot{Q} + 2C^2 \frac{d^5 Q}{d\sigma^5}} \right) + \frac{5}{3} Q^2 \frac{d^4 Q}{d\sigma^4} = 0. \quad (177)$$

Given a solution to this one can then deduce Γ from (37). Finding all solutions to (177) would lead to the classification of all allowed near-horizon geometries of extremal vacuum black holes with $R \times U(1)^2$ symmetry in AdS₅. Curiously all explicit dependence in Λ has cancelled from this sixth-order ODE (although we emphasize it is only valid when $\Lambda \neq 0$)—it is thus more convenient

to work with the coupled pair of ODEs (37) and (86). We will now present some results which follow from these equations.

Lemma: The most general polynomial solution to (37) and (86) is given by $\Gamma = a_0 + a_1\sigma$ and $Q = -\Lambda a_1\sigma^3 - (C^2 + 3\Lambda a_0)\sigma^2 + c_1\sigma + c_2$.

Proof: First observe that (37) implies that Q is a polynomial if and only if Γ is a polynomial. Suppose Γ is a polynomial of order $n=2$. The ODE (86) then implies $Q^2=0$ and then (37) implies $\Gamma=0$, a contradiction. Now suppose Γ is a polynomial of order $n \geq 3$. For $\sigma \rightarrow \infty$ we have $\Gamma \sim a_n\sigma^n$ for some nonzero constant a_n . The ODE (37) then implies $Q \sim -6\Lambda a_n\sigma^{n+2}/[(n+1)(n+2)]$. Then, examining the $\sigma \rightarrow \infty$ limit of the ODE (86) implies $n=4/7$ which is a contradiction. This leaves $\Gamma = a_1\sigma + a_0$ which is indeed a solution with the Q given above.

As we showed in an earlier section $\Gamma = a_1\sigma + a_0$ gives the near-horizon geometry of the known extremal rotating AdS₅ black hole,⁴⁰ which has spherical horizon topology. An interesting question is whether there exists a near-horizon geometry with $S^1 \times S^2$ topology, thus providing a candidate extremal AdS black ring near-horizon geometry. Recall that the near-horizon limit of the asymptotically flat black ring has $\Gamma = a_0 + a_2\sigma^2$. But the above lemma tells us that this cannot be the case when one has a cosmological constant. This is perhaps surprising as the near-horizon limits of the topologically spherical Myers–Perry black hole and its generalization to include a negative cosmological constant both have Γ of the same form (a linear polynomial).

If an extremal vacuum AdS black ring does exist, one might expect it to be continuously connected to the asymptotically flat extremal vacuum black ring as one turns off the cosmological constant. It is thus of interest to investigate the existence of “small” AdS black rings, in the sense that both the radii of the S^1 , say R_1 , and the S^2 , say R_2 , are much smaller than the AdS length scale ℓ ($\Lambda = -4/\ell^2$). For the asymptotically flat extremal black ring $R_2 \sim a$ (where a is the Kerr parameter in the corresponding boosted Kerr string solution) and R_1 is just proportional to the period of z (which does not appear explicitly in the near-horizon geometry, only implicitly through identification of z). Therefore, we will consider linearizing the pair of ODEs about the solution corresponding to the boosted Kerr string near-horizon geometry (which includes that of the extremal black ring) for small a/ℓ (or equivalently small ΛC^{-2}). In our formalism a near-horizon geometry is specified by the data $(C^2, \Gamma, Q, \gamma_{ij})$ (recall we set $k^1=1, k^2=0$) and thus these are the data which we must linearize about.

Expand⁷⁵

$$Q(\sigma) = Q_0(\sigma) + \epsilon Q_1(\sigma) + O(\epsilon^2), \quad \Gamma(\sigma) = \Gamma_0(\sigma) + \epsilon \Gamma_1(\sigma) + O(\epsilon^2), \quad C^2 = C_0^2(1 + A_1\epsilon + O(\epsilon^2)), \quad (178)$$

where $\epsilon \equiv \Lambda C_0^{-2}$ is a dimensionless expansion parameter, Q_1, Γ_1 , and A_1 are dimensionless functions and constant respectively, and

$$Q_0 = C_0^2(1 - \sigma^2), \quad \Gamma_0 = \frac{(1 + \sigma^2)}{2c_\beta}, \quad C_0^2 = \frac{1}{2a^2c_\beta}, \quad (179)$$

are the zeroth order data corresponding to the Kerr string (which we denote with a 0 subscript). Plugging this into the ODEs (37) and (86) gives

$$Q_1 + 2C_0^2A_1 + 6\Gamma_0C_0^2 = 0, \quad \frac{d}{d\sigma} \left(\frac{Q_0^3 d^3\Gamma_1}{\Gamma_0 d\sigma^3} \right) - \frac{10C_0^2Q_0^2}{c_\beta} = 0, \quad (180)$$

which determines Q_1 and Γ_1 . Explicitly

$$Q_1 = C_0^2 \left[-\frac{\sigma^4}{4c_\beta} - \left(A_1 + \frac{3}{2c_\beta} \right) \sigma^2 + d_1\sigma + d_2 \right], \quad (181)$$

where d_1, d_2 are integration constants and

$$\frac{d^3\Gamma_1}{d\sigma^3} = \frac{10C_0^2\Gamma_0}{c_\beta Q_0^3} \int^\sigma d\sigma' Q_0(\sigma')^2, \quad (182)$$

which determines Γ_1 up to quadratures. We find

$$\begin{aligned} \Gamma_1 = & -\frac{\sigma^4}{24c_\beta^2} + \frac{\sigma^2}{2} \left(\frac{1}{c_\beta^2} + e_2 \right) + \sigma \left(e_3 - \frac{5e_1}{8C_0^4 c_\beta^2} \right) + e_4 \\ & + \frac{(1+\sigma^2)}{2} \left[\left(\frac{5e_1}{8C_0^4 c_\beta^2} - \frac{1}{3c_\beta^2} \right) \log(1+\sigma) + \left(-\frac{5e_1}{8C_0^4 c_\beta^2} - \frac{1}{3c_\beta^2} \right) \log(1-\sigma) \right], \end{aligned} \quad (183)$$

