A CLASSIFICATION OF SEPARABLE ROSENTHAL COMPACTA AND ITS APPLICATIONS

SPIROS ARGYROS, PANDELIS DODOS AND VASSILIS KANELLOPOULOS

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1. Introduction

The theory of Rosenthal compacta, namely of compact subsets of the first Baire class on a Polish space X, was initiated with the pioneering work of Rosenthal [Ro2]. Significant contributions of many researchers coming from divergent areas have revealed the deep structural properties of this class. Our aim is to study some aspects of separable Rosenthal compacta, as well as, to present some of their applications.

The present work consists of three parts. In the first part we determine the prototypes of separable Rosenthal compacta and we provide a classification theorem. The second part concerns an extension of a theorem of Todorčević included in his profound study of Rosenthal compacta [To1]. The last part is devoted to applications.

Our results, concerning the first part, are mainly included in Theorems 2 and 3 below. Roughly speaking, we assert that there exist seven separable Rosenthal compacta such that every $\mathcal{K}$ in the same class contains one of them in a very canonical way. We start with the following definition.

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Definition 1. (a) *Let I be a countable set and let X,Y be Polish spaces. Let $\{f_i\}_{i\in I}$ and $\{g_i\}_{i\in I}$ be two pointwise bounded families of real-valued functions on X and Y respectively, indexed by the set I. We say that $\{f_i\}_{i\in I}$ and $\{g_i\}_{i\in I}$ are* equivalent *if the natural map $f_i \mapsto g_i$ is extended to a topological homeomorphism between $\overline{\{f_i\}}^p_{i\in I}$ and $\overline{\{g_i\}}^p_{i\in I}$.*
(b) *Let X be a Polish space and let $\{f_t\}_{t\in 2^{<\mathbb{N}}}$ be relatively compact in $\mathcal{B}_1(X)$. We say that $\{f_t\}_{t\in 2^{<\mathbb{N}}}$ is* minimal *if for every dyadic subtree $S=(s_t)_{t\in 2^{<\mathbb{N}}}$ of the Cantor tree $2^{<\mathbb{N}}$ the families $\{f_t\}_{t\in 2^{<\mathbb{N}}}$ and $\{f_{s_t}\}_{t\in 2^{<\mathbb{N}}}$ are equivalent.*

Related to the above notions the following theorem is proved.

Theorem 2. *The following hold.*

(a) *Up to equivalence, there are exactly seven minimal families.*
(b) *For every family $\{f_t\}_{t\in 2^{<\mathbb{N}}}$ relatively compact in $\mathcal{B}_1(X)$, with X Polish, there exists a regular dyadic subtree $S=(s_t)_{t\in 2^{<\mathbb{N}}}$ of the Cantor tree $2^{<\mathbb{N}}$ such that $\{f_{s_t}\}_{t\in 2^{<\mathbb{N}}}$ is equivalent to one of the seven minimal families.*

For any of the seven minimal families the corresponding pointwise closure is a separable Rosenthal compact containing the family as a discrete set. We denote them as follows

$$A(2^{<\mathbb{N}}),\ 2^{\leqslant\mathbb{N}},\ \hat{S}_+(2^{\mathbb{N}}),\ \hat{S}_-(2^{\mathbb{N}}),\ \hat{A}(2^{\mathbb{N}}),\ \hat{D}(2^{\mathbb{N}}) \text{ and } \hat{D}\big(S(2^{\mathbb{N}})\big).$$

The precise description of the families and the corresponding compacta is given in Subsection 4.3. The first two in the above list are metrizable spaces. The next two are hereditarily separable, non-metrizable and mutually homeomorphic (thus, the above defined notion of equivalence of families is stronger than saying that the corresponding closures are homeomorphic). The space $\hat{S}_+(2^{\mathbb{N}})$, and so the space $\hat{S}_-(2^{\mathbb{N}})$ as well, can be realized as a closed subspace of the split interval $S(I)$. Following [E], we shall denote by $A(2^{\mathbb{N}})$ the one point compactification of the Cantor set $2^{\mathbb{N}}$. The space $\hat{A}(2^{\mathbb{N}})$ is the standard separable extension of $A(2^{\mathbb{N}})$ (see [Po2, Ma]). This is the only not first countable space from the above list. The space $\hat{D}(2^{\mathbb{N}})$ is the separable extension of the Alexandroff duplicate of the Cantor set $D(2^{\mathbb{N}})$ as it was described in [To1]. Finally, the space $\hat{D}\big(S(2^{\mathbb{N}})\big)$ can be realized as a closed subspace of the Helly space. Its accumulation points is the closure of the standard uncountable discrete subset of the Helly space.

Theorem 2 is essentially a success of the infinite-dimensional Ramsey theory for trees and perfect sets. There is a long history on the interaction between Ramsey theory and Rosenthal compacta which can be traced back to Farahat's proof [F] of Rosenthal's ℓ_1 theorem [Ro1] and its tree extension due to Stern [Ste]. This interaction was further expanded by Todorčević in [To1] with the use of the parameterized Ramsey theory for perfect sets.

The new Ramsey theoretic ingredient in the proof of Theorem 2 is a result concerning partitions of two classes of antichains of the Cantor tree which we call *increasing* and *decreasing*. We will briefly comment on the proof of Theorem 2 and the critical role of this result. One starts with a family $\{f_t\}_{t\in 2^{<\mathbb{N}}}$ relatively compact in $\mathcal{B}_1(X)$. A first topological reduction shows that in order to understand the closure of $\{f_t\}_{t\in 2^{<\mathbb{N}}}$ in $\mathbb{R}^X$ it is enough to determine all subsets of the Cantor tree for which the corresponding subsequence of $\{f_t\}_{t\in 2^{<\mathbb{N}}}$ is pointwise convergent. A second reduction shows that it is enough to determine only a cofinal subset of convergent subsequences. One is then led to analyze which classes of subsets of the Cantor tree are Ramsey and cofinal. First we observe that every infinite subset of $2^{<\mathbb{N}}$ either contains an infinite chain or an infinite antichain. It is well-known, and goes back to Stern, that chains are Ramsey. On the other hand, the set of all antichains is not. However, the classes of increasing and decreasing antichains are Ramsey and, moreover, they are cofinal in the set of all antichains. Using these properties of chains and of increasing and decreasing antichains, we are able to have a satisfactory control over the convergent subsequences of $\{f_t\}_{t\in 2^{<\mathbb{N}}}$. Finally, repeated applications of Galvin's theorem on partitions of doubletons of perfect sets of reals permit us to fully canonize the topological behavior of $\{f_t\}_{t\in 2^{<\mathbb{N}}}$ yielding the proof of Theorem 2.

A direct consequence of part (b) of Theorem 2 is that for every separable Rosenthal compact and for every countable dense subset $\{f_t\}_{t\in 2^{<\mathbb{N}}}$ of it there exists a regular dyadic subtree $S=(s_t)_{t\in 2^{<\mathbb{N}}}$ such that the pointwise closure of $\{f_{s_t}\}_{t\in 2^{<\mathbb{N}}}$ is homeomorphic to one of the above described compacta. In general, for a given countable dense subset $\{f_n\}$ of a separable Rosenthal compact $\mathcal{K}$, we say that one of the minimal families *canonically embeds into $\mathcal{K}$ with respect to* $\{f_n\}$ if there exists an increasing injection $\phi\colon 2^{<\mathbb{N}}\to\mathbb{N}$ such that the family $\{f_{\phi(t)}\}_{t\in 2^{<\mathbb{N}}}$ is equivalent to it. The next theorem is a supplement of Theorem 2 and shows that the minimal families characterize certain topological properties of $\mathcal{K}$.

Theorem 3. *Let $\mathcal{K}$ be a separable Rosenthal compact and let $\{f_n\}$ be a countable dense subset of $\mathcal{K}$.*

(a) *If $\mathcal{K}$ consists of bounded functions in $\mathcal{B}_1(X)$, is metrizable and non-separable in the supremum norm, then $2^{\leqslant\mathbb{N}}$ canonically embeds into $\mathcal{K}$ with respect to $\{f_n\}$ such that its image is norm non-separable.*
(b) *If $\mathcal{K}$ is non-metrizable and hereditarily separable, then either $\hat{S}_+(2^{\mathbb{N}})$ or $\hat{S}_-(2^{\mathbb{N}})$ canonically embeds into $\mathcal{K}$ with respect to $\{f_n\}$.*
(c) *If $\mathcal{K}$ is not hereditarily separable and first countable, then either $\hat{D}(2^{\mathbb{N}})$ or $\hat{D}\big(S(2^{\mathbb{N}})\big)$ canonically embeds into $\mathcal{K}$ with respect to $\{f_n\}$.*
(d) *If $\mathcal{K}$ is not first countable, then $\hat{A}(2^{\mathbb{N}})$ canonically embeds into $\mathcal{K}$ with respect to $\{f_n\}$.*

In particular, if $\mathcal{K}$ is non-metrizable, then one of the non-metrizable prototypes canonically embeds into $\mathcal{K}$ with respect to any dense subset of $\mathcal{K}$.

Part (a) is an extension of the classical result of Stegall [St] which led to the characterization of the Radon–Nikodym property in dual Banach spaces. We mention that Todorčević [To1] has shown that in case (b) above the split interval $S(I)$ embeds into $\mathcal{K}$. It is an immediate consequence of the above theorem that every not hereditarily separable $\mathcal{K}$ contains an uncountable discrete subspace of the size of the continuum, a result which is due to Pol [Po1]. The proofs of parts (a), (b) and (c) use variants of Stegall's fundamental construction similar in spirit as in the work of Godefroy and Talagrand [GT]. Part (d) is a consequence of a more general structural result concerning non-G_δ points which we are about to describe. To this end we start with the following definition.

Definition 4. *Let $\mathcal{K}$ be a separable Rosenthal compact on a Polish space X and let $\mathcal{C}$ be a closed subspace of $\mathcal{K}$. We say that $\mathcal{C}$ is an* analytic subspace *if there exist a countable dense subset $\{f_n\}$ of $\mathcal{K}$ and an analytic subset A of $[\mathbb{N}]^\infty$ such that the following are satisfied.*

1. *For every $L \in A$ the accumulation points of the set $\{f_n : n \in L\}$ in $\mathbb{R}^X$ is a subset of $\mathcal{C}$.*
2. *For every $g \in \mathcal{C}$ which is an accumulation point of $\mathcal{K}$ there exists $L \in A$ with $g \in \overline{\{f_n\}}^p_{n\in L}$.*

Observe that every separable Rosenthal compact $\mathcal{K}$ is an analytic subspace of itself with respect to any countable dense set. Let us point out that while the class of analytic subspaces is strictly wider than the class of separable ones, it shares all structural properties of separable Rosenthal compacta. This will become clear in the sequel.

A natural question raised by the above definition is whether the concept of an analytic subspace depends on the choice of the countable dense subset of $\mathcal{K}$. We believe that it is independent. This is supported by the fact that this is indeed the case for analytic subspaces of separable Rosenthal compacta in $\mathcal{B}_1(X)$ with X compact metrizable.

To state our results concerning analytic subspaces we need to introduce the following definition.

Definition 5. *Let $\mathcal{K}$ be a separable Rosenthal compact, let $\{f_n\}$ be a countable dense subset of $\mathcal{K}$ and let $\mathcal{C}$ be a closed subspace of $\mathcal{K}$. We say that one of the prototypes $\mathcal{K}_i$ ($1 \leqslant i \leqslant 7$)* canonically embeds into $\mathcal{K}$ with respect to $\{f_n\}$ and $\mathcal{C}$ *if there exists a subfamily $\{f_t\}_{t\in 2^{<\mathbb{N}}}$ of $\{f_n\}$ which is equivalent to the canonical dense family of $\mathcal{K}_i$ and such that all accumulation points of $\{f_t\}_{t\in 2^{<\mathbb{N}}}$ are in $\mathcal{C}$.*

The following theorem describes the structure of not first countable analytic subspaces.

Theorem 6. *Let $\mathcal{K}$ be a separable Rosenthal compact, let $\mathcal{C}$ be an analytic subspace of $\mathcal{K}$ and let $\{f_n\}$ be a countable dense subset of $\mathcal{K}$ witnessing the analyticity of $\mathcal{C}$.*

Also let $f \in \mathcal{C}$ be a non-G_δ point of $\mathcal{C}$. Then $\hat{A}(2^{\mathbb{N}})$ canonically embeds into $\mathcal{K}$ with respect to $\{f_n\}$ and $\mathcal{C}$ and such that f is the unique non-G_δ point of its image.

Theorem 6 is the last step of a serie of results initiated by a fruitful problem concerning the character of points in separable Rosenthal compacta posed by Pol [Po1]. The first decisive step towards the solution of this problem was made by Krawczyk [Kr]. He proved that a point $f \in \mathcal{K}$ is non-G_δ if and only if the set

$$\mathcal{L}_{\mathbf{f},f} \coloneqq \{L \in [\mathbb{N}]^\infty : (f_n)_{n\in L} \text{ is pointwise convergent to } f\}$$

is co-analytic non-Borel. His analysis revealed a fundamental construction which we call *Krawczyk tree* (K-tree) with respect to the given point f and any countable dense subset $\mathbf{f} = \{f_n\}$ of $\mathcal{K}$. He showed that there exists a subfamily $\{f_t\}_{t\in\mathbb{N}^{<\mathbb{N}}}$ of $\{f_n\}$ such that the following are satisfied.

(P1) For every $\sigma \in \mathbb{N}^{\mathbb{N}}$ we have $f \notin \overline{\{f_{\sigma|n}\}}^p$.

(P2) If $A \subseteq \mathbb{N}^{<\mathbb{N}}$ is such that $f \notin \overline{\{f_t\}}^p_{t\in A}$, then for every $n \in \mathbb{N}$ there exist $t_0, \dots, t_k \in \mathbb{N}^n$ such that A is almost included in the set of the successors of the t_i's.

Using K-trees, the second named author has shown that the set

$$\mathcal{L}_{\mathbf{f}} \coloneqq \{L \in [\mathbb{N}]^\infty : (f_n)_{n\in L} \text{ is pointwise convergent}\}$$

is complete co-analytic if there exists a non-G_δ point $f \in \mathcal{K}$ ([Do]). Let us also point out that the deep effective version of Debs' theorem [De] yields that for any separable Rosenthal compact the set $\mathcal{L}_{\mathbf{f}}$ contains a Borel cofinal subset.

There are strong evidences, as Debs' theorem mentioned above, that separable Rosenthal compacta are definable objects and, consequently, they are connected to descriptive set theory (see, also, [ADK1, B, Do]). One of the first results illustrating this connection was proved in the late 1970s by Godefroy [Go] and asserts that a separable compact $\mathcal{K}$ is Rosenthal if and only if $C(\mathcal{K})$ is an analytic subset of $\mathbb{R}^D$ for every countable dense subset D of $\mathcal{K}$. Related to this, Pol has conjectured that a separable Rosenthal compact $\mathcal{K}$ embeds into $\mathcal{B}_1(2^{\mathbb{N}})$ if and only if $C(\mathcal{K})$ is a Borel subset of $\mathbb{R}^D$ (see [Ma, Po2]). It is worth mentioning that for a separable $\mathcal{K}$ in $\mathcal{B}_1(2^{\mathbb{N}})$, for every countable dense subset $\{f_n\}$ of $\mathcal{K}$ and every $f \in \mathcal{K}$, there exists a Borel cofinal subset of the corresponding set $\mathcal{L}_{\mathbf{f},f}$, a property not shared by all separable Rosenthal compacta.

The last step to the solution of Pol's problem was made by Todorčević [To1]. He proved that if f is a non-G_δ point of $\mathcal{K}$, then the space $A(2^{\mathbb{N}})$ is homeomorphic to a closed subset of $\mathcal{K}$ with f as the unique limit point. His remarkable proof uses metamathematical arguments (in particular, forcing and absoluteness).

We proceed to discuss the proof of Theorem 6. The first decisive step is the following theorem concerning the existence of K-trees.

Theorem 7. *Let $\mathcal{K}$, $\mathcal{C}$, $\{f_n\}$ and $f \in \mathcal{C}$ be as in Theorem 6. Then there exists a K-tree $\{f_t\}_{t\in\mathbb{N}^{<\mathbb{N}}}$ with respect to the point f and the dense sequence $\{f_n\}$ such that for every $\sigma \in \mathbb{N}^{\mathbb{N}}$ all accumulation points of the set $\{f_{\sigma|n} : n \in \mathbb{N}\}$ are in $\mathcal{C}$.*

The proof of the above result is a rather direct extension of the results of Krawczyk [Kr] and is based on the key property of bisequentiality established for separable Rosenthal compacta by Pol [Po3]. We will briefly comment on some further properties of the K-tree $\{f_t\}_{t\in\mathbb{N}^{<\mathbb{N}}}$ obtained by Theorem 7. To this end, let us call an antichain $\{t_n\}$ of $\mathbb{N}^{<\mathbb{N}}$ a *fan* if there exist $s \in \mathbb{N}^{<\mathbb{N}}$ and a strictly increasing sequence (m_n) in $\mathbb{N}$ such that $s^\frown m_n \sqsubseteq t_n$ for every $n \in \mathbb{N}$. We also say that an antichain $\{t_n\}$ *converges* to $\sigma \in \mathbb{N}^{\mathbb{N}}$ if for every $k \in \mathbb{N}$ the set $\{t_n\}$ is almost contained in the set of the successors of $\sigma|k$. Property (P2) of K-trees implies that for every fan $\{t_n\}$ of $\mathbb{N}^{<\mathbb{N}}$ the sequence (f_{t_n}) must be pointwise convergent to f. This fact combined with the bisequentiality of separable Rosenthal compacta yields the following property.

(P3) For every $\sigma \in \mathbb{N}^{\mathbb{N}}$ there exists an antichain $\{t_n\}$ of $\mathbb{N}^{<\mathbb{N}}$ which converges to σ and such that the sequence (f_{t_n}) is pointwise convergent to f.

In the second crucial step, we use the infinite dimensional extension of Hindman's theorem, due to Milliken [Mil1], to select an infinitely splitting subtree T of $\mathbb{N}^{<\mathbb{N}}$ such that for every $\sigma \in [T]$ the corresponding antichain $\{t_n\}$, described in property (P3) above, is found in a canonical way. We should point out that, although Milliken's theorem is a result concerning partitions of block sequences, it can be also considered as a partition theorem for a certain class of infinitely splitting subtrees of $\mathbb{N}^{<\mathbb{N}}$. This fact was first realized by Henson, in his alternative proof of Stern's theorem (see [Od]); it is used in the proof of Theorem 6 in a similar spirit. The proof of Theorem 6 is completed by choosing an appropriate dyadic subtree S of T and applying the canonization method (Theorem 2) to the family $\{f_s\}_{s\in S}$.

The following consequence of Theorem 6 describes the universal property of $\hat{A}(2^{\mathbb{N}})$ among all fundamental prototypes.

Corollary 8. *Let $\mathcal{K}$ be a non-metrizable separable Rosenthal compact and $D = \{f_n\}$ a countable dense subset of $\mathcal{K}$. Then the space $\hat{A}(2^{\mathbb{N}})$ canonically embeds into $\mathcal{K}-\mathcal{K}$ with respect to $D-D$ and with the constant function 0 as the unique non-G_δ point.*

We notice that the above corollary remains valid within the class of analytic subspaces.

The embedding of $\hat{A}(2^{\mathbb{N}})$ in an analytic subspace $\mathcal{C}$ of a separable Rosenthal compact $\mathcal{K}$ yields unconditional families of elements of $\mathcal{C}$ as follows.

Theorem 9. *Let $\mathcal{K}$ be a separable Rosenthal compact on a Polish space X consisting of bounded functions. Also let $\mathcal{C}$ be an analytic subspace of $\mathcal{K}$ having the constant function 0 as a non-G_δ point. Then there exists a family $\{f_\sigma : \sigma \in 2^{\mathbb{N}}\}$ in $\mathcal{C}$ which is 1-unconditional in the supremum norm, pointwise discrete and has 0 as unique accumulation point.*

The proof of Theorem 9 follows from Theorem 6 and the "perfect unconditionality theorem" form [ADK2].

A second application concerns representable Banach spaces, a class introduced in [GT] and closely related to separable Rosenthal compacta.

Theorem 10. *Let X be a non-separable representable Banach space. Then X^* contains an unconditional family of size $|X^*|$.*

We also introduce the concept of spreading and level unconditional tree bases. This notion is implicitly contained in [ADK2] where their existence was established in every separable Banach space not containing ℓ_1 and with non-separable dual. We present some extensions of this result in the framework of separable Rosenthal compacta.

We proceed to discuss how this work is organized. In Section 2 we set up our notation concerning trees and we present the Ramsey theoretic preliminaries needed in the rest of this paper. In the next section we define and study the classes of increasing and decreasing antichains. The main result in Section 3 is Theorem 10 which establishes the Ramsey properties of these classes. Section 4 is exclusively devoted to the proof of Theorem 2. It consists of four subsections. In the first subsection we prove a theorem (Theorem 16 in the main text) which is the first step towards the proof of Theorem 2. Theorem 16 is a consequence of the Ramsey and structural properties of chains and of increasing and decreasing antichains. In Subsection 4.2 we introduce the notion of equivalence of families of functions and we provide a criterion for establishing it. As we have already mentioned, in Subsection 4.3 we describe the seven minimal families. The proof of Theorem 2 is completed in Subsection 4.4.

In Subsection 5.1 we introduce the class of analytic subspaces of separable Rosenthal compacta and we present some of their properties, and in Subsection 5.2 we study separable Rosenthal compacta in $\mathcal{B}_1(2^{\mathbb{N}})$. In Section 6 we present parts (a), (b) and (c) of Theorem 3. Actually, Theorem 3 is proved for the wider class of analytic subspaces and within the context of Definition 5. More precisely, we prove the following theorem.

Theorem 11. *Let $\mathcal{K}$ be a separable Rosenthal compact, let $\mathcal{C}$ be an analytic subspace of $\mathcal{K}$ and let $\{f_n\}$ be a countable dense subset of $\mathcal{K}$ witnessing the analyticity of $\mathcal{C}$.*

(a) *If $\mathcal{C}$ is metrizable in the pointwise topology, consists of bounded functions and is non-separable in the supremum norm of $\mathcal{B}_1(X)$, then $2^{\leqslant\mathbb{N}}$ canonically embeds into $\mathcal{K}$ with respect to $\{f_n\}$ and $\mathcal{C}$ such that its image is norm non-separable.*

(b) *If $\mathcal{C}$ is hereditarily separable and non-metrizable, then either $\hat{S}_+(2^{\mathbb{N}})$ or $\hat{S}_-(2^{\mathbb{N}})$ canonically embeds into $\mathcal{K}$ with respect to $\{f_n\}$ and $\mathcal{C}$.*

(c) *If $\mathcal{C}$ is not hereditarily separable and first countable, then either $\hat{D}(2^{\mathbb{N}})$ or $\hat{D}\big(S(2^{\mathbb{N}})\big)$ canonically embeds into $\mathcal{K}$ with respect to $\{f_n\}$ and $\mathcal{C}$.*

Section 7 is devoted to the study of not first countable analytic subspaces. In Subsection 7.1 we prove Theorem 7, and in Subsection 7.2 we present the proof of Theorem 6. The last section is devoted to applications (in particular to the proofs of Theorems 9 and 10).

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2. Ramsey properties of perfect sets and of subtrees of the Cantor tree

The aim of this section is to present the Ramsey theoretic preliminaries needed in the rest of the paper, as well as, to set up our notation concerning trees.

Ramsey theory for trees was initiated with the fundamental Halpern–Läuchli partition theorem [HL]. The original proof was based on metamathematical arguments. The proof avoiding metamathematics was given in [AFK]. Partition theorems related to the ones presented in this section can be found in the work of Milliken [Mil2], Blass [Bl], and Louveau, Shelah and Veličković [LSV].

2.1. **Notation.** Let $\mathbb{N} = \{0, 1, 2, \dots\}$ denote the set of natural numbers. By $[\mathbb{N}]^\infty$ we denote the set of all infinite subsets of $\mathbb{N}$, and for every $L \in [\mathbb{N}]^\infty$ by $[L]^\infty$ we denote the set of all infinite subsets of L. If $k \geqslant 1$ and $L \in [\mathbb{N}]^\infty$, then $[L]^k$ stands for the set of all subsets of L of cardinality k.

2.1.1. By $2^{<\mathbb{N}}$ we denote the set of all finite sequences of 0's and 1's (the empty sequence is included). We view $2^{<\mathbb{N}}$ as a tree equipped with the (strict) partial order $\sqsubset$ of extension. If $t \in 2^{<\mathbb{N}}$, then the length $|t|$ of t is defined to be the cardinality of the set $\{s : s \sqsubset t\}$. If $s, t \in 2^{<\mathbb{N}}$, then by $s^\frown t$ we denote their concatenation. Two nodes s, t are said to be *comparable* if either $s \sqsubseteq t$ or $t \sqsubseteq s$; otherwise, they are said to be *incomparable*. A subset of $2^{<\mathbb{N}}$ consisting of pairwise comparable nodes is said to be a *chain* while a subset of $2^{<\mathbb{N}}$ consisting of pairwise incomparable nodes is said to be an *antichain*. For every $x \in 2^{\mathbb{N}}$ and every $n \geqslant 1$ we set $x|n = \big(x(0), \dots, x(n-1)\big) \in 2^{<\mathbb{N}}$ while $x|0 = \emptyset$. If $x, y \in (2^{<\mathbb{N}} \cup 2^{\mathbb{N}})$ with $x \neq y$, then by $x \wedge y$ we denote the $\sqsubset$-maximal node t of $2^{<\mathbb{N}}$ with $t \sqsubseteq x$ and $t \sqsubseteq y$. Moreover, we write $x \prec y$ if $w^\frown 0 \sqsubseteq x$ and $w^\frown 1 \sqsubseteq y$ where $w = x \wedge y$. The ordering $\prec$ restricted on $2^{\mathbb{N}}$ is the usual lexicographical ordering of the Cantor set.

2.1.2. We view every subset of $2^{<\mathbb{N}}$ as a *subtree* with the induced partial ordering. A subtree T of $2^{<\mathbb{N}}$ is said to be *pruned* if for every $t \in T$ there exists $s \in T$ with $t \sqsubset s$. It is said to be *downwards closed* if for every $t \in T$ and every $s \sqsubset t$ we have $s \in T$. If T is a subtree of $2^{<\mathbb{N}}$ (not necessarily downwards closed), then we set $\hat{T} \coloneqq \{s : \exists t \in T \text{ with } s \sqsubseteq t\}$. If T is downwards closed, then the *body* $[T]$ of T is the set $\{x \in 2^{\mathbb{N}} : x|n \in T \ \forall n\}$.

2.1.3. Let T be a (not necessarily downwards closed) subtree of $2^{<\mathbb{N}}$. For every $t \in T$ by $|t|_T$ we denote the cardinality of the set $\{s \in T : s \sqsubset t\}$ and for every $n \in \mathbb{N}$ we set $T(n) \coloneqq \{t \in T : |t|_T = n\}$. Moreover, for every $t_1, t_2 \in T$ by

$t_1 \wedge_T t_2$ we denote the $\sqsubset$-maximal node w of T such that $w \sqsubseteq t_1$ and $w \sqsubseteq t_2$. Notice that $t_1 \wedge_T t_2 \sqsubseteq t_1 \wedge t_2$. Given two subtrees S and T of $2^{<\mathbb{N}}$, we say that S is a *regular* subtree of T if $S \subseteq T$ and for every $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $S(n) \subseteq T(m)$. For a regular subtree T of $2^{<\mathbb{N}}$ the *level set* L_T of T is the set $\{l_n : T(n) \subseteq 2^{l_n}\} \subseteq \mathbb{N}$. Notice that for every $x \in [\hat{T}]$ and every $m \in \mathbb{N}$ we have that $x|m \in T$ if and only if $m \in L_T$. Hence, the chains of T are naturally identified with the product $[\hat{T}] \times [L_T]^{\infty}$. A pruned subtree T of $2^{<\mathbb{N}}$ is said to be *skew* if for every $n \in \mathbb{N}$ there exists at most one splitting node of T in $T(n)$ with exactly two immediate successors in T; it is said to be *dyadic* if every $t \in T$ has exactly two immediate successors in T. We observe that a subtree T of the Cantor tree is regular dyadic if there exists a (necessarily unique) bijection $i_T \colon 2^{<\mathbb{N}} \to T$ such that the following are satisfied.

