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A CLASSIFICATION OF SIMPLE LIE MODULES HAVING A 1-DIMENSIONAL WEIGHT SPACE

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ABSTRACT. Let L denote a simple Lie algebra over the complex numbers. In this paper, we classify and construct all simple L modules which may be infinite dimensional but have at least one 1-dimensional weight space. This completes the study begun earlier by the authors for the case of $L = A_n$. The approach presented here relies heavily on the results of Suren Fernando whose dissertation dealt with simple weight modules and their weight systems.

0. Introduction. Let L be a finite-dimensional simple Lie algebra over the complex field $\mathbf{C}$ having a Cartan subalgebra H and denote by $C(L)$ the centralizer of H in the universal enveloping algebra U of L. If λ: $H \to \mathbf{C}$ is a weight function of a 1-dimensional weight space M_λ in a simple L module M then η: $C(L) \to \mathbf{C}$, defined by $\eta(c)v = cv$ for $v \in M_\lambda$ and $c \in C(L)$, is an algebra homomorphism called a mass function of M. Clearly η restricted to H is equal to λ. Conversely, given any algebra homomorphism η: $C(L) \to \mathbf{C}$ one can construct a unique simple L module which admits η as a mass function **[4, 10]**. In **[4]** the authors determined all algebra homomorphisms η: $C(L) \to \mathbf{C}$ for the simple Lie algebras L of type A_n and using these classified all "pointed" A_n modules (where we call a module pointed if it is simple and has at least one 1-dimensional weight space). In this paper we complete the classification of all pointed L modules for arbitrary simple Lie algebras. The collection of all pointed L modules clearly includes the highest weight L modules and is included in the collection of all Harish-Chandra L modules relative to the Cartan subalgebra H (-i.e. simple L modules having a weight space decomposition and finite-dimensional weight spaces relative to H). The latter inclusion is strict since there exist examples of Harish-Chandra A_2 modules in which every weight space is two dimensional. In the special case of A_1 modules, every Harish-Chandra A_1 module is pointed.

Our approach to this problem makes heavy use of the results from Fernando's thesis **[8]**. In particular, from Fernando's results we see that in place of determining all algebra homomorphisms η: $C(L) \to \mathbf{C}$ it suffices to find only those algebra homomorphisms η which are associated with so-called pointed "torsion free" L

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modules (these are defined in §1) and further that such modules can only occur for simple Lie algebras of type A_n or C_n.

In §1 we outline the relevant results from Fernando's thesis and other background material which will be used in this paper. §2 contains the construction of canonical examples of pointed torsion free A_n and C_n modules. The main results of this paper are contained in §§3 and 4 where we show that the examples given in §2 exhaust all pointed torsion free A_n and C_n modules and hence by Fernando's results exhaust all such modules. Finally in §5 we provide an alternate construction of all pointed L modules by constructing a mass function associated with each one. This allows us to realize all pointed L modules directly as quotient modules of the universal enveloping algebra $U(L)$.

1. Structure of simple weight modules. In this section, we review the results of Fernando [**8**] on simple weight modules, and establish some basic facts about pointed modules. Fernando's results reduce the classification of pointed modules of simple Lie algebras to the classification of pointed, torsion free modules of the simple Lie algebras of types A and C.

Throughout this section L denotes a reductive Lie algebra, H a Cartan subalgebra of L, R the root system of (L, H), B a base of R, R_B^+ the positive roots relative to B, and for $B_1 \subseteq B$, $\langle B_1 \rangle$ denotes the integer linear span of B_1 intersected with R. Also, we express the root space decomposition of L by $L = H \oplus \sum_{\alpha \in R} \oplus L_\alpha$, and denote a root vector of L belonging to $\alpha \in R$ by X_α. For notational convenience, occasionally we use Y_α to denote $X_{-\alpha}$.

DEFINITION 1.1. M is an (L, H) *weight module* provided

(i) M is a finitely generated L module, and

(ii) $M = \sum_{\lambda \in H^*} \oplus M_\lambda$ where for $\lambda \in H^*$, $M_\lambda = \{v \in M \mid hv = \lambda(h)v$ for all $h \in H\}$ is finite dimensional.

DEFINITION 1.2. Let M be an (L, H) weight module. M is *torsion free* provided the action of X_α on M, X_α: $M \to M$, is injective for all $\alpha \in R$. M is said to be *pointed* provided it is simple and has a 1-dimensional weight space.

THEOREM 1.3 [FERNANDO]. *If L is a finite-dimensional complex reductive Lie algebra which admits a nonzero, torsion free weight module then the simple ideals of L are of two possible types; type A and type C.*

Let M be an arbitrary simple (L, H) weight module. Let $T(M)$ be the set of all roots $\alpha \in R$ such that X_α: $M \to M$ is injective. As shown in [**8**], for some base B of R and some subset B_1 of B the parabolic subset $P = \langle B_1 \rangle \cup R_B^+$ is related to $T(M)$ by the following two properties:

(i) $P \cap (-P) = T(M) \cap (-T(M))$, and

(ii) $T(M) \cap (R \setminus -T(M)) \subseteq P \cap (R \setminus -P)$.

Setting $P_s = P \cap (-P)$ and $P_a = P \cap (R \setminus -P)$, we define the following subalgebras of L:

$$p^- = H \oplus \sum_{\alpha \in P} \oplus L_\alpha, \quad g = H \oplus \sum_{\alpha \in P_s} \oplus L_\alpha, \quad u^- = \sum_{\alpha \in P_a} \oplus L_\alpha,$$

$$u^+ = \sum_{\alpha \in -P_a} \oplus L_\alpha, \quad p^+ = g \oplus u^+.$$

DEFINITION 1.4. $M^{u^+} = \{v \in M \mid u^+ v = 0\}$.