where e_i are four constants of integration. Now, using (39) we may compute γ_{11} to linear order in ϵ , which for later convenience we write as

$$\gamma_{11} = \frac{Q}{c_\beta^2 \Gamma} [1 + \epsilon F + O(\epsilon^2)],$$

$$\begin{aligned} F = & \frac{[2\sigma^4 + \sigma^2(6c_\beta(A_1 - d_2) - 6c_\beta^2(2e_4 + 3e_2) - 17) + 11 + 6c_\beta(A_1 - d_2) + 6c_\beta^2(e_2 + 6e_4)]}{12c_\beta(1 - \sigma^2)} \\ & + \frac{(15e_1 - 8C_0^4)}{12c_\beta C_0^4} \log(1 + \sigma) - \frac{(15e_1 + 8C_0^4)}{12c_\beta C_0^4} \log(1 - \sigma). \end{aligned} \quad (184)$$

We now turn to determining $\omega = \gamma_{12}/\gamma_{11}$. Equation (80) determines $\dot{\omega}^2$ in terms of (Γ, Q, γ_{11}) which can now be calculated to linear order in ϵ . Recall that we also showed that $\dot{\omega} \gamma_{11}^2 \Gamma = k$, where k is a constant (90). Using (80) we compute this quantity to linear order and find

$$\dot{\omega}^2 \gamma_{11}^4 \Gamma^2 = \frac{4\epsilon C_0^6}{3c_\beta^4} [3c_\beta(A_1 - d_2) + 3c_\beta^2(-e_2 + 2e_4) - 1] + O(\epsilon^2), \quad (185)$$

which is indeed a constant; in fact, note that for generic parameter values $k = O(\sqrt{\epsilon})$. Integrating for ω gives

$$\omega = k \left(\frac{\sigma}{1 - \sigma^2} + O(\epsilon) \right) + \omega_0, \quad (186)$$

where the constant ω_0 is the $\epsilon=0$ value of the boosted Kerr string.

Let us now analyze regularity of this perturbative solution. First, observe that the location of the roots of Q change, so write them as $\sigma_\pm = \pm 1 + \epsilon \delta\sigma_\pm + O(\epsilon^2)$, where we have written $\sigma_+ = \sigma_2$ and $\sigma_- = \sigma_1$ for convenience. Inserting into Q gives

$$\delta\sigma_\pm = \pm \frac{Q_1(\pm 1)}{2C_0^2} = \pm \frac{1}{2} \left(-\frac{7}{4c_\beta} + d_2 - A_1 \pm d_1 \right) \quad (187)$$

and regularity requires $\delta\sigma_+ > 0$ and $\delta\sigma_- < 0$ to ensure that $\log(1 \pm \sigma)$ is regular in the relevant interval $[\sigma_-, \sigma_+]$ (note $\epsilon < 0$). For consistency of our perturbation series we require that the various metric functions evaluated at the endpoints σ_\pm coincide with those of the boosted Kerr string as $\epsilon \rightarrow 0$. It turns out that $\Gamma(\sigma_\pm) = 1/c_\beta + O(\epsilon \log|\epsilon|)$ due to the logarithm terms. However, the function F (appearing in γ_{11}) and ω both contain factors of $1/(1 - \sigma^2)$ which at the end points contribute $O(\epsilon^{-1})$ as a result for generic parameter values $F(\sigma_\pm) = O(\epsilon^{-1})$ and $\omega = O(\epsilon^{-1/2})$. Both of these are not acceptable: we must choose parameters such that the factor of $1 - \sigma^2$ in the denominator of the first term of F cancels with its numerator and also impose that the constant $k = O(\epsilon)$ so $\omega = O(1)$. Demanding that the $O(\epsilon)$ term in (185) vanishes gives

$$A_1 - d_2 = \frac{1}{3c_\beta} - c_\beta(2e_4 - e_2), \quad (188)$$

which is equivalent to $k=O(\epsilon)$. Using this to eliminate $A_1 - d_2$ in F (184) implies, remarkably, that the numerator of the first term has a factor of $1 - \sigma^2$ which thus cancels the unwanted factor in the denominator, leaving

$$F = c_\beta(e_2 + 2e_4) + \frac{13 - 2\sigma^2}{12c_\beta} + \frac{(15e_1 - 8C_0^4)}{12c_\beta C_0^4} \log(1 + \sigma) - \frac{(15e_1 + 8C_0^4)}{12c_\beta C_0^4} \log(1 - \sigma). \quad (189)$$

Therefore, we have $\epsilon F(\sigma_\pm) = O(\epsilon \log|\epsilon|)$. We conclude that the near-horizon solution we have is valid to order $O(\epsilon^2)$ for $\sigma_- \leq \sigma \leq \sigma_+$.

We must also ensure the absence of conical singularities in the horizon metric which reads

$$\gamma_{ab} dx^a dx^b = \frac{\Gamma d\sigma^2}{Q} + \frac{Q}{c_\beta^2 \Gamma} (1 + \epsilon F + O(\epsilon^2))(dx^1 + \omega dx^2)^2 + c_\beta^2 (1 - \epsilon F + O(\epsilon^2))(dx^2)^2. \quad (190)$$

Simultaneous removal of conical singularities is equivalent to

$$\frac{\dot{Q}(\sigma_+) \left(1 + \frac{\epsilon}{2} F(\sigma_+) + O(\epsilon^2)\right)}{\Gamma(\sigma_+)} = - \frac{\dot{Q}(\sigma_-) \left(1 + \frac{\epsilon}{2} F(\sigma_-) + O(\epsilon^2)\right)}{\Gamma(\sigma_-)}. \quad (191)$$

It is easily seen that this can be satisfied if Q, F, Γ are even⁷⁶ functions in σ . This can be achieved by setting $d_1 = e_1 = e_3 = 0$.