(1) For every $t_1, t_2 \in 2^{<\mathbb{N}}$ we have $|t_1| = |t_2|$ if and only if $|i_T(t_1)|_T = |i_T(t_2)|_T$.
(2) For every $t_1, t_2 \in 2^{<\mathbb{N}}$ we have $t_1 \sqsubset t_2$ (respectively, $t_1 \prec t_2$) if and only if $i_T(t_1) \sqsubset i_T(t_2)$ (respectively, $i_T(t_1) \prec i_T(t_2)$).

When we write $T = (s_t)_{t \in 2^{<\mathbb{N}}}$, where T is a regular dyadic subtree of $2^{<\mathbb{N}}$, we mean that $s_t = i_T(t)$ for every $t \in 2^{<\mathbb{N}}$. Finally, we notice the following property. If T is a regular dyadic subtree of $2^{<\mathbb{N}}$ and R is a regular dyadic subtree of T, then R is also a regular dyadic subtree of $2^{<\mathbb{N}}$.

2.2. **Partitions of trees.** We begin by recalling the following notion from [Ka].

Definition 1. *Let T be a skew subtree of $2^{<\mathbb{N}}$. We define $f_T \colon \mathbb{N} \to \{1,2\}^{<\mathbb{N}}$ as follows. For every $n \in \mathbb{N}$ let $\{s_0 \prec \cdots \prec s_{m-1}\}$ be the $\prec$-increasing enumeration of $T(n)$. We set $f_T(n) = (e_0, \dots, e_{m-1}) \in \{1,2\}^m$ where e_i denotes the cardinality of the set of the immediate successors of s_i in T for every $i \in \{0, \dots, m-1\}$. The function f_T will be called the* code *of the tree T. If $f \colon \mathbb{N} \to \{1,2\}^{<\mathbb{N}}$ is a function such that there exists a skew tree T with $f = f_T$, then f will be called a* skew tree code.

For instance, if $f_T(n) = (1)$ for every $n \in \mathbb{N}$, then the tree T is a chain. On the other hand, if $f_T(0) = (2)$ and $f_T(n) = (1,1)$ for all $n \geqslant 1$, then T consists of two chains. Moreover, observe that if T and S are two skew subtrees of $2^{<\mathbb{N}}$ with $f_T = f_S$, then T and S are isomorphic with respect to both $\prec$ and $\sqsubset$. If f is a skew tree code and T is a regular dyadic subtree of $2^{<\mathbb{N}}$, then by $[T]_f$ we denote the set of all regular skew subtrees of T of code f. It is easy to see that the set $[T]_f$ is a Polish subspace of 2^T. Also observe that if R is a regular dyadic tree of T, then $[R]_f = [T]_f \cap 2^R$. We will need the following theorem which is a consequence of [Ka, Theorem 46].

Theorem 2. *Let T be a regular dyadic subtree of $2^{<\mathbb{N}}$, let f be a skew tree code and let A be an analytic subset of $[T]_f$. Then there exists a regular dyadic subtree R of T such that either $[R]_f \subseteq A$ or $[R]_f \cap A = \emptyset$.*

For a regular dyadic subtree T of $2^{<\mathbb{N}}$ by $[T]_{\text{chains}}$ we denote the set of all infinite chains of T. Theorem 2 includes the following result due to Stern [Ste], Miller, Todorčević [Mi] and Pawlikowski [Pa].

Theorem 3. *Let T be a regular dyadic subtree of $2^{<\mathbb{N}}$ and let A be an analytic subset of $[T]_{\text{chains}}$. Then there exists a regular dyadic subtree R of T such that either $[R]_{\text{chains}} \subseteq A$ or $[R]_{\text{chains}} \cap A = \emptyset$.*

Theorem 2 will be applied to the following classes of skew subtrees.

Definition 4. *Let T be a regular dyadic subtree of $2^{<\mathbb{N}}$. A subtree S of T will be called* increasing (*respectively,* decreasing) *if the following are satisfied.*

(a) *The tree S is uniquely rooted, regular, skew and pruned.*
(b) *For every $n \in \mathbb{N}$ there exists a splitting node of S in $S(n)$ which is the $\prec$-maximum (respectively, $\prec$-minimum) node of $S(n)$ and it has two immediate successors in S.*

The class of increasing (respectively, decreasing) subtrees of T will be denoted by $[T]_{\text{Incr}}$ (respectively, $[T]_{\text{Decr}}$).

It is easy to see that every increasing (respectively, decreasing) subtree is of fixed code. Thus Theorem 2 can be applied to yield the following corollary.

Corollary 5. *Let T be a regular dyadic subtree of $2^{<\mathbb{N}}$ and let A be an analytic subset of $[T]_{\text{Incr}}$. Then there exists a regular dyadic subtree R of T such that either $[R]_{\text{Incr}} \subseteq A$ or $[R]_{\text{Incr}} \cap A = \emptyset$. Similarly for the case of $[T]_{\text{Decr}}$.*

Corollary 5 may be considered as a parameterized version of the Louveau–Shelah–Veličković theorem [LSV].

2.3. **Partitions of perfect sets.** For every subset X of $2^{\mathbb{N}}$ by $[X]^2$ we denote the set of all doubletons of X. We identify $[X]^2$ with the set of all $(\sigma,\tau) \in X^2$ with $\sigma \prec \tau$. We will need the following partition theorem due to Galvin (see, e.g., [Ke, Theorem 19.7]).

Theorem 6. *Let P be a perfect subset of $2^{\mathbb{N}}$. If A is a subset of $[P]^2$ with the Baire property, then there exists a perfect subset Q of P such that either $[Q]^2 \subseteq A$ or $[Q]^2 \cap A = \emptyset$.*

3. Increasing and decreasing antichains of a regular dyadic tree

In this section we define the increasing and decreasing antichains and we establish their fundamental Ramsey properties.

As we have already seen in Section 2 the class of infinite chains of the Cantor tree is Ramsey. On the other hand an analogue of Theorem 3 for infinite antichains is not valid. For instance, color an antichain (t_n) of $2^{<\mathbb{N}}$ red if $t_0 \prec t_1$; otherwise color it blue. It is easy to see that this is an open partition, yet there is no dyadic subtree of $2^{<\mathbb{N}}$ all of whose antichains are monochromatic. So, it is necessary, in order to have a Ramsey result for antichains, to restrict our attention to those which are monotone with respect to $\prec$. Still, however, this is not enough. To see this, consider the set of all $\prec$-increasing antichains and color such an antichain (t_n) red if $|t_0| \leqslant |t_1 \wedge t_2|$; otherwise color it blue. Again, we see that this is an open partition which is not Ramsey.

The following definition incorporates all the restrictions indicated in the above discussion and which are, as we shall see, essentially the only obstacles to a Ramsey result for antichains.

Definition 7. *Let T be a regular dyadic subtree of the Cantor tree $2^{<\mathbb{N}}$. An infinite antichain (t_n) of T will be called* increasing *if the following conditions are satisfied.*

(1) *For every $n, m \in \mathbb{N}$ with $n < m$ we have $|t_n|_T < |t_m|_T$.*
(2) *For every $n, m, l \in \mathbb{N}$ with $n < m < l$ we have $|t_n|_T \leqslant |t_m \wedge_T t_l|_T$.*
(3I) *For every $n, m \in \mathbb{N}$ with $n < m$ we have $t_n \prec t_m$.*

The set of all increasing antichains of T will be denoted by $\mathrm{Incr}(T)$. *Similarly, an infinite antichain (t_n) of T will be called decreasing if* (1) *and* (2) *above are satisfied and* (3I) *is replaced by the following.*

(3D) *For every $n, m \in \mathbb{N}$ with $n < m$ we have $t_m \prec t_n$.*

The set of all decreasing antichains of T will be denoted by $\mathrm{Decr}(T)$.

The classes of increasing and decreasing antichains of T have the following crucial stability properties.

Lemma 8. *Let T be a regular dyadic subtree of $2^{<\mathbb{N}}$. Then the following hold.*

(1) (Hereditariness) *Let $(t_n) \in \mathrm{Incr}(T)$ and let $L = \{l_0 < l_1 < \cdots\}$ be an infinite subset of $\mathbb{N}$. Then $(t_{l_n}) \in \mathrm{Incr}(T)$. Similarly, if $(t_n) \in \mathrm{Decr}(T)$, then $(t_{l_n}) \in \mathrm{Decr}(T)$.*
(2) (Cofinality) *Let (t_n) be an infinite antichain of T. Then there exists $\{l_0 < l_1 < \cdots\} \in [\mathbb{N}]^\infty$ such that either $(t_{l_n}) \in \mathrm{Incr}(T)$ or $(t_{l_n}) \in \mathrm{Decr}(T)$.*
(3) (Coherence) *We have $\mathrm{Incr}(T) = \mathrm{Incr}(2^{<\mathbb{N}}) \cap 2^T$, and similarly for the decreasing antichains.*

Proof. (1) It is straightforward.

(2) The point is that all three properties in the definition of increasing and decreasing antichains are cofinal in the set of all antichains of T. Indeed, let (t_n) be an infinite antichain of T. Clearly, there exists $N \in [\mathbb{N}]^\infty$ such that the sequence $(|t_n|_T)_{n\in N}$ is strictly increasing. Moreover, by Ramsey's theorem, there exists $M \in [N]^\infty$ such that the sequence $(t_n)_{n\in M}$ is either $\prec$-increasing or $\prec$-decreasing. Finally, to see that condition (2) in Definition 7 is cofinal, set

$$A \coloneqq \big\{(n,m,l) \in [M]^3 : |t_n|_T \leqslant |t_m \wedge_T t_l|_T\big\}.$$

By Ramsey's theorem again, there exists $L \in [M]^\infty$ such that either $[L]^3 \subseteq A$ or $[L]^3 \cap A = \emptyset$. We claim that $[L]^3 \subseteq A$ which clearly completes the proof. Assume not, that is, $[L]^3 \cap A = \emptyset$. Set $n := \min(L)$ and $L' := L \setminus \{n\} \in [L]^\infty$. Also set $k := |t_n|_T$. Then for every $(m,l) \in [L']^2$ we have $|t_m \wedge_T t_l|_T < k$. The set $\{t \in T : |t|_T < k\}$ is finite. Hence, by another application of Ramsey's theorem, there exist $s \in T$ with $|s|_T < k$ and $L'' \in [L']^\infty$ such that for every $(m,l) \in [L'']^2$ we have that $s = t_m \wedge_T t_l$. But this is impossible since the tree T is dyadic.

(3) First we observe the following. As the tree T is regular, for every $t, s \in T$ we have $|t|_T < |s|_T$ (respectively, $|t|_T = |s|_T$) if and only if $|t| < |s|$ (respectively, $|t| = |s|$).

Now, let $(t_n) \in \mathrm{Incr}(T)$. In order to show that $(t_n) \in \mathrm{Incr}(2^{<\mathbb{N}}) \cap 2^T$ it is enough to prove that for every $n < m < l$ we have $|t_n| \leqslant |t_m \wedge t_l|$. By the previous remark, we have $|t_n| \leqslant |t_m \wedge_T t_l|$. Since $t_m \wedge_T t_l \sqsubseteq t_m \wedge t_l$, we are done.

Conversely assume that $(t_n) \in \mathrm{Incr}(2^{<\mathbb{N}}) \cap 2^T$. Again it is enough to check that condition (2) in Definition 7 is satisfied. So, let $n < m < l$. There exist $s_m, s_l \in T$ with $|s_m|_T = |s_l|_T = |t_n|_T$, $s_m \sqsubseteq t_m$ and $s_l \sqsubseteq t_l$. We claim that $s_m = s_l$. Indeed, if not, then $|t_m \wedge t_l| = |s_m \wedge s_l| < |t_n|$ contradicting the fact that the antichain (t_n) is increasing in $2^{<\mathbb{N}}$. It follows that $t_m \wedge_T t_l \sqsupseteq s_m$, and so, $|s_m|_T = |t_n|_T \leqslant |t_m \wedge_T t_l|_T$ as desired. The proof for the decreasing antichains is identical. □

By part (3) of Lemma 8, we obtain the following corollary.

Corollary 9. *Let T be a regular dyadic subtree of $2^{<\mathbb{N}}$ and let R be a regular dyadic subtree of T. Then $\mathrm{Incr}(R) = \mathrm{Incr}(T) \cap 2^R$ and $\mathrm{Decr}(R) = \mathrm{Decr}(T) \cap 2^R$.*

We notice that for every regular dyadic subtree T of the Cantor tree $2^{<\mathbb{N}}$ the sets $\mathrm{Incr}(T)$ and $\mathrm{Decr}(T)$ are Polish subspaces of 2^T. The main result in this section is the following theorem.

Theorem 10. *Let T be a regular dyadic subtree of $2^{<\mathbb{N}}$ and let A be an analytic subset of $\mathrm{Incr}(T)$ (respectively, of $\mathrm{Decr}(T)$). Then there exists a regular dyadic subtree R of T such that either $\mathrm{Incr}(R) \subseteq A$ or $\mathrm{Incr}(R) \cap A = \emptyset$ (respectively, either $\mathrm{Decr}(R) \subseteq A$ or $\mathrm{Decr}(R) \cap A = \emptyset$).*

We notice that, after a first draft of the present paper, Todorčević informed us that he is also aware of the above result with a proof based on Milliken's theorem for strong subtrees ([To2]).

The proof of Theorem 10 is based on Corollary 5. The method is to reduce the coloring of $\mathrm{Incr}(T)$ in Theorem 10 to a coloring of the class $[T]_{\mathrm{Incr}}$ of all increasing regular subtrees of T introduced in Definition 4. (The case of decreasing antichains is similar.) To this end, we need the following easy fact concerning the classes $[T]_{\mathrm{Incr}}$ and $[T]_{\mathrm{Decr}}$.

Fact 11. *Let T be a regular dyadic subtree of $2^{<\mathbb{N}}$. If $S \in [T]_{\mathrm{Incr}}$ or $S \in [T]_{\mathrm{Decr}}$, then for every $n \in \mathbb{N}$ we have $|S(n)| = n + 1$.*

As we have indicated, the crucial fact in the present setting is that there is a canonical correspondence between $[T]_{\mathrm{Incr}}$ and $\mathrm{Incr}(T)$ (and similarly for the decreasing antichains) which we are about to describe. For every $S \in [2^{<\mathbb{N}}]_{\mathrm{Incr}}$ or $S \in [2^{<\mathbb{N}}]_{\mathrm{Decr}}$ and every $n \in \mathbb{N}$ let $\{s_0^n \prec \cdots \prec s_n^n\}$ be the $\prec$-increasing enumeration of $S(n)$. Define $\Phi\colon [2^{<\mathbb{N}}]_{\mathrm{Incr}} \to \mathrm{Incr}(2^{<\mathbb{N}})$ by

$$\Phi(S) = (s_n^{n+1}).$$

It is easy to see that Φ is a well-defined continuous map. Respectively, define $\Psi\colon [2^{<\mathbb{N}}]_{\mathrm{Decr}} \to \mathrm{Decr}(2^{<\mathbb{N}})$ by $\Psi(S) = (s_1^{n+1})$. Again it is easy to see that Ψ is well-defined and continuous.

Lemma 12. *Let T be a regular dyadic subtree of $2^{<\mathbb{N}}$. Then $\Phi\big([T]_{\mathrm{Incr}}\big) = \mathrm{Incr}(T)$ and $\Psi\big([T]_{\mathrm{Decr}}\big) = \mathrm{Decr}(T)$.*

Proof. We shall give the proof only for the case of increasing subtrees. (The proof of the other case is similar.) First, we notice that for every $S \in [T]_{\mathrm{Incr}}$ we have $\Phi(S) \in \mathrm{Incr}(2^{<\mathbb{N}}) \cap 2^T$, and so, by part (3) of Lemma 8, we obtain that $\Phi\big([T]_{\mathrm{Incr}}\big) \subseteq \mathrm{Incr}(T)$. Conversely, let $(t_n) \in \mathrm{Incr}(T)$.

Claim 1. *For every $n < m < l$ we have $t_n \wedge_T t_m = t_n \wedge_T t_l$.*

Proof of the claim. Let $n < m < l$. By condition (2) in Definition 7, there exists $s \in T$ with $|s|_T = |t_n|_T$ and such that $s \sqsubseteq t_m \wedge_T t_l$. Moreover, observe that $t_n \prec s$ since $t_n \prec t_m$. It follows that $t_n \wedge_T t_m = t_n \wedge_T s = t_n \wedge_T t_l$, as claimed. $\square$

For every $n \in \mathbb{N}$ we set $c_n \coloneqq t_n \wedge_T t_{n+1}$.

Claim 2. *For every $n < m$ we have $c_n \sqsubset c_m$. That is, the sequence (c_n) is an infinite chain of T.*

Proof of the claim. Let $n < m$. By Claim 1, we see that c_n and c_m are compatible since $c_n = t_n \wedge_T t_m$ and, by definition, $c_m = t_m \wedge_T t_{m+1}$. Finally, notice that $|c_n|_T < |t_n|_T \leqslant |t_m \wedge_T t_{m+1}|_T = |c_m|_T$. $\square$

For every $n \geqslant 1$ let c'_n denote the unique node of T such that $c'_n \sqsubseteq c_n$ and $|c'_n|_T = |t_{n-1}|_T$. Recursively, we define $S \in [T]_{\mathrm{Incr}}$ as follows. We set $S(0) \coloneqq \{c_0\}$ and $S(1) = \{t_0, c'_1\}$. Assume that $S(n) = \{s_0^n \prec \cdots \prec s_n^n\}$ has been defined so as $s_{n-1}^n = t_{n-1}$ and $s_n^n = c'_n$. For every $i \in \{0, \ldots, n-1\}$ we select a node s_i^{n+1} such that $s_i^n \sqsubset s_i^{n+1}$ and $|s_i^{n+1}|_T = |t_n|_T$. We set

$$S(n+1) \coloneqq \{s_0^{n+1} \prec \cdots \prec s_{n-1}^{n+1} \prec t_n \prec c'_{n+1}\}.$$

It is easy to check that $S \in [T]_{\mathrm{Incr}}$ and $\Phi(S) = (t_n)$. The proof of Lemma 12 is completed. $\square$

We are ready to give the proof of Theorem 10.

Proof of Theorem 10. Let A be an analytic subset of $\mathrm{Incr}(T)$. By Lemma 12, the set $B=\Phi^{-1}(A)\cap[T]_{\mathrm{Incr}}$ is an analytic subset of $[T]_{\mathrm{Incr}}$. By Corollary 5, there exists a regular dyadic subtree R of T such that either $[R]_{\mathrm{Incr}}\subseteq B$ or $[R]_{\mathrm{Incr}}\cap B=\emptyset$. By Lemma 12, the first case implies that $\mathrm{Incr}(R)=\Phi\big([R]_{\mathrm{Incr}}\big)\subseteq\Phi(B)\subseteq A$ while the second case yields that $\mathrm{Incr}(R)\cap A=\Phi\big([R]_{\mathrm{Incr}}\big)\cap A=\emptyset$. The proof for the case of decreasing antichains is similar. □

4. Canonizing sequential compactness of trees of functions

The present section consists of four subsections. In the first one, using the Ramsey properties of chains and of increasing and decreasing antichains, we prove a strengthening of a result of Stern [Ste]. In the second one, we introduce the notion of equivalence of families of functions and we provide a criterion for establishing it. In the third subsection, we define the seven minimal families. The last subsection is devoted to the proof of the main result of the section, concerning the canonical embedding in any separable Rosenthal compact of one of the minimal families.

4.1. **Sequential compactness of trees of functions.** We start with the following definition.

Definition 13. *Let $L\subseteq 2^{<\mathbb{N}}$ be infinite and $\sigma\in 2^{\mathbb{N}}$. We say that L* converges *to σ if for every $k\in\mathbb{N}$ the set L is almost included in the set $\{t\in 2^{<\mathbb{N}}:\sigma|k\sqsubseteq t\}$. The element σ will be called the* limit *of the set L. We write $L\to\sigma$ to denote the fact that L converges to σ.*

It is clear that the limit of a subset L of $2^{<\mathbb{N}}$ is unique, if it exists.

Fact 14. *Let (t_n) be an increasing (respectively, decreasing) antichain of $2^{<\mathbb{N}}$. Then (t_n) converges to σ where σ is the unique element of $2^{\mathbb{N}}$ determined by the chain (c_n) with $c_n=t_n\wedge t_{n+1}$ (see the proof of Lemma 12).*

We also need to introduce some pieces of notation.

Notation. For every infinite $L\subseteq 2^{<\mathbb{N}}$ and every $\sigma\in 2^{\mathbb{N}}$ we write $L\prec^*\sigma$ if the set L is almost included in the set $\{t:t\prec\sigma\}$. Respectively, we write $L\preceq^*\sigma$ if L is almost included in the set $\{t:t\prec\sigma\}\cup\{\sigma|n:n\in\mathbb{N}\}$. The notation $\sigma\prec^* L$ (respectively, $\sigma\preceq^* L$) has the obvious meaning. We also write $L\subseteq^*\sigma$ if for all but finitely many $t\in L$ we have $t\sqsubset\sigma$. Finally, we write $L\perp\sigma$ to denote the fact that the set $L\cap\{\sigma|n:n\in\mathbb{N}\}$ is finite.

The following fact is essentially a consequence of part (2) of Lemma 8.

Fact 15. *If L is an infinite subset of $2^{<\mathbb{N}}$ and $\sigma\in 2^{\mathbb{N}}$ are such that $L\to\sigma$ and $L\prec^*\sigma$ (respectively, $\sigma\prec^* L$), then every infinite subset of L contains an increasing (respectively, decreasing) antichain which converges to σ.*

Our goal in this subsection is to give a proof of the following theorem.

Theorem 16. *Let X be a Polish space and let $\{f_t\}_{t\in 2^{<\mathbb{N}}}$ be a family relatively compact in $\mathcal{B}_1(X)$. Then there exist a regular dyadic subtree T of $2^{<\mathbb{N}}$ and a family $\{g^0_\sigma, g^+_\sigma, g^-_\sigma : \sigma\in P\}$, where $P=[\hat{T}]$, such that for every $\sigma\in P$ the following are satisfied.*

1. *The sequence $(f_{\sigma|n})_{n\in L_T}$ converges pointwise to g^0_σ (recall that L_T stands for the level set of T).*
2. *For every sequence (σ_n) in P converging to σ with $\sigma_n\prec\sigma$ for all $n\in\mathbb{N}$, the sequence $(g^{\varepsilon_n}_{\sigma_n})$ converges pointwise to g^+_σ for any choice of $\varepsilon_n\in\{0,+,-\}$. If such a sequence (σ_n) does not exist, then $g^+_\sigma = g^0_\sigma$.*
3. *For every sequence (σ_n) in P converging to σ with $\sigma\prec\sigma_n$ for all $n\in\mathbb{N}$, the sequence $(g^{\varepsilon_n}_{\sigma_n})$ converges pointwise to g^-_σ for any choice of $\varepsilon_n\in\{0,+,-\}$. If such a sequence (σ_n) does not exist, then $g^-_\sigma = g^0_\sigma$.*
4. *For every infinite subset L of T converging to σ with $L\prec^*\sigma$, the sequence $(f_t)_{t\in L}$ converges pointwise to g^+_σ.*
5. *For every infinite subset L of T converging to σ with $\sigma\prec^* L$, the sequence $(f_t)_{t\in L}$ converges pointwise to g^-_σ.*

Moreover, the functions $0,+,-\colon P\times X\to\mathbb{R}$ defined by

$$0(\sigma,x)=g^0_\sigma(x),\ +(\sigma,x)=g^+_\sigma(x),\ -(\sigma,x)=g^-_\sigma(x)$$

are all Borel.

For the proof of Theorem 16 we will need the following simple fact (the proof of which is left to the reader).

Fact 17. *The following hold.*

1. *Let $A_1=(t^1_n)$, $A_2=(t^2_n)$ be two increasing (respectively, decreasing) antichains of $2^{<\mathbb{N}}$ converging to the same $\sigma\in 2^{\mathbb{N}}$. Then there exists an increasing (respectively, decreasing) antichain (t_n) of $2^{<\mathbb{N}}$ converging to σ such that $t_{2n}\in A_1$ and $t_{2n+1}\in A_2$ for every $n\in\mathbb{N}$.*
2. *Let (σ_n) be a sequence in $2^{\mathbb{N}}$ converging to $\sigma\in 2^{\mathbb{N}}$. For every $n\in\mathbb{N}$ let $N_n=(t^n_k)$ be a sequence in $2^{<\mathbb{N}}$ converging to σ_n. If $\sigma_n\prec\sigma$ (respectively, $\sigma_n\succ\sigma$) for every n, then there exist an increasing (respectively, decreasing) antichain (t_m) and $L=\{n_m : m\in\mathbb{N}\}\in[\mathbb{N}]^\infty$ such that (t_m) converges to σ and $t_m\in N_{n_m}$ for every $m\in\mathbb{N}$.*

We proceed to the proof of Theorem 16.

Proof of Theorem 16. Our hypotheses imply that for every sequence (g_n) belonging to the closure of $\{f_t\}_{t\in 2^{<\mathbb{N}}}$ in $\mathbb{R}^X$ there exists a subsequence of (g_n) which is pointwise convergent. Consider the following subset Π_1 of $[2^{<\mathbb{N}}]_{\mathrm{chains}}$ defined by

$$\Pi_1 := \big\{c\in[2^{<\mathbb{N}}]_{\mathrm{chains}} : \text{the sequence } (f_t)_{t\in c} \text{ is pointwise convergent}\big\}.$$

Then Π_1 is a co-analytic subset of $[2^{<\mathbb{N}}]_{\mathrm{chains}}$ (see [Ste]). Applying Theorem 3 and invoking our hypotheses, we obtain a regular dyadic subtree T_1 of $2^{<\mathbb{N}}$ such that

$[T_1]_{\text{chains}} \subseteq \Pi_1$. Now consider the subset Π_2 of $\text{Incr}(T_1)$ defined by

$$\Pi_2 := \big\{(t_n) \in \text{Incr}(T_1) : \text{the sequence } (f_{t_n}) \text{ is pointwise convergent}\big\}.$$

Again Π_2 is co-analytic (this can be checked with similar arguments as in [Ste]). Applying Theorem 10, we obtain a regular dyadic subtree T_2 of T_1 such that $\text{Incr}(T_2) \subseteq \Pi_2$. Finally, applying Theorem 10 for the decreasing antichains of T_2 and the color

$$\Pi_3 := \big\{(t_n) \in \text{Decr}(T_2) : \text{the sequence } (f_{t_n}) \text{ is pointwise convergent}\big\},$$

we obtain a regular dyadic subtree T of T_2 such that, setting $P = [\hat{T}]$, the following are satisfied.

(i) For every increasing antichain (t_n) of T the sequence (f_{t_n}) is pointwise convergent.

(ii) For every decreasing antichain (t_n) of T the sequence (f_{t_n}) is pointwise convergent.

(iii) For every $\sigma \in P$ the sequence $(f_{\sigma|n})_{n\in L_T}$ is pointwise convergent to a function g^0_σ.

We notice the following. By part (1) of Fact 17, if (t^1_n) and (t^2_n) are two increasing (respectively, decreasing) antichains of T converging to the same σ, then $(f_{t^1_n})$ and $(f_{t^2_n})$ are both pointwise convergent to the same function. For every $\sigma \in P$ we define g^+_σ as follows. If there exists an increasing antichain (t_n) of T converging to σ, then we set g^+_σ to be the pointwise limit of (f_{t_n}). (By the previous remark, g^+_σ is independent of the choice of (t_n).) Otherwise, we set $g^+_\sigma = g^0_\sigma$. Similarly, we define g^-_σ to be the pointwise limit of (f_{t_n}) where (t_n) is a decreasing antichain of T converging to σ, if such an antichain exists. Otherwise, we set $g^-_\sigma = g^0_\sigma$. By Fact 15 and the above discussion, properties (i) and (ii) can be strengthened as follows.