THEOREM 1.5 [FERNANDO]. *Let $W(M)$ denote the set of weights of M and $\mathbf{Z}$ the set of integers. If $\alpha \in R$ and $\lambda \in W(M)$ then $(\lambda + \mathbf{Z}\alpha) \cap W(M)$ is an interval and X_α is locally nilpotent if and only if $(\lambda + \mathbf{Z}^+\alpha) \cap W(M)$ is finite. Moreover, if X_α is not locally nilpotent then it is torsion free.*

THEOREM 1.6 [FERNANDO]. *M^{u^+} is a nonzero simple, torsion free (g, H) weight module. Moreover, M^{u^+} is equal to a sum of complete weight spaces of M.*

THEOREM 1.7 [FERNANDO]. *M is equivalent to the unique simple quotient of the (L, H) weight module $U(L) \otimes_{U(p^+)} M^{u^+}$.*

We close this section with an outline of some basic results on pointed, torsion free L modules M which are used in the subsequent sections of this paper. In particular, it is clear that all nonzero weight spaces of M are 1 dimensional and for $\mu_i \in R$ the monomial $X_{\mu_1} \cdots X_{\mu_k}$ acts bijectively on M mapping M_λ onto $M_{\lambda + \mu_1 + \cdots + \mu_k}$. Therefore, it follows that if μ, ν, $\mu + \nu \in R$ and M_λ is a nonzero weight space of M, there exists a nonzero scalar $K_\lambda(\mu, \nu)$ such that $(X_\mu X_\nu - K_\lambda(\mu, \nu) X_{\mu+\nu}) M_\lambda = \{0\}$. If $C(L)$ denotes the cycle subalgebra of $U(L)$, i.e. the centralizer of the Cartan subalgebra H in the universal enveloping algebra $U(L)$ of L, then each weight function $\lambda \in H^*$ of M can be extended uniquely to an algebra homomorphism η: $C(L) \to \mathbf{C}$ referred to as a mass function of M. Its action is defined by $cv_\lambda = \eta(c) v_\lambda$ for all $c \in C(L)$. Evidently, the existence of this algebra homomorphism requires only that $\dim M_\lambda = 1$ and not that M is torsion free.

THEOREM 1.8 **[4, 10]**. *A pointed module M is determined, up to equivalence, by any of its mass functions.*

In the case of pointed, torsion free modules we can strengthen this result to read

THEOREM 1.9. *Let M_1 and M_2 be pointed, torsion free L modules admitting a common weight function λ. Let η_1 and η_2 be mass functions for M_1 and M_2 with respect to λ such that $\eta_1(c) = \eta_2(c)$ for all monomials $c \in C(L)$ of degree less than or equal to three. Then M_1 is equivalent to M_2.*

PROOF. Let $(M_1)_\lambda = \mathbf{C}v_1$ and $(M_2)_\lambda = \mathbf{C}v_2$. As previously pointed out, if μ, ν, $\mu + \nu \in R$, then there are nonzero complex numbers K_1 and K_2 such that

$$(X_\mu X_\nu - K_1 X_{\mu+\nu}) v_1 = 0$$

and

$$(X_\mu X_\nu - K_2 X_{\mu+\nu}) v_2 = 0.$$

Since

$$\eta_1(X_{-(\mu+\nu)}X_\mu X_\nu) = \eta_2(X_{-(\mu+\nu)}X_\mu X_\nu)$$

and

$$\eta_1(X_{-(\mu+\nu)}X_{\mu+\nu}) = \eta_2(X_{-(\mu+\nu)}X_{\mu+\nu}),$$

we know that $K_1 = K_2$.

Let $c = z_1 \cdots z_k$ be a monomial of degree k in the Poincaré-Birkhoff-Witt basis which is in $C(L)$. We prove $\eta_1(c) = \eta_2(c)$ by induction on k. This is true for $k \leqslant 3$ by assumption. The induction assumption, that $\eta_1(c) = \eta_2(c)$ for all c of degree less than k, permits us to permute the z_i's since such a permutation is done at the expense of adding lower degree terms. Either c is an element in the universal enveloping algebra of H in which case the result is clear or

$$z_1 \cdots z_k = X_{\mu_1} \cdots X_{\mu_{k-2}}X_\mu X_\nu + \text{polynomial of degree } < k$$

where $\mu_1, \ldots, \mu_{k-2}$, μ, ν, $\mu + \nu \in R$. Our induction assumption implies we need only show that

$$\eta_1(X_{\mu_1} \cdots X_{\mu_{k-2}}X_\mu X_\nu) = \eta_2(X_{\mu_1} \cdots X_{\mu_{k-2}}X_\mu X_\nu).$$

But for $i = 1, 2$ we know that

$$\begin{aligned}\eta_i(X_{\mu_1} \cdots X_{\mu_{k-2}}X_\mu X_\nu)v_i = X_{\mu_1} \cdots X_{\mu_{k-2}}X_\mu X_\nu v_i &= K_i X_{\mu_1} \cdots X_{\mu_{k-2}}X_{\mu+\nu}v_i \\ &= K_i\eta_i(X_{\mu_1} \cdots X_{\mu_{k-2}}X_{\mu+\nu})v_i.\end{aligned}$$

Therefore,

$$\begin{aligned}\eta_1(X_{\mu_1} \cdots X_{\mu_{k-2}}X_\mu X_\nu) &= K_1\eta_1(X_{\mu_1} \cdots X_{\mu_{k-2}}X_{\mu+\nu}) = K_2\eta_2(X_{\mu_1} \cdots X_{\mu_{k-2}}X_{\mu+\nu}) \\ &= \eta_2(X_{\mu_1} \cdots X_{\mu_{k-2}}X_\mu X_\nu). \qquad \square\end{aligned}$$

2. Construction of simple torsion free modules. In this section, we construct certain simple, torsion free modules of the Lie algebras of types A and C which we show later exhaust all such modules. These are slight variants of those presented by Fernando in [**8**].

The Weyl algebra W_n of order n can be realized as the associative subalgebra of $\mathrm{End}_{\mathbf{C}}\mathbf{C}[x_1, \ldots, x_n]$ generated by $\{x_i, \delta_i \mid i = 1, \ldots, n\}$ where x_i and δ_i are viewed as left multiplication by x_i and partial differentiation with respect to x_i, respectively. Following [**6**, Chapter 4.6] the simple Lie algebra C_n can be embedded in the subalgebra of degree two elements of W_n as indicated below.