With the choices the various functions simplify

$$\Gamma_1 = -\frac{\sigma^4}{24c_\beta^2} + \frac{\sigma^2}{2} \left(\frac{1}{c_\beta^2} + e_2\right) + e_4 - \frac{(1 + \sigma^2)}{6c_\beta^2} \log(1 - \sigma^2),$$

$$F = c_\beta(e_2 + 2e_4) + \frac{13 - 2\sigma^2}{12c_\beta} - \frac{2}{3c_\beta} \log(1 - \sigma^2). \quad (192)$$

Note that determining ω (i.e., the constant k) requires a higher order calculation: one needs the $O(\epsilon^2)$ term in (185) which we will not pursue here. The perturbation we have constructed is parametrized by e_2, e_4, d_2 (on top of three parameters of boosted Kerr string) with A_1 determined by (188) and $2e_4 - e_2 > 25/12c_\beta^2$ (this is equivalent to the $\pm \delta\sigma_\pm > 0$ condition).

For a boost value given by $\sinh^2 \beta = 1$ the near-horizon geometry of the Kerr string is isometric to that of the asymptotically flat extremal black ring which is a two-parameter family of solutions (these can be taken to be the two angular momenta J_i). One would expect an AdS extremal black ring to also have two parameters. However, the regular perturbations we have derived depend on more parameters. Presumably these extra parameters must be fixed somehow (perhaps asymptotic information) for our perturbative solution to be interpreted as the near-horizon geometry of a “small” AdS black ring.

We can introduce coordinates (ϕ, z) where $\phi = d_1 x^1$ where d_1 is chosen to ensure ϕ has period 2π and $z = x^2$ runs along a periodic direction (corresponding to that of the string in the unperturbed case). As explained in Refs. 22 and 23, we expect ∂_ψ generating the S^1 of the presumptive black ring solution to be given by a linear combination of ∂_ϕ and ∂_z , while ∂_ϕ can be taken to be the generator of the $U(1)$ in the transverse S^2 . From our linearized solution above, we can readily compute J_ϕ via a Komar integral.²³ However, to determine J_z and hence J_ψ , we require knowledge of the $O(\epsilon^2)$ term in (185) which is not available from our first order calculation. Physically, one would expect that a black ring in AdS₅ would have greater angular momenta in the S^1 direction, relative to the corresponding asymptotically flat solution, in order to prevent self-collapse.

To summarize we have constructed an approximate solution to the vacuum near-horizon equations with a negative cosmological constant by perturbing about the near-horizon geometry of the boosted Kerr string. To this level of approximation it describes a regular near-horizon geometry with horizon topology $S^1 \times S^2$. Taking the boost to be that of the asymptotically flat black ring $\sinh^2 \beta = 1$ provides a candidate for a near-horizon geometry of a “small” extremal ring in AdS₅.

VI. DISCUSSION

In this paper we have shown how one may determine all possible vacuum near-horizon geometries of extremal (but nonsupersymmetric) black holes under in four and five dimensions under the following assumptions. In 4D we assume axisymmetry and that the horizon has compact sections of nontoroidal topology. In 5D we assume there are two commuting rotational symmetries and the horizon has compact sections of nontoroidal topology.

Our results in 4D are unsurprising. We find that the only solution is the near-horizon limit of the extremal Kerr black hole. In fact, in the context of isolated horizons the same result has been established.⁴⁴ Observe that uniqueness of Kerr has only been proven for nonextremal black holes; therefore our result can be viewed as a first step toward proving uniqueness of extremal Kerr among asymptotically flat black holes with degenerate horizons. Pleasingly, our method in 4D worked just as easily with a negative cosmological constant showing that the only regular solution is the near-horizon geometry of extremal Kerr-AdS₄. It should be noted that there are no known uniqueness theorems for asymptotically AdS black holes even in 4D; perhaps our result will be useful in proving uniqueness of extremal Kerr-AdS₄.

In 5D we were able to find all solutions in the pure vacuum, i.e., zero cosmological constant. Naturally the results are more complicated than in 4D. We found three families of near-horizon geometries: two spherical topology horizons and one $S^1 \times S^2$ horizon. Further, we identified how all the known vacuum extremal black hole solutions fit into these families: i.e., extremal boosted Kerr string, extremal vacuum black ring, extremal Myers–Perry, and the extremal KK black holes (both slow and fast rotating). Our results are summarized in detail in Sec. II. A number of things may be deduced from our classification.

For example, one expects a vacuum doubly spinning black ring which is asymptotic to the KK monopole to exist (i.e. a “Taub NUT” black ring).⁷⁷ Such a solution would have four parameters (roughly J_i, M, P). Presumably like other doubly spinning solutions in 5D it admits an extremal limit, which would be a three-parameter family. One can then consider its near-horizon limit. From our Theorem 2, it follows that its near-horizon geometry is contained in our family of $S^1 \times S^2$ horizons. A reasonable guess is that it is simply given by the near-horizon limit of the extremal boosted Kerr string (which is a three-parameter subfamily of our solution). The boost then would be related to the NUT parameter P and as $P \rightarrow \infty$ (flat space limit) one must have $\sinh^2 \beta \rightarrow 1$ in order to get the near-horizon geometry of the asymptotically flat black ring, see Ref. 22. In fact, for the asymptotically flat extremal black ring both the infinite radius limit and the near-horizon limit simplify to the tensionless (i.e., $\sinh^2 \beta = 1$) boosted Kerr string.²³ In view of our near-horizon results it is thus natural to expect that the infinite radius limit of a KK black ring is the boosted Kerr string for arbitrary boost.