(iv) For every $\sigma \in P$ and every infinite $L \subseteq T$ converging to σ with $L \prec^* \sigma$, the sequence $(f_t)_{t\in L}$ is pointwise convergent to g^+_σ.

(v) For every $\sigma \in P$ and every infinite $L \subseteq T$ converging to σ with $\sigma \prec^* L$, the sequence $(f_t)_{t\in L}$ is pointwise convergent to g^-_σ.

We claim that the tree T and the family $\{g^0_\sigma, g^+_\sigma, g^-_\sigma : \sigma \in P\}$ are as desired. First we check that properties (1)–(5) are satisfied. Clearly we only have to check (2) and (3). As the argument is symmetric, we will prove only property (2). We argue by contradiction. So, assume that there exist a sequence (σ_n) in P, $\sigma \in P$ and $\varepsilon_n \in \{0, +, -\}$ such that $\sigma_n \prec \sigma$ and (σ_n) converges to σ while the sequence $(g^{\varepsilon_n}_{\sigma_n})$ does not converge pointwise to g^+_σ. Hence, there exist $L \in [\mathbb{N}]^\infty$ and an open neighborhood V of g^+_σ in $\mathbb{R}^X$ such that $g^{\varepsilon_n}_{\sigma_n} \notin \overline{V}$ for every $n \in L$. By definition, for every $n \in L$ we may select a sequence (t^n_k) in T such that for every $n \in L$ the following hold.

(a) The sequence $N_n = (t^n_k)$ converges to σ_n.

(b) The sequence $(f_{t^n_k})$ converges pointwise to $g^{\varepsilon_n}_{\sigma_n}$.

(c) For every $k \in \mathbb{N}$ we have $f_{t^n_k} \notin \overline{V}$.

(d) The sequence $(\sigma_n)_{n\in L}$ converges to σ and $\sigma_n \prec \sigma$.

By part (2) of Fact 17, there exist a diagonal increasing antichain (t_m) converging to σ. By (c) above, we see that (f_{t_m}) is not pointwise convergent to g^+_σ. This leads to a contradiction by the definition of g^+_σ.

We will now show that the maps $0, +$ and $-$ are Borel. Let $\{l_0 < l_1 < \cdots\}$ be the increasing enumeration of the level set L_T of T. For every $n \in \mathbb{N}$ define $h_n \colon P \times X \to \mathbb{R}$ by $h_n(\sigma, x) = f_{\sigma|l_n}(x)$. Clearly h_n is Borel. Since for every $(\sigma, x) \in P \times X$ we have

$$0(\sigma, x) = g^0_\sigma(x) = \lim h_n(\sigma, x)$$

the Borelness of 0 is clear. We will only check the Borelness of the function $+$ (the argument for the map $-$ is identical). For every $n \in \mathbb{N}$ and every $\sigma \in P$ let $l_n(\sigma)$ be the lexicographically minimum of the closed set $\{\tau \in P : \sigma|l_n \sqsubset \tau\}$. The function $P \ni \sigma \mapsto l_n(\sigma) \in P$ is clearly continuous. Invoking the definition of g^+_σ and property (2) in the statement of the theorem, we see that for every $(\sigma, x) \in P \times X$ we have

$$+(\sigma, x) = g^+_\sigma(x) = \lim g^0_{l_n(\sigma)}(x) = \lim 0\big(l_n(\sigma), x\big).$$

Thus $+$ is Borel too, and the proof of the theorem is completed. □

Remark 1. We would like to point out that in order to apply the Ramsey theory for trees in the present setting one has to know that all the colors are sufficiently definable. This is also the reason why the Borelness of the functions $0, +$ and $-$ is emphasized in Theorem 16. As a matter of fact, we will need the full strength of the Ramsey theory for trees and perfect sets in the sense that in certain cases the color will belong to the σ-algebra generated by the analytic sets. It should be noted that this is in contrast with the classical Silver theorem [Si] for which, most applications, involve Borel partitions.

4.2. **Equivalence of families of functions.** We start with the following definition.

Definition 18. *Let I be a countable set and let X, Y be Polish spaces. Also let $\{f_i\}_{i\in I}$ and $\{g_i\}_{i\in I}$ be two pointwise bounded families of real-valued functions on X and Y respectively, indexed by the set I. We say that $\{f_i\}_{i\in I}$ is* equivalent *to $\{g_i\}_{i\in I}$ if the map*

$$f_i \mapsto g_i$$

is extended to a topological homeomorphism between $\overline{\{f_i\}}^p_{i\in I}$ and $\overline{\{g_i\}}^p_{i\in I}$.

The equivalence of the families $\{f_i\}_{i\in I}$ and $\{g_i\}_{i\in I}$ is stronger than saying that $\overline{\{f_i\}}^p_{i\in I}$ is homeomorphic to $\overline{\{g_i\}}^p_{i\in I}$ (an example illustrating this fact will be given in the next subsection). The crucial point in Definition 18 is that the equivalence of $\{f_i\}_{i\in I}$ and $\{g_i\}_{i\in I}$ gives a natural homeomorphism between their closures.

The following lemma provides an efficient criterion for checking the equivalence of families of Borel functions. We mention that in its proof we will often make use of the Bourgain–Fremlin–Talagrand theorem [BFT] without making an explicit reference. From the context it will be clear that this is what we use.

Lemma 19. *Let I be a countable set and let X,Y be Polish spaces. Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two separable Rosenthal compacta on X and Y respectively. Let $\{f_i\}_{i\in I}$ and $\{g_i\}_{i\in I}$ be two dense families of $\mathcal{K}_1$ and $\mathcal{K}_2$ respectively. Assume that for every $i\in I$ the functions f_i and g_i are isolated in $\mathcal{K}_1$ and $\mathcal{K}_2$ respectively. Then the following are equivalent.*

1. *The families $\{f_i\}_{i\in I}$ and $\{g_i\}_{i\in I}$ are equivalent.*
2. *For every infinite subset L of I the sequence $(f_i)_{i\in L}$ converges pointwise if and only if the sequence $(g_i)_{i\in L}$ does.*

Proof. The direction (1) $\Rightarrow$ (2) is obvious. What remains is to prove the converse. So assume that (2) holds. For every infinite $M\subseteq I$ we set $\mathcal{K}_1^M=\overline{\{f_i\}}^p_{i\in M}$ and $\mathcal{K}_2^M=\overline{\{g_i\}}^p_{i\in M}$. Notice that $\mathcal{K}_1^M$ and $\mathcal{K}_2^M$ are separable Rosenthal compacta. Our assumptions imply that the isolated points of $\mathcal{K}_1^M$ is precisely the set $\{f_i : i\in M\}$, and similarly for $\mathcal{K}_2^M$. Define $\Phi_M\colon \mathcal{K}_1^M\to\mathcal{K}_2^M$ as follows. First, for every $i\in M$ we set $\Phi_M(f_i)=g_i$. If $h\in\mathcal{K}_1^M$ with $h\notin\{f_i : i\in M\}$, then there exists an infinite subset L of M such that h is the pointwise limit of the sequence $(f_i)_{i\in L}$. Define $\Phi_M(h)$ to be the pointwise limit of the sequence $(g_i)_{i\in L}$ (by our assumptions this limit exists). To simplify notation, we set $\Phi=\Phi_I$.

Claim. *Let $M\subseteq I$ be infinite. Then the following hold.*

1. *The map Φ_M is well-defined, one-to-one and onto.*
2. *We have $\Phi|_{\mathcal{K}_1^M}=\Phi_M$.*

Proof of the claim. (1) Fix an infinite subset M of I. To see that Φ_M is well-defined, notice that for every $h\in\mathcal{K}_1^M$ with $h\notin\{f_i : i\in M\}$ and every pair L_1,L_2 of infinite subsets of M with $h=\lim_{i\in L_1} f_i=\lim_{i\in L_2} f_i$ we have that $\lim_{i\in L_1} g_i=\lim_{i\in L_2} g_i$. For if not, we would have that the sequence $(f_i)_{i\in L_1\cup L_2}$ converges pointwise while the sequence $(g_i)_{i\in L_1\cup L_2}$ does not, contradicting our assumptions.

We observe the following consequence of our assumptions and the definition of the map Φ_M. For every $h\in\mathcal{K}_1^M$ the point h is isolated in $\mathcal{K}_1^M$ if and only if $\Phi_M(h)$ is isolated in $\mathcal{K}_2^M$. Using this we will show that Φ_M is one-to-one. Indeed, let $h_1,h_2\in\mathcal{K}_1^M$ with $\Phi_M(h_1)=\Phi_M(h_2)$. Then, either $\Phi_M(h_1)$ is isolated in $\mathcal{K}_2^M$ or not. In the first case, there exists an $i_0\in M$ with $\Phi_M(h_1)=g_{i_0}=\Phi_M(h_2)$. Thus, $h_1=f_{i_0}=h_2$. So assume that $\Phi_M(h_1)$ is not isolated in $\mathcal{K}_2^m$. Hence, neither $\Phi_M(h_2)$ is. It follows that both h_1 and h_2 are not isolated points of $\mathcal{K}_1^M$. We select two infinite subsets L_1,L_2 of M with $h_1=\lim_{i\in L_1} f_i$ and $h_2=\lim_{i\in L_2} f_i$. Since the sequence $(g_i)_{i\in L_1\cup L_2}$ is pointwise convergent to $\Phi_M(h_1)=\Phi_M(h_2)$, our assumptions yield that

$$h_1 = \lim_{i\in L_1} f_i = \lim_{i\in L_1\cup L_2} f_i = \lim_{i\in L_2} f_i = h_2$$

which proves that Φ_M is one-to-one. Finally, to see that Φ_M is onto, let $w\in\mathcal{K}_2^M$ with $w\notin\{g_i : i\in M\}$. Let $L\subseteq M$ be infinite with $w=\lim_{i\in L} g_i$. By our assumptions, the sequence $(f_i)_{i\in L}$ converges pointwise to an $h\in\mathcal{K}_1^M$ and clearly $\Phi_M(h)=w$.

(2) Using similar arguments as in (1). □

By the above claim, it is enough to show that the map Φ is continuous. Notice that it is enough to show that if (h_n) is a sequence in $\mathcal{K}_1$ that converges pointwise to an $h\in\mathcal{K}_1$, then the sequence $\big(\Phi(h_n)\big)$ converges to $\Phi(h)$. Assume not. Then there exist a sequence (h_n) in $\mathcal{K}_1$, $h\in\mathcal{K}_1$ and $w\in\mathcal{K}_2$ such that $h=\lim h_n$, $w=\lim\Phi(h_n)$ and $w\neq\Phi(h)$. Since the map Φ is onto, there exists $z\in\mathcal{K}_1$ such that $z\neq h$ and $\Phi(z)=w$. We select $x\in X$ and $\varepsilon>0$ such that $|h(x)-z(x)|>\varepsilon$. The sequence (h_n) converges pointwise to h, and so, we may assume that for every $n\in\mathbb{N}$ we have $|h_n(x)-z(x)|>\varepsilon$. Set

$$M := \big\{ i\in I : |f_i(x)-z(x)| \geqslant \frac{\varepsilon}{2}\big\}.$$

Observe that

(O1) for every $n\in\mathbb{N}$ we have $h_n\in\mathcal{K}_1^M$, and
(O2) $z\notin\mathcal{K}_1^M$.

By part (2) of the above claim and (O1), we have that $\Phi(h_n)=\Phi_M(h_n)\in\mathcal{K}_2^M$ for every $n\in\mathbb{N}$, and so, $w\in\mathcal{K}_2^M$. Since Φ_M is onto, there exists $h'\in\mathcal{K}_1^M$ such that $\Phi_M(h')=w$. Therefore, by (O2) and invoking the above claim once again, we see that $z\neq h'$ while $\Phi_M(h')=\Phi(h')=\Phi(z)$, contradicting the fact that Φ is one-to-one. The proof of the lemma is completed. □

4.3. **Seven families of functions.** Our aim in this subsection is to describe seven families

$$\{d^i_t : t\in 2^{<\mathbb{N}}\} \ \ (1\leqslant i\leqslant 7)$$

of functions indexed by the Cantor tree. For every $i\in\{1,\dots,7\}$ the closure of the family $\{d^i_t : t\in 2^{<\mathbb{N}}\}$ in the pointwise topology is a separable Rosenthal compact $\mathcal{K}_i$. Each one of them is *minimal*, namely, for every dyadic (not necessarily regular) subtree $S=(s_t)_{t\in 2^{<\mathbb{N}}}$ of $2^{<\mathbb{N}}$ and every $i\in\{1,\dots,7\}$ the families $\{d^i_t\}_{t\in 2^{<\mathbb{N}}}$ and $\{d^i_{s_t}\}_{t\in 2^{<\mathbb{N}}}$ are equivalent in the sense of Definition 18. Although the families are mutually non-equivalent, the corresponding compacta might be homeomorphic. In all cases the family $\{d^i_t : t\in 2^{<\mathbb{N}}\}$ will be discrete in its closure. For any of the corresponding compacta $\mathcal{K}_i$ $(1\leqslant i\leqslant 7)$ by $\mathcal{L}(\mathcal{K}_i)$ we shall denote the set of all infinite subsets L of $2^{<\mathbb{N}}$ for which the sequence $(d^i_t)_{t\in L}$ is pointwise convergent. We will name the corresponding compacta (all of them are homeomorphic to closed subspaces of well-known compacta—see [AU, E]) and we will refer to the families of functions as the canonical dense sequences of them. We will use the following pieces of notation.

If $\sigma \in 2^{\mathbb{N}}$, then δ_σ is the Dirac function at σ. By x^+_σ we denote the characteristic function of the set $\{\tau \in 2^{\mathbb{N}} : \sigma \preceq \tau\}$, and by x^-_σ we denote the characteristic function of the set $\{\tau \in 2^{\mathbb{N}} : \sigma \prec \tau\}$. Notice that if $t \in 2^{<\mathbb{N}}$, then $t^\frown 0^\infty \in 2^{\mathbb{N}}$, and so, the function $x^+_{t^\frown 0^\infty}$ is well-defined. It is useful at this point to isolate the following property of the functions x^+_σ and x^-_σ which will justify the notation g^+_σ and g^-_σ in Theorem 16. *If (σ_n) is a sequence in $2^{\mathbb{N}}$ converging to σ with $\sigma_n \prec \sigma$ (respectively, $\sigma \prec \sigma_n$) for every $n \in \mathbb{N}$, then the sequence $(x^{\varepsilon_n}_{\sigma_n})$ converges pointwise to x^+_σ (respectively, to x^-_σ) for any choice of $\varepsilon_n \in \{+,-\}$.*

By identifying the Cantor set with a subset of the unit interval, we will identify every $\sigma \in 2^{\mathbb{N}}$ with the real-valued function on $2^{\mathbb{N}}$ which is equal everywhere with σ. Notice that for every $t \in 2^{<\mathbb{N}}$ we have $t^\frown 0^\infty \in 2^{\mathbb{N}}$, and so, the function $t^\frown 0^\infty$ is well-defined. For every $t \in 2^{<\mathbb{N}}$ be v_t we denote the characteristic function of the clopen set $V_t := \{\sigma \in 2^{\mathbb{N}} : t \sqsubset \sigma\}$. By 0 we denote the constant function on $2^{\mathbb{N}}$ which is equal everywhere with zero. We will also consider real-valued functions on $2^{\mathbb{N}} \oplus 2^{\mathbb{N}}$. In this case when we write, for instance, $(\delta_\sigma, x^+_\sigma)$ we mean that this function is the function δ_σ on the first copy of $2^{\mathbb{N}}$ and it is the function x^+_σ on the second copy of $2^{\mathbb{N}}$.

We also fix a regular dyadic subtree $R = (s_t)_{t \in 2^{<\mathbb{N}}}$ of $2^{<\mathbb{N}}$ with the following property.

(Q) For every $s, s' \in R$ we have $s^\frown 0^\infty \neq s'^\frown 0^\infty$ and $s^\frown 1^\infty \neq s'^\frown 1^\infty$. Therefore, the set $[\hat{R}]$ does not contain the eventually constant sequences.

In what follows by P we shall denote the perfect set $[\hat{R}]$. By P^+ we shall denote the subset of P consisting of all σ's for which there exists an increasing antichain (s_n) of R converging to σ in the sense of Definition 13. Respectively, by P^- we shall denote the subset of P consisting of all σ's for which there exists a decreasing antichain (s_n) of R converging to σ.

4.3.1. *The Alexandroff compactification of the Cantor tree $A(2^{<\mathbb{N}})$.* It is the pointwise closure of the family

$$\Big\{\frac{1}{|t|+1} v_t : t \in 2^{<\mathbb{N}}\Big\}.$$

Clearly the space $A(2^{<\mathbb{N}})$ is countable compact as the whole family accumulates to 0. Setting $d^1_t := \frac{1}{|t|+1} v_t$ for every $t \in 2^{<\mathbb{N}}$, we see that the family $\{d^1_t : t \in 2^{<\mathbb{N}}\}$ is a dense discrete subset of $A(2^{<\mathbb{N}})$. In this case the description of $\mathcal{L}\big(A(2^{<\mathbb{N}})\big)$ is trivial, since

$$L \in \mathcal{L}\big(A(2^{<\mathbb{N}})\big) \Leftrightarrow L \subseteq 2^{<\mathbb{N}}.$$

4.3.2. *The space $2^{\leqslant\mathbb{N}}$.* It is the pointwise closure of the family

$$\{s^\frown 0^\infty : s \in R\}.$$

The accumulation points of $2^{\leqslant\mathbb{N}}$ is the set

$$\{\sigma : \sigma \in P\}$$

which is clearly homeomorphic to $2^{\mathbb{N}}$. Thus, the space $2^{\leqslant\mathbb{N}}$ is uncountable compact metrizable. Setting $d^2_t := s_t{}^\frown 0^\infty$ for every $t \in 2^{<\mathbb{N}}$ and invoking property (Q) above, we see that the family $\{d^2_t : t \in 2^{<\mathbb{N}}\}$ is a dense discrete subset of $2^{\leqslant\mathbb{N}}$. The description of $\mathcal{L}\big(2^{\leqslant\mathbb{N}}\big)$ is given by

$$L \in \mathcal{L}\big(2^{\leqslant\mathbb{N}}\big) \Leftrightarrow \exists \sigma \in 2^{\mathbb{N}} \text{ with } L \to \sigma.$$

4.3.3. *The extended split Cantor set $\hat{S}_+(2^{\mathbb{N}})$.* It is the pointwise closure of the family

$$\{x^+_{s^\frown 0^\infty} : s \in R\}.$$

Notice that $\hat{S}_+(2^{\mathbb{N}})$ can be realized as a closed subspace of the split interval $S(I)$. Thus, it is hereditarily separable. For every $\sigma \in P$ the function x^+_σ belongs to $\hat{S}_+(2^{\mathbb{N}})$. However, for an element $\sigma \in P$, the function x^-_σ belongs to $\hat{S}_+(2^{\mathbb{N}})$ if and only if there exists a decreasing antichain (s_n) of R converging to σ. Finally observe that the family $\{x^+_{s^\frown 0^\infty} : s \in R\}$ is a discrete subset of $\hat{S}_+(2^{\mathbb{N}})$ (this is essentially a consequence of property (Q) above). Therefore, the accumulation points of $\hat{S}_+(2^{\mathbb{N}})$ is the set

$$\{x^+_\sigma : \sigma \in P\} \cup \{x^-_\sigma : \sigma \in P^-\}.$$

Setting $d^3_t := x^+_{s_t{}^\frown 0^\infty}$ for every $t \in 2^{<\mathbb{N}}$, we see that the family $\{d^3_t : t \in 2^{<\mathbb{N}}\}$ is a dense discrete subset of $\hat{S}_+(2^{\mathbb{N}})$. Moreover, we have

$$L \in \mathcal{L}\big(\hat{S}_+(2^{\mathbb{N}})\big) \Leftrightarrow \exists \sigma \in 2^{\mathbb{N}} \text{ with } L \to \sigma \text{ and (either } L \preceq^* \sigma \text{ or } \sigma \prec^* L).$$

4.3.4. *The mirror image $\hat{S}_-(2^{\mathbb{N}})$ of the extended split Cantor set.* The space $\hat{S}_+(2^{\mathbb{N}})$ has a natural "mirror image" $\hat{S}_-(2^{\mathbb{N}})$ which is the pointwise closure of the set

$$\{x^-_{s^\frown 1^\infty} : s \in R\}.$$

The spaces $\hat{S}_+(2^{\mathbb{N}})$ and $\hat{S}_-(2^{\mathbb{N}})$ are homeomorphic. To see this, for every $t \in 2^{<\mathbb{N}}$ let $\bar{t} \in 2^{<\mathbb{N}}$ be the finite sequence obtained by reversing 0 with 1 and 1 with 0 in the finite sequence t. Define $\phi\colon R \to R$ by setting $\phi(s_t) = s_{\bar{t}}$ for every $t \in 2^{<\mathbb{N}}$. Then it is easy to see that the map

$$\hat{S}_+(2^{\mathbb{N}}) \ni x^+_{s_t{}^\frown 0^\infty} \mapsto x^-_{\phi(s_t)^\frown 1^\infty} \in \hat{S}_-(2^{\mathbb{N}})$$

is extended to a topological homeomorphism between $\hat{S}_+(2^{\mathbb{N}})$ and $\hat{S}_-(2^{\mathbb{N}})$. However, the canonical dense sequences in them are *not* equivalent. Notice that for every $\sigma \in P$ the function x^-_σ belongs to $\hat{S}_-(2^{\mathbb{N}})$ while the function x^+_σ belongs to $\hat{S}_-(2^{\mathbb{N}})$ if and only if there exists an increasing antichain (s_n) of R converging to σ. It follows that the accumulation points of $\hat{S}_-(2^{\mathbb{N}})$ is the set

$$\{x^-_\sigma : \sigma \in P\} \cup \{x^+_\sigma : \sigma \in P^+\}.$$

As before, setting $d_t^4 := x^-_{s_t^\frown 1^\infty}$ for every $t \in 2^{<\mathbb{N}}$, the family $\{d_t^4 : t \in 2^{<\mathbb{N}}\}$ is a dense discrete subset of $\mathcal{L}\big(\hat{S}_-(2^{\mathbb{N}})\big)$ and, moreover,

$$L \in \mathcal{L}\big(\hat{S}_-(2^{\mathbb{N}})\big) \Leftrightarrow \exists \sigma \in 2^{\mathbb{N}} \text{ with } L \to \sigma \text{ and (either } L \prec^* \sigma \text{ or } \sigma \preceq^* L).$$

4.3.5. *The extended Alexandroff compactification of the Cantor set $\hat{A}(2^{\mathbb{N}})$.* The space $\hat{A}(2^{\mathbb{N}})$ is the pointwise closure of the family

$$\{v_t : t \in 2^{<\mathbb{N}}\}.$$

For every $\sigma \in 2^{\mathbb{N}}$ the function δ_σ belongs in $\hat{A}(2^{\mathbb{N}})$, the family $\{\delta_\sigma : \sigma \in 2^{\mathbb{N}}\}$ is discrete and accumulates to 0. The function 0 is the only non-G_δ point of $\hat{A}(2^{\mathbb{N}})$ and this is witnessed in the most extreme way. The accumulation points of $\hat{A}(2^{\mathbb{N}})$ is the set

$$\{\delta_\sigma : \sigma \in 2^{\mathbb{N}}\} \cup \{0\}$$

Setting $d_t^5 := v_t$ for every $t \in 2^{<\mathbb{N}}$, the family $\{d_t^5 : t \in 2^{<\mathbb{N}}\}$ is a dense discrete subset of $\hat{A}(2^{\mathbb{N}})$ and

$$L \in \mathcal{L}\big(\hat{A}(2^{\mathbb{N}})\big) \Leftrightarrow (\exists \sigma \in 2^{\mathbb{N}} \text{ with } L \subseteq^* \sigma) \text{ or } (\forall \sigma \in 2^{\mathbb{N}}\ L \perp \sigma).$$

4.3.6. *The extended duplicate of the Cantor set $\hat{D}(2^{\mathbb{N}})$.* The space $\hat{D}(2^{\mathbb{N}})$ is the pointwise closure of the family

$$\{(v_t, t^\frown 0^\infty) : t \in 2^{<\mathbb{N}}\}.$$

This is the separable extension of the space $D(2^{\mathbb{N}})$ as it was described in [To1]. The accumulation points of $\hat{D}(2^{\mathbb{N}})$ is the set

$$\{(\delta_\sigma, \sigma) : \sigma \in 2^{\mathbb{N}}\} \cup \{(0, \sigma) : \sigma \in 2^{\mathbb{N}}\}$$

which is homeomorphic to the Alexandroff duplicate of the Cantor set. Todorčević was the first to realize that this classical construction can be represented as a compact subset of the first Baire class. The space $\hat{D}(2^{\mathbb{N}})$ is not only first countable but it is also pre-metric of degree at most two in the sense of [To1]. As in the previous cases, setting $d_t^6 := (v_t, t^\frown 0^\infty)$ for every $t \in 2^{<\mathbb{N}}$, we see that the family $\{d_t^6 : t \in 2^{<\mathbb{N}}\}$ is a dense discrete subset of $\hat{D}(2^{\mathbb{N}})$, and

$$L \in \mathcal{L}\big(\hat{D}(2^{\mathbb{N}})\big) \Leftrightarrow \exists \sigma \in 2^{\mathbb{N}} \text{ with } L \to \sigma \text{ and (either } L \subseteq^* \sigma \text{ or } L \perp \sigma).$$

4.3.7. *The extended duplicate of the split Cantor set $\hat{D}\big(S(2^{\mathbb{N}})\big)$.* It is the pointwise closure of the family

$$\{(v_s, x^+_{s^\frown 0^\infty}) : s \in R\}.$$

The space $\hat{D}\big(S(2^{\mathbb{N}})\big)$ is homeomorphic to a subspace of the Helly space $\mathcal{H}$. To see this let $\{(a_t, b_t) : t \in 2^{<\mathbb{N}}\}$ be a family in $[0,1]^2$ such that

(i) $a_t = a_{t^\frown 0} < b_{t^\frown 0} < a_{t^\frown 1} < b_{t^\frown 1} = b_t$, and
(ii) $b_t - a_t \leqslant \frac{1}{3^{|t|}}$

for every $t \in 2^{<\mathbb{N}}$. Define $h_t\colon [0,1] \to [0,1]$ by

$$h_t(x) = \begin{cases} 1 & \text{if } b_t < x, \\ \frac{1}{2} & \text{if } a_t \leqslant x \leqslant b_t, \\ 0 & \text{if } x < a_t. \end{cases}$$

It is easy to see that the map

$$\hat{D}\big(S(2^{\mathbb{N}})\big) \ni (v_{s_t}, x^+_{s_t^\frown 0^\infty}) \mapsto h_t \in \mathcal{H}$$

is extended to a homeomorphic embedding. It is follows that the space $\hat{D}\big(S(2^{\mathbb{N}})\big)$ is first countable. Notice, however, that it is not pre-metric of degree at most two.

As in all previous cases, we will describe the accumulation points of $\hat{D}\big(S(2^{\mathbb{N}})\big)$. First we observe that if (s_n) is a chain of R converging to $\sigma \in P$, then the sequence $\big((v_{s_n}, x^+_{s_n^\frown 0^\infty})\big)$ is pointwise convergent to $(\delta_\sigma, x^+_\sigma)$. If (s_n) is an increasing antichain of R converging to σ, then the sequence $\big((v_{s_n}, x^+_{s_n^\frown 0^\infty})\big)$ is pointwise convergent to $(0, x^+_\sigma)$, and if it is decreasing, then it is pointwise convergent to $(0, x^-_\sigma)$. Thus, the accumulation points of $\hat{D}\big(S(2^{\mathbb{N}})\big)$ is the set

$$\{(\delta_\sigma, x^+_\sigma) : \sigma \in P\} \cup \{(0, x^+_\sigma) : \sigma \in P^+\} \cup \{(0, x^-_\sigma) : \sigma \in P^-\}.$$

Finally, setting $d^7_t := (v_{s_t}, x^+_{s_t^\frown 0^\infty})$ for every $t \in 2^{<\mathbb{N}}$, we see that the family $\{d^7_t : t \in 2^{<\mathbb{N}}\}$ is a dense discrete subset of $\hat{D}\big(S(2^{\mathbb{N}})\big)$, and

$$L \in \mathcal{L}\big(\hat{D}\big(S(2^{\mathbb{N}})\big)\big) \Leftrightarrow \exists \sigma \in 2^{\mathbb{N}} \text{ with } L \to \sigma \text{ and } (L \prec^* \sigma \text{ or } L \subseteq^* \sigma \text{ or } \sigma \prec^* L).$$

We close this subsection by noticing the following minimality property of the above described families.