The root system R of C_n can be identified with the set of vectors

$$\{\pm(\varepsilon_i \pm \varepsilon_j) \mid 1 \leqslant i < j \leqslant n\} \cup \{\pm 2\varepsilon_i \mid i = 1, \ldots, n\}$$

in $\mathbf{R}^n$ where ε_i denotes the ith standard basis vector of $\mathbf{R}^n$. A base B for R is given by $B = \{\varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{n-1} - \varepsilon_n, 2\varepsilon_n\}$. For computational purposes throughout this paper, we fix a Chevalley basis of C_n which we call *standard*. This is one of the form

$$\{X_{\pm(\varepsilon_i \pm \varepsilon_j)} \mid 1 \leqslant i < j \leqslant n\} \cup \{X_{\pm 2\varepsilon_i} \mid i = 1, \ldots, n\} \cup \{h_{\alpha_i} \mid \alpha_i \in B\}$$

having the property that the map $\phi: C_n \to W_n$ given by

$$\begin{aligned}
&\phi\left(X_{\varepsilon_i-\varepsilon_j}\right) = x_i\delta_j && \text{for } 1 \leqslant i \neq j \leqslant n,\\
&\phi\left(X_{\varepsilon_i+\varepsilon_j}\right) = x_i x_j && \text{for } i,j = 1,\dots,n,\\
&\phi\left(X_{-(\varepsilon_i+\varepsilon_j)}\right) = \delta_i\delta_j && \text{for } i,j = 1,\dots,n,\\
&\phi\left(h_{\varepsilon_i-\varepsilon_{i+1}}\right) = x_i\delta_i - x_{i+1}\delta_{i+1} && \text{for } i = 1,\dots,n-1,\\
&\phi\left(h_{2\varepsilon_n}\right) = (x_n\delta_n + \delta_n x_n)/2
\end{aligned}$$

is a Lie algebra isomorphism.

A modification of the C_n module $\mathbf{C}[x_1,\dots,x_n]$ produces the module $M(a)$ which we seek. For each n-tuple $a = (a_1,\dots,a_n) \in \mathbf{C}^n$ with $a_i \notin \mathbf{Z}$ for all i, we define $M(a)$ to be the complex linear space spanned by

$$\left\{ x^b = x_1^{b_1} \cdots x_n^{b_n} \,|\, b_i - a_i \in \mathbf{Z} \text{ for all } i \text{ and } \sum_{i=1}^{n} (b_i - a_i) \in 2\mathbf{Z} \right\}.$$

The linear space $M(a)$ can be viewed as a C_n module through the natural action of $\phi(C_n)$ on $M(a)$. One can easily show that distinct $x^b \in M(a)$ form a basis for distinct weight spaces for $M(a)$. The C_n module $M(a)$ is torsion free, since

$$\begin{aligned}
&X_{\varepsilon_i-\varepsilon_j}x^b = b_j x^{b+\varepsilon_i-\varepsilon_j} && \text{for } 1 \leqslant i \neq j \leqslant n,\\
&X_{\varepsilon_i+\varepsilon_j}x^b = x^{b+\varepsilon_i+\varepsilon_j} && \text{for } i,j = 1,\dots,n,\\
&X_{-(\varepsilon_i+\varepsilon_j)}x^b = b_i b_j x^{b-\varepsilon_i-\varepsilon_j} && \text{for } i \neq j = 1,\dots,n,\\
&X_{-2\varepsilon_i}x^b = b_i(b_i - 1)x^{b-2\varepsilon_i} && \text{for } i = 1,\dots,n,
\end{aligned}$$

and none of the b_i's are integers.

The following argument shows that $M(a)$ is a simple C_n module. For x^b, $x^c \in M(a)$ we have $b - c = \sum_{i=1}^n k_i\varepsilon_i$ where $k_i \in \mathbf{Z}$ and $\sum_{i=1}^n k_i \in 2\mathbf{Z}$. Therefore,

$$\begin{aligned}
\sum_{i=1}^{n} k_i\varepsilon_i &= k_1(\varepsilon_1 - \varepsilon_2) + (k_1 + k_2)(\varepsilon_2 - \varepsilon_3)\\
&\quad + \cdots + \left(\sum_{i=1}^{n-1} k_i\right)(\varepsilon_{n-1} - \varepsilon_n) + \left(\frac{1}{2}\right)\left(\sum_{i=1}^{n} k_i\right)2\varepsilon_n
\end{aligned}$$

so that

$$X^{e_1}_{s_1(\varepsilon_1-\varepsilon_2)} \cdots X^{e_n}_{s_n(2\varepsilon_n)} x^c$$

is a nonzero multiple of x^b where $e_i = |\sum_{j=1}^i k_j|$ for $i = 1,\dots,n-1$ and $e_n = \frac{1}{2}|\sum_{j=1}^n k_j|$ and $s_i = \operatorname{sgn}\sum_{j=1}^i k_j$. Summarizing these results we have

THEOREM 2.1. *For each n-tuple $a = (a_1,\dots,a_n) \in \mathbf{C}^n$ with $a_i \notin \mathbf{Z}$ for all i, $M(a)$ is a pointed, torsion free C_n module.*

The root system R of C_n contains a subsystem $\{\pm(\varepsilon_i - \varepsilon_j) | 1 \leqslant i < j \leqslant n\}$ which is equivalent to a root system of the simple Lie algebra A_{n-1}. The basal vectors in a standard base of C_n corresponding to this subsystem form a base for A_{n-1} which we call a standard base for A_{n-1}. When discussing simple Lie algebras of type A, we use this base. For each n-tuple $a = (a_1, \ldots, a_n) \in \mathbf{C}^n$ with $a_i \notin \mathbf{Z}$ for all i, $N(a)$ is defined to be the complex linear space spanned by

$$\left\{ x^b = x_1^{b_1} \cdots x_n^{b_n} \mid b_i - a_i \in \mathbf{Z} \text{ for all } i \text{ and } \sum_{i=1}^{n} (b_i - a_i) = 0 \right\}.$$

$N(a)$ can be viewed as an A_{n-1} module through the natural action of $\phi(A_{n-1})$ on $N(a)$. By an analysis similar to that described for the previous case, we have

THEOREM 2.2. *For each n-tuple $a = (a_1, \ldots, a_n) \in \mathbf{C}^n$ with $a_i \notin \mathbf{Z}$ for all i, $N(a)$ is a pointed, torsion free A_{n-1} module.*

3. Pointed, torsion free A_n modules. As indicated by Theorem 1.8, the determination of pointed modules can be achieved by determining the mass functions defined on $C(L)$. In [**4**], the authors constructed all possible mass functions on $C(A_n)$. In this section, we show that the mass functions of the pointed torsion free A_n modules of the form $N(a)$ constructed in §2 exhaust all mass functions of pointed torsion A_n modules. It follows that every pointed torsion free A_n module is equivalent to one of the form $N(a)$.