We also remark that a curious output of our analysis is that in some cases the near-horizon geometries we derived are isometric to the near-horizon limit of known black holes only in a subregion of parameter space. This occurs both for the $S^1 \times S^2$ family and the second spherical topology case. It is possible that these other regions of parameter space are occupied by unknown black hole solutions (e.g., KK black ring) but it seems more likely that such bounds on the parameters are invisible from the near-horizon geometry alone (e.g., as for the near horizon of the asymptotically flat extremal ring which actually is only isometric to the tensionless boosted Kerr string in a subregion of its parameter space, see Ref. 23).

Other interesting consequences of our results regards uniqueness of near-horizon geometries. Our analysis has revealed there are two distinct classes of S^3 horizon geometries in 5D vacuum gravity. Also the same near-horizon geometry can arise as the near-horizon limit of different black holes although in all known examples the black holes have different asymptotics (i.e., KK or

asymptotically flat). Furthermore, it seems clear that not all near-horizon geometries arise as near-horizon limits of black holes with a given asymptotics. For example, one can ask whether our second class of S^3 topology horizon geometries can ever arise as the near-horizon limit of an asymptotically flat extremal black hole. Due to its S^3 topology one can identify the correct $U(1)$ generators which must match onto those in the orthogonal two-planes as asymptotic infinity. One can therefore calculate the angular momenta via a Komar integral over the horizon²³ which gives

$$J_{\phi_1} = -\frac{4\pi\sqrt{2}\kappa^2}{G_5 C^8 \Gamma(\sigma_2) \sqrt{C^2 a_0 - a_2 c_2}}, \quad J_{\phi_2} = -\frac{4\pi\sqrt{2}\kappa^2}{G_5 C^8 \Gamma(\sigma_1) \sqrt{C^2 a_0 - a_2 c_2}}. \quad (193)$$

It is clear that one can have $J_{\phi_1} = J_{\phi_2}$ [this occurs if and only if $\Gamma(\sigma_2) = \Gamma(\sigma_1)$ which is equivalent to the parameter $c_1 = 0$]. Observe that the near-horizon geometry always possesses exactly a $U(1)^2$ rotational symmetry group (i.e., it is never enhanced even when $J_{\phi_1} = J_{\phi_2}$). However, from group theoretic reasoning one might expect⁷⁸ asymptotically flat black holes (with a single horizon) with equal angular momenta to possess an enhanced rotational symmetry group $SU(2) \times U(1)$ [recall the rotation group $SO(4) \sim SU(2) \times SU(2)$]. This leads us to conclude that this near-horizon geometry does not correspond to that of an asymptotically flat black hole. It should also be noted that in the nonextremal case it has been shown⁶⁴ that the Myers–Perry black hole is the unique asymptotically flat black hole with two rotational symmetries and S^3 topology horizon, and one expects this result to go over in the extremal case (and its near-horizon geometry is, in fact, given by our other class of S^3 horizon geometries).

Another useful aspect of this analysis is that the explicit metrics for the various near-horizon geometries appear simple in the coordinates we have derived. In contrast, the metrics one obtains by taking the near-horizon limits of known solutions tend to be far more complicated, as can be seen from the appendices. This should make the problem of generalizing our results to include gauge fields more tractable. It would be interesting to classify the near-horizon geometries of extremal, nonsupersymmetric black holes in ungauged supergravity theories. We intend to investigate this problem in the near future.

One of the main motivations for this work was to investigate the existence of asymptotically AdS black rings. Unfortunately, we were not able to solve the vacuum near-horizon equations in the presence of a negative cosmological constant, in general, even with the assumption of two rotational symmetries. This is in contrast to 4D where using the assumption of axisymmetry it was possible for us to do so. However, we did reduce the problem to solving a single sixth-order ODE of one function. We found one set of solutions to this equation which correspond to the near-horizon geometry of the known, topologically S^3 , extremal rotating AdS₅ black hole.⁴⁰ It would be interesting to find a solution which gives rise to the near-horizon geometry of an extremal AdS₅ black ring. By perturbing the near-horizon geometry of the asymptotically flat black ring, we were able to construct an approximate near-horizon geometry corresponding to the near-horizon limit of a small (i.e., the sizes of the S^1 and S^2 are small compared to the AdS length scale) extremal black ring in AdS. The fact that the perturbation can always be made regular and preserves the $S^1 \times S^2$ topology appears to be nontrivial; perhaps this provides some evidence for the existence of, at least a small, extremal vacuum black ring in AdS₅.

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APPENDIX A: GLOBAL ARGUMENT

In this section we prove the following results quoted in the main text: $Q^2\mathcal{P}$ (needed in 4D) and $Q^3 d^3\Gamma/d\sigma^3$ (needed in 5D) are globally defined functions on \mathcal{H} which vanish where Q vanishes.

This is not actually obvious as $\dot{\Gamma}$, $\ddot{\Gamma}$, and $d^3\Gamma/d\sigma^3$ need not be globally defined, although Γ is. To see this note that the norm of $\partial/\partial\sigma$ is Γ/Q which is regular everywhere except at the points where Q vanishes. However, we know that Q must vanish at two distinct points, and thus we conclude that this vector field is not globally defined and thus $(\partial/\partial\sigma)\Gamma=\dot{\Gamma}$ and higher derivatives are not guaranteed to be globally defined. Note that this argument relies crucially on Q vanishing somewhere. Recall this comes from the fact that σ is a globally defined smooth nonconstant⁷⁹ function on a compact space and thus $d\sigma$ vanishes at two distinct points (the max and min of σ). Then the invariant $(d\sigma)^2=Q/\Gamma$ tells us $Q\geq 0$ and vanishes at these two points.