Proposition 20. *Let $\{d^i_t : t \in 2^{<\mathbb{N}}\}$ with $i \in \{1, \dots, 7\}$ be one of the seven families of functions and let $S = (s_t)_{t \in 2^{<\mathbb{N}}}$ be a dyadic (not necessarily regular) subtree of $2^{<\mathbb{N}}$. Then the family $\{d^i_t : t \in 2^{<\mathbb{N}}\}$ and the corresponding family $\{d^i_{s_t} : t \in 2^{<\mathbb{N}}\}$ determined by the tree S are equivalent.*

We also observe that any two of the seven families are *not* equivalent. Moreover, beside the case of $\hat{S}_+(2^{\mathbb{N}})$ and $\hat{S}_-(2^{\mathbb{N}})$, the corresponding compacta are not mutually homeomorphic either.

4.4. **Canonization.** The following theorem is the main result of this section.

Theorem 21. *Let $\{f_t\}_{t \in 2^{<\mathbb{N}}}$ be a family of real-valued functions on a Polish space X which is relatively compact in $\mathcal{B}_1(X)$. Also let $\{d^i_t\}_{t \in 2^{<\mathbb{N}}}$ $(1 \leqslant i \leqslant 7)$ be the families described in the previous subsection. Then there exist a regular dyadic subtree $S = (s_t)_{t \in 2^{<\mathbb{N}}}$ of $2^{<\mathbb{N}}$ and $i_0 \in \{1, \dots, 7\}$ such that $\{f_{s_t}\}_{t \in 2^{<\mathbb{N}}}$ is equivalent to $\{d^{i_0}_t\}_{t \in 2^{<\mathbb{N}}}$.*

Proof. The family $\{f_t\}_{t\in 2^{<\mathbb{N}}}$ satisfies the hypotheses of Theorem 16. Thus, there exist a regular dyadic subtree T of $2^{<\mathbb{N}}$ and a family $\{g^0_\sigma, g^+_\sigma, g^-_\sigma : \sigma \in P\}$ of functions, where $P=[\hat{T}]$, as described in Theorem 16. Let $0,+$ and $-$ denote the corresponding Borel functions. We recall that for every subset X of $2^{\mathbb{N}}$ we identify the set $[X]^2$ of all doubletons of X with the set of all $(\sigma,\tau)\in X^2$ with $\sigma\prec\tau$. For every $\varepsilon\in\{0,+,-\}$ we set

$$A_{\varepsilon,\varepsilon} \coloneqq \{(\sigma_1,\sigma_2)\in[P]^2 : g^\varepsilon_{\sigma_1}\neq g^\varepsilon_{\sigma_2}\}.$$

Then $A_{\varepsilon,\varepsilon}$ is an analytic subset of $[P]^2$. To see this notice that

$$\begin{aligned}(\sigma_1,\sigma_2)\in A_{\varepsilon,\varepsilon} \quad &\Leftrightarrow \quad \exists x\in X \text{ with } g^\varepsilon_{\sigma_1}(x)\neq g^\varepsilon_{\sigma_2}(x)\\ &\Leftrightarrow \quad \exists x\in X \text{ with } \varepsilon(\sigma_1,x)\neq\varepsilon(\sigma_2,x).\end{aligned}$$

Invoking the Borelness of the functions $0,+,-$ we see that $A_{\varepsilon,\varepsilon}$ is analytic, as desired. Notice that for every perfect subset Q of P and every $\varepsilon\in\{0,+,-\}$ the set $A_{\varepsilon,\varepsilon}\cap[Q]^2$ is analytic in $[Q]^2$. Thus, applying Theorem 6 successively three times, we obtain a perfect subset Q_0 of P such that for every $\varepsilon\in\{0,+,-\}$ we have

$$\text{either } [Q_0]^2\subseteq A_{\varepsilon,\varepsilon} \text{ or } A_{\varepsilon,\varepsilon}\cap[Q_0]^2=\emptyset.$$

Case 1: $A_{0,0}\cap[Q_0]^2=\emptyset$. In this case we have $g^0_{\sigma_1}=g^0_{\sigma_2}$ for every $(\sigma_1,\sigma_2)\in[Q_0]^2$. Thus, there exists a function g such that $g^0_\sigma=g$ for every $\sigma\in Q_0$. By properties (2) and (3) in Theorem 16 and the homogeneity of Q_0, we see that $g^+_\sigma=g^-_\sigma=g^0_\sigma=g$ for every $\sigma\in Q_0$. We select a regular dyadic subtree $S=(s_t)_{t\in 2^{<\mathbb{N}}}$ of T such that $[\hat{S}]\subseteq Q_0$ and $f_s\neq g$ for every $s\in S$. Invoking properties (1), (4) and (5) of Theorem 16 and part (2) of Lemma 8, we see that for every infinite subset A of S the sequence $(f_t)_{t\in A}$ accumulates to g. It follows that $\overline{\{f_s\}}^p_{s\in S}=\{f_s\}_{s\in S}\cup\{g\}$, and so, $\{f_{s_t}\}_{t\in 2^{<\mathbb{N}}}$ is equivalent to the canonical dense family of $A(2^{<\mathbb{N}})$.

Case 2: $[Q_0]^2\subseteq A_{0,0}$. Notice that for every $(\sigma_1,\sigma_2)\in[Q_0]^2$ we have $g^0_{\sigma_1}\neq g^0_{\sigma_2}$. By passing to a further perfect subset of Q_0 if necessary, we may also assume that

(P1) $g^0_\sigma\neq f_t$ for every $\sigma\in Q_0$ and every $t\in T$.

Case 2.1: *Either* $A_{+,+}\cap[Q_0]^2=\emptyset$ *or* $A_{-,-}\cap[Q_0]^2=\emptyset$. Assume, first, that $A_{+,+}\cap[Q_0]^2=\emptyset$. In this case, there exists a function g such that $g^+_\sigma=g$ for every $\sigma\in Q_0$. By property (3) in Theorem 16 and the homogeneity of Q_0, we also have that $g^-_\sigma=g$ for every $\sigma\in Q_0$. This means that $A_{-,-}\cap[Q_0]^2=\emptyset$. Thus, by symmetry, this case is equivalent to saying that $A_{+,+}\cap[Q_0]^2=\emptyset$ and $A_{-,-}\cap[Q_0]^2=\emptyset$. It follows that there exists a function g such that $g^+_\sigma=g^-_\sigma=g$ for every $\sigma\in Q_0$. By passing to a further perfect subset of Q_0 if necessary, we may also assume that $g^0_\sigma\neq g$ for every $\sigma\in Q_0$. We select a regular dyadic subtree $S=(s_t)_{t\in 2^{<\mathbb{N}}}$ of T such that $[\hat{S}]\subseteq Q_0$ and $f_s\neq g$ for every $s\in S$. This property and (P1) implies that for every $s\in S$ the function f_s is isolated in $\overline{\{f_s\}}^p_{s\in S}$.

We claim that $\{f_{s_t}\}_{t\in 2^{<\mathbb{N}}}$ is equivalent to the canonical dense family of $\hat{A}(2^{\mathbb{N}})$. We will give a detailed exposition of the argument which will serve as a prototype for the other cases as well. First, we notice that, by Lemma 19 and the description of $\mathcal{L}\big(\hat{A}(2^{\mathbb{N}})\big)$, it is enough to show that for a subset A of S, the sequence $(f_s)_{s\in A}$ converges pointwise if and only if either A is almost included in a chain or A does not contain an infinite chain. For the if part we observe that if A is almost contained in a chain, then, by property (1) of Theorem 16, the sequence $(f_s)_{s\in A}$ is pointwise convergent. Assume that A does not contain an infinite chain. Since $g^+_\sigma = g^-_\sigma = g$ for every $\sigma\in Q_0$, we see that for every increasing and every decreasing antichain (s_n) of S, the sequence (f_{s_n}) converges pointwise to g. Thus, $(f_s)_{s\in A}$ is pointwise convergent to g. For the only if part we argue by contradiction. If there exist $\sigma_1\neq\sigma_2$ contained in $[\hat{S}]$ such that $A\cap\{\sigma_1|n : n\in\mathbb{N}\}$ and $A\cap\{\sigma_2|n : n\in\mathbb{N}\}$ are both infinite, then the fact that $g^0_{\sigma_1}\neq g^0_{\sigma_2}$ implies that the sequence $(f_s)_{s\in A}$ is not pointwise convergent. Finally, if A contains an infinite chain and an infinite antichain, then the fact that $g^0_\sigma\neq g$ for every $\sigma\in[\hat{S}]$ implies that $(f_s)_{s\in A}$ is not pointwise convergent either.

Case 2.2: $[Q_0]^2\subseteq A_{+,+}$ *and* $[Q_0]^2\subseteq A_{-,-}$. In this case we have

(P2) $g^\varepsilon_{\sigma_1}\neq g^\varepsilon_{\sigma_2}$ for every $(\sigma_1,\sigma_2)\in[Q_0]^2$ and every $\varepsilon\in\{0,+,-\}$.

Moreover, by passing to a further perfect subset of Q_0, we may strengthen (P1) to

(P3) $g^\varepsilon_\sigma\neq f_t$ for every $\sigma\in Q_0$, every $\varepsilon\in\{0,+,-\}$ and every $t\in T$.

Observe that (P3) implies the following. For every regular dyadic subtree S of T with $[\hat{S}]\subseteq Q_0$ and every $s\in S$ the function f_s is isolated in the closure of $\{f_s\}_{s\in S}$ in $\mathbb{R}^X$. Thus, as in Case 2.1, in what follows Lemma 19 will be applicable.

For every $\varepsilon_1,\varepsilon_2\in\{0,+,-\}$ with $\varepsilon_1\neq\varepsilon_2$ we set

$$A_{\varepsilon_1,\varepsilon_2} := \{(\sigma_1,\sigma_2)\in[Q_0]^2 : g^{\varepsilon_1}_{\sigma_1}\neq g^{\varepsilon_2}_{\sigma_2}\}.$$

Notice that $A_{\varepsilon_1,\varepsilon_2}$ is an analytic subset of $[Q_0]^2$. Applying Theorem 6 successively six times, we find a perfect subset Q_1 of Q_0 such that for every $\varepsilon_1,\varepsilon_2\in\{0,+,-\}$ with $\varepsilon_1\neq\varepsilon_2$ we have

$$\text{either } [Q_1]^2\subseteq A_{\varepsilon_1,\varepsilon_2} \text{ or } A_{\varepsilon_1,\varepsilon_2}\cap[Q_1]^2=\emptyset.$$

We claim that for every pair $\varepsilon_1,\varepsilon_2$ the first alternative must occur. Assume, on the contrary, that there exist $\varepsilon_1,\varepsilon_2$ with $\varepsilon_1\neq\varepsilon_2$ such that $A_{\varepsilon_1,\varepsilon_2}\cap[Q_1]^2=\emptyset$. Let τ be the lexicographical minimum of Q_1. Then for every $\sigma,\sigma'\in Q_1$ with $\tau\prec\sigma\prec\sigma'$ we have $g^{\varepsilon_2}_\sigma = g^{\varepsilon_1}_\tau = g^{\varepsilon_2}_{\sigma'}$ which contradicts (P2). Summing up, by passing to Q_1, we have strengthen (P2) to

(P4) $g^{\varepsilon_1}_{\sigma_1}\neq g^{\varepsilon_2}_{\sigma_2}$ for every $(\sigma_1,\sigma_2)\in[Q_1]^2$ and every $\varepsilon_1,\varepsilon_2\in\{0,+,-\}$.

For every $\varepsilon\in\{+,-\}$ we define $B_{0,\varepsilon}\subseteq Q_1$ by

$$B_{0,\varepsilon} := \{\sigma\in Q_1 : g^0_\sigma\neq g^\varepsilon_\sigma\}.$$

It is easy to see that $B_{0,\varepsilon}$ is an analytic subset of Q_1. Thus, by the classical perfect set theorem, there exists a perfect subset Q_2 of Q_1 such that for every $\varepsilon\in\{+,-\}$

we have

$$\text{either } Q_2 \subseteq B_{0,\varepsilon} \text{ or } B_{0,\varepsilon} \cap Q_2 = \emptyset.$$

CASE 2.2.1: *$B_{0,+} \cap Q_2 = \emptyset$ and $B_{0,-} \cap Q_2 = \emptyset$.* In this case for every $\sigma \in Q_2$ there exists a function g_σ such that $g_\sigma = g^0_\sigma = g^+_\sigma = g^-_\sigma$. Moreover, $g_{\sigma_1} \neq g_{\sigma_2}$ for every $\sigma_1 \neq \sigma_2$ in Q_2 since $Q_2 \subseteq Q_1$. Invoking properties (2) and (3) in Theorem 16, we see that the set $\{g_\sigma : \sigma \in Q_2\}$ is homeomorphic to Q_2. We select a regular dyadic subtree $S = (s_t)_{t \in 2^{<\mathbb{N}}}$ of T such that $[\hat{S}] \subseteq Q_2 \subseteq Q_0$. It follows that $\overline{\{f_s\}}^p_{s \in S} = \{f_s\}_{s\in S} \cup \{g_\sigma : \sigma \in [\hat{S}]\}$, and so, the family $\{f_{s_t}\}_{t \in 2^{<\mathbb{N}}}$ is equivalent to the canonical dense family of $2^{\leqslant \mathbb{N}}$.

CASE 2.2.2: *$B_{0,+} \cap Q_2 = \emptyset$ and $Q_2 \subseteq B_{0,-}$.* This means that $g^0_\sigma = g^+_\sigma$ and $g^0_\sigma \neq g^-_\sigma$ for every $\sigma \in Q_2$. Let $S = (s_t)_{t \in 2^{<\mathbb{N}}}$ be a regular dyadic subtree of T such that $[\hat{S}] \subseteq Q_2 \subseteq Q_0$. Invoking (P3) and the remarks following it, the description of $\mathcal{L}\big(\hat{S}_+(2^{\mathbb{N}})\big)$, Lemma 19 and arguing precisely as in Case 2.1, we see that $\{f_{s_t}\}_{t \in 2^{<\mathbb{N}}}$ is equivalent to the canonical dense family of $\hat{S}_+(2^{\mathbb{N}})$.

CASE 2.2.3: *$Q_2 \subseteq B_{0,+}$ and $B_{0,-} \cap Q_2 = \emptyset$.* This means that $g^0_\sigma = g^-_\sigma$ and $g^0_\sigma \neq g^+_\sigma$ for every $\sigma \in Q_2$. As in the previous case, let $S = (s_t)_{t \in 2^{<\mathbb{N}}}$ be a regular dyadic subtree of T such that $[\hat{S}] \subseteq Q_2 \subseteq Q_0$. In this case $\{f_{s_t}\}_{t \in 2^{<\mathbb{N}}}$ is equivalent to canonical dense family of the mirror image $\hat{S}_-(2^{\mathbb{N}})$ of the extended split Cantor set; the argument is similar to that in the proof of Case 2.1.

CASE 2.2.4: *$Q_2 \subseteq B_{0,+}$ and $Q_2 \subseteq B_{0,-}$.* In this case we have

(P5) $g^0_\sigma \neq g^+_\sigma$ and $g^0_\sigma \neq g^-_\sigma$ for every $\sigma \in Q_2$.

Set

$$B_{+,-} := \{\sigma \in Q_2 : g^+_\sigma \neq g^-_\sigma\}$$

Again we see that $B_{+,-}$ is an analytic subset of Q_2. Thus, there exists a perfect subset Q_3 of Q_2 such that either $Q_3 \subseteq B_{+,-}$ or $Q_3 \cap B_{+,-} = \emptyset$.

CASE 2.2.4.1: *$Q_3 \cap B_{+,-} = \emptyset$.* This means that for every $\sigma \in Q_3$ there exists a function g_σ such that $g_\sigma = g^+_\sigma = g^-_\sigma$ and $g_\sigma \neq g^0_\sigma$. Moreover since $Q_3 \subseteq Q_2 \subseteq Q_1$, by property (P4) above, we have $g_{\sigma_1} \neq g_{\sigma_2}$ and $g^0_{\sigma_1} \neq g^0_{\sigma_2}$ for every $(\sigma_1, \sigma_2) \in [Q_3]^2$. Let $S = (s_t)_{t \in 2^{<\mathbb{N}}}$ be a regular dyadic subtree of T such that $[\hat{S}] \subseteq Q_3 \subseteq Q_0$. In this case $\{f_{s_t}\}_{t \in 2^{<\mathbb{N}}}$ is equivalent to the canonical dense family of $\hat{D}(2^{\mathbb{N}})$. The verification is similar to the previous cases.

CASE 2.2.4.2: *$Q_3 \subseteq B_{+,-}$.* This means that $g^+_\sigma \neq g^-_\sigma$ for every $\sigma \in Q_3$. Combining this property with (P4) and (P5), we see that $g^{\varepsilon_1}_{\sigma_1} \neq g^{\varepsilon_2}_{\sigma_2}$ if either $\varepsilon_1 \neq \varepsilon_2$ or $\sigma_1 \neq \sigma_2$. As before, let $S = (s_t)_{t \in 2^{<\mathbb{N}}}$ be a regular dyadic subtree of T such that $[\hat{S}] \subseteq Q_3 \subseteq Q_0$. It follows that the family $\{f_{s_t}\}_{t \in 2^{<\mathbb{N}}}$ is equivalent to the canonical dense family of $\hat{D}\big(S(2^{\mathbb{N}})\big)$.

The above cases are exhaustive and the proof of the theorem is completed. □

By Theorem 21 and Proposition 20, we obtain the following corollary.

Corollary 22. *Let X be a Polish space and let $\{f_t\}_{t\in 2^{<\mathbb{N}}}$ be a family of functions relatively compact in $\mathcal{B}_1(X)$. Then for every regular dyadic subtree T of $2^{<\mathbb{N}}$ there exist a regular dyadic subtree S of T and $i_0 \in \{1,\dots,7\}$ such that for every regular dyadic subtree $R=(r_t)_{t\in 2^{<\mathbb{N}}}$ of S the family $\{f_{r_t}\}_{t\in 2^{<\mathbb{N}}}$ is equivalent to $\{d_t^{i_0}\}_{t\in 2^{<\mathbb{N}}}$.*

5. Analytic subspaces of separable Rosenthal compacta

In this section we introduce a class of subspaces of separable Rosenthal compacta and we present some of their basic properties.

5.1. **Definitions and basic properties.** Let $\mathcal{K}$ be a separable Rosenthal compact on a Polish space X. For every subset $\mathcal{F}$ of $\mathcal{K}$ by $Acc(\mathcal{F})$ we denote the set of accumulation points of $\mathcal{F}$ in $\mathbb{R}^X$. We start with the following definition.

Definition 23. *Let $\mathcal{K}$ be a separable Rosenthal compact on a Polish space X and let $\mathcal{C}$ be a closed subspace of $\mathcal{K}$. We say that $\mathcal{C}$ is an* analytic subspace *of $\mathcal{K}$ if there exist a countable dense subset $\{f_n\}$ of $\mathcal{K}$ and an analytic subset A of $[\mathbb{N}]^\infty$ such that the following are satisfied.*

1. *For every $L\in A$ we have $Acc(\{f_n : n\in L\}) \subseteq \mathcal{C}$.*
2. *For every $g\in \mathcal{C}\cap Acc(\mathcal{K})$ there exists $L\in A$ with $g\in \overline{\{f_n\}}^p_{n\in L}$.*

Let us make some remarks concerning the above notion. First we notice that the analytic set A witnessing the analyticity of $\mathcal{C}$ can always be assumed to be hereditary. Also observe that an analytic subspace of $\mathcal{K}$ is not necessarily separable. For instance, if $\mathcal{K}=\hat{A}(2^{\mathbb{N}})$ and $\mathcal{C}=A(2^{\mathbb{N}})$, then it is easy to see that $\mathcal{C}$ is an analytic subspace of $\mathcal{K}$. In the following proposition we give some examples of analytic subspaces.

Proposition 24. *Let $\mathcal{K}$ be a separable Rosenthal compact. Then the following hold.*

1. *$\mathcal{K}$ is analytic with respect to any countable dense subset $\{f_n\}$ of $\mathcal{K}$.*
2. *Every closed G_δ subspace $\mathcal{C}$ of $\mathcal{K}$ is analytic.*
3. *Every closed separable subspace $\mathcal{C}$ of $\mathcal{K}$ is analytic.*

Proof. (1) Take $A=[\mathbb{N}]^\infty$.

(2) Let (U_k) be a sequence of open subsets of $\mathcal{K}$ such that $\overline{U}_{k+1}\subseteq U_k$ for every $k\in\mathbb{N}$ and with $\mathcal{C}=\bigcap_k U_k$. Also let $\{f_n\}$ be a countable dense subset of $\mathcal{K}$. For every $k\in\mathbb{N}$ set $M_k := \{n\in\mathbb{N} : f_n\in U_k\}$. Notice that the sequence (M_k) is decreasing. Let $A\subseteq[\mathbb{N}]^\infty$ be defined by

$$L\in A \Leftrightarrow \forall k\in\mathbb{N}\ (L\subseteq^* M_k).$$

Clearly, the set A is Borel. It is easy to see that A satisfies condition (1) of Definition 23 for $\mathcal{C}$. To see that condition (2) is also satisfied let $g\in\mathcal{C}\cap Acc(\mathcal{K})$. By the Bourgain–Fremlin–Talagrand theorem [BFT], there exists an infinite subset L on $\mathbb{N}$ such that g is the pointwise limit of the sequence $(f_n)_{n\in L}$. Since $g\in U_k$

for every $k \in \mathbb{N}$, we see that $L \subseteq^* M_k$ for every k. Therefore, the set A witness the analyticity of $\mathcal{C}$.

(3) Let D_1 be a countable dense subset of $\mathcal{K}$ and D_2 a countable dense subset of $\mathcal{C}$. Let $\{f_n\}$ be an enumeration of the set $D_1 \cup D_2$ and set $L := \{n \in \mathbb{N} : f_n \in D_2\}$. Also set $M := \{k \in L : f_k \in Acc(\mathcal{K})\}$, and for every $k \in M$ we select $L_k \in [\mathbb{N}]^\infty$ such that the function f_k is the pointwise limit of the sequence $(f_n)_{n \in L_k}$. We define $A := [L]^\infty \cup \big(\bigcup_{k \in M}[L_k]^\infty\big)$. The countable dense subset $\{f_n\}$ of $\mathcal{K}$ and the set A verify the analyticity of $\mathcal{C}$. □

To proceed with our discussion on the properties of analytic subspaces we need to introduce some pieces of notation. Let $\mathcal{K}$ be a separable Rosenthal compact and let $\mathbf{f} = \{f_n\}$ be a countable dense subset of $\mathcal{K}$. We set

$$\mathcal{L}_{\mathbf{f}} := \{L \in [\mathbb{N}]^\infty : (f_n)_{n \in L} \text{ is pointwise convergent}\}.$$

Moreover, for every accumulation point f of $\mathcal{K}$ we set

$$\mathcal{L}_{\mathbf{f},f} := \{L \in [\mathbb{N}]^\infty : (f_n)_{n \in L} \text{ is pointwise convergent to } f\}.$$

We notice that both $\mathcal{L}_{\mathbf{f}}$ and $\mathcal{L}_{\mathbf{f},f}$ are co-analytic. The first result relating the topological behavior of a point f in $\mathcal{K}$ with the descriptive set-theoretic properties of the set $\mathcal{L}_{\mathbf{f},f}$ is the result of Krawczyk from [Kr] asserting that a point $f \in \mathcal{K}$ is G_δ if and only if the set $\mathcal{L}_{\mathbf{f},f}$ is Borel. Another important structural property is the following consequence of the effective version of the Bourgain–Fremlin–Talagrand theorem proved by Debs in [De].

Theorem 25. *Let $\mathcal{K}$ be a separable Rosenthal compact. Then for every countable dense subset $\mathbf{f} = \{f_n\}$ of $\mathcal{K}$ there is a Borel, hereditary and cofinal subset C of $\mathcal{L}_{\mathbf{f}}$.*

We refer the reader to [Do] for an explanation of how Debs' theorem yields the above result.

Let $\mathcal{K}$ and $\mathbf{f} = \{f_n\}$ be as above. For every $A \subseteq \mathcal{L}_{\mathbf{f}}$ we set

$$\mathcal{K}_{A,\mathbf{f}} := \{g \in \mathcal{K} : \exists L \in A \text{ with } g = \lim_{n \in L} f_n\}.$$

We have the following characterization of analytic subspaces which is essentially a consequence of Theorem 25.

Proposition 26. *Let $\mathcal{K}$ be a separable Rosenthal compact and $\mathcal{C}$ a closed subspace of $\mathcal{K}$. Then $\mathcal{C}$ is analytic if and only if there exist a countable dense subset $\mathbf{f} = \{f_n\}$ of $\mathcal{K}$ and a hereditary and analytic subset A' of $\mathcal{L}_{\mathbf{f}}$ such that $\mathcal{K}_{A',\mathbf{f}} = \mathcal{C} \cap Acc(\mathcal{K})$.*

Proof. The direction ($\Leftarrow$) is immediate. Conversely, assume that $\mathcal{C}$ is analytic and let $\mathbf{f} = \{f_n\}$ and $A \subseteq [\mathbb{N}]^\infty$ verifying its analyticity. As we have already remarked, we may assume that A is hereditary. By Theorem 25, there exists a Borel, hereditary and cofinal subset C of $\mathcal{L}_{\mathbf{f}}$. We set $A' := A \cap C$. We claim that A' is the desired set. Clearly A' is a hereditary and analytic subset of $\mathcal{L}_{\mathbf{f}}$. Also observe that, by condition (1) in Definition 23, for every $L \in A'$ the sequence $(f_n)_{n\in L}$ must be pointwise convergent to a function $g \in \mathcal{C}$. Therefore, $\mathcal{K}_{A',\mathbf{f}} \subseteq \mathcal{C} \cap Acc(\mathcal{K})$. Conversely, let $g \in \mathcal{C} \cap Acc(\mathcal{K})$. There exists $M \in A$ with $g \in \overline{\{f_n\}}^p_{n\in M}$. By the Bourgain–Fremlin–Talagrand theorem, there exists $N \in [M]^\infty$ such that g is the pointwise limit of the sequence $(f_n)_{n\in N}$. Clearly we have $N \in \mathcal{L}_{\mathbf{f}}$. Since C is cofinal in $\mathcal{L}_{\mathbf{f}}$, there exists $L \in [N]^\infty$ with $L \in C$. As A is hereditary, we see that $L \in A \cap C = A'$. The proof is completed. □

5.2. **Separable Rosenthal compacta in $\mathcal{B}_1(2^{\mathbb{N}})$.** Let $\mathcal{K}$ be separable Rosenthal compact on a Polish space X and let $\mathbf{f} = \{f_n\}$ be a countable dense subset of $\mathcal{K}$. By Theorem 25, there exists a Borel cofinal subset of $\mathcal{L}_{\mathbf{f}}$. The following proposition shows that if X is compact metrizable, then this global property of $\mathcal{L}_{\mathbf{f}}$ (namely, that it contains a Borel cofinal set) is also valid locally. We notice that in the argument below we make use of the Arsenin–Kunugui theorem in a spirit similar as in [Po2].