First, we treat the case of M being an A_2 module. Let $\{\alpha, \beta\}$ be a base for the root system of A_2. Let $M_\lambda = \mathbf{C}v_\lambda$ be a weight space of M. The basic cycles of $C(A_2)$ are h_α, h_β,

$$c_1 = Y_\alpha X_\alpha, \quad c_2 = Y_\beta X_\beta, \quad c_3 = Y_{\alpha+\beta} X_{\alpha+\beta},$$
$$c_4 = Y_{\alpha+\beta} X_\alpha X_\beta, \quad c_5 = Y_\beta Y_\alpha X_{\alpha+\beta}.$$

Each of the c_i's acts on v_λ to produce a nonzero scalar multiple of v_λ. Define r, s, and z_i for $1 \leqslant i \leqslant 5$ according to $h_\alpha v_\lambda = r v_\lambda$, $h_\beta v_\lambda = s v_\lambda$ and $c_i v_\lambda = z_i v_\lambda$, $1 \leqslant i \leqslant 5$. Since $M_{\lambda+\alpha+\beta}$ is 1 dimensional, we know that there is some $K \in \mathbf{C}$ such that $X_\alpha X_\beta v_\lambda = K X_{\alpha+\beta} v_\lambda$. We show that we can express the z_i's in terms of r, s, and K and hence M is uniquely determined by these values.

LEMMA 3.1. *Let r, s, and K be defined as above, then*

$$z_1 = (K + r)(K - 1), \tag{3.2}$$

$$z_2 = K(K - s - 1), \tag{3.3}$$

$$z_3 = (K + r)(K - s - 1), \tag{3.4}$$

$$z_4 = z_5 = K(K + r)(K - s - 1). \tag{3.5}$$

Moreover, if $\lambda \neq 0$ (i.e. $r \neq 0$ or $s \neq 0$) then the mass function η: $C(A_2) \to \mathbf{C}$ defined by r, s, and z_i, $1 \leqslant i \leqslant 5$, is in fact determined by r, s, and z_i, $1 \leqslant i \leqslant 3$ (i.e. K can be found from these values).

PROOF. One can show that the following identities hold in the universal enveloping algebra of A_2:

$$c_1 c_2 = c_2 c_1 + c_5 - c_4, \tag{3.6}$$

$$c_1 c_4 = c_4 c_1 - c_2 c_1 + c_3 c_1 - c_4 h_\alpha + c_3 h_\alpha - c_5 + c_4, \tag{3.7}$$

$$c_2 c_4 = c_4 c_2 + c_2 c_1 - c_3 c_2 - c_4 h_\beta + c_5 - c_4. \tag{3.8}$$

The definition of r, s, and K and these three identities give us respectively

$$z_4 = z_5, \tag{3.9}$$

$$z_2 z_1 = z_3(z_1 - Kr + r), \tag{3.10}$$

$$z_2 z_1 = z_3(z_2 + Ks). \tag{3.11}$$

The definition of K produces immediately $z_4 = Kz_3$. Also, computing $c_1 c_2 v_\lambda$ we get

$$\begin{aligned} c_1 c_2 v_\lambda &= Y_\alpha X_\alpha Y_\beta X_\beta v_\lambda = Y_\alpha Y_\beta X_\alpha X_\beta v_\lambda = K Y_\alpha Y_\beta X_{\alpha+\beta} v_\lambda \\ &= K\left(Y_\beta Y_\alpha X_{\alpha+\beta} - Y_{\alpha+\beta} X_{\alpha+\beta}\right) v_\lambda = K(c_5 - c_3) v_\lambda \\ &= K(c_4 - c_3) v_\lambda = K(K-1) c_3 v_\lambda \end{aligned}$$

or in terms of the z_i's we have

$$z_1 z_2 = K(K-1) z_3. \tag{3.12}$$

Since none of the z_i's can be zero, (3.11) and (3.12) yield

$$z_2 = K(K - s - 1). \tag{3.13}$$

Equating the right-hand sides of (3.10) and (3.11) and using (3.13) produces

$$z_1 = (K + r)(K - 1). \tag{3.14}$$

Finally, we obtain the formula for z_3 from (3.12), (3.13) and (3.14).

The closing statement of this lemma is proved by using (3.2) and (3.4) to solve for K, if $s \neq 0$, or by using (3.2) and (3.3) to solve for K, if $s = 0$ and $r \neq 0$. □

Let $M_\lambda = \mathbf{C} v_\lambda$ be a weight space of the pointed, torsion free A_n module M. We show

Lemma 3.15. *Let $B = \{\alpha_1, \ldots, \alpha_n\}$ be a base for the root system R of A_n. The values $r_1, \ldots, r_n$ and K defined by $h_{\alpha_i} v_\lambda = r_i v_\lambda$ and $X_{\alpha_1} X_{\alpha_2} v_\lambda = K X_{\alpha_1 + \alpha_2} v_\lambda$ uniquely determine the action of the basic cycles c_i on v_λ.*