To proceed we introduce the vector field $S=Q(\partial/\partial\sigma)$. Its norm squared is ΓQ which is globally defined and vanishes at the zeroes of Q . S is certainly regular everywhere except possibly at the zeroes of Q . Let the zeroes of Q be $\sigma_1<\sigma_2$. Then, assuming regularity, we have $Q=\dot{Q}_i(\sigma-\sigma_i)+\dots$ near $\sigma=\sigma_i$, and since $Q\geq 0$ we learn that $\dot{Q}_1>0$ and $\dot{Q}_2<0$. This allows us to define $r_1^2=\sigma-\sigma_1$ and $r_2^2=\sigma_2-\sigma$. Then, near σ_i we have $S\sim(|\dot{Q}_i|/2\Gamma_i)r_i(\partial/\partial r_i)$ which is regular at $r_i=0$ and vanishes there, as can be seen by using the Cartesian coordinates x_i, y_i associated with r_i . We deduce that S is a globally defined vector field on \mathcal{H} which vanishes at the zeroes of Q .

Thus we now employ the globally defined vector S to construct invariants, e.g., $S(\Gamma)=Q\dot{\Gamma}$ is globally defined (and vanishes where Q does). Note the following identity:

$$Q^2\ddot{\Gamma}\equiv S(S(\Gamma))-S(\Gamma)\dot{Q} \quad (\text{A1})$$

proves that $Q^2\ddot{\Gamma}$ is globally defined as \dot{Q} must be (this is because \dot{Q} is regular at the only potential problem points $\sigma=\sigma_i$ as $Q=\dot{Q}_i(\sigma-\sigma_i)+\dots$). Therefore $Q^2\ddot{\Gamma}$ is an invariant of the solution which vanishes at the zeroes of Q since S vanishes at those points. Since $Q^2\mathcal{P}=2Q^2\ddot{\Gamma}-S(\Gamma)^2/\Gamma-Q^2/\Gamma$ this proves that $Q^2\mathcal{P}$ is indeed globally defined and vanishes at the zeroes of Q . This establishes the result needed for the 4D analysis. The 5D case may be treated similarly using the identity

$$Q^3\frac{d^3\Gamma}{d\sigma^3}=S(Q^2\ddot{\Gamma})-2\dot{Q}Q^2\ddot{\Gamma}, \quad (\text{A2})$$

which proves that $Q^3d^3\Gamma/d\sigma^3$ is globally defined (using the fact that $Q^2\ddot{\Gamma}$ is). Therefore, $Q^3d^3\Gamma/d\sigma^3$ is an invariant which vanishes at the zeroes of Q as claimed.

APPENDIX B: NEAR-HORIZON GEOMETRY OF KERR-AdS₄

Use the form of the Kerr-AdS₄ metric as in Ref. 65 which satisfies $R_{\mu\nu}=-3g^2g_{\mu\nu}$ (our g is the same as their α). The angular velocity is given by $\Omega=a/(r_+^2+a^2)$, where r_+ is the largest zero of $\Delta_r=(r^2+a^2)(1+g^2r^2)-2mr$. Define

$$\rho^2=r^2+a^2\cos^2\theta, \quad \Delta_\theta=1-a^2g^2\cos^2\theta, \quad \Xi=1-g^2a^2. \quad (\text{B1})$$

Using the algorithm presented in Ref. 22 to determine the near-horizon data, we find

$$k^\Phi=\frac{2ar_+}{(r_+^2+a^2)^2}, \quad \Gamma=\frac{\rho_+^2}{\Xi(r_+^2+a^2)}, \quad A_0=\frac{-\Delta_+}{2\Xi(r_+^2+a^2)}, \quad (\text{B2})$$

$$\gamma_{ab}dx^a dx^b=\frac{\rho_+^2}{\Delta_\theta}d\theta^2+\frac{\sin^2\theta\Delta_\theta(r_+^2+a^2)^2}{\rho_+^2\Xi^2}d\Phi^2. \quad (\text{B3})$$

where $\Delta_+''=(\Delta_r'')_{r=r_+}$, etc. Notice that in the flat space limit $g\rightarrow 0$ limit $r_+\rightarrow a$ and thus the above near-horizon metric reduces correctly to that of Kerr as given in Ref. 22.

APPENDIX C: NEAR-HORIZON GEOMETRY OF ROTATING AdS₅ BLACK HOLE

In this appendix we present the near-horizon geometry of the known, topologically S^3 , rotating AdS₅ black hole.⁴⁰

1. Self-dual case

We first consider the self-dual case that occurs if the two independent angular momenta are set equal ($J_1=J_2$). In this case the full solution exhibits symmetry enhancement, and it is convenient to treat it separately to the general case studied below. The self-dual solution can be written in corotating coordinates as

$$ds^2 = -\frac{V(r)}{w(r)^2}dT^2 + \frac{dr^2}{V(r)} + \frac{r^2w(r)^2}{4}[d\psi + \cos\theta d\phi - (\Omega(r) - \Omega_+)dT]^2 + \frac{r^2}{4}(d\theta^2 + \sin^2\theta d\phi^2), \quad (C1)$$

where

$$V = 1 + g^2r^2 - \frac{2M\Xi}{r^2} + \frac{2Ma^2}{r^4}, \quad w(r)^2 = 1 + \frac{2Ma^2}{r^4}, \quad \Omega(r) = \frac{4Ma}{r^4w^2}. \quad (C2)$$

The horizon is located at the largest real root of $V(r)$, $r=r_+$ so $V(r_+)=0$. Extremality implies $V'(r_+)=0$. The near-horizon limit of this metric is given by the data

$$\Gamma = \frac{1}{w_+}, \quad k^\psi = -\Omega'_+, \quad A_0 = -\frac{V''_+}{2w_+}, \quad (C3)$$

$$\gamma_{ab}dx^a dx^b = \frac{r_+^2w_+^2}{4}(d\psi + \cos\theta d\phi)^2 + \frac{r_+^2}{4}(d\theta^2 + \sin^2\theta d\phi^2). \quad (C4)$$