Proposition 27. *Let X be a compact metrizable space, $\mathcal{K}$ a separable Rosenthal compact on X and $\mathbf{f} = \{f_n\}$ a countable dense subset of $\mathcal{K}$. Then for every $f \in \mathcal{K}$ there exists an analytic hereditary subset B of $\mathcal{L}_{\mathbf{f},f}$ which is cofinal in $\mathcal{L}_{\mathbf{f},f}$.*

Proof. We apply Theorem 25 and we obtain a hereditary, Borel and cofinal subset C of $\mathcal{L}_{\mathbf{f}}$. Consider the function $\Phi\colon C \times X \to \mathbb{R}$ defined by $\Phi(L,x) = f_L(x)$ where by f_L we denote the pointwise limit of the sequence $(f_n)_{n\in L}$. Notice that Φ is Borel. Indeed, for every $n \in \mathbb{N}$ let $\Phi_n\colon C \times X \to \mathbb{R}$ be defined by $\Phi_n(L,x) = f_{l_n}(x)$ where l_n is the n-th element of the increasing enumeration of L. Clearly the function Φ_n is Borel. Since $\Phi(L,x) = \lim \Phi_n(L,x)$ for every $(L,x) \in C \times X$, the Borelness of Φ follows. For every $m \in \mathbb{N}$ define $P_m \subseteq C \times X$ by

$$\begin{aligned}(L,x) \in P_m \quad &\Leftrightarrow \quad |f_L(x) - f(x)| > \frac{1}{m+1} \\ &\Leftrightarrow \quad (c,x) \in \Phi^{-1}\big((-\infty, -\frac{1}{m+1}) \cup (\frac{1}{m+1}, +\infty)\big).\end{aligned}$$

Clearly, P_m is Borel. For every $L \in C$ the function $x \mapsto |f_L(x) - f(x)|$ is Baire-1. Hence, for every $L \in C$ the section $(P_m)_L = \{x \in X : (L,x) \in P_m\}$ of P_m at L is F_σ, and since X is compact metrizable, it is K_σ. By the Arsenin–Kunugui theorem (see, e.g., [Ke, Theorem 35.46]), the set

$$G_m = \mathrm{proj}_C P_m$$

is Borel. It follows that the set $G = \bigcup_m G_m$ is a Borel subset of C. We set $D \coloneqq C \setminus G$. Now observe that for every $L \in C$ we have that $L \in \mathcal{L}_{\mathbf{f},f}$ if and only if $L \notin G$. Hence, the set D is a Borel subset of $\mathcal{L}_{\mathbf{f},f}$, and as C is cofinal, we see that D is cofinal in $\mathcal{L}_{\mathbf{f},f}$. Therefore, setting B to be the hereditary closure of D, we conclude that B is as desired. □

Remark 2. (1) We notice that Proposition 27 is not valid for an arbitrary separable Rosenthal compact. A counterexample, taken from [Po2] (see also [Ma]), is the following. Let A be an analytic non-Borel subset of $2^{\mathbb{N}}$ and denote by $\mathcal{K}_A$ the separable Rosenthal compact obtained by restrict every function of $\hat{A}(2^{\mathbb{N}})$ on A. Clearly, the function $0|_A$ belongs to $\mathcal{K}_A$ and is a non-G_δ point of $\mathcal{K}_A$. It is easy to check that, in this case, there does not exist a Borel cofinal subset of $\mathcal{L}_{0|_A}$.

(2) We should point out that the hereditary and cofinal subset B of $\mathcal{L}_{\mathbf{f},f}$ obtained by Proposition 27, can be chosen to be Borel. To see this, start with an analytic and cofinal subset A_0 of $\mathcal{L}_{\mathbf{f},f}$. Using Souslin's separation theorem, we construct two sequences (B_n) and (C_n) such that B_n is Borel, C_n is the hereditary closure of B_n and $A_0 \subseteq B_n \subseteq C_n \subseteq B_{n+1} \subseteq \mathcal{L}_{\mathbf{f},f}$ for every $n \in \mathbb{N}$. Setting $B := \bigcup_n B_n$, we see that B is as desired.

The argument in the proof of Proposition 27 can be used to derive certain properties of analytic subspaces of separable Rosenthal compacta. To state these properties we need to introduce some pieces of notation. If $\mathcal{K}$ is a separable Rosenthal compact $\mathcal{K}$ on a Polish space X, $\mathbf{f} = \{f_n\}$ is a countable dense subset of $\mathcal{K}$ and $\mathcal{C}$ is a closed subspace of $\mathcal{K}$, then we set

$$\mathcal{L}_{\mathbf{f},\mathcal{C}} := \{L \in [\mathbb{N}] : \exists g \in \mathcal{C} \text{ with } g = \lim_{n\in L} f_n\}.$$

Clearly $\mathcal{L}_{\mathbf{f},\mathcal{C}}$ is a subset of $\mathcal{L}_{\mathbf{f}}$. Also notice that if $\mathcal{C} = \{f\}$ for some $f \in \mathcal{K}$, then $\mathcal{L}_{\mathbf{f},\mathcal{C}} = \mathcal{L}_{\mathbf{f},f}$.

Part (1) of the following proposition extends Proposition 27 for analytic subspaces. The second part shows that the notion of an analytic subspace of $\mathcal{K}$ is independent of the choice of the dense sequence for every separable Rosenthal compact $\mathcal{K}$ in $\mathcal{B}_1(2^{\mathbb{N}})$.

Proposition 28. *Let X be a compact metrizable space, $\mathcal{K}$ a separable Rosenthal compact on X and $\mathcal{C}$ and analytic subspace of $\mathcal{K}$. Let $\mathbf{f} = \{f_n\}$ be a countable dense subset of $\mathcal{K}$ and let $A \subseteq [\mathbb{N}]^\infty$ be analytic witnessing the analyticity of $\mathcal{C}$. Then the following hold.*

1. *There exists an analytic cofinal subset A_1 of $\mathcal{L}_{\mathbf{f},\mathcal{C}}$.*
2. *For every countable dense subset $\mathbf{g} = \{g_n\}$ of $\mathcal{K}$ there exists an analytic subset A_2 of $\mathcal{L}_{\mathbf{g}}$ such that $\mathcal{K}_{A_2,\mathbf{g}} = \mathcal{C} \cap Acc(\mathcal{K})$.*

Proof. (1) By Proposition 26, there exists a hereditary and analytic subset A' of $\mathcal{L}_{\mathbf{f}}$ such that $\mathcal{K}_{A',\mathbf{f}} = \mathcal{C} \cap Acc(\mathcal{K})$. Applying Theorem 25, we obtain a Borel, hereditary and cofinal subset C of $\mathcal{L}_{\mathbf{f}}$. As in Proposition 27, for every $L \in C$ by f_L we denote the pointwise limit of the sequence $(f_n)_{n\in L}$. We set $A'' := A' \cap C$. Clearly A'' is analytic and hereditary. Moreover, it is easy to see that $\mathcal{K}_{A'',\mathbf{f}} = \mathcal{C} \cap Acc(\mathcal{K})$ (that is, the set A'' codes all function in $Acc(\mathcal{K}) \cap \mathcal{C}$). Consider the following equivalence relation $\sim$ on C defined by the rule

$$L \sim M \Leftrightarrow f_L = f_M \Leftrightarrow \forall x \in X \ f_L(x) = f_M(x).$$

We claim that $\sim$ is Borel. To see this notice that the map

$$C \times C \times X \ni (L, M, x) \mapsto |f_L(x) - f_M(x)|$$

is Borel (this can be easily checked arguing as in the proof of Proposition 27). Moreover, for every $(L, M) \in C \times C$ the map $x \mapsto |f_L(x) - f_M(x)|$ is Baire-1. Observe that

$$\neg(L \sim M) \Leftrightarrow \exists x \in X \ \exists \varepsilon > 0 \text{ with } |f_L(x) - f_M(x)| > \varepsilon.$$

By the fact that X is compact metrizable and the Arsenin–Kunugui theorem, we see that $\sim$ is Borel. We set A_1 to be the $\sim$ saturation of A'', that is,

$$A_1 \coloneqq \{M \in C : \exists L \in A'' \text{ with } M \sim L\}.$$

Since A'' is analytic and $\sim$ is Borel, we see that A_1 is analytic. As C is cofinal, it is easy to check that A_1 is cofinal in $\mathcal{L}_{\mathbf{f},\mathcal{C}}$. Thus, the set A_1 is the desired one.

(2) Let C_1 and C_2 be two hereditary, Borel and cofinal subsets of $\mathcal{L}_{\mathbf{f}}$ and $\mathcal{L}_{\mathbf{g}}$ respectively. By part (1), there exists a hereditary and analytic subset A_1 of $\mathcal{L}_{\mathbf{f}}$ which is cofinal in $\mathcal{L}_{\mathbf{f},\mathcal{C}}$. We set $A'_1 = A_1 \cap C_1$. Consider the following subset S of $C_1 \times C_2$ defined by setting

$$(L, M) \in S \Leftrightarrow f_L = g_M \Leftrightarrow \forall x \in X \ f_L(x) = g_M(x)$$

where f_L denotes the pointwise limit of the sequence $(f_n)_{n \in L}$ and g_M denotes the pointwise limit of the sequence $(g_n)_{n \in M}$. Since X is compact metrizable, arguing as in part (1), it is easy to see that S is Borel. We set

$$A_2 \coloneqq \{M \in C_2 : \exists L \in A'_1 \text{ with } (L, M) \in S\}.$$

The set A_2 is as desired. □

We close this subsection with the following proposition which provides further examples of analytic subspaces.

Proposition 29. *Let $\mathcal{K}$ be a separable Rosenthal compact on a Polish space X. Also let F be a K_σ subset of X. Then the subspace $\mathcal{C}_F \coloneqq \{f \in \mathcal{K} : f|_F = 0\}$ of $\mathcal{K}$ is analytic with respect to any countable dense subset $\mathbf{f} = \{f_n\}$ of $\mathcal{K}$.*

Proof. Let C be a hereditary, Borel and cofinal subset of $\mathcal{L}_{\mathbf{f}}$. Let Z be the subset of $C \times X$ defined by

$$(L, x) \in Z \Leftrightarrow (x \in F) \text{ and } (\exists \varepsilon > 0 \text{ with } |f_L(x)| > \varepsilon).$$

The set Z is Borel. Since F is K_σ, we see that for every $L \in C$ the section $Z_L = \{x \in X : (L, x) \in Z\}$ of Z at L is K_σ. Thus, setting $A \coloneqq C \setminus \mathrm{proj}_C Z$ and invoking the Arsenin–Kunugui theorem, we see that the set A witnesses the analyticity of $\mathcal{C}_F$ with respect to $\{f_n\}$. □

Related to the concept of an analytic subspace of $\mathcal{K}$ and the above propositions, the following questions are open to us.

Problem 1. Is it true that the concept of an analytic subspace is independent of the choice of the countable dense subset of $\mathcal{K}$? More precisely, if $\mathcal{C}$ is an analytic subspace of a separable Rosenthal compact $\mathcal{K}$ on a Polish space X and $\mathbf{f} = \{f_n\}$ is an arbitrary countable dense subset of $\mathcal{K}$, then does there exists $A \subseteq \mathcal{L}_{\mathbf{f}}$ analytic with $\mathcal{K}_{A,\mathbf{f}} = \mathcal{C} \cap Acc(\mathcal{K})$?

Problem 2. Let $\mathcal{K}$ be a separable Rosenthal compact on a Polish space X and let $B \subseteq X$ be Borel. Is the subspace $\mathcal{C}_B := \{f \in \mathcal{K} : f|_B = 0\}$ analytic?

6. Canonical embeddings in analytic subspaces

This section is devoted to the canonical embedding of the most representative prototype, among the seven minimal families, into a given analytic subspace of a separable Rosenthal compact. The section is divided into two subsections. The first subsection concerns metrizable Rosenthal compacta, and the second subsection the non-metrizable ones. We start with the following definitions.

Definition 30. *An injection $\phi\colon 2^{<\mathbb{N}} \to \mathbb{N}$ is said to be* canonical *provided that $\phi(s) < \phi(t)$ if either $|s| < |t|$, or $|s| = |t|$ and $s \prec t$. By ϕ_0 we denote the unique canonical bijection between $2^{<\mathbb{N}}$ and $\mathbb{N}$.*

Definition 31. *Let $\mathcal{K}$ be a separable Rosenthal compact, $\{f_n\}$ a countable dense subset of $\mathcal{K}$ and $\mathcal{C}$ a closed subspace of $\mathcal{K}$. Also let $\{d^i_t\}_{t\in 2^{<\mathbb{N}}}$ $(1 \leqslant i \leqslant 7)$ be the canonical families described in Subsection 4.3 and let $\mathcal{K}_i$ $(1 \leqslant i \leqslant 7)$ be the corresponding separable Rosenthal compacta. For every $i \in \{1, \dots, 7\}$ we say that $\mathcal{K}_i$* canonically embeds into $\mathcal{K}$ with respect to $\{f_n\}$ and $\mathcal{C}$ *if there exists a canonical injection $\phi\colon 2^{<\mathbb{N}} \to \mathbb{N}$ such that the families $\{d^i_t\}_{t\in 2^{<\mathbb{N}}}$ and $\{f_{\phi(t)}\}_{t\in 2^{<\mathbb{N}}}$ are equivalent, that is, if the map*

$$\mathcal{K}_i \ni d^i_t \mapsto f_{\phi(t)} \in \mathcal{K}$$

is extended to a homeomorphism between $\mathcal{K}_i$ and $\overline{\{f_{\phi(t)}\}}^p_{t\in 2^{<\mathbb{N}}}$ and, moreover,

$$Acc\big(\{f_{\phi(t)} : t \in 2^{<\mathbb{N}}\}\big) \subseteq \mathcal{C}.$$

If $\mathcal{C} = \mathcal{K}$, then we simply say that $\mathcal{K}_i$ canonical embeds into $\mathcal{K}$ with respect to $\{f_n\}$.

6.1. **Metrizable Rosenthal compacta.** We have the following theorem.

Theorem 32. *Let $\mathcal{K}$ be a separable Rosenthal compact on a Polish space X consisting of bounded functions. Also let $\{f_n\}$ be a countable dense subset of $\mathcal{K}$. Assume that $\mathcal{K}$ is metrizable in the pointwise topology and non-separable in the supremum norm of $\mathcal{B}_1(X)$. Then there exists a canonical embedding of $2^{\leqslant\mathbb{N}}$ into $\mathcal{K}$ with respect to $\{f_n\}$ whose accumulation points are ε-separated in the supremum norm for some $\varepsilon > 0$. In particular, its image is non-separable in the supremum norm.*

Proof. Fix a compatible metric ρ for the pointwise topology of $\mathcal{K}$. Our assumptions on $\mathcal{K}$ yield that there exist $\varepsilon > 0$ and a family $\Gamma = \{f_\xi : \xi < \omega_1\} \subseteq \mathcal{K}$ such that Γ is ε-separated in the supremum norm and each f_ξ is a condensation point of the family Γ in the pointwise topology.

By recursion on the length of finite sequences in $2^{<\mathbb{N}}$, we shall construct

(C1) a family $(B_t)_{t\in 2^{<\mathbb{N}}}$ of open subsets of $\mathcal{K}$,
(C2) a family $(x_t)_{t\in 2^{<\mathbb{N}}}$ in X,
(C3) two families $(r_t)_{t\in 2^{<\mathbb{N}}}$ and $(q_t)_{t\in 2^{<\mathbb{N}}}$ of reals, and
(C4) a canonical injection $\phi\colon 2^{<\mathbb{N}} \to \mathbb{N}$

such that for every $t \in 2^{<\mathbb{N}}$ the following are satisfied.

(P1) We have $\overline{B}_{t^\frown 0} \cap \overline{B}_{t^\frown 1} = \emptyset$, $\overline{B}_{t^\frown 0} \cup \overline{B}_{t^\frown 1} \subseteq B_t$ and $\rho - \mathrm{diam}(B_t) \leqslant \frac{1}{|t|+1}$.
(P2) We have $|B_t \cap \Gamma| = \aleph_1$.
(P3) We have $r_t < q_t$ and $q_t - r_t > \varepsilon$.
(P4) If $f \in \overline{B}_{t^\frown 0}$, then $f(x_t) < r_t$, and if $f \in \overline{B}_{t^\frown 1}$, then $f(x_t) > q_t$.
(P5) We have $f_{\phi(t)} \in B_t$.

We set $B_\emptyset = \mathcal{K}$ and $\phi(\emptyset) = 0$. We select $f, g \in \Gamma$ and we fix $x \in X$ and $r, q \in \mathbb{R}$ such that $f(x) < r < q < g(x)$ and $q - r > \varepsilon$. We set $x_\emptyset := x$, $r_\emptyset := r$ and $q_\emptyset := q$. We select $B_{(0)}, B_{(1)}$ open subsets of $\mathcal{K}$ such that $f \in B_{(0)} \subseteq \{h \in \mathbb{R}^X : h(x_\emptyset) < r_\emptyset\}$, $g \in B_{(1)} \subseteq \{h \in \mathbb{R}^X : h(x_\emptyset) > q_\emptyset\}$, $\rho - \mathrm{diam}(B_{(0)}) < \frac{1}{2}$ and $\rho - \mathrm{diam}(B_{(1)}) < \frac{1}{2}$. Observe that $x_\emptyset, r_\emptyset, q_\emptyset, B_{(0)}$ and $B_{(1)}$ satisfy properties (P1)–(P4) above. Also notice that $B_{(0)}, B_{(1)}$ are uncountable, hence, they intersect the dense set $\{f_n\}$ at an infinite set. Therefore, we may select $\phi(\emptyset) < \phi\big((0)\big) < \phi\big((1)\big)$ satisfying (P5). The general inductive step proceeds similarly assuming that

(a) x_t, r_t and q_t have been selected for every $t \in 2^{<\mathbb{N}}$ with $|t| < n-1$, and
(b) B_t and $\phi(t)$ have been selected for every $t \in 2^{<\mathbb{N}}$ with $|t| < n$

so that (P1)–(P5) are satisfied. This completes the recursive construction.

Now notice that for every $\sigma \in 2^{\mathbb{N}}$ we have that $\bigcap_n B_{\sigma|n} = \{f_\sigma\}$ and the map $2^{\mathbb{N}} \ni \sigma \mapsto f_\sigma \in \mathcal{K}$ is a homeomorphic embedding. Moreover, for every $\sigma \in 2^{\mathbb{N}}$ the sequence $(f_{\phi(\sigma|n)})$ is pointwise convergent to f_σ. Also observe the following consequence of properties (P3) and (P4). If $\sigma < \tau \in 2^{\mathbb{N}}$, then, setting $t = \sigma \wedge \tau$, we have that $f_\sigma(x_t) \leqslant r_t < q_t \leqslant f_\tau(x_t)$, and so, $\|f_\sigma - f_\tau\|_\infty > \varepsilon$. Since there are at most countable many $\sigma \in 2^{\mathbb{N}}$ with $f_\sigma \in \{f_n\}$, by passing to a regular dyadic subtree of $2^{<\mathbb{N}}$ if necessary, we may assume that for every $t \in 2^{<\mathbb{N}}$ the function $f_{\phi(t)}$ is isolated in $\overline{\{f_{\phi(t)}\}}^p_{t\in 2^{<\mathbb{N}}}$. This easily yields that the family $\{f_{\phi(t)}\}_{t\in 2^{<\mathbb{N}}}$ is equivalent to the canonical dense family of $2^{\leqslant\mathbb{N}}$. The proof is completed. □

6.2. **Non-metrizable separable Rosenthal compacta.** This subsection is devoted to the proofs of the following theorems.

Theorem 33. *Let $\mathcal{K}$ be a separable Rosenthal compact on a Polish space X and let $\mathcal{C}$ be an analytic subspace of $\mathcal{K}$. Also let $\{f_n\}$ be a countable dense subset of*

$\mathcal{K}$ and let $A \subseteq [\mathbb{N}]^\infty$ be analytic witnessing the analyticity of $\mathcal{C}$. Assume that $\mathcal{C}$ is not hereditarily separable. Then either $\hat{A}(2^{\mathbb{N}})$, or $\hat{D}(2^{\mathbb{N}})$, or $\hat{D}\big(S(2^{\mathbb{N}})\big)$ canonically embeds into $\mathcal{K}$ with respect to $\{f_n\}$ and $\mathcal{C}$.

In particular, if $\mathcal{K}$ is first countable and not hereditarily separable, then either $\hat{D}(2^{\mathbb{N}})$ or $\hat{D}\big(S(2^{\mathbb{N}})\big)$ canonically embeds into $\mathcal{K}$ with respect to every countable dense subset $\{f_n\}$ of $\mathcal{K}$.

As it is shown in Corollary 45, if $\mathcal{K}$ is not first countable, then $\hat{A}(2^{\mathbb{N}})$ canonically embeds into $\mathcal{K}$.

Theorem 34. *Let $\mathcal{K}$ be a separable Rosenthal compact on a Polish space X and let $\{f_n\}$ be a countable dense subset of $\mathcal{K}$. Assume that $\mathcal{K}$ is hereditarily separable and non-metrizable. Then either $\hat{S}_+(2^{\mathbb{N}})$ or $\hat{S}_-(2^{\mathbb{N}})$ canonically embeds into $\mathcal{K}$ with respect to $\{f_n\}$.*

6.2.1. *Proof of Theorem* 33. The main goal is to prove the following proposition.

Proposition 35. *Let $\mathcal{K}$, $\mathcal{C}$ and $\{f_n\}$ be as in Theorem* 33. *Then there exists a canonical injection $\psi\colon 2^{<\mathbb{N}} \to \mathbb{N}$ such that, setting*

$$\mathcal{K}_\sigma = \overline{\{f_{\psi(\sigma|n)}\}}^p \setminus \{f_{\psi(\sigma|n)}\}$$

for every $\sigma \in 2^{\mathbb{N}}$, there exists an open subset V_σ of $\mathbb{R}^X$ with $\mathcal{K}_\sigma \subseteq V_\sigma \cap \mathcal{C}$ and such that $\mathcal{K}_\tau \cap V_\sigma = \emptyset$ for every $\tau \in 2^{\mathbb{N}}$ with $\tau \neq \sigma$.

Granting Proposition 35, we complete the proof as follows. Let ψ be the canonical injection obtained by the above proposition, and define $f_t = f_{\psi(t)}$ for every $t \in 2^{<\mathbb{N}}$. We apply Theorem 21 and we obtain a regular dyadic subtree $S = (s_t)_{t\in 2^{<\mathbb{N}}}$ of $2^{<\mathbb{N}}$ and $i_0 \in \{1, \dots, 7\}$ such that $\{f_{s_t}\}_{t\in 2^{<\mathbb{N}}}$ is equivalent to $\{d_t^{i_0}\}_{t\in 2^{<\mathbb{N}}}$. By Proposition 35, we see that the closure of $\{f_{s_t}\}_{t\in 2^{<\mathbb{N}}}$ in $\mathbb{R}^X$ contains an uncountable discrete set. Thus, $\{f_{s_t}\}_{t\in 2^{<\mathbb{N}}}$ is equivalent to the canonical dense family of either $\hat{A}(2^{\mathbb{N}})$, or $\hat{D}(2^{\mathbb{N}})$, or $\hat{D}\big(S(2^{\mathbb{N}})\big)$. Setting $\phi = \psi \circ i_S$, we see that ϕ is an injection imposing a canonical embedding of either $\hat{A}(2^{\mathbb{N}})$, or $\hat{D}(2^{\mathbb{N}})$, or $\hat{D}\big(S(2^{\mathbb{N}})\big)$ into $\mathcal{K}$ with respect to $\{f_n\}$ and $\mathcal{C}$.

We proceed to the proof of Proposition 35. By enlarging the topology on X if necessary (see, e.g., [Ke]), we may assume that the functions $\{f_n\}$ are continuous. We may also assume that the set A is hereditary. By condition (2) of Definition 23 and the Bourgain–Fremlin–Talagrand theorem, for every $g \in \mathcal{C} \cap Acc(\mathcal{K})$ there exists $L \in A$ such that g is the pointwise limit of the sequence $(f_n)_{n\in L}$. We fix a continuous map $\Phi\colon \mathbb{N}^{\mathbb{N}} \to [\mathbb{N}]^\infty$ with $\Phi(\mathbb{N}^{\mathbb{N}}) = A$.

We need to introduce some pieces of notation. For every $m \in \mathbb{N}$, every $y = (x_1, \dots, x_m) \in X^m$, every $\lambda = (\lambda_1, \dots, \lambda_m) \in \mathbb{R}^m$ and every $\varepsilon > 0$ we set

$$V(y, \lambda, \varepsilon) := \{g \in \mathbb{R}^X : \lambda_i - \varepsilon < g(x_i) < \lambda_i + \varepsilon \ \forall i = 1, \dots, m\}.$$

By $\overline{V}(y, \lambda, \varepsilon)$, we denote the closure of $V(y, \lambda, \varepsilon)$ in $\mathbb{R}^X$.

Using the fact that $\mathcal{C}$ is not hereditarily separable, by recursion on countable ordinals, we select

(1) $m \in \mathbb{N}$, $\lambda = (\lambda_1, \dots, \lambda_m) \in \mathbb{Q}^m$ and positive rationals ε and δ,
(2) a family $\Gamma = \{y_\xi = (x_1^\xi, \dots, x_m^\xi) : \xi < \omega_1\} \subseteq X^m$,
(3) a family $\{f_\xi : \xi < \omega_1\} \subseteq \mathcal{C}$,
(4) a family $\{M_\xi : \xi < \omega_1\} \subseteq [\mathbb{N}]^\infty$, and
(5) a family $\{b_\xi : \xi < \omega_1\} \subseteq \mathbb{N}^\mathbb{N}$

such that for every $\xi < \omega_1$ the following are satisfied.

(i) We have $f_\xi \in Acc(\mathcal{K})$.
(ii) We have $f_\xi \in V(y_\xi, \lambda, \varepsilon)$, while for every $\zeta < \xi$ we have $f_\zeta \notin \overline{V}(y_\xi, \lambda, \varepsilon + \delta)$.
(iii) y_ξ is a condensation point of Γ in X^m.
(iv) We have $\Phi(b_\xi) = M_\xi$ and f_ξ is the pointwise limit of the sequence $(f_n)_{n \in M_\xi}$.

Next, by induction on the length of the finite sequences in $2^{<\mathbb{N}}$, we shall construct

(C1) a canonical injection $\psi \colon 2^{<\mathbb{N}} \to \mathbb{N}$,
(C2) a family $(B_t)_{t \in 2^{<\mathbb{N}}}$ of open balls in X^m taken with respect to a compatible complete metric ρ of X^m, and
(C3) a family $(\Delta_t)_{t \in 2^{<\mathbb{N}}}$ of uncountable subsets of ω_1.

The construction is done so that for every $t \in 2^{<\mathbb{N}}$ the following are satisfied.

(P1) If $t \neq \emptyset$, then $f_{\psi(t)} \in V(y, \lambda, \varepsilon)$ for every $y \in B_t$.
(P2) For every $t', t \in 2^{<\mathbb{N}}$ with $|t'| = |t|$ and $t' \neq t$ we have $f_{\psi(t)} \notin \overline{V}(y, \lambda, \varepsilon + \delta)$ for every $y \in B_{t'}$.
(P3) We have $\overline{B}_{t^\frown 0} \cap \overline{B}_{t^\frown 1} = \emptyset$, $\overline{B}_{t^\frown 0} \cup \overline{B}_{t^\frown 1} \subseteq B_t$ and $\rho - \mathrm{diam}(B_t) \leqslant \frac{1}{|t|+1}$.
(P4) We have $\Delta_{t^\frown 0} \cap \Delta_{t^\frown 1} = \emptyset$ and $\Delta_{t^\frown 0} \cup \Delta_{t^\frown 1} \subseteq \Delta_t$.
(P5) We have $\mathrm{diam}\big(\{b_\xi : \xi \in \Delta_t\}\big) \leqslant \frac{1}{2^{|t|}}$.
(P6) We have $\{y_\xi : \xi \in \Delta_t\} \subseteq B_t$.
(P7) If $t \neq \emptyset$, then $\psi(t) \in M_\xi$ for every $\xi \in \Delta_t$.