Proof. According to Lemma 3.1, this lemma is true for $n = 2$. Now assume $n > 2$. We may assume that this lemma is true for all basic cycles contained in the cycle subalgebra of A_{n-1} determined by the simple roots $\alpha_1, \ldots, \alpha_{n-1}$. If the lemma is not true for the basic cycles of A_n, then according to Lemma 1.8, there must be some A_2 in A_n whose basic cycles are not uniquely determined by r_i, $1 \leqslant i \leqslant n$, and K. Evidently, the root system of this A_2 has a base of the form $\beta = \alpha_j + \cdots + \alpha_{k-1}$, $\gamma = \alpha_k + \cdots + \alpha_n$. These two roots can be incorporated into a base for the root system of A_n whose first $n - 1$ elements form a base for the root system of A_{n-1}, e.g. $-\alpha_{n-1}, -\alpha_{n-2}, \ldots, -\alpha_{k+1}, -(\alpha_k + \alpha_{k-1}), -\alpha_{k-2}, \ldots, -\alpha_{j+1}, -(\alpha_j + \alpha_{j-1}), -\alpha_{j-2}, \ldots, -\alpha_1, \alpha_1 + \cdots + \alpha_{j-1}, \alpha_j + \cdots + \alpha_{k-1}, \alpha_k + \cdots + \alpha_n$. Label the last three roots in this list α, β, γ, respectively.

We now have the problem reduced to a problem concerning A_3 where A_3 is the subalgebra determined by α, β, γ. Let r, s, S be defined by $h_\alpha v_\lambda = r v_\lambda$, $h_\beta v_\lambda = s v_\lambda$ and $X_\alpha X_\beta v_\lambda = S X_{\alpha+\beta} v_\lambda$. Clearly r, s are uniquely expressible in terms of the values of the r_i's originally given to us. Also, according to Lemma 3.1, $Y_{\alpha+\beta} X_{\alpha+\beta} v_\lambda = (S + r)(S - s - 1) v_\lambda$ and hence by our assumption on A_{n-1}, S is uniquely determined.

We introduce some notation which helps us handle the A_3 case. Whenever μ, ν, $\mu+\nu$ are roots of A_3 we define $K(\mu,\nu)$ by $X_\mu X_\nu v_\lambda = K(\mu,\nu)X_{\mu+\nu}v_\lambda$. Let

$$K(\alpha,\beta)=S,\quad K(\beta,\gamma)=T,$$
$$K(\alpha+\beta,\gamma)=U,\quad K(\alpha,\beta+\gamma)=V. \tag{3.16}$$

Since our inductive assumption implies S is uniquely determined by the r_i's and K, it suffices to prove that T is uniquely determined by r, s and S. Using relations in the universal enveloping algebra of A_3, one can readily show that corresponding to the base $-\alpha$, $\alpha+\beta$, γ the values analogous to (3.16) are

$$K(-\alpha,\alpha+\beta)=S+r,\quad K(\alpha+\beta,\gamma)=U,$$
$$K(\beta,\gamma)=T,\quad K(-\alpha,\alpha+\beta+\gamma)=V+s. \tag{3.17}$$

Also, since $TX_\alpha X_{\beta+\gamma}=X_\alpha X_\beta X_\gamma = X_\alpha X_{\beta+\gamma}+X_\gamma X_\alpha X_\beta$ we have $(T-1)X_\alpha X_{\beta+\gamma} = X_\gamma X_\alpha X_\beta$ so that

$$\begin{aligned} V(T-1)X_{\alpha+\beta+\gamma} &= (T-1)X_\alpha X_{\beta+\gamma} = SX_\gamma X_{\alpha+\beta} \\ &= S(X_{\alpha+\beta}X_\gamma - X_{\alpha+\beta+\gamma}) = S(U-1)X_{\alpha+\beta+\gamma} \end{aligned}$$

and hence the definitions given by equations (3.16) produce the relationship

$$S(U-1)=V(T-1). \tag{3.18}$$

The following series of computations yield linear relationships among S, T, U, and V.

$$\begin{aligned} X_\alpha X_{\beta+\gamma}X_\beta v_\lambda &= (X_{\alpha+\beta+\gamma}X_\beta + X_{\beta+\gamma}X_\alpha X_\beta)v_\lambda \\ &= X_{\alpha+\beta+\gamma}X_\beta v_\lambda + SX_{\beta+\gamma}X_{\alpha+\beta}v_\lambda, \end{aligned} \tag{3.19}$$

and

$$\begin{aligned} X_\alpha X_{\beta+\gamma}X_\beta v_\lambda &= (X_{\alpha+\beta}X_{\beta+\gamma} + X_\beta X_\alpha X_{\beta+\gamma})v_\lambda \\ &= X_{\alpha+\beta}X_{\beta+\gamma}v_\lambda + VX_{\alpha+\beta+\gamma}X_\beta v_\lambda. \end{aligned} \tag{3.20}$$

Subtracting (3.20) from (3.19) and using (3.16), we get

$$\begin{aligned} 0 &= (S-1)X_{\alpha+\beta}X_{\beta+\gamma}v_\lambda + (1-V)X_{\alpha+\beta+\gamma}X_\beta v_\lambda \\ &= [(S-1)/T]X_{\alpha+\beta}X_\beta X_\gamma v_\lambda + (1-V)X_{\alpha+\beta+\gamma}X_\beta v_\lambda \\ &= [[U(S-1)/T]+(1-V)]X_{\alpha+\beta+\gamma}X_\beta v_\lambda \end{aligned} \tag{3.21}$$

which implies

$$(V-1)T=(S-1)U. \tag{3.22}$$

When we subtract (3.18) from (3.22) and simplify, we obtain

$$S-U=V-T. \tag{3.23}$$

The equation analogous to (3.23) derived from (3.17) is

$$S-T=V-U. \tag{3.24}$$

These last two equations tell us

$$S=V \quad\text{and}\quad T=U. \tag{3.25}$$

As we did in going from (3.16) to (3.17), we choose a new base for our A_3 subalgebra, namely β, γ, $-(\alpha+\beta+\gamma)$. Equation (3.24) implies $K(\beta,\gamma) = K(\beta, -(\alpha+\beta))$, by analogy with equation (3.16). By Lemma 3.1, the coefficient $K(\beta,-(\alpha+\beta))$ is uniquely determined by r, s and S. This implies that T is uniquely determined by r, s, and S, since $T = K(\beta,\gamma)$. □