We will now show that the near-horizon metric (101) derived in the main text is identical to the near-horizon of self-dual solution above. Consider (101) and define

$$M \equiv \frac{4\Gamma(C^2 + 2\Lambda\Gamma)^2}{(C^2 + 3\Lambda\Gamma)^3}, \quad a^2 \equiv \frac{2\Gamma(C^2 + \Lambda\Gamma)}{(C^2 + 2\Lambda\Gamma)^2}, \quad r_+^2 \equiv \frac{4\Gamma}{C^2 + 3\Lambda\Gamma}. \quad (C5)$$

Observe that these definitions imply $V(r_+)=0$, $V'(r_+)=0$, $V''(r_+)=2C^2/\Gamma$, and that

$$w_+^2 \equiv w(r_+)^2 = \frac{2(C^2 + 2\Lambda\Gamma)}{C^2 + 3\Lambda\Gamma}, \quad (C6)$$

where V and w are defined as above. It now follows that the horizon metric (101) agrees exactly with that of self-dual solution. Now, using the definition of Ω above compute

$$\Omega'(r_+) = -(C^2 + 3\Lambda\Gamma) \sqrt{\frac{C^2 + \Lambda\Gamma}{C^2 + 2\Lambda\Gamma}} \times \sqrt{\frac{C^2 + 3\Lambda\Gamma}{2\Gamma^2(C^2 + 2\Lambda\Gamma)}}. \quad (C7)$$

Next, use the scaling freedom $\Gamma \rightarrow K\Gamma$ to set $\Gamma=1/w_+$, which is equivalent to

$$\frac{C^2 + 3\Lambda\Gamma}{2\Gamma^2(C^2 + 2\Lambda\Gamma)} = 1. \quad (C8)$$

This then implies that in (101) $k^\psi = -\Omega'(r_+)$ and $C^2 = V''(r_+)/(2w_+)$, both of which coincide with those for the self-dual solution given above. This completes the proof of equivalence.

2. General angular momenta

We now consider the general case for which the two independent angular momenta are not equal, i.e., $J_1 \neq J_2$. The solution satisfies $R_{\mu\nu} = -4g^2 g_{\mu\nu}$ (we have set the parameter l used in Ref. 40 to g^{-1}). The near horizon geometry is parametrized by three parameters (r_+, a, b) subject to the extremality constraint,

$$2g^2 r_+^6 + r_+^4(1 + g^2 b^2 + g^2 a^2) - a^2 b^2 = 0. \quad (\text{C9})$$

Following the procedure given in Ref. 22, it is straightforward to compute the near-horizon limit and we omit the details. The near-horizon metric can be written in the form (22) with horizon metric given by

$$\gamma_{ab} dx^a dx^b = \frac{\rho_+^2 d\theta^2}{\Delta_\theta} + \gamma_{ij} dx^i dx^j, \quad (\text{C10})$$

with

$$\begin{aligned} \gamma_{ij} dx^i dx^j = & \frac{\Delta_\theta}{\rho_+^2} \left[\frac{(r_+^2 + a^2)^2 \sin^2 \theta d\phi^2}{\Xi_a^2} + \frac{(r_+^2 + b^2)^2 \cos^2 \theta d\psi^2}{\Xi_b^2} \right] \\ & + \frac{1 + r_+^2 g^2}{r_+^2 \rho_+^2} \left[\frac{b(r_+^2 + a^2) \sin^2 \theta d\phi}{\Xi_a} + \frac{a(r_+^2 + b^2) \cos^2 \theta d\psi}{\Xi_b} \right]^2, \end{aligned} \quad (\text{C11})$$

where

$$\Delta_\theta = 1 + g^2 r_+^2 - g^2 \rho_+^2, \quad \rho_+^2 = r_+^2 + a^2 \cos^2 \theta + b^2 \sin^2 \theta, \quad \Xi_a = 1 - a^2 g^2, \quad \Xi_b = 1 - b^2 g^2. \quad (\text{C12})$$

The remaining near-horizon data are

$$\begin{aligned} \Gamma = & \frac{\rho_+^2 r_+^2}{(r_+^2 + a^2)(r_+^2 + b^2)}, \quad A_0 = - \frac{4r_+^2(1 + 3g^2 r_+^2 + g^2 a^2 + g^2 b^2)}{(r_+^2 + a^2)(r_+^2 + b^2)}, \\ k^\phi = & \frac{2ar_+ \Xi_a}{(r_+^2 + a^2)^2}, \quad k^\psi = \frac{2br_+ \Xi_b}{(r_+^2 + b^2)^2}. \end{aligned} \quad (\text{C13})$$

Note that the above formulas simplify in the zero cosmological constant case $g=0$, in particular, $r_+^2 = |ab|$. We should also note that there is no loss of generality in assuming $a > b > 0$.

APPENDIX D: NEAR-HORIZON GEOMETRY OF KK BLACK HOLE

In this section we give the near-horizon geometries of the extremal KK black holes found in Ref. 37 (see also Ref. 38). We will use the form of the solution as given in Ref. 26. The nonextremal solution carries the 4D conserved charges (M, Q, P, J) (i.e., it is a rotating dyonic black hole) and we will choose an orientation for rotation such that $J \geq 0$. In 5D when $P \neq 0$ it has horizon topology S^3 and is asymptotic to the KK monopole. When $P=0$ it is merely the boosted Kerr string, and thus we only consider the $P \neq 0$ case in this section. As is well known there are two different extremal limits of this black hole called slowly rotating (since $G_4 J < PQ$) and fast rotating (since $G_4 J > PQ$).