Assume that the construction has been carried out. We set $y_\sigma := \bigcap_n B_{\sigma|n}$ and $V_\sigma := V(y_\sigma, \lambda, \varepsilon + \frac{\delta}{2})$ for every $\sigma \in 2^\mathbb{N}$. Using (P1) and (P2), it is easy to see that $\mathcal{K}_\sigma \subseteq V_\sigma$ and $\mathcal{K}_\sigma \cap V_\tau = \emptyset$ if $\sigma \neq \tau$. We only need to check that $\mathcal{K}_\sigma \subseteq \mathcal{C}$ for every $\sigma \in 2^\mathbb{N}$. So, let $\sigma \in 2^\mathbb{N}$ be arbitrary. We set $M := \{\psi(\sigma|n) : n \geqslant 1\} \in [\mathbb{N}]^\infty$. It is enough to show that $M \in A$. For every $k \geqslant 1$ we select $\xi_k \in \Delta_{\sigma|k}$. By properties (P4), (P5) and (P7), the sequence $(b_{\xi_k})_{k \geqslant 1}$ converges to a unique $b \in \mathbb{N}^\mathbb{N}$ and, moreover, $\psi(\sigma|n) \in M_{\xi_k} = \Phi(b_{\xi_k})$ for every $1 \leqslant n \leqslant k$. By the continuity of Φ, we see that $M_{\xi_k} \to \Phi(b)$, and so $M \subseteq \Phi(b)$. Since A is hereditary, we conclude that $M \in A$ as desired.

We proceed to the construction. We set $\psi(\emptyset) := 0$, $B_\emptyset := X^m$ and $\Delta_\emptyset := \omega_1$. Assume that for some $n \geqslant 1$ and for every $t \in 2^{<n}$ the values $\psi(t) \in \mathbb{N}$, the open balls B_t and the sets Δ_t have been constructed. Refining if necessary, we may assume that for every $t \in 2^{<n}$ and every $\xi \in \Delta_t$ the point y_ξ is a condensation point of the set $\{y_\zeta : \zeta \in \Delta_t\}$.

Let $\{t_0 \prec \cdots \prec t_{2^{n-1}-1}\}$ be the $\prec$-increasing enumeration of 2^{n-1}. For every $j \in \{0,\dots,2^n-1\}$ we select an open ball B_j^{-1} in X^m and an uncountable subset Δ_j^{-1} of ω_1 such that $\rho-\mathrm{diam}(B_j^{-1}) < \frac{1}{n+1}$, $\{y_\xi : \xi \in \Delta_j^{-1}\} \subseteq B_j^{-1}$ and, moreover, $\mathrm{diam}\{b_\xi : \xi \in \Delta_j^{-1}\} \leqslant \frac{1}{2^n}$. The selection is done so that for j even we have that $\overline{B_j^{-1}} \cap \overline{B_{j+1}^{-1}} = \emptyset$, $\overline{B_j^{-1}} \cup \overline{B_{j+1}^{-1}} \subseteq B_{t_{j/2}}$ and $\Delta_j^{-1} \cup \Delta_{j+1}^{-1} \subseteq \Delta_{t_{j/2}}$. Also we set $m_{-1} := \max\{\psi(t) : t \in 2^{<n}\}$.

By recursion on $k \in \{0,\dots,2^n-1\}$, we will select a family $\{B_j^k : j=0,\dots,2^n-1\}$ of open balls of X^m, a family $\{\Delta_j^k : j=0,\dots,2^n-1\}$ of uncountable subsets of ω_1 and a positive integer m_k such that for every $k \in \{0,\dots,2^n-1\}$ the following are satisfied.

(a) For every $j \in \{0,\dots,2^n-1\}$ we have that $B_j^{k-1} \supseteq B_j^k$, $\Delta_j^{k-1} \supseteq \Delta_j^k$ and $\{y_\xi : \xi \in \Delta_j^k\} \subseteq B_j^k$. Moreover, for every j and every $\xi \in \Delta_j^k$ the point y_ξ is a condensation point of $\{y_\zeta : \zeta \in \Delta_j^k\}$.

(b) We have $m_{k-1} < m_k$.

(c) For every $y \in B_k^k$ we have $f_{m_k} \in V(y,\lambda,\varepsilon)$, and for every $j \in \{0,\dots,2^n-1\}$ with $j \neq k$ and every $y \in B_j^k$ we have $f_{m_k} \notin \overline{V}(y,\lambda,\varepsilon+\delta)$.

(d) We have $m_k \in M_\xi$ for every $\xi \in \Delta_k^k$.

The first step of the recursive selection is identical to the general one, and so we may assume that the selection has been carried out for every $k' < k$ where $k \in \{0,\dots,2^n-1\}$. Fix a countable base $\mathcal{B}$ of open balls of X^m. We first observe that for every $\xi \in \Delta_k^{k-1}$ and every $j \in \{0,\dots,2^n-1\}$ there exist

(e) a basic open ball $B_j^{k,\xi} \subseteq B_j^{k-1}$, and

(f) a positive integer $m_\xi \in M_\xi$ with $m_{k-1} < m_\xi$

such that the following holds.

(g) For every $y \in B_k^{k,\xi}$ we have $f_{m_\xi} \in V(y,\lambda,\varepsilon)$, and for every $j \neq k$ and every $y \in B_j^{k,\xi}$ we have $f_{m_\xi} \notin \overline{V}(y,\lambda,\varepsilon+\delta)$.

To see that such choices are possible, fix $\xi \in \Delta_k^{k-1}$. We select distinct countable ordinals $\xi_0,\dots,\xi_{2^n-1}$ such that

(h) for every $j \in \{0,\dots,2^n-1\}$ we have $\xi_j \in \Delta_j^{k-1}$, and

(k) $\xi = \xi_k = \min\big\{\xi_j : j \in \{0,\dots,2^n-1\}\big\}$.

By (ii) and (k) above, we have that $f_{\xi_k} \in V(y_{\xi_k},\lambda,\varepsilon)$ while $f_{\xi_k} \notin \overline{V}(y_{\xi_j},\lambda,\varepsilon+\delta)$ for every $j \neq k$. By (iv), we can select $m_\xi \in M_\xi$ with $m_{k-1} < m_\xi$ (thus, condition (f) above is satisfied) and such that $f_{m_\xi} \in V(y_\xi,\lambda,\varepsilon)$ while $f_{m_\xi} \notin \overline{V}(y_{\xi_j},\lambda,\varepsilon+\delta)$ for every $j \neq k$. Since f_{m_ξ} is continuous, for every $j \in \{0,\dots,2^n-1\}$ there exists a basic open ball $B_j^{k,\xi}$ in X^m containing y_{ξ_j} such that conditions (e) and (g) above are satisfied.

By cardinality arguments, we see that there exist an uncountable subset Δ_k^k of Δ_k^{k-1}, a positive integer m_k and for every j a ball B_j^k such that $m_\xi = m_k$ and $B_j^{k,\xi} = B_j^k$ for every $\xi \in \Delta_k^k$. Setting $\Delta_j^k := \{\xi : y_\xi \in B_j^k\} \cap \Delta_j^{k-1}$, we see that the recursive selection, described in (a)–(d) above, is completed.

Now let $\{t_0 \prec \cdots \prec t_{2^n-1}\}$ be the $\prec$-increasing enumeration of 2^n. We set $\psi(t_k) \coloneqq m_k$, $B_{t_k} \coloneqq B_k^{2^n-1}$ and $\Delta_{t_k} \coloneqq \Delta_k^{2^n-1}$ for every $k \in \{0, \dots, 2^n-1\}$. It is easy to check that (P1)–(P7) are satisfied. This completes the construction described in (C1), (C2) and (C3). As we have already indicated, having completed the proof of Proposition 35, the proof of Theorem 33 is also completed.

6.2.2. *Proof of Theorem* 34. As in the proof of Theorem 33, the main goal is to prove the following proposition.

Proposition 36. *Let $\{f_n\}$ be a family of continuous functions relatively compact in $\mathcal{B}_1(X)$. If the closure $\mathcal{K}$ of $\{f_n\}$ in $\mathbb{R}^X$ is non-metrizable, then there exist canonical injections $\psi_1 \colon 2^{<\mathbb{N}} \to \mathbb{N}$ and $\psi_2 \colon 2^{<\mathbb{N}} \to \mathbb{N}$ such that following holds. Setting*

$$\mathcal{K}^1_\sigma = \overline{\{f_{\psi_1(\sigma|n)}\}}^p \setminus \{f_{\psi_(\sigma|n)}\} \quad \textit{and} \quad \mathcal{K}^2_\sigma = \overline{\{f_{\psi_2(\sigma|n)}\}}^p \setminus \{f_{\psi_2(\sigma|n)}\}$$

for every $\sigma \in 2^{\mathbb{N}}$, there exists an open subset V_σ of $\mathbb{R}^X$ with $(\mathcal{K}^1_\sigma - \mathcal{K}^2_\sigma) \subseteq V_\sigma$ and such that $(\mathcal{K}^1_\tau - \mathcal{K}^2_\tau) \cap V_\sigma = \emptyset$ for every $\tau \in 2^{\mathbb{N}}$ with $\tau \neq \sigma$.

Granting Proposition 36, we complete the proof of Theorem 34 as follows. Let $\{f_n\}$ be the countable dense subset of $\mathcal{K}$. As we have already remarked, we may assume that the functions $\{f_n\}$ are continuous. We apply Proposition 36 to the family $\{f_n\}$ and we obtain two canonical injections $\psi_1 \colon 2^{<\mathbb{N}} \to \mathbb{N}$ and $\psi_2 \colon 2^{<\mathbb{N}} \to \mathbb{N}$ as described above. We define $g_t = f_{\psi_1(t)}$ and $h_t = f_{\psi_2(t)}$ for every $t \in 2^{<\mathbb{N}}$. Applying Corollary 22 successively two times, we obtain a regular dyadic subtree $S = (s_t)_{t \in 2^{<\mathbb{N}}}$ of $2^{<\mathbb{N}}$ such that the families $\{g_{s_t}\}_{t \in 2^{<\mathbb{N}}}$ and $\{h_{s_t}\}_{t \in 2^{<\mathbb{N}}}$ are canonized. The fact that $\mathcal{K}$ is hereditarily separable implies that each of the above families must be equivalent to the canonical dense family of either $A(2^{<\mathbb{N}})$, or $2^{\leqslant\mathbb{N}}$, or $\hat{S}_+(2^{\mathbb{N}})$, or $\hat{S}_-(2^{\mathbb{N}})$. By Proposition 36, we see that it cannot be the case that both $\overline{\{g_{s_t}\}}^p_{t \in 2^{<\mathbb{N}}}$ and $\overline{\{h_{s_t}\}}^p_{t \in 2^{<\mathbb{N}}}$ are metrizable. Thus, at least one of them is equivalent to either $\hat{S}_+(2^{\mathbb{N}})$ or $\hat{S}_-(2^{\mathbb{N}})$. Clearly, this implies Theorem 34.

We proceed to the proof of Proposition 36 which is similar to the proof of Proposition 35. It relies on the fact that $\mathcal{K}$ is metrizable if and only if there exists $D \subseteq X$ countable such that the map $\mathrm{Acc}(\mathcal{K}) \ni f \mapsto f|_D \in \mathbb{R}^D$ is one-to-one. Thus, by our assumptions and by transfinite recursion on countable ordinals, we obtain

(1) two rationals $p < q$,
(2) a set $\Gamma = \{x_\xi : \xi < \omega_1\} \subseteq X$, and
(3) two families $\{g_\xi : \xi < \omega_1\}$ and $\{h_\xi : \xi < \omega_1\}$ in $\mathrm{Acc}(\mathcal{K})$

such that for every $\xi < \omega_1$ the following are satisfied.

(i) We have $g_\xi(x_\zeta) = h_\xi(x_\zeta)$ for every $\zeta < \xi$.
(ii) We have $g_\xi(x_\xi) < p < q < h_\xi(x_\xi)$.
(iii) x_ξ is a condensation point of Γ.

By recursion on the length of the finite sequences in $2^{<\mathbb{N}}$, we shall construct

(C1) two canonical injections $\psi_1 \colon 2^{<\mathbb{N}} \to \mathbb{N}$ and $\psi_2 \colon 2^{<\mathbb{N}} \to \mathbb{N}$, and

(C2) a family $(B_t)_{t\in 2^{<\mathbb{N}}}$ of open balls in X taken with respect to a compatible complete metric ρ of X.

The construction is done so that for every $t\in 2^{<\mathbb{N}}$ the following are satisfied.

(P1) If $t\neq\emptyset$, then $f_{\psi_1(t)}(x)<p<q<f_{\psi_2(t)}(x)$ for every $x\in B_t$.
(P2) For every $t,t'\in 2^{<\mathbb{N}}$ with $|t'|=|t|$ and $t'\neq t$ and every $x'\in B_{t'}$ we have $|f_{\psi_1(t)}(x')-f_{\psi_2(t)}(x')|<\frac{1}{|t|+1}$.
(P3) We have $\overline{B}_{t^\frown 0}\cap\overline{B}_{t^\frown 1}=\emptyset$, $\overline{B}_{t^\frown 0}\cup\overline{B}_{t^\frown 1}\subseteq B_t$ and $\rho-\mathrm{diam}(B_t)\leqslant\frac{1}{|t|+1}$.
(P4) We have $|B_t\cap\Gamma|=\aleph_1$.

Assuming that the construction has been carried out, setting $x_\sigma\coloneqq\bigcap_n B_{\sigma|n}$ and

$$V_\sigma\coloneqq\{w\in\mathbb{R}^X:|w(x_\sigma)|>(q-p)/2\}$$

for every $\sigma\in 2^{\mathbb{N}}$, it is easy to see that ψ_1, ψ_2 and $\{V_\sigma:\sigma\in 2^{\mathbb{N}}\}$ satisfy the requirements of Proposition 36.

We proceed to the construction. We set $\psi_1(\emptyset)\coloneqq 0$, $\psi_2(\emptyset)\coloneqq 0$ and $B_\emptyset\coloneqq X$. Assume that for some $n\geqslant 1$ and for every $t\in 2^{<n}$ the values $\psi_1(t),\psi_2(t)\in\mathbb{N}$ and the open balls B_t have been constructed. As in Proposition 35, in order to determine $\psi_1(t),\psi_2(t)$ and B_t for every $t\in 2^n$ we will follow a finite selection.

Let $\{t_0\prec\cdots\prec t_{2^{n-1}-1}\}$ be the $\prec$-increasing enumeration of 2^{n-1}. For every $j\in\{0,\dots,2^n-1\}$ we select an open ball B_j^{-1} in X such that $\rho-\mathrm{diam}(B_j^{-1})<\frac{1}{n+1}$ and $|B_j^{-1}\cap\Gamma|=\aleph_1$. Moreover, the selection is done so that for j even we have $\overline{B_j^{-1}}\cap\overline{B_{j+1}^{-1}}=\emptyset$ and $\overline{B_j^{-1}}\cup\overline{B_{j+1}^{-1}}\subseteq B_{t_{j/2}}$. We set $m_{-1}\coloneqq\max\{\psi_1(t):t\in 2^{<n}\}$ and $l_{-1}\coloneqq\max\{\psi_2(t):t\in 2^{<n}\}$.

By recursion on $k\in\{0,\dots,2^n-1\}$, we will select a family $\{B_j^k:j=0,\dots,2^n-1\}$ of open balls of X and a pair $m_k,l_k\in\mathbb{N}$ such that for every $k\in\{0,\dots,2^n-1\}$ the following are satisfied.

(a) For every $j\in\{0,\dots,2^n-1\}$ we have $B_j^{k-1}\supseteq B_j^k$.
(b) We have $m_{k-1}<m_k$ and $l_{k-1}<l_k$.
(c) For every $x\in B_k^k$ we have $f_{m_k}(x)<p<q<f_{l_k}(x)$; on the other hand, for every $j\in\{0,\dots,2^n-1\}$ with $j\neq k$ and every $x'\in B_j^k$ we have $|f_{m_k}(x')-f_{l_k}(x')|<\frac{1}{n+1}$.
(d) For every $j\in\{0,\dots,2^n-1\}$ we have $|B_j^k\cap\Gamma|=\aleph_1$.

We omit this recursive selection since it is similar to the selection in the proof of Proposition 35. We only notice that condition (k) is replaced by

(k)′ $\xi_k=\max\big\{\xi_j:j\in\{0,\dots,2^n-1\}\big\}$.

Let $\{t_0\prec\cdots\prec t_{2^n-1}\}$ be the $\prec$-increasing enumeration of 2^n. We set $\psi_1(t_k)\coloneqq m_k$, $\psi_2(t_k)\coloneqq l_k$ and $B_{t_k}\coloneqq B_k^{2^n-1}$ for every $k\in\{0,\dots,2^n-1\}$. It is easy to check that with these choices properties (P1)–(P4) are satisfied. The proof of Proposition 36 is completed.

Remark 3. We notice that Theorem 32 (respectively, Theorem 34) is valid for an analytic and metrizable (respectively, hereditarily separable) subspace $\mathcal{C}$ of $\mathcal{K}$. In particular, we have the following theorem.

Theorem 37. *Let $\mathcal{K}$ be a separable Rosenthal compact and let $\mathcal{C}$ be an analytic subspace of $\mathcal{K}$. Let $\{f_n\}$ be a countable dense subset of $\mathcal{K}$ and let $A \subseteq [\mathbb{N}]^\infty$ be analytic witnessing the analyticity of $\mathcal{C}$.*

If $\mathcal{C}$ is metrizable in the pointwise topology, consists of bounded functions and it is norm non-separable, then $2^{\leqslant\mathbb{N}}$ canonically embeds into $\mathcal{K}$ with respect to $\{f_n\}$ and $\mathcal{C}$ such that its image is norm non-separable.

Respectively, if $\mathcal{C}$ is hereditarily separable and not metrizable, then either $\hat{S}_+(2^\mathbb{N})$ or $\hat{S}_-(2^\mathbb{N})$ canonically embeds into $\mathcal{K}$ with respect to $\{f_n\}$ and $\mathcal{C}$.

The additional information provided by Theorem 37 is that the canonical embedding of the corresponding prototype is found with respect to the dense subset $\{f_n\}$ of $\mathcal{K}$ witnessing the analyticity of $\mathcal{C}$ which is not necessarily a subset of $\mathcal{C}$. The proof of Theorem 37 follows the lines of Theorems 32 and 34 using the arguments of the proof of Proposition 35.

7. Non-G_δ points in analytic subspaces

This section is devoted to the study of the structure of not first countable analytic subspaces. The first subsection is devoted to the presentation of an extension of a result of Krawczyk [Kr]. The proof follows the same lines as in [Kr]. In the second subsection we show that $\hat{A}(2^\mathbb{N})$ canonically embeds into any not first countable analytic subspace $\mathcal{C}$ of a separable Rosenthal compact $\mathcal{K}$ and with respect to any countable subset D of $\mathcal{K}$ witnessing the analyticity of $\mathcal{C}$.

7.1. **Krawczyk trees.** We begin by introducing some pieces of notation and recalling some standard terminology. By Σ we denote the set of all nonempty strictly increasing finite sequences of $\mathbb{N}$. We view Σ as a tree equipped with the (strict) partial order $\sqsubset$ of extension. We view, however, every $t \in \Sigma$ not only as a finite increasing sequence but also as a finite subset of $\mathbb{N}$. Thus, for every $t, s \in \Sigma$ with $\max(s) < \min(t)$ we will frequently denote by $s \cup t$ the concatenation of s and t. By $[\Sigma]$ we denote the branches of Σ, that is, the set $\{\sigma \in \mathbb{N}^\mathbb{N} : \sigma|n \in \Sigma \ \forall n \geqslant 1\}$. For every $t \in \Sigma$ by Σ_t we denote the set $\{s \in \Sigma : t \sqsubseteq s\}$.

For every $A, B \in [\mathbb{N}]^\infty$ we write $A \subseteq^* B$ if the set $A \setminus B$ is finite. If $\mathcal{A} \subseteq [\mathbb{N}]^\infty$, then we set $\mathcal{A}^* := \{\mathbb{N} \setminus A : A \in \mathcal{A}\}$. For a pair $\mathcal{A}, \mathcal{B} \subseteq [\mathbb{N}]^\infty$ we say that $\mathcal{A}$ is $\mathcal{B}$-*generated* if for every $A \in \mathcal{A}$ there exist $B_0, \dots, B_k \in \mathcal{B}$ such that $A \subseteq B_0 \cup \dots \cup B_k$. We say that $\mathcal{A}$ is *countably $\mathcal{B}$-generated* if there exists a sequence (B_n) is $\mathcal{B}$ such that $\mathcal{A}$ is $\{B_n : n \in \mathbb{N}\}$-generated. An ideal $\mathcal{I}$ on $\mathbb{N}$ is said to be *bisequential* if for every $p \in \beta\mathbb{N}$ with $\mathcal{I} \subseteq p^*$ the family $\mathcal{I}$ is countably p^*-generated. Finally, for every family $\mathcal{F}$ of subsets of $\mathbb{N}$ and every $A \subseteq \mathbb{N}$ let $\mathcal{F}[A] := \{L \cap A : L \in \mathcal{F}\}$ denote the *trace* of

$\mathcal{F}$ on A. Observe that if $\mathcal{F}$ is hereditary, then $\mathcal{F}[A] = \mathcal{F} \cap \mathcal{P}(A) = \{L \in \mathcal{F} : L \subseteq A\}$. The following lemma is essentially Lemma 1 from [Kr].

Lemma 38. *Let $\mathcal{I}$ be a bisequential ideal. Also let $\mathcal{F} \subseteq \mathcal{I}$ and $A \in [\mathbb{N}]^\infty$. Assume that $\mathcal{F}[A]$ is not countably $\mathcal{I}$-generated. Then there exists a sequence (A_n) of pairwise disjoint infinite subsets of A such that, setting $\mathcal{A} = \{A_n : n \in \mathbb{N}\}$, we have that $\mathcal{I}[A]$ is $\mathcal{A}$-generated, while $\mathcal{F}[A_n]$ is not countably $\mathcal{I}$-generated for every $n \in \mathbb{N}$.*

Proof. (Sketch) It suffices to prove the lemma for $A = \mathbb{N}$. We set

$$\mathcal{J} \coloneqq \{C \subseteq \mathbb{N} : \mathcal{F}[C] \text{ is countably } \mathcal{I} - \text{generated}\}.$$

Then $\mathcal{J}$ is an ideal, $\mathbb{N} \notin \mathcal{J}$ and $\mathcal{I} \subseteq \mathcal{J}$. We select $p \in \beta\mathbb{N}$ with $\mathcal{J} \subseteq p^*$. By the bisequentiality of $\mathcal{I}$, there exists a sequence (D_n) in p^* such that $\mathcal{I}$ is $\{D_n : n \in \mathbb{N}\}$-generated. Since p^* is an ideal, we may assume that $D_n \cap D_m = \emptyset$ if $n \neq m$. Define $M \coloneqq \{n \in \mathbb{N} : D_n \in \mathcal{J}\}$. By the fact that $\mathcal{I}$ is $\{D_n : n \in \mathbb{N}\}$-generated and $\mathcal{F} \subseteq \mathcal{I}$, we see that $\mathcal{F}$ is $\{D_n : n \in \mathbb{N}\}$-generated. This observation and the fact that (D_n) are pairwise disjoint yield that the set $D = \bigcup_{n \in M} D_n$ belongs to $\mathcal{J}$. Moreover, the set $\mathbb{N} \setminus M$ is infinite (for if not, we would obtain that $\mathbb{N} \in p^*$). Let $\{k_0 < k_1 < \cdots\}$ be the increasing enumeration of $\mathbb{N} \setminus M$ and define $A_0 \coloneqq D \cup D_{k_0}$ and $A_n \coloneqq D_{k_n}$ for $n \geqslant 1$. It is easy to see that the sequence (A_n) is as desired. □

The main result of this subsection is the following theorem which corresponds to Lemma 2 in [Kr]. We notice that it is one of the basic ingredients in the proof of the embedding of $\hat{A}(2^{\mathbb{N}})$ in not first countable separable Rosenthal compacta.

Theorem 39. *Let $\mathcal{I}$ be a bisequential ideal and let $\mathcal{F} \subseteq \mathcal{I}$ be analytic and hereditary. Assume that $\mathcal{F}$ is not countably $\mathcal{I}$-generated. Then there exists a one-to-one map $\kappa \colon \Sigma \to \mathbb{N}$ such that, setting $\mathcal{J}_{\mathcal{F}} \coloneqq \{\kappa^{-1}(L) : L \in \mathcal{F}\}$ and $\mathcal{J} \coloneqq \{\kappa^{-1}(M) : M \in \mathcal{I}\}$, the following are satisfied.*

1. *For every $\sigma \in [\Sigma]$ we have $\{\sigma|n : n \geqslant 1\} \in \mathcal{J}_{\mathcal{F}}$.*
2. (Domination property) *For every $B \in \mathcal{J}$ and every $n \geqslant 1$ there exist $t_0, \ldots, t_k \in \Sigma$ with $|t_0| = \cdots = |t_k| = n$ such that $B \subseteq^* \Sigma_{t_0} \cup \cdots \cup \Sigma_{t_k}$.*

It is easy to see that property (2) in Theorem 39 is equivalent to saying that B is contained in a finitely splitting subtree of Σ.

Proof. We fix a continuous map $\phi \colon \mathbb{N}^{\mathbb{N}} \to [\mathbb{N}]^\infty$ with $\phi(\mathbb{N}^{\mathbb{N}}) = \mathcal{F}$. Recursively, we shall construct

- (C1) a family $(A_s)_{s \in \mathbb{N}^{<\mathbb{N}}}$ of infinite subsets of $\mathbb{N}$,
- (C2) a family $(a_s)_{s \in \mathbb{N}^{<\mathbb{N}}}$ of finite subsets of $\mathbb{N}$, and
- (C3) a family $(U_s)_{s \in \mathbb{N}^{<\mathbb{N}}}$ of basic clopen subsets of $\mathbb{N}^{\mathbb{N}}$.

The construction is done so that the following are satisfied.

- (P1) We have that $A_s \subseteq A_t$ if $t \sqsubset s$, and $A_s \cap A_t = \emptyset$ if s and t are incomparable.

(P2) For every $s \in \mathbb{N}^{<\mathbb{N}}$ we have $|a_s| = |s|$ and $\max(a_s) \in A_s$ for every $s \in \mathbb{N}^{<\mathbb{N}}$ with $s \neq \emptyset$. Moreover, $a_s \sqsubset a_t$ if and only if $s \sqsubset t$.
(P3) We have $U_s \subseteq U_t$ if $t \sqsubset s$ and $\mathrm{diam}(U_s) \leqslant \frac{1}{2^{|s|}}$. Moreover, $U_s \cap U_t = \emptyset$ if s and t are incomparable.
(P4) For every $s \in \mathbb{N}^{<\mathbb{N}}$ we have that $\phi(U_s)[A_s]$ is not countably $\mathcal{I}$-generated.
(P5) For every $s \in \mathbb{N}^{<\mathbb{N}}$ and every $\tau \in U_s$ we have $a_s \subseteq \phi(\tau)$.
(P6) For every $s \in \mathbb{N}^{<\mathbb{N}}$ we have that $\mathcal{I}[\bigcup_n A_{s^\frown n}]$ is $\{A_{s^\frown n} : n \in \mathbb{N}\}$-generated.