The main result of this section is

THEOREM 3.24. *If M is a pointed torsion free A_n module, then there is an complex $(n+1)$-tuple a such that $M \cong N(a)$.*

PROOF. Let $\mathbf{C}v_\lambda$ be a weight space of M and define r_i, $K \in \mathbf{C}$ for $1 \leqslant i \leqslant n$ by setting

$$h_{\alpha_i} v_\lambda = r_i v_\lambda \quad \text{for } 1 \leqslant i \leqslant n$$

and

$$X_{\alpha_1} X_{\alpha_2} v_\lambda = K X_{\alpha_1+\alpha_2} v_\lambda.$$

For any complex $n+1$ tuple $a = (a_1, \ldots, a_{n+1})$ we have

$$h_{\alpha_i} x^a = (a_i - a_{i+1}) x^a \quad \text{for } 1 \leqslant i \leqslant n,$$

$$X_{\alpha_1} X_{\alpha_2} x^a = a_3(a_2+1) x^{a+\varepsilon_1-\varepsilon_2} \quad \text{and} \quad X_{\alpha_1+\alpha_2} x^a = a_3 x^{a+\varepsilon_1-\varepsilon_2}.$$

Setting $a_2 = K-1$, $a_1 = r_1 + K - 1$ and $a_{i+1} = a_i - r_i$ for $i = 2, \ldots, n$ we have

$$h_{\alpha_i} x^a = r_i x^a \quad \text{for } 1 \leqslant i \leqslant n,$$

and

$$X_{\alpha_1} X_{\alpha_2} x^a = K X_{\alpha_1+\alpha_2} x^a.$$

Therefore by Lemma 3.15 $M \cong N(a)$. □

4. Pointed torsion free C_n modules. In this section we show that every pointed, torsion free C_n module is equivalent to one of the modules $M(a)$ constructed in §2. Our approach is similar to the method used in §3 to study A_n modules. We first treat the case of $n = 2$ and then use this result along with the results of §3 to establish the general result.

THEOREM 4.1. *If M is a pointed, torsion free C_2 **module and** $M_\lambda = \mathbf{C}v_\lambda$ is one of its weight spaces then the action of the cycles of C_2 on $\mathbf{M}_\lambda$ **is completely determined by** λ.*

PROOF. Let $B = \{\alpha, \beta\}$ be a base for the root system R of C_2 and let $\{X_\mu \mid \mu \in R\} \cup \{h_\alpha, h_\beta\}$ be the corresponding standard basis. The elements h_α, h_β together with the basic cycles of C_2 given by

$$\begin{aligned}
c_1 &= Y_\beta X_\beta, & c_5 &= Y_{\alpha+\beta} X_\beta X_\alpha, & c_9 &= Y_{2\alpha+\beta} X_\beta X_\alpha X_\alpha, \\
c_2 &= Y_\alpha X_\alpha, & c_6 &= Y_\alpha Y_\beta X_{\alpha+\beta}, & c_{10} &= Y_\alpha Y_\alpha Y_\beta X_{2\alpha+\beta}, \\
c_3 &= Y_{\alpha+\beta} X_{\alpha+\beta}, & c_7 &= Y_{2\alpha+\beta} X_\alpha X_{\alpha+\beta}, & c_{11} &= Y_{2\alpha+\beta} Y_\beta X_{\alpha+\beta} X_{\alpha+\beta}, \\
c_4 &= Y_{2\alpha+\beta} X_{2\alpha+\beta}, & c_8 &= Y_{\alpha+\beta} Y_\alpha X_{2\alpha+\beta}, & c_{12} &= Y_{\alpha+\beta} Y_{\alpha+\beta} X_\beta X_{2\alpha+\beta}
\end{aligned}$$

form a generating set for the cycle subalgebra $C(C_2)$ of C_2.

Let $0 \neq v_\lambda \in M_\lambda$ and let $K = K(\beta, \alpha)$ be the coefficient defined by $X_\beta X_\alpha v_\lambda = KX_{\alpha+\beta}v_\lambda$. Multiplying this equation on the left by $Y_{\alpha+\beta}$, $Y_\alpha Y_\beta$, $Y_{2\alpha+\beta}X_\alpha$ and $Y_{2\alpha+\beta}Y_\beta X_{\alpha+\beta}$ respectively and letting $c_i v_\lambda = z_i v_\lambda$ for $i = 1, 2, \ldots, 12$ we obtain the following relations:

$$(4.2) \qquad \begin{aligned} z_5 &= Kz_3, & (K+2)z_6 &= z_1 z_2, \\ z_9 &= (K-2)z_7 + 2z_4, & Kz_{11} &= z_1(z_7 - z_4). \end{aligned}$$

The following identities hold in the universal enveloping algebra of C_2:

(4.3) $[c_1, c_2] = 2c_5 - 2c_6$

(4.4) $[c_2, c_4] = 2c_7 - 2c_8,$

(4.5) $[c_1, c_7] = -2c_{11} + 2c_9 + 8c_7 - 4c_4,$

(4.6) $[c_4, c_5] = 2c_{12} - 2c_9 - 4c_8 - 4c_7 + 4c_4,$

(4.7) $[c_4, c_6] = -2c_{11} + 2c_{10} + 4c_8 + 4c_7 - 4c_4,$

(4.8) $[c_1, c_5] = -2c_2c_1 - 2c_1c_3 + 4h_\beta(c_5 + 2c_3),$

(4.9) $[c_4, c_{11}] = -4(h_\alpha + h_\beta)c_{11} + 4c_4c_6 + 8c_4c_3 - 2c_4c_1,$

(4.10) $[c_2, c_7] = -2c_2c_3 - 2c_6 + c_2c_4 - h_\alpha(c_7 - c_4) + c_9 + 2c_8 - 2c_4,$

(4.11) $[c_2, c_5] = -c_2c_1 + 2c_2c_3 - c_9 - 2c_8 - (h_\alpha + 2)c_5 + 2c_4 + 4c_6.$