1. Slowly rotating solution

This extremal limit of the KK black hole is given by $a, m \rightarrow 0$ with $\eta = a/m < 1$ fixed. This extremal solution can be parametrized by three positive constants (p, q, η) . In this case the angular velocities are

$$\Omega_y = \sqrt{\frac{p+q}{q}}, \quad \Omega_\phi = 0. \quad (\text{D1})$$

After some calculation one can show that the near-horizon is of the form (22) with the metric on H given by

$$\gamma_{ab}dx^a dx^b = H_p d\theta^2 + \frac{H_q}{H_p} (dy + A_\phi d\phi)^2 + \frac{(pq)^3(1-\eta^2)\sin^2\theta d\phi^2}{4(p+q)^2 H_q}, \quad (\text{D2})$$

where

$$H_p = \frac{p^2 q}{2(p+q)}(1 + \eta \cos \theta), \quad H_q = \frac{pq^2}{2(p+q)}(1 - \eta \cos \theta), \quad A_\phi = \frac{q^2 p^{5/2}}{2(p+q)^{3/2} H_q}(\eta - \cos \theta) \quad (\text{D3})$$

and regularity of the horizon demands $y \sim y + 8\pi P$ (or quotients) and $\phi \sim \phi + 2\pi$, where $P = \sqrt{p^3/4(p+q)}\ell$ and $0 \leq \theta \leq \pi$. Coordinates which are adapted to the $U(1)^2$ rotational symmetry can be defined by $\phi = \phi_1 + \phi_2$ and $y = 2P(\phi_2 - \phi_1)$; the absence of conical singularities then implies ϕ_1, ϕ_2 are 2π periodic with $\partial/\partial\phi_1$ vanishing at $\theta = \pi$ and $\partial/\partial\phi_2$ vanishing at $\theta = 0$, i.e., one must have S^3 topology. The other near-horizon data are

$$A_0 = -\frac{2(p+q)}{(pq)^{3/2}(1-\eta^2)^{1/2}}, \quad \Gamma = \frac{2(p+q)}{(pq)^{3/2}(1-\eta^2)^{1/2}} H_p, \quad (\text{D4})$$

and

$$k^\phi = -\frac{2(p+q)\eta}{(pq)^{3/2}(1-\eta^2)}, \quad k^y = \frac{2}{1-\eta^2} \sqrt{\frac{p+q}{q^3}}. \quad (\text{D5})$$

There is a special case which simplifies considerably, $\eta = 0$ (note this gives $J = 0$). Defining $y = p\sqrt{p/(p+q)}\psi$ one gets

$$\gamma_{ab}dx^a dx^b = \frac{p^2 q}{2(p+q)} [d\theta^2 + \sin^2\theta d\phi^2 + 2(d\psi - \cos\theta d\phi)^2] \quad (\text{D6})$$

and

$$\Gamma = \sqrt{\frac{p}{q}}, \quad C^2 = \frac{2(p+q)}{(pq)^{3/2}}, \quad (\text{D7})$$

and

$$k = \frac{2(p+q)}{(pq)^{3/2}} \frac{\partial}{\partial\psi} = C^2 \frac{\partial}{\partial\psi}. \quad (\text{D8})$$

Noting that

$$\Gamma C^{-2} = \frac{p^2 q}{2(p+q)}, \quad (\text{D9})$$

it is easy to see this is of the form of the $\Gamma = a_0$ case we derived in the main text. To prove complete equivalence one needs to invert the parameter change which is easily done,

$$p = C^{-1} \sqrt{2\Gamma(1+\Gamma^2)}, \quad q = C^{-1} \sqrt{\frac{2(1+\Gamma^2)}{\Gamma^3}}. \quad (\text{D10})$$

2. Fast rotating solution

This extremal limit of the KK black hole is given by $m=a>0$. This extremal solution can be parametrized by three positive constants (p, q, a) which satisfy $p, q \geq 2a$. In this case the angular velocities are

$$\Omega_y = \sqrt{\frac{(q^2 - 4a^2)}{q(p+q)}}, \quad \Omega_\phi = \frac{1}{\sqrt{pq}} \quad (\text{D11})$$

After some calculation one can show that the near-horizon is of the form (22) with the metric on H given by

$$\gamma_{ab} dx^a dx^b = H_p d\theta^2 + \frac{H_q}{H_p} (dy + A_\phi d\phi)^2 + \frac{pq a^2 \sin^2 \theta}{H_q} d\Phi^2, \quad (\text{D12})$$

where

$$H_p = -a^2 \sin^2 \theta + \frac{p(pq + 4a^2)}{2(p+q)} + \frac{2pQP}{\sqrt{pq}} \cos \theta, \quad H_q = -a^2 \sin^2 \theta + \frac{q(pq + 4a^2)}{2(p+q)} - \frac{2qQP}{\sqrt{pq}} \cos \theta \quad (\text{D13})$$

and

$$A_\phi = -\frac{2P}{H_q} (H_q + a^2 \sin^2 \theta) \cos \theta + \sqrt{\frac{p}{q}} \frac{Q(2a^2(p+q) + q(p^2 - 4a^2)) \sin^2 \theta}{(p+q)H_q} \quad (\text{D14})$$

and

$$P = \sqrt{\frac{p(p^2 - 4a^2)}{4(p+q)}}, \quad Q = \sqrt{\frac{q(q^2 - 4a^2)}{4(p+q)}}. \quad (\text{D15})$$

Regularity of the horizon demands $y \sim y + 8\pi P$ (or quotients) and $\phi \sim \phi + 2\pi$. Coordinates which are adapted to the $U(1)^2$ rotational symmetry can be defined by $\phi = \phi_1 + \phi_2$ and $y = 2P(\phi_2 - \phi_1)$; the absence of conical singularities then implies ϕ_1, ϕ_2 are 2π periodic with $\partial/\partial\phi_1$ vanishing at $\theta = \pi$ and $\partial/\partial\phi_2$ vanishing at $\theta=0$, i.e., one must have S^3 topology. The other near-horizon data are

$$A_0 = -\frac{1}{a\sqrt{pq}}, \quad \Gamma = \frac{H_p}{a\sqrt{pq}}, \quad (\text{D16})$$

and

$$k^\phi = \frac{pq + 4a^2}{2a^2\sqrt{pq}(p+q)}, \quad k^y = -\frac{(p^2 - 4a^2)Q}{qa^2(p+q)}. \quad (\text{D17})$$

APPENDIX E: D=5, $\Lambda=0$ SPECIAL CASES

In this appendix we provide details concerning special cases arising in the $\Lambda=0$ and $\Gamma=a_0 + a_2\sigma^2$ case analyzed in the main text.