Assuming that the construction has been carried out, we complete the proof as follows. We define $\lambda\colon \mathbb{N}^{<\mathbb{N}} \setminus \{\emptyset\} \to \mathbb{N}$ by $\lambda(s) = \max(a_s)$. By (P1) and (P2) above, we see that λ is one-to-one. Let $\sigma \in \mathbb{N}^{\mathbb{N}}$. We claim that $\{\lambda(\sigma|n) : n \geqslant 1\} = \bigcup_n a_{\sigma|n} \in \mathcal{F}$. To see this, by (P3), let τ be the unique element of $\bigcap_n U_{\sigma|n}$. Then, by (P5), we have that $a_{\sigma|n} \subseteq \phi(\tau)$ for every $n \in \mathbb{N}$. Thus, $\bigcup_n a_{\sigma|n} \subseteq \phi(\tau) \in \mathcal{F}$. Since the family $\mathcal{F}$ is hereditary, our claim is proved. Now let $B \subseteq \mathbb{N}^{<\mathbb{N}} \setminus \{\emptyset\}$ be such that $\{\lambda(t) : t \in B\} \in \mathcal{I}$. We claim that B must be dominated, that is, for every $n \geqslant 1$ there exist $s_0, \dots, s_k \in \mathbb{N}^n$ such that B is almost included in the set of the successors of the s_i's in $\mathbb{N}^{<\mathbb{N}}$. If not, then we may find a subset $\{t_n\}$ of B, a node s of $\mathbb{N}^{<\mathbb{N}}$ and a subset $\{m_n : n \in \mathbb{N}\}$ of $\mathbb{N}$ such that $s^\frown m_n \sqsubseteq t_n$ for every $n \in \mathbb{N}$. Notice that $\{\lambda(t_n) : n \in \mathbb{N}\} \in \mathcal{I}$ since $\mathcal{I}$ is hereditary. Moreover, by the definition of λ and properties (P1) and (P2) above, we see that $\lambda(t_n) \in A_{s^\frown m_n}$ for every $n \in \mathbb{N}$. Therefore, $\{\lambda(t_n) : n \in \mathbb{N}\} \in \mathcal{I}[\bigcup_n A_{s^\frown n}]$. This leads to a contradiction by properties (P1) and (P6) above. We set $\kappa \coloneqq \lambda|_\Sigma$. Clearly κ is as desired.

We proceed to the construction. We set $A_\emptyset = \mathbb{N}$, $a_\emptyset = \emptyset$ and $U_\emptyset = \mathbb{N}^{\mathbb{N}}$. Assume that A_s, a_s and U_s have been constructed for some $s \in \mathbb{N}^{<\mathbb{N}}$. We set $\mathcal{F}_s \coloneqq \phi(U_s)$. By property (P4) above and Lemma 38, there exists a sequence (A_n) of pairwise disjoint infinite subsets of A_s such that $\mathcal{F}_s[A_n]$ is not countably $\mathcal{I}$-generated for every $n \in \mathbb{N}$, while $\mathcal{I}[A_s]$ is $\{A_n : n \in \mathbb{N}\}$-generated. Recursively, we select a subset $\{\tau_n : n \in \mathbb{N}\}$ in U_s such that for every $n \in \mathbb{N}$ the following are satisfied.

(i) The set $\phi(\tau_n) \cap A_n$ is infinite.
(ii) The family $\phi(V_{\tau_n|k})[A_n]$ is not countably $\mathcal{I}$-generated for every $k \in \mathbb{N}$ where $V_{\tau_n|k} = \{\sigma \in \mathbb{N}^{\mathbb{N}} : \tau_n|k \sqsubset \sigma\}$.

This can be easily done since $\phi(U_s)[A_n] = \mathcal{F}_s[A_n]$ is not countably $\mathcal{I}$-generated for every $n \in \mathbb{N}$. We select $L = \{l_0 < l_1 < \cdots\} \in [\mathbb{N}]^\infty$ and a sequence (k_n) in $\mathbb{N}$ such that, setting $\sigma_n \coloneqq \tau_{l_n}$ for every n, the following are satisfied.

(iii) We have $V_{\sigma_n|k_n} \cap V_{\sigma_m|k_m} = \emptyset$ if $n \neq m$.
(iv) For every $n \in \mathbb{N}$ we have $V_{\sigma_n|k_n} \subseteq U_s$.
(v) For every $n \in \mathbb{N}$ we have $\mathrm{diam}(V_{\sigma_n|k_n}) < \frac{1}{2^{|s|+1}}$.

Using the continuity of the map ϕ, for every $n \in \mathbb{N}$ we may also select $k'_n, i_n \in \mathbb{N}$ such that the following are satisfied.

(vi) We have $i_n \in \phi(\sigma_n) \cap A_{l_n}$.
(vii) We have $\max(a_s) < i_n$ and $k_n < k'_n$.

(viii) For every $\tau \in V_{\sigma_n|k'_n}$ we have $i_n \in \phi(\tau)$.

For every $n \in \mathbb{N}$ we set $a_{s^\frown n} := a_s \cup \{i_n\}$, $A_{s^\frown n} := A_{l_n}$ and $U_{s^\frown n} := V_{\sigma_n|k'_n}$. It is easy to verify that properties (P1)–(P6) are satisfied. The proof of the theorem is thus completed. $\square$

7.2. **The embedding of $\hat{A}(2^{\mathbb{N}})$ in analytic subspaces.** The main result in this subsection is the following theorem.

Theorem 40. *Let $\mathcal{K}$ be a separable Rosenthal compact and let $\mathcal{C}$ be an analytic subspace of $\mathcal{K}$. Let $\{f_n\}$ be a countable dense subset of $\mathcal{K}$ and let $A \subseteq [\mathbb{N}]^\infty$ be analytic witnessing the analyticity of $\mathcal{C}$. Also let $f \in \mathcal{C}$ be a non-G_δ point of $\mathcal{C}$. Then there exists a canonical homeomorphic embedding of $\hat{A}(2^{\mathbb{N}})$ into $\mathcal{K}$ with respect to $\{f_n\}$ and $\mathcal{C}$ which sends 0 to f.*

For the proof we need to do some preparatory work. Let $\mathcal{K}$, $\mathcal{C}$, $\{f_n\}$ and $f \in \mathcal{C}$ be as in Theorem 40. We may assume that $f_n \neq f$ for every $n \in \mathbb{N}$. We set

$$\mathcal{I}_f := \{L \in [\mathbb{N}]^\infty : f \notin \overline{\{f_n\}}^p_{n\in L}\}.$$

Then $\mathcal{I}_f$ is an analytic ideal on $\mathbb{N}$ (see [Kr]). A fundamental property of $\mathcal{I}_f$ is that it is bisequential. This is due to Pol [Po3]. We notice that the bisequentiality of $\mathcal{I}_f$ can be also derived by the non-effective proof of the Bourgain–Fremlin–Talagrand theorem due to Debs (see [De] or [AGR]).

Next, let $A \subseteq [\mathbb{N}]^\infty$ be as in Theorem 40. As we have already pointed out, we may assume that A is hereditary. We set

$$\mathcal{F} := A \cap \mathcal{I}_f.$$

Clearly $\mathcal{F}$ is an analytic and hereditary subset of $\mathcal{I}_f$. The assumption that f is a non-G_δ point of $\mathcal{C}$ yields (and, in fact, is equivalent to saying) that $\mathcal{F}$ is not countably $\mathcal{I}_f$-generated, that is, there does exist a sequence (M_k) in $\mathcal{I}_f$ such that for every $L \in \mathcal{F}$ there exists $k \in \mathbb{N}$ with $L \subseteq M_0 \cup \cdots \cup M_k$. To see this, assume on the contrary that such a sequence (M_k) existed. We set $N_k := M_0 \cup \cdots \cup M_k$ for every $k \in \mathbb{N}$. Since $\mathcal{I}_f$ is an ideal, we see that $N_k \in \mathcal{I}_f$ for every k. We set $F_k := \overline{\{f_n\}}^p_{n\in N_k} \cup \{f_k\}$. The fact that $N_k \in \mathcal{I}_f$ implies that $f \notin F_k$ for every $k \in \mathbb{N}$. Let $g \in \mathcal{C} \cap Acc(\mathcal{K})$ with $g \neq f$. By condition (2) of Definition 23, there exists $L \in A$ with $g \in \overline{\{f_n\}}^p_{n\in L}$. Hence, there exists $M \in [L]^\infty$ such that g is the pointwise limit of the sequence $(f_n)_{n\in M}$. Since A is hereditary, we see that $M \in \mathcal{F}$, and so, there exists $k_0 \in \mathbb{N}$ with $M \subseteq N_{k_0}$. This implies that $g \in F_{k_0}$. It follows by the above discussion that $\{f\} = \bigcap_k (\mathcal{C} \setminus F_k)$, that is, the point f is G_δ in $\mathcal{C}$, a contradiction.

Summarizing, we see that $\mathcal{I}_f$ is bi-sequential, $\mathcal{F} \subseteq \mathcal{I}_f$ is analytic, hereditary and not countably $\mathcal{I}_f$-generated. Thus, we may apply Theorem 39 and we obtain the one-to-one map $\kappa\colon \Sigma \to \mathbb{N}$ as described above. Setting $f_t := f_{\kappa(t)}$ for every $t \in \Sigma$ and invoking condition (1) of Definition 23, we obtain the following corollary.

Corollary 41. *There is a sub-family $\{f_t\}_{t\in\Sigma}$ of $\{f_n\}$ with the following properties.*

(1) *For every* $\sigma \in [\Sigma]$ *we have* $f \notin \overline{\{f_{\sigma|n}\}}^p$ *and* $Acc\big(\{f_{\sigma|n} : n \in \mathbb{N}\}\big) \subseteq \mathcal{C}$.
(2) *For every* $B \subseteq \Sigma$ *with* $f \notin \overline{\{f_t\}}^p_{t\in B}$ *and every* $n \geqslant 1$ *there exist* $t_0, \dots, t_k \in \Sigma$ *with* $|t_0| = \cdots = |t_k| = n$ *such that* $B \subseteq^* \Sigma_{t_0} \cup \cdots \cup \Sigma_{t_k}$.

We call the family $\{f_t\}_{t\in\Sigma}$ obtained by Corollary 41 as the *Krawczyk tree* of f with respect to $\{f_n\}$ and $\mathcal{C}$. The following property of the Krawczyk tree $\{f_t\}_{t\in\Sigma}$ will be needed later on.

(P) Let $i \in \Sigma$ and let (b_n) be a sequence in Σ such that $\max(i) < \min(b_n)$ and $\max(b_n) < \min(b_{n+1})$ for every $n \in \mathbb{N}$. Then, setting $s_n := i \cup b_n$ for every $n \in \mathbb{N}$, the sequence (f_{s_n}) is pointwise convergent to f. Indeed, by property (2) in Corollary 41, every subsequence of the sequence (f_{s_n}) accumulates to f. Hence, the sequence (f_{s_n}) is pointwise convergent to f.

We will also need the following well-known consequence of the bisequentiality of $\mathcal{I}_f$. For the sake of completeness we include a proof.

Lemma 42. *Let* (A_l) *be a sequence in* $[\mathbb{N}]^\infty$ *such that* $\lim_{n\in A_l} f_n = f$ *for every* $l \in \mathbb{N}$. *Then there exists* $D \in [\mathbb{N}]^\infty$ *with* $\lim_{n\in D} f_n = f$ *and* $D \subseteq \bigcup_l A_l$, *and such that* $D \cap A_l \neq \emptyset$ *for infinitely many* $l \in \mathbb{N}$.

Proof. For every $k \in \mathbb{N}$ we set $B_k := \bigcup_{l\geqslant k} A_l$. Then (B_k) is a decreasing sequence of infinite subsets of $\mathbb{N}$. We may select $p \in \beta\mathbb{N}$ such that $p-\lim f_n = f$ and $B_k \in p$ for every $k \in \mathbb{N}$. By the bisequentiality of $\mathcal{I}_f$, there exists a sequence (C_m) of elements of p converging to f. We select a strictly increasing sequence (l_k) in $\mathbb{N}$ such that $l_k \in B_k \cap C_0 \cap \cdots \cap C_k$ for every $k \in \mathbb{N}$, and we set $D := \{l_k : k \in \mathbb{N}\}$. It is easy to see that D is as desired. □

In the sequel we will apply Milliken's theorem [Mil1]. To this end we need to recall some pieces of notation. Given $b, b' \in \Sigma$ we write $b < b'$ provided that $\max(b) < \min(b')$. By $\mathbf{B}$ we denote the subset of $\Sigma^{\mathbb{N}}$ consisting of all sequences (b_n) which are increasing, that is, $b_n < b_{n+1}$ for every $n \in \mathbb{N}$. It is easy to see that $\mathbf{B}$ is a closed subspace of $\Sigma^{\mathbb{N}}$ where Σ is equipped with the discrete topology and $\Sigma^{\mathbb{N}}$ with the product topology. For every $\mathbf{b} = (b_n) \in \mathbf{B}$ we set

$$\langle \mathbf{b} \rangle := \Big\{ \bigcup_{n\in F} b_n : F \subseteq \mathbb{N} \text{ is nonempty finite} \Big\} \text{ and } [\mathbf{b}] := \{(c_n) \in \mathbf{B} : c_n \in \langle \mathbf{b} \rangle \ \forall n\}.$$

Notice that for every block sequence $\mathbf{b}$ the set $\langle \mathbf{b} \rangle$ corresponds to an infinitely branching subtree of Σ denoted by $\mathcal{T}_{\mathbf{b}}$. Also observe that the chains of $\mathcal{T}_{\mathbf{b}}$ are in one-to-one correspondence with the set $[\mathbf{b}]$ of all block subsequences of $\mathbf{b}$. More precisely, if (t_n) is a chain of $\mathcal{T}_{\mathbf{b}}$, then $(t_0, t_1 \setminus t_0, \dots, t_{n+1} \setminus t_n, \dots)$ is the block subsequence of $\mathbf{b}$ which corresponds to the chain (t_n). This observation was used by Henson to derive an alternative proof of Stern's theorem (see, e.g., [Od]). If $\beta = (b_0, \dots, b_k)$ with $b_0 < \cdots < b_k$ and $\mathbf{d} \in \mathbf{B}$, then we set

$$[\beta, \mathbf{d}] := \{(c_n) \in \mathbf{B} : c_n = b_n \ \forall n \leqslant k \text{ and } c_n \in \langle \mathbf{d} \rangle \ \forall n > k\}.$$

We will use the following consequence of Milliken's theorem.

Theorem 43. *For every* $\mathbf{b} \in \mathbf{B}$ *and every analytic subset* A *of* $\mathbf{B}$ *there exists* $\mathbf{c} \in [\mathbf{b}]$ *such that either* $[\mathbf{c}] \subseteq A$ *or* $[\mathbf{c}] \cap A = \emptyset$.

For every $\mathbf{b} = (b_n) \in \mathbf{B}$ and every $n \in \mathbb{N}$ we set $i_n \coloneqq b_0 \cup \cdots \cup b_n$. We define $C \colon \mathbf{B} \to \Sigma^{\mathbb{N}}$ and $A \colon \mathbf{B} \to \Sigma^{\mathbb{N}}$ by

$$C\big((b_n)\big) = (i_0, \dots, i_n, \dots) \ \text{ and } \ A\big((b_n)\big) = (i_0 \cup b_2, \dots, i_{3n} \cup b_{3n+2}, \dots).$$

Note that for every $\mathbf{b} \in \mathbf{B}$ the sequence $C(\mathbf{b})$ is a chain of Σ, while $A(\mathbf{b})$ is an antichain of Σ converging, in the sense of Definition 13, to $\sigma = \bigcup_n i_n \in [\Sigma]$. Also observe that the functions C and A are continuous.

Lemma 44. *Let* $\{f_t\}_{t\in\Sigma}$ *be a Krawczyk tree of* f *with respect to* $\{f_n\}$ *and* $\mathcal{C}$. *Then there exists a block sequence* $\mathbf{b} = (b_n)$ *such that for every* $\mathbf{c} \in [\mathbf{b}]$ *the sequence* $(f_t)_{t\in C(\mathbf{c})}$ *is pointwise convergent to a function which belongs to* $\mathcal{C}$ *and is different from* f, *while the sequence* $(f_t)_{t\in A(\mathbf{c})}$ *is pointwise convergent to* f.

Proof. Let

$$C_1 \coloneqq \{\mathbf{c} \in \mathbf{B} : \text{the sequence } (f_t)_{t\in C(\mathbf{c})} \text{ is pointwise convergent}\}.$$

It is easy to see that C_1 is a co-analytic subset of $\mathbf{B}$. By Theorem 43 and the sequential compactness of $\mathcal{K}$, there exists $\mathbf{d} \in \mathbf{B}$ such that $[\mathbf{d}]$ is a subset of C_1. As we have already remarked, for every block sequence $\mathbf{c}$ the sequence $C(\mathbf{c})$ is a chain of Σ. Hence, by part (1) of Corollary 41, we see that for every $\mathbf{c} \in [\mathbf{d}]$ the sequence $(f_t)_{t\in C(\mathbf{c})}$ must be pointwise convergent to a function which belongs to $\mathcal{C}$ and is different from f.

Now let

$$C_2 \coloneqq \{\mathbf{c} \in [\mathbf{d}] : \text{the sequence } (f_t)_{t\in A(\mathbf{c})} \text{ is pointwise convergent to } f\}.$$

Again by Milliken's theorem, there exists $\mathbf{b} = (b_n) \in [\mathbf{d}]$ such that either $[\mathbf{b}] \subseteq C_2$ or $[\mathbf{b}] \cap C_2 = \emptyset$. We claim that $[\mathbf{b}]$ is subset of C_2. It is enough to show that $[\mathbf{b}] \cap C_2 \neq \emptyset$. To this end we argue as follows. Recall that for every $l \in \mathbb{N}$ we have that $i_l = b_0 \cup \cdots \cup b_l$. Set

$$A_l \coloneqq \{i_l \cup b_m : m > l+1\} \subseteq \Sigma.$$

Since the sequence (b_n) is block, by property (P) above, we see that the sequence $(f_t)_{t\in A_l}$ is pointwise convergent to f. By Lemma 42, there exists $D \subseteq \bigcup_l A_l$ such that the sequence $(f_t)_{t\in D}$ is pointwise convergent to f and $D \cap A_l \neq \emptyset$ for infinitely many l. We may select $L = \{l_0 < l_1 < \cdots\}, M = \{m_0 < m_1 < \cdots\} \in [\mathbb{N}]^\infty$ such that $l_n + 1 < m_n < l_{n+1}$ and $i_{l_n} \cup b_{m_n} \in D$ for every $n \in \mathbb{N}$. Next we define $\mathbf{c} = (c_n) \in [\mathbf{b}]$ as follows. We set $c_0 \coloneqq i_{l_0}$, $c_1 \coloneqq b_{l_0+1} \cup \cdots \cup b_{m_0-1}$ and $c_2 \coloneqq b_{m_0}$. For every $n \in \mathbb{N}$ with $n \geqslant 1$ let $I_n = [m_{n-1}+1, l_n]$ and $J_n = [l_n, m_n - 1]$, and set

$$c_{3n} \coloneqq \bigcup_{i\in I_n} b_i, \ c_{3n+1} \coloneqq \bigcup_{i\in J_n} b_i \ \text{ and } \ c_{3n+2} \coloneqq b_{m_n}.$$

It is easy to see that $\mathbf{c} \in [\mathbf{b}]$ and $A(\mathbf{c}) = (i_{l_n} \cup b_{m_n}) \subseteq D$. Hence, the sequence $(f_t)_{t \in A(\mathbf{c})}$ is pointwise convergent to f. It follows that $[\mathbf{b}] \cap C_2 \neq \emptyset$ and the proof is completed. □

We are ready to proceed to the proof of Theorem 40.

Proof of Theorem 40. Let $\mathbf{b} = (b_n)$ be the block sequence obtained by Lemma 44. If $\beta = (b_{n_0}, \dots, b_{n_k})$ with $n_0 < \cdots < n_k$ is a nonempty finite subsequence of $\mathbf{b}$, then we set $\cup\beta = b_{n_0} \cup \cdots \cup b_{n_k} \in \Sigma$. Recursively, we will select a family $(\beta_s)_{s \in 2^{<\mathbb{N}}}$ such that the following are satisfied.

(C1) For every $s \in 2^{<\mathbb{N}}$ we have that β_s is a finite subsequence of $\mathbf{b}$.
(C2) For every $s, s' \in 2^{<\mathbb{N}}$ we have $s \sqsubset s'$ if and only if $\beta_s \sqsubset \beta_{s'}$.
(C3) For every $s \in 2^{<\mathbb{N}}$ and every $\mathbf{c} \in [\beta_{s^\frown 0}, \mathbf{b}]$ we have $\cup\beta_{s^\frown 1} \in A(\mathbf{c})$.

We set $\beta_\emptyset := \emptyset$ and we proceed as follows. For every $M = \{m_0 < m_1 < \cdots\} \in [\mathbb{N}]^\infty$ let $\mathbf{b}_M = (b_{m_n})$ be the subsequence of $\mathbf{b}$ determined by M. Assume that for some $s \in 2^{<\mathbb{N}}$ the finite sequence β_s has been defined. Let $M = M_s \in [\mathbb{N}]^\infty$ be such that $\beta_s \sqsubset \mathbf{b}_M$. The set $A(\mathbf{b}_M)$ converges to the unique branch of Σ determined by the infinite chain $C(\mathbf{b}_M)$. Therefore, we may select a finite subsequence $\beta_{s^\frown 1}$ such that $\beta_s \sqsubset \beta_{s^\frown 1}$ and $\cup\beta_{s^\frown 1} \in A(\mathbf{b}_M)$. The function $A\colon [\mathbf{b}] \to \Sigma^{\mathbb{N}}$ is continuous, and so, there exists a finite subsequence $\beta_{s^\frown 0}$ of $\mathbf{b}$ with $\beta_{s^\frown 0} \sqsubset \mathbf{b}_M$ such that condition (C3) above is satisfied. Finally, notice that $\beta_{s^\frown 0}$ and $\beta_{s^\frown 1}$ are incomparable with respect to the partial order $\sqsubset$ of extension.

One can also explicitly define a family $(\beta_s)_{s \in 2^{<\mathbb{N}}}$ satisfying conditions (C1)–(C3). Specifically, set $\beta_\emptyset := \emptyset$, $\beta_{(0)} := (b_0, b_1, b_2)$ and $\beta_{(1)} := (b_0, b_2)$. Assume that β_s has been defined for some $s \in 2^{<\mathbb{N}}$ and set $n_s := \max\{n : b_n \in \beta_s\}$. If s ends with 0, then we set

$$\beta_{s^\frown 0} := \beta_s^\frown(b_{n_s+1}, b_{n_s+2}, b_{n_s+3}) \ \text{ and } \ \beta_{s^\frown 1} := \beta_s^\frown(b_{n_s+1}, b_{n_s+3}).$$

On the other hand, if s ends with 1, then we set

$$\beta_{s^\frown 0} := \beta_s^\frown(b_{n_s+1}, b_{n_s+2}, b_{n_s+3}, b_{n_s+4}) \ \text{ and } \beta_{s^\frown 1} := \beta_s^\frown(b_{n_s+1}, b_{n_s+2}, b_{n_s+4}).$$

It is easy to see that, with the above choices, conditions (C1)–(C3) are satisfied.

Having defined the family $(\beta_s)_{s \in 2^{<\mathbb{N}}}$, for every $s \in 2^{<\mathbb{N}}$ we set

$$t_s := \cup\beta_s \in \Sigma \ \text{ and } \ h_s := f_{t_s}.$$

Clearly, the family $\{h_s\}_{s \in 2^{<\mathbb{N}}}$ is a dyadic subtree of the Krawczyk tree $\{f_t\}_{t \in \Sigma}$ of f with respect to $\{f_n\}$ and $\mathcal{C}$. The basic properties of the family $\{h_s\}_{s \in 2^{<\mathbb{N}}}$ are summarized in the following claim.

Claim 1. *The following hold.*

(1) *For every $\sigma \in 2^{\mathbb{N}}$ the sequence $(h_{\sigma|n})$ is pointwise convergent to a function $g_\sigma \in \mathcal{C}$ with $g_\sigma \neq f$.*

(2) *For every perfect subset P of $2^{\mathbb{N}}$ the function f belongs to the closure of the family $\{g_\sigma : \sigma \in P\}$.*

Proof of the claim. (1) Let $\sigma \in 2^{\mathbb{N}}$ and set $\mathbf{b}_\sigma \coloneqq \bigcup_n \beta_{\sigma|n} \in [\mathbf{b}]$. It is easy to see that the sequence $(t_{\sigma|n})$ is a subsequence of the sequence $C(\mathbf{b}_\sigma)$. Therefore, the result follows by Lemma 44.

(2) Assume not. Then there exist a perfect subset P of $2^{\mathbb{N}}$ and a neighborhood V of f in $\mathbb{R}^X$ such that $g_\sigma \notin \overline{V}$ for every $\sigma \in P$. By part (1), for every $\sigma \in P$ there exists $n_\sigma \in \mathbb{N}$ such that $h_{\sigma|n} \notin V$ for every $n \geqslant n_\sigma$. For every $n \in \mathbb{N}$ let $P_n = \{\sigma \in P : n_\sigma \leqslant n\}$. Then each P_n is a closed subset of P and, clearly, $P = \bigcup_n P_n$. Thus, there exist $n_0 \in \mathbb{N}$ and a perfect subset Q of $2^{\mathbb{N}}$ with $Q \subseteq P_{n_0}$. It follows that $h_{\sigma|n} \notin V$ for every $\sigma \in Q$ and every $n \geqslant n_0$. Let τ be the lexicographical minimum of Q. We may select a sequence (σ_k) in Q such that, setting $s_k = \tau \wedge \sigma_k$ for every $k \in \mathbb{N}$, we have $\sigma_k \to \tau$, $\tau \prec \sigma_k$ and $|s_k| > n_0$. Notice that $s_k^\frown 0 \sqsubset \tau$, while $s_k^\frown 1 \sqsubset \sigma_k$ and $|s_k^\frown 1| > n_0$. Hence, by our assumptions on the set Q and the definition of $\{h_s\}_{s\in 2^{<\mathbb{N}}}$, we obtain that

$$h_{s_k^\frown 1} = f_{t_{s_k^\frown 1}} \notin V \text{ for every } k \in \mathbb{N}. \tag{1}$$

We are ready to derive the contradiction. Set $\mathbf{b}_\tau \coloneqq \bigcup_n \beta_{\tau|n} \in [\mathbf{b}]$. Since $\beta_{s_k^\frown 0} \sqsubset \mathbf{b}_\tau$, by property (C3) in the above construction, we see that $t_{s_k^\frown 1} = \cup \beta_{s_k^\frown 1} \in A(\mathbf{b}_\tau)$ for every $k \in \mathbb{N}$. By Lemma 44, the sequence $(f_t)_{t \in A(\mathbf{b}_\tau)}$ is pointwise convergent to the function f. It follows that the sequence $(f_{t_{s_k^\frown 1}})$ is also pointwise convergent to f and this clearly contradicts (1) above. The proof of the clam is completed. □

We apply Theorem 21 to the family $\{h_s\}_{s\in 2^{<\mathbb{N}}}$ and we obtain a regular dyadic subtree $T = (s_t)_{t\in 2^{<\mathbb{N}}}$ of $2^{<\mathbb{N}}$ such that the family $\{h_{s_t}\}_{t\in 2^{<\mathbb{N}}}$ is canonized. The main claim is the following.

Claim 2. *$\{h_{s_t}\}_{t\in 2^{<\mathbb{N}}}$ is equivalent to the canonical dense family of $\hat{A}(2^{\mathbb{N}})$.*

Proof of the claim. In order to prove the claim we will isolate a property of the whole family $\{h_s\}_{s\in 2^{<\mathbb{N}}}$ (property (Q) below). Let S be an arbitrary regular dyadic subtree of $2^{<\mathbb{N}}$. Notice that $g_\sigma \in \overline{\{h_s\}}^p_{s\in S}$ for every $\sigma \in [\hat{S}]$. By part (2) of Claim 1, we see that the function f belongs to the pointwise closure of $\{h_s\}_{s\in S}$ in $\mathbb{R}^X$. By the Bourgain–Fremlin–Talagrand theorem, there exists an infinite subset A of S such that the sequence $(h_s)_{s\in A}$ is pointwise convergent to f. By part (1) of Claim 1, we see that A can be chosen to be an antichain converging to some $\sigma \in [\hat{S}]$. Since all these facts hold for every regular dyadic subtree S of $2^{<\mathbb{N}}$, we obtain the following property of the family $\{h_s\}_{s\in 2^{<\mathbb{N}}}$.