Define r, $s \in \mathbf{C}$ by $h_\alpha v_\lambda = rv_\lambda$ and $h_\beta v_\lambda = sv_\lambda$. Applying the elements represented by (4.3) through (4.11) to v_λ, we get a series of relations (4.3)′ through (4.11)′ respectively, involving r, s and the z_i's. Using equations (4.3)′ and (4.4)′, we obtain

$$(4.12) \qquad z_5 = z_6, \quad \text{and}$$

$$(4.13) \qquad z_7 = z_8.$$

Adding (4.5)′ to (4.6)′ and using (4.13) gives us

$$(4.14) \qquad z_{11} = z_{12}.$$

Similarly using (4.6)′, (4.7)′ and (4.14), we find

$$(4.15) \qquad z_9 = z_{10}.$$

Now we use (4.8)′ to express z_1 in terms of K and s.

$$\begin{aligned} 0 &= -2z_1z_2 - 2z_1z_3 + 4s(z_5 + 2z_3) \quad \text{by } (4.8)' \\ &= -2(K+2)z_6 - 2z_1z_3 + (4sK + 8s)z_3 \quad \text{by } (4.2) \\ &= 2z_3[-(K+2)K - z_1 + 2s(K+2)] \quad \text{by } (4.2) \text{ and } (4.12) \end{aligned}$$

and hence

$$(4.16) \qquad z_1 = -(K+2)(K-2s).$$

We can express z_7 in terms of z_4, K and s as follows

$$\begin{aligned} 0 &= -2z_{11} + 2z_9 + 8z_7 - 4z_4 \quad \text{by } (4.5)' \\ &= -2z_{11} + 2[(K-2)z_7 + 2z_4] + 8z_7 - 4z_4 \quad \text{by } (4.2), \end{aligned}$$

which simplifies after multiplication by $K/2$ to

$$\begin{aligned} 0 &= -Kz_{11} + K(K+2)z_7 \\ &= [K(K+2) + (K+2)(K-2s)]z_7 - (K+2)(K-2s)z_4 \end{aligned}$$

with the last equality following from (4.2)′ and (4.16)′ and simplifying to give

(4.17) $$z_7 = [(K-2s)/2(K-s)]z_4.$$

We use this relationship to express z_{11} in terms of s, K and z_4 by modifying (4.6)′

$$\begin{aligned} 0 &= 2z_{12} - 2z_9 - 4z_8 - 4z_7 + 4z_4 \quad \text{by } (4.6)' \\ &= 2z_{11} - 2(K-2)z_7 - 4z_4 - 4z_8 - 4z_7 + 4z_4, \quad \text{by (4.2) and (4.14), or} \\ 0 &= z_{11} - (K+2)z_7, \quad \text{by (4.13).} \end{aligned}$$

This expression and (4.17) give us

(4.18) $$z_{11} = [(K+2)(K-2s)/2(K-s)]z_4.$$

Substitute this value as well as the one given by (4.17) into (4.6)′ to get

(4.19) $$\begin{aligned} z_9 &= [(K+2)(K-2s)/2(K-s) - 4(K-2s)/2(K-s) + 2]z_4 \\ &= [K(K-2s+2)/2(K-s)]z_4. \end{aligned}$$

We see now that determining z_4 determines several of the z_i values we need. Before trying to find z_4, we first find z_3. From (4.9)′ we get

$$\begin{aligned} z_6 &= (r+s)z_{11}/z_4 - 2z_3 + z_1/2 \\ &= (r+s)(K+2)(K-2s)/2(K-s) - (K+2)(K-2s)/2 - 2z_3 \\ &= -(K+2)(K-2s)(K-2s-r)/2(K-s) - 2z_3 \quad \text{by (4.16) and (4.18)} \\ &= Kz_3 \quad \text{by (4.2)} \end{aligned}$$

and hence

(4.20) $$z_3 = -(K-2s)(K-2s-r)/2(K-s) \quad \text{and}$$

(4.21) $$z_6 = -K(K-2s)(K-2s-r)/2(K-s).$$

Moreover, we can now use (4.2) to determine z_2.

(4.22) $$z_2 = (K+2)z_6/z_1 = K(K-2s-r)/2(K-s) \quad \text{by (4.21).}$$

Our remaining unused primed equations (4.10)′ and (4.11)′ allow us to express z_4 and K in terms of r and s. First add these equations together to get

(4.23)

$$\begin{aligned} 0 &= -z_2z_1 - rz_5 + z_2z_4 + rz_4 - rz_7 - 2z_5 + 2z_6 \\ &= [K(K-2s-r)(K+2)(K-2s) + rK(K-2s)(K-2s-r)]/2(K-s) \\ &\quad + [[K(K-2s-r) + r(2(K-s)) - r(K-2s)]/2(K-s)]z_4 \end{aligned}$$

which implies

(4.24) $$z_4 = -(K-2s-r)(K+2+r).$$

Now multiply equation (4.11)′ by $2(K-s)^2$ and use the values we have obtained for the z_i's as follows:

$$\begin{aligned}0 &= 2(K-s)^2[-z_2z_1 + 2z_2z_3 - z_9 - 2z_8 - (r-2)z_6 + 2z_4]\\ &= K(K-2s)(K-2s-r)\\ &\quad\cdot[(K-s)(K+2) - (K-2s-r) + (r-2)(K-s)]\\ &\quad-(K-s)[K(K-2s+2) + 2(K-2s) - 4(K-s)]z_4\\ &= K(K-2s)(K-2s-r)[(K-s)(K+r) - (K-2s-r)]\\ &\quad+K(K-s)(K-2s)(K-2s-r)(K+2+r), \quad \text{or}\end{aligned}$$

$$(4.25)\quad 0 = 2(K-s)[K+r+1] - (K-2s-r) = 2[K-(2s-1)/2][K+r].$$

Therefore, $K = (2s-1)/2$, or $K = -r$. To see that the second possibility cannot occur, note that if $K = -r$ then $z_2 = K$ and we arrive at a contradiction by computing

$$(4.26)\qquad X_\alpha Y_\alpha v_\lambda = (h_\alpha + Y_\alpha X_\alpha)v_\lambda = (r+K)v_\lambda = 0.$$