1. Exclusion of a special case

In this subsection we show that the case

$$c_1^2 a_0 a_2 + (C^2 a_0 + a_2 c_2)^2 = 0 \quad (\text{E1})$$

is not compatible with having a compact horizon. Observe that this case implies that the polynomial $P(\sigma) = \alpha\sigma^2 + \beta\sigma + \gamma$ has vanishing discriminant. From (147), $\dot{\omega} = 0$, and we may shift x^1 to set $\omega = 0$. Since $(C^2 a_0 + a_2 c_2) = \pm c_1 \sqrt{-a_0 a_2}$, it follows that

$$\gamma_{11} = \frac{2\alpha(\sigma - \sigma_0)^2}{\Gamma}, \quad (\text{E2})$$

$\alpha = \mp a_2 c_1 \sqrt{-a_0 a_2}$ and

$$\sigma_0 = \pm \left(-\frac{a_0}{a_2} \right)^{1/2}. \quad (\text{E3})$$

Further, $\Gamma = a_2(\sigma - \sigma_0)(\sigma + \sigma_0)$ and hence the horizon metric is

$$\gamma_{ab} dx^a dx^b = \frac{\Gamma d\sigma^2}{Q} + \frac{2\alpha(\sigma - \sigma_0)(dx^1)^2}{a_2(\sigma + \sigma_0)} + \frac{Q(dx^2)^2}{2\alpha(\sigma - \sigma_0)^2}. \quad (\text{E4})$$

Having obtained the local form of the horizon metric, we turn to its regularity. The roots of Q in this case are easily seen to be

$$\sigma_{\pm} = \frac{c_1}{2C^2} \pm \frac{C^2 a_0 - a_2 c_2}{2C^2 \sqrt{-a_0 a_2}}. \quad (\text{E5})$$

Now suppose $\sigma_0 > 0$; then it is easy to show $\sigma_+ = \sigma_0$. Similarly $\sigma_0 < 0$ implies $\sigma_- = \sigma_0$. Therefore in either case, $(d\sigma)^2 = Q/\Gamma$ vanishes only at one point. This implies that (E4) cannot describe a compact manifold and hence we exclude this case.

2. $\alpha = 0$

Consider now the special case $\alpha = 0$. Note that since $\alpha = 0$, $a_0 = -a_2 c_2 C^{-2}$, which implies $\gamma = 0$, and therefore $\beta \neq 0$. Observe that another way of writing the solution to (147), valid when $\beta \neq 0$ (and any α), is

$$\omega = \pm \left[\frac{\kappa(a_2 \sigma^2 - a_0)}{\beta P(\sigma)} + c_3' \right]. \quad (\text{E6})$$

The advantage of this expression is that it is valid when $\alpha = 0$ and it is related to (155) by $c_3 = c_3' + (\kappa a_2)/(\beta \alpha)$. Thus, setting $\alpha = 0$ gives

$$\omega = \pm \left[\frac{C}{4\sqrt{c_1^2 c_2^2 a_2}} \frac{C^2 \sigma^2 + c_2}{a_2 \sigma} + c_3' \right] \quad (\text{E7})$$

and $a_2 > 0$. Also note that $\Gamma = a_2(\sigma^2 - c_2 C^{-2})$ and

$$\gamma_{11} = \frac{4a_0 a_2 c_1 \sigma}{\Gamma} = 4a_0 a_2 \left(\frac{Q(\sigma)}{\Gamma} + C^2 \right) \quad (\text{E8})$$

so $a_0 > 0$ and thus $c_2 < 0$. We must have $c_1 \sigma > 0$, and without loss of generality, we choose $\sigma > 0$ so $c_1 > 0$. Therefore, $\gamma_{11} > 0$ for $\sigma_1 \leq \sigma \leq \sigma_2$ and hence the horizon metric is nondegenerate everywhere except at the points σ_i where there are conical singularities. The rest of the analysis is identical to the $\alpha \neq 0$ case, and one obtains the same values for d_i noting that $P(\sigma_i) = 2a_0 a_2 c_1 \sigma_i$.

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- ⁶⁹ A static near-horizon geometry is defined by $V \wedge dV=0$, see Ref. 22.
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- ⁷¹ In fact, the $c_1 \neq 0$ solution is the near-horizon limit of Kerr-AdS₄-NUT whose horizon suffers from conical singularities.
- ⁷² We could consider the two cases simultaneously, however, for clarity we have chosen not to.
- ⁷³ In this case, one can solve for $c_1 = \Lambda c_2 C^{-2}$ from which it follows that $Q = (C^2 \sigma^2 - c_2)(-\Lambda \sigma C^{-2} - 1)$. Therefore if $\Lambda=0$ there is no root $\sigma_2 > \sigma_1$. If $\Lambda < 0$ then $\sigma_2 = -C^2 \Lambda^{-1}$, however, $Q(\sigma) < 0$ for $\sigma_1 < \sigma < \sigma_2$.
- ⁷⁴ The expression for e_i is valid for $\alpha \neq 0$. For $\alpha=0$ one must use an expression for ω valid when $\alpha=0$. This is given in the appendices and simply amounts to a shift in ω which can be generated by shifting x^1 . It thus suffices to check $\alpha \neq 0$ as then the $\alpha=0$ case follows by the appropriate shift in x^1 .
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