(Q) For every regular dyadic subtree S of $2^{<\mathbb{N}}$ there exist two antichains A_1, A_2 of S and $\sigma_1, \sigma_2 \in [\hat{S}]$ with $\sigma_1 \neq \sigma_2$ such that A_1 converges to σ_1, A_2 converges to σ_2 and the sequences $(h_s)_{s\in A_1}$ and $(h_s)_{s\in A_2}$ are both pointwise convergent to f.

Now let $T = (s_t)_{t\in 2^{<\mathbb{N}}}$ be the regular dyadic subtree of $2^{<\mathbb{N}}$ such that the family $\{h_{s_t}\}_{t\in 2^{<\mathbb{N}}}$ is canonized. Invoking property (Q) above and referring to the description of the families $\{d^i_t : t \in 2^{<\mathbb{N}}\}$ $(1 \leqslant i \leqslant 7)$ in Subsection 4.3, we see that $\{h_{s_t}\}_{t\in 2^{<\mathbb{N}}}$ must be equivalent either to the canonical dense family of $A(2^{<\mathbb{N}})$ or the canonical dense family of $\hat{A}(2^{\mathbb{N}})$. By part (1) of Claim 1, the first case is impossible. It follows that $\{h_{s_t}\}_{t\in 2^{<\mathbb{N}}}$ must be equivalent to the canonical dense family of $\hat{A}(2^{\mathbb{N}})$ and the claim is proved. □

Let $T = (s_t)_{t\in 2^{<\mathbb{N}}}$ and $\{h_{s_t}\}_{t\in 2^{<\mathbb{N}}}$ be as above. Observe that for every $t \in 2^{<\mathbb{N}}$ there exists a unique $n_t \in \mathbb{N}$ with $h_{s_t} = f_{n_t}$. Thus, by passing to dyadic subtree of T if necessary and invoking the minimality of the canonical dense family of $\hat{A}(2^{\mathbb{N}})$, we see that the function $2^{<\mathbb{N}} \ni t \mapsto n_t \in \mathbb{N}$ is a canonical injection and, moreover, the map

$$\hat{A}(2^{\mathbb{N}}) \ni v_t \mapsto f_{n_t} \in \mathcal{K}$$

is extended to homeomorphism Φ between $\hat{A}(2^{\mathbb{N}})$ and $\overline{\{f_{n_t}\}}^p_{t\in 2^{<\mathbb{N}}}$. The fact that this homeomorphism sends 0 to f is an immediate consequence of property (Q) in Claim 2 above. Moreover, by part (1) of Claim 1, we see that $\Phi(\delta_\sigma) \in \mathcal{C}$ for every $\sigma \in 2^{\mathbb{N}}$. The proof of the theorem is completed. □

By Theorem 40 and part (1) of Proposition 24, we obtain the following corollary.

Corollary 45. *Let $\mathcal{K}$ be a separable Rosenthal compact on a Polish space X, let $\{f_n\}$ be a countable dense subset of $\mathcal{K}$ and let $f \in \mathcal{K}$. If f is a non-G_δ point of $\mathcal{K}$, then there exists a canonical homeomorphic embedding of $\hat{A}(2^{\mathbb{N}})$ into $\mathcal{K}$ with respect to $\{f_n\}$ which sends 0 to f.*

After a first draft of the present paper, Todorčević informed us ([To3]) that he is aware of the above corollary with a proof based on his approach in [To1].

We notice that if $\mathcal{K}$ is a non-metrizable separable Rosenthal compact on a Polish space X, then the constant function 0 is a non-G_δ point of $\mathcal{K} - \mathcal{K}$. Indeed, since $\mathcal{K}$ is non-metrizable, for every $D \subseteq X$ countable there exist $f, g \in \mathcal{K}$ with $f \neq g$ and such that $f|_D = g|_D$. This easily yields that 0 is a non-G_δ point of $\mathcal{K} - \mathcal{K}$. By Corollary 45, we see that there exists a homeomorphic embedding of $\hat{A}(2^{\mathbb{N}})$ into $\mathcal{K} - \mathcal{K}$ with 0 as the unique non-G_δ point of its image. This fact can be lifted to the class of analytic subspaces as follows.

Corollary 46. *Let $\mathcal{K}$ be a separable Rosenthal compact and $\mathcal{C}$ an analytic subspace of $\mathcal{K}$ which is non-metrizable. Also let $D = \{f_n\}$ be a countable dense subset of $\mathcal{K}$ witnessing the analyticity of $\mathcal{C}$. Then there exists a family $\{f_t\}_{t\in 2^{<\mathbb{N}}} \subseteq D - D$ which is equivalent to the canonical dense family of $\hat{A}(2^{\mathbb{N}})$. Moreover, we have $Acc\big(f_t : t \in 2^{<\mathbb{N}}\}\big) \subseteq \mathcal{C} - \mathcal{C}$ and the constant function 0 is the unique non-G_δ point of $\overline{\{f_t\}}^p_{t\in 2^{<\mathbb{N}}}$.*

Proof. Let $\{g_n\}$ be an enumeration of the set $D-D$ which is dense in $\mathcal{K}-\mathcal{K}$. It is easy to see that $\mathcal{C}-\mathcal{C}$ is analytic subspace of $\mathcal{K}-\mathcal{K}$ and this is witnessed by the family $\{g_n\}$. Moreover, by the fact that $\mathcal{C}$ is non-metrizable, we see that the constant function 0 belongs to $\mathcal{C}-\mathcal{C}$ and is a non-G_δ point of $\mathcal{C}-\mathcal{C}$. By Theorem 40, the result follows. □

8. Connections with Banach space theory

This section is devoted to applications, motivated by the results obtained in [ADK2], of the embedding of $\hat{A}(2^{\mathbb{N}})$ in analytic subspaces of separable Rosenthal compacta containing 0 as a non-G_δ point. The first application concerns the existence of unconditional families. The second deals with spreading and level unconditional tree bases.

8.1. **Existence of unconditional families.** We recall that a family $(x_i)_{i\in I}$ in a Banach space X is said to be 1-*unconditional* if for every $F\subseteq G\subseteq I$ and every $(a_i)_{i\in G}\in\mathbb{R}^G$ we have

$$\Big\|\sum_{i\in F}a_ix_i\Big\|\leqslant\Big\|\sum_{i\in G}a_ix_i\Big\|.$$

We will need the following reformulation of Theorem 4 in [ADK2] where we also refer the reader for a proof.

Theorem 47. *Let X be a Polish space and let $\{f_\sigma:\sigma\in 2^{\mathbb{N}}\}$ be a bounded family of real-valued functions on X which is pointwise discrete and having the constant function 0 as the unique accumulation point in $\mathbb{R}^X$. Assume, moreover, that the map $\Phi\colon 2^{\mathbb{N}}\times X\to\mathbb{R}$ defined by $\Phi(\sigma,x)=f_\sigma(x)$ is Borel. Then there exists a perfect subset P of $2^{\mathbb{N}}$ such that the family $\{f_\sigma:\sigma\in P\}$ is 1-unconditional in the supremum norm.*

In [ADK2] it is shown that if X is a separable Banach space not containing ℓ_1 and with non-separable dual, then X^{**} contains an 1-unconditional family of the size of the continuum. This result can be lifted to the setting of separable Rosenthal compacta as follows.

Theorem 48. *Let $\mathcal{K}$ be a separable Rosenthal compact on a Polish space X. Also let $\mathcal{C}$ be an analytic subspace of $\mathcal{K}$ consisting of bounded functions.*

(a) *If $\mathcal{C}$ contains the function 0 as a non-G_δ point, then there exists a family $\{f_\sigma:\sigma\in 2^{\mathbb{N}}\}$ in $\mathcal{C}$ which is 1-unconditional in the supremum norm, pointwise discrete and having 0 as unique accumulation point.*

(b) *If $\mathcal{C}$ is non-metrizable, then there exists a family $\{f_\sigma-g_\sigma:\sigma\in 2^{\mathbb{N}}\}$, where $f_\sigma,g_\sigma\in\mathcal{C}$ for every $\sigma\in 2^{\mathbb{N}}$, which is 1-unconditional in the supremum norm.*

Proof. (a) Let $D=\{f_n\}$ be a countable dense subset of $\mathcal{K}$ witnessing the analyticity of $\mathcal{C}$. As 0 is a non-G_δ point of $\mathcal{C}$, by Theorem 40, there exists a family $\{f_t\}_{t\in 2^{<\mathbb{N}}}\subseteq D$

equivalent to the canonical dense family of $\hat{A}(2^{\mathbb{N}})$ with $Acc\big(\{f_t : t \in 2^{<\mathbb{N}}\}\big) \subseteq \mathcal{C}$ and such that the constant function 0 is the unique non-G_δ point of $\overline{\{f_t\}}^p_{t\in 2^{<\mathbb{N}}}$. For every $\sigma \in 2^{\mathbb{N}}$ let f_σ be the pointwise limit of the sequence $(f_{\sigma|n})$. Clearly the family $\{f_\sigma : \sigma \in 2^{\mathbb{N}}\}$ is pointwise discrete and has 0 as the unique accumulation point. Moreover, it is easy to see that the map $\Phi\colon 2^{\mathbb{N}} \times X \to \mathbb{R}$ defined by $\Phi(\sigma, x) = f_\sigma(x)$ is Borel. By Theorem 47, the result follows.

(b) It follows by Corollary 46 and Theorem 47. □

Actually, we can strengthen the properties of the family $\{f_\sigma : \sigma \in 2^{\mathbb{N}}\}$ obtained by part (a) of Theorem 48 as follows.

Theorem 49. *Let $\mathcal{K}$ be a separable Rosenthal compact on a Polish space X and let $\mathcal{C}$ be an analytic subspace of $\mathcal{K}$ consisting of bounded functions. Assume that $\mathcal{C}$ contains the function 0 as a non-G_δ point. Then there exist a family $\{(g_\sigma, x_\sigma) : \sigma \in 2^{\mathbb{N}}\} \subseteq \mathcal{C} \times X$ and $\varepsilon > 0$ satisfying $|g_\sigma(x_\sigma)| > \varepsilon$ and $g_\sigma(x_\tau) = 0$ if $\sigma \neq \tau$, and such that the family $\{g_\sigma : \sigma \in 2^{\mathbb{N}}\}$ is 1-unconditional in the supremum norm and has 0 as the unique accumulating point.*

Proof. Let $\{f_\sigma : \sigma \in 2^{\mathbb{N}}\} \subseteq \mathcal{C}$ be the family obtained by part (a) of Theorem 48. Notice that, by the proof of Theorem 48, we have that the map $\Phi\colon 2^{\mathbb{N}} \times X \to \mathbb{R}$ defined by $\Phi(\sigma, x) = f_\sigma(x)$ is Borel. Using this fact and by passing to a perfect subset of $2^{\mathbb{N}}$ if necessary, we may select $\varepsilon > 0$ such that $\|f_\sigma\|_\infty > \varepsilon$ for every $\sigma \in 2^{\mathbb{N}}$. Define $N \subseteq 2^{\mathbb{N}} \times X$ by the rule

$$(\sigma, z) \in N \Leftrightarrow |f_\sigma(z)| > \varepsilon.$$

The set N is Borel since the map Φ is Borel. Moreover, by the choice of ε, we have that for every $\sigma \in 2^{\mathbb{N}}$ the section $N_\sigma = \{z : (\sigma, z) \in N\}$ of N at σ is nonempty. By the Yankov–von Neumann uniformization theorem (see, e.g., [Ke, Theorem 18.1]), there exists a map

$$2^{\mathbb{N}} \ni \sigma \mapsto z_\sigma \in X$$

which is measurable with respect to the σ-algebra generated by the analytic sets and such that $(\sigma, z_\sigma) \in N$ for every $\sigma \in 2^{\mathbb{N}}$. Invoking the classical fact that analytic sets have the Baire property, by [Ke, Theorem 8.38] and by passing to a further perfect subset of $2^{\mathbb{N}}$ if necessary, we may assume that the map $\sigma \mapsto z_\sigma$ is continuous.

For every $m \in \mathbb{N}$ define $A_m \subseteq 2^{\mathbb{N}} \times 2^{\mathbb{N}}$ by setting

$$(\sigma, \tau) \in A_m \Leftrightarrow |f_\tau(z_\sigma)| > \frac{1}{m+1}.$$

Notice that the set A_m is Borel. Since the family $\{f_\sigma : \sigma \in 2^{\mathbb{N}}\}$ accumulates to 0, for every $\sigma \in 2^{\mathbb{N}}$ the section $(A_m)_\sigma = \{\tau : (\sigma, \tau) \in A_m\}$ of A_m at σ is finite, and so, it is meager in $2^{\mathbb{N}}$. By the Kuratowski–Ulam theorem (see [Ke, Theorem 8.41]), we see that the set A_m is meager in $2^{\mathbb{N}} \times 2^{\mathbb{N}}$. Hence, so is the set

$$A := \bigcup_{m\in\mathbb{N}} A_m.$$

By a result of Mycielski (see, e.g., [Ke, Theorem 19.1]), there exists a perfect subset P of $2^{\mathbb{N}}$ such that for every $\sigma,\tau \in P$ with $\sigma \neq \tau$ we have that $(\sigma,\tau) \notin A$ and $(\tau,\sigma) \notin A$. This implies that $f_\tau(z_\sigma)=0$ and $f_\sigma(z_\tau)=0$. We fix a homeomorphism $h\colon 2^{\mathbb{N}} \to P$ and we set $g_\sigma = f_{h(\sigma)}$ and $x_\sigma = z_{h(\sigma)}$ for every $\sigma \in 2^{\mathbb{N}}$. Clearly the family $\{(g_\sigma, x_\sigma) : \sigma \in 2^{\mathbb{N}}\}$ is as desired. $\square$

The proof of the corresponding result in [ADK2] is based on Ramsey and Banach space tools, avoiding the embedding of $\hat{A}(2^{\mathbb{N}})$ into $(B_{X^{**}}, w^*)$.

We recall that a Banach space X is said to be *representable* if X isomorphic to a subspace of $\ell_\infty(\mathbb{N})$ which is analytic in the weak* topology (see [GT, GL, AGR]). We close this subsection with the following theorem.

Theorem 50. *Let X be a non-separable representable Banach space. Then X^* contains an unconditional family of size $|X^*|$.*

Proof. Identify X with its isomorphic copy in $\ell_\infty(\mathbb{N})$. Then B_X is an analytic subset of (B_{ℓ_∞}, w^*). Let $f\colon \mathbb{N}^{\mathbb{N}} \to B_X$ be an onto continuous map. Let $\{x_n\}$ be a norm dense subset of $\ell_1(\mathbb{N})$. Viewing ℓ_1 as a subspace of ℓ_∞^*, we define $f_n\colon \mathbb{N}^{\mathbb{N}} \to \mathbb{R}$ by setting $f_n = x_n \circ f$. Then $\{f_n\}$ is a uniformly bounded sequence of continuous real-valued functions on $\mathbb{N}^{\mathbb{N}}$. Notice that $\overline{\{f_n\}}^p = \{x^* \circ f : x^* \in B_{X^*}\}$ which can be naturally identified with $\{x^*|_{B_X} : x^* \in B_{X^*}\}$. By the non-effective version of Debs' theorem (see, e.g., [AGR]), one of the following cases must occur.

Case 1: *There exist an increasing sequence (n_k), a continuous map $\phi\colon 2^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ and real numbers $a<b$ such that for every $\sigma \in 2^{\mathbb{N}}$ and every $k \in \mathbb{N}$ if $\sigma(k)=0$ then $f_{n_k}\big(\phi(\sigma)\big) < a$, while if $\sigma(k)=1$, then $f_{n_k}\big(\phi(\sigma)\big) > b$.* In this case for every $p \in \beta\mathbb{N}$ we set

$$g_p \coloneqq p - \lim f_{n_k}.$$

Then $g_p = x_p^*|_{B_X}$ for some $x_p^* \in X^*$. We claim that the family $\{x_p^* : p \in \beta\mathbb{N}\}$ is equivalent to the natural basis of $\ell_1(2^{\mathfrak{c}})$. To see this observe that $g_p\big(\phi(\sigma)\big) \leqslant a$ if and only if $\{k : \sigma(k)=0\} \in p$, and $g_p\big(\phi(\sigma)\big) \geqslant b$ if and only if $\{k : \sigma(k)=1\} \in p$. Setting $A_p \coloneqq [g_p \leqslant a]$ and $B_p \coloneqq [g_p \geqslant b]$ for every $p \in \beta\mathbb{N}$, we see that the family $(A_p, B_p)_{p\in\beta\mathbb{N}}$ is an independent family of disjoint pairs. By Rosenthal's criterion, the family $\{g_p : p \in \beta\mathbb{N}\}$ is equivalent to $\ell_1(2^{\mathfrak{c}})$. Thus, so is the family $\{x_p^* : p \in \beta\mathbb{N}\}$.

Case 2: *The sequence $\{f_n\}$ is relatively compact in $\mathcal{B}_1(\mathbb{N}^{\mathbb{N}})$.* In this case since X is non-separable, we see that $0 \in \overline{\{f_n\}}^p$ is a non-G_δ point. By part (a) of Theorem 48, there exists an 1-unconditional family in X^* of the size of the continuum. $\square$

It can also be shown that every representable Banach space has a separable quotient (see [ADK2, Theorem 15]). For further applications of the existence of unconditional families we refer the reader to [ADK2].

8.2. **Spreading and level unconditional tree bases.** We start with the following definition.

Definition 51. *Let X be a Banach space.*

(1) *A* tree basis *is a bounded family $\{x_t\}_{t\in 2^{<\mathbb{N}}}$ in X which is a basic sequence when enumerated according to the canonical bijection ϕ_0 between $2^{<\mathbb{N}}$ and $\mathbb{N}$.*
(2) *A tree basis $\{x_t\}_{t\in 2^{<\mathbb{N}}}$ is said to be* spreading *if there exists a sequence $(\varepsilon_n)\searrow 0$ such that for every $n,m\in\mathbb{N}$ with $n<m$, every $0\leqslant d<2^n$ and every pair $\{s_i\}_{i=0}^d\subseteq 2^n$ and $\{t_i\}_{i=0}^d\subseteq 2^m$ with $s_i\sqsubset t_i$ for all $i\in\{0,\dots,d\}$, we have $\|T\|\cdot\|T^{-1}\|<1+\varepsilon_n$ where*
$$T\colon \mathrm{span}\{x_{s_i}: i=0,\dots,d\}\to \mathrm{span}\{x_{t_i}: i=0,\dots,d\}$$
is the natural one-to-one and onto linear operator.
(3) *A tree basis $\{x_t\}_{t\in 2^{<\mathbb{N}}}$ is said to be* level unconditional *if there exists a sequence $(\varepsilon_n)\searrow 0$ such that for every $n\in\mathbb{N}$ the family $\{x_t: t\in 2^n\}$ is $(1+\varepsilon_n)$-unconditional.*

In [ADK2] the existence of spreading and level unconditional tree bases was established for every separable Banach space X not containing ℓ_1 and with non-separable dual. This result can be extended in the context of separable Rosenthal compacta as follows.

Theorem 52. *Let $\mathcal{K}$ be a uniformly bounded separable Rosenthal compact on a compact metrizable space X and having a countable dense subset D of continuous functions. Also let (ε_n) be a decreasing sequence of positive reals with $\lim\varepsilon_n=0$. Assume that the constant function 0 is a non-G_δ point of $\mathcal{K}$. Then there exists a family $\{u_t\}_{t\in 2^{<\mathbb{N}}}\subseteq \mathrm{conv}(D)$ which is equivalent to the canonical dense family of $\hat{A}(2^{\mathbb{N}})$ and such that, setting $g_\sigma:=\lim u_{\sigma|n}$ for every $\sigma\in 2^{\mathbb{N}}$, the following hold.*

(1) *The function 0 is the unique non-G_δ point of $\overline{\{u_t\}}^p_{t\in 2^{<\mathbb{N}}}$.*
(2) *The family $\{u_t\}_{t\in 2^{<\mathbb{N}}}$ is a tree basis with respect to the supremum norm.*
(3) *The family $\{g_\sigma:\sigma\in 2^{\mathbb{N}}\}$ is a subset of $\mathcal{K}$ and 1-unconditional.*
(4) *For every $n\in\mathbb{N}$ if $\{t_0\prec\cdots\prec t_{2^n-1}\}$ is the $\prec$-increasing enumeration of 2^n, then for every collection $\{\sigma_0,\dots,\sigma_{2^n-1}\}\subseteq 2^{\mathbb{N}}$ with $t_i\sqsubset\sigma_i$ for all $i\in\{0,\dots,2^n-1\}$ we have that $(g_{\sigma_i})_{i=0}^{2^n-1}$ is $(1+\varepsilon_n)$-equivalent to $(u_{t_i})_{i=0}^{2^n-1}$.*

The proof of the above result is a slight modification of [ADK2, Theorem 17] where we also refer the reader for more information.

We close this subsection with the following result whose proof is based on Stegall's construction [St].

Theorem 53. *Let X be a Banach space such that X^* is separable and X^{**} is non-separable. Also let $\varepsilon>0$. Then there exists a family $\{u_t\}_{t\in 2^{<\mathbb{N}}}\subseteq B_X$ such that the following are satisfied.*

(i) *The family $\{u_t\}_{t\in 2^{<\mathbb{N}}}$ is equivalent to the canonical dense family of $2^{\leqslant\mathbb{N}}$.*

(ii) *For every $\sigma \in 2^{\mathbb{N}}$ if y^{**}_σ is the weak* limit of the sequence $(u_{\sigma|n})$, then there exists $y^{***}_\sigma \in X^{***}$ with $\|y^{***}_\sigma\| \leqslant 1+\varepsilon$ and such that $y^{***}_\sigma(y^{**}_\sigma)=1$, while $y^{***}_\sigma(y^{**}_\tau)=0$ for every $\tau \neq \sigma$.*

(iii) *For every $n \in \mathbb{N}$ if $\{t_0 \prec \cdots \prec t_{2^n-1}\}$ is the $\prec$-increasing enumeration of 2^n, then for every collection $\{\sigma_0,\dots,\sigma_{2^n-1}\} \subseteq 2^{\mathbb{N}}$ with $t_i \sqsubset \sigma_i$ for all $i \in \{0,\dots,2^n-1\}$ we have that $(y^{**}_{\sigma_i})_{i=0}^{2^n-1}$ is $(1+\frac{1}{n})$-equivalent to $(u_{t_i})_{i=0}^{2^n-1}$.*

Proof. Since X^* is separable, the space $(B_{X^{**}}, w^*)$ is compact metrizable. We fix a compatible metric ρ for $(B_{X^{**}}, w^*)$. Using Stegall's construction [St], we obtain

(C1) a family $\{x^*_t\}_{t\in 2^{<\mathbb{N}}} \subseteq X^*$, and
(C2) a family $\{B_t\}_{t\in 2^{<\mathbb{N}}}$ of open subsets of $(B_{X^{**}}, w^*)$

such that for every $t \in 2^{<\mathbb{N}}$ the following properties are satisfied.

(P1) We have $1 < \|x^*_t\| < 1+\varepsilon$.
(P2) We have $\overline{B}_{t^\frown 0} \cap \overline{B}_{t^\frown 1} = \emptyset$, $\overline{B}_{t^\frown 0} \cup \overline{B}_{t^\frown 1} \subseteq B_t$ and $\rho-\mathrm{diam}(B_t) \leqslant \frac{1}{|t|+1}$.
(P3) For every $x^{**} \in B_t$ we have $|x^{**}(x^*_t)-1| < \frac{1}{|t|+1}$.
(P4) For every $t' \neq t$ with $|t|=|t'|$ and every $x^{**} \in B_{t'}$ we have $|x^{**}(x^*_t)| < \frac{1}{|t|+1}$.

By property (P2), we see that $\bigcap_n B_{\sigma|n} = \{x^{**}_\sigma\}$ for every $\sigma \in 2^{\mathbb{N}}$ and the map $2^{\mathbb{N}} \ni \sigma \mapsto x^{**}_\sigma \in (B_{X^{**}}, w^*)$ is a homeomorphic embedding. By Goldstine's theorem, for every $t \in 2^{<\mathbb{N}}$ we may select $x_t \in B_t \cap X$. Notice that $w^*-\lim x_{\sigma|n} = x^{**}_\sigma$ for every $\sigma \in 2^{\mathbb{N}}$. For every $\sigma \in 2^{\mathbb{N}}$ we select $x^{***}_\sigma \in \bigcap_n \overline{\{x^*_{\sigma|k} : k \geqslant n\}}^{w^*}$. By property (P3), we see that $x^{***}_\sigma(x^{**}_\sigma)=1$ while, by property (P4), we have $x^{***}_\sigma(x^{**}_\tau)=0$ for every $\tau \neq \sigma$. Moreover,

$$\sup\{|\lambda_i| : i=0,\dots,n\} \leqslant (1+\varepsilon)\,\Big\| \sum_{i=0}^{n} \lambda_i x^{**}_{\sigma_i} \Big\|$$

for every $n \in \mathbb{N}$, every $\{\sigma_0,\dots,\sigma_n\} \subseteq 2^{\mathbb{N}}$ and every $(\lambda_i)_{i=0}^n \in \mathbb{R}^{n+1}$. Arguing as in the proof of [ADK2, Theorem 17], we select a family $\{u_t\}_{t\in 2^{<\mathbb{N}}} \subseteq \mathrm{conv}\{x_t : t \in 2^{<\mathbb{N}}\}$ and a regular dyadic subtree $S=(s_t)_{t\in 2^{<\mathbb{N}}}$ of $2^{<\mathbb{N}}$ such that the following are satisfied.

(1) For every $\sigma \in 2^{\mathbb{N}}$ the sequence $(u_{\sigma|n})$ is weak* convergent to y^{**}_σ where
$$y^{**}_\sigma := \lim x_{s_{\sigma|n}}.$$
(2) For every $n \in \mathbb{N}$ if $\{t_0 \prec \cdots \prec t_{2^n-1}\}$ is the $\prec$-increasing enumeration of 2^n, then for every $\{\sigma_0,\dots,\sigma_{2^n-1}\} \subseteq 2^{\mathbb{N}}$ with $t_i \sqsubset \sigma_i$ for all $i \in \{0,\dots,2^n-1\}$ we have that $(y^{**}_{\sigma_i})_{i=0}^{2^n-1}$ is $(1+\frac{1}{n})$-equivalent to $(u_{t_i})_{i=0}^{2^n-1}$.

For every $\sigma \in 2^{\mathbb{N}}$ let $\bar{\sigma} = \bigcup_n s_{\sigma|n} \in 2^{\mathbb{N}}$. Setting $y^{***}_\sigma = x^{***}_{\bar{\sigma}}$ for every $\sigma \in 2^{\mathbb{N}}$, we see that properties (ii) and (iii) in the statement of the theorem are satisfied. Finally, by passing to a regular dyadic subtree if necessary, we may also assume that the family $\{u_t\}_{t\in 2^{<\mathbb{N}}}$ is equivalent to the canonical dense family of $2^{\leqslant\mathbb{N}}$, that is, property (i) is satisfied. The proof is completed. □

Remark 4. (1) We do not know if the family $\{u_t\}_{t\in 2^{<\mathbb{N}}}$ obtained by Theorem 53 can be chosen to be basic or an FDD. Also note that it seems to be unknown whether for every Banach space X with X^* separable and X^{**} non-separable, there exists a subspace Y of X with a Schauder basis such that Y^{**} is non-separable.

(2) The family $\{y^{**}_\sigma : \sigma \in 2^{\mathbb{N}}\}$ obtained by Theorem 53 cannot be chosen to be unconditional as the examples of non-separable HI spaces show (see [AAT, AT]). However, all these second duals, non-separable HI spaces have quotients with separable kernel which contain unconditional families of the cardinality of the continuum. The following problem is motivated by the previous observation.

Problem. Let X be a separable Banach space with X^{**} non-separable. Does there exist a quotient Y of X^{**} containing an unconditional family of size $|X^{**}|$?

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National Technical University of Athens, Faculty of Applied Sciences, Department of Mathematics, Zografou Campus, 157 80, Athens, Greece.

E-mail address: `sargyros@math.ntua.gr, pdodos@math.ntua.gr, bkanel@math.ntua.gr`