Equation (4.26) implies that either X_α or Y_α is not torsion free contrary to assumption. □

THEOREM 4.27. *If M is a pointed, torsion free C_n module then there exists a complex n-tuple a such that $M \cong M(a)$.*

PROOF. Let λ be a weight function of M such that $\lambda(h_\alpha) \neq 0$ for all roots α of C_n. Let v_λ be a nonzero vector in M_λ. If $a = (a_1, \ldots, a_n)$ is the unique complex n tuple such that $h_{\alpha_i} v_\lambda = (a_i - a_{i+1})v_\lambda$ for $i = 1, \ldots, n-1$ and $h_{\alpha_n} v_\lambda = ((a_n+1)/2)v_\lambda$, then $M(a)$ is a pointed, torsion free C_n module admitting λ as a weight function. In order to verify that $M \cong M(a)$, it suffices by Theorem 1.9 to show that the action of the C_n basic cycles of degree less than or equal to 3 on M_λ are completely determined by λ.

Since each basic cycle c of degree 2 belongs to a C_2 subalgebra of C_n, Lemma 4.1 implies that the value of cv_λ is uniquely determined by λ. Any basic cycle c' of degree 3 belongs to either a C_2 or an A_2 subalgebra of C_n. If c' belongs to a C_2 subalgebra, Lemma 4.1 again implies that $c'v_\lambda$ is determined by λ. If c' belongs to an A_2 subalgebra, Lemma 3.13 states that $c'v_\lambda$ is determined by the values of the basic cycles of degree less than or equal to 2. However from the argument above, these values are uniquely determined by λ and hence so is $c'v_\lambda$. □

5. General construction of simple pointed modules. Fernando [8] has shown that all simple (L, H) weight modules are equivalent to simple quotients of modules induced from torsion free modules M^{u^+} of certain reductive subalgebras (g, H) of (L, H). The problem then is to determine all possible torsion free modules. In §§3 and 4 we have constructed all pointed torsion free modules. One can use these torsion free modules and apply parabolic induction as presented by Fernando to construct pointed modules. We can conclude from the results below that in fact this process yields *all* pointed modules. However, in this section we present an alternate construction of these modules which realizes all pointed modules as simple quotients of $U(L)$.

We first require some general results concerning the classification of the roots of L as torsion free or locally nilpotent for a given (L, H) weight module M. Fernando **[8]** has shown that the set of torsion free roots $T(M)$ is a "convex" subset of R, i.e., all roots which are positive rational linear combinations of roots in $T(M)$ are contained in $T(M)$. We now use this result to prove

LEMMA 5.1. *$N = \{\mu \in R \mid \pm\mu \notin T(M)\}$ is a root subsystem of R.*

PROOF. We first show that if $\overline{R}$ is any connected, rank 2 subsystem of R with $\overline{R} \cap N$ containing more than two linearly independent roots then $\overline{R} \subseteq N$. We proceed by contradiction assuming that $\overline{R} \not\subseteq N$—i.e. there exists at least one torsion free root in $\overline{R}$.

Since $\overline{R}$ is of rank 2 and contains both torsion free and locally nilpotent roots we may select two roots μ, $\nu \in \overline{R}$ such that μ is torsion free, ν is locally nilpotent and no positive linear combination of μ and ν are roots. Clearly $\{\mu, -\nu\}$ forms a base of $\overline{R}$. If $-\nu$ is torsion free then by the convexity of $T(M)$ all roots in $\overline{R}$ which are positive with respect to $\{\mu, -\nu\}$ are torsion free and hence $\overline{R} \cap N = \varnothing$ contrary to assumption. We may therefore assume that $-\nu$ is locally nilpotent.

Since ν is locally nilpotent there exists a nonzero vector $v \in M$ such that $X_\nu v = 0$. Since $-\nu$ is locally nilpotent there exists a nonnegative integer j such that $h_\nu v = jv$ where h_ν denotes the element of the Cartan subalgebra which is dual to ν with respect to the Killing form. Then for any nonnegative integer m we have

(i) $X_\mu^m v \neq 0$ since μ is torsion free,

(ii) $X_\nu X_\mu^m v = 0$ since $[X_\nu, X_\mu] = 0$,

(iii) $h_\nu X_\mu^m v = (m\langle \mu, \nu\rangle + j)X_\mu^m v$ where $\langle \cdot\,, \cdot\rangle$ denotes the Killing form on H^*.

It follows that $X_{-\nu}^p X_\mu^m v \neq 0$ for all integers p with $0 \leqslant p \leqslant m\langle \mu, \nu\rangle + j$.

For any root $r\mu + s(-\nu) \in \overline{R}$ with $r \geqslant 1$ we have that $\langle r\mu + s(-\nu), -\nu\rangle \leqslant s$ which implies that $s \leqslant r\langle \mu, \nu\rangle$. Therefore, for all nonnegative integers k, $sk \leqslant rk\langle \mu, \nu\rangle + j$ and hence $X_{-\nu}^{sk} X_\mu^{rk} v \neq 0$. Then by Theorem 1.5 all roots $r\mu + s(-\nu)$ with $r \geqslant 1$ are torsion free and hence $\overline{R} \cap N = \{\pm\nu\}$ contrary to assumption. It follows then that $\overline{R} \subseteq N$ as required.

Since N is a subset of a root system R, in order to show that N is a root subsystem it suffices to show that α, $\beta \in N$ implies $\sigma_\alpha(\beta) \in N$ where σ_α is the Weyl reflection in the hyperplane perpendicular to α. If $\alpha = \pm\beta$ the $\sigma_\alpha(\beta) = -\beta \in N$. If $\langle \alpha, \beta\rangle = 0$, the $\sigma_\alpha(\beta) = \beta \in N$. Finally if $\langle \alpha, \beta\rangle \neq 0$ and $\alpha \neq \pm\beta$, then $\{\alpha, \beta\}$ spans a connected rank 2 subsystem $\overline{R}$ of R with $\{\pm\alpha, \pm\beta\} \subseteq \overline{R}$ and hence by our previous argument $\sigma_\alpha(\beta) \in \overline{R} \subseteq N$ as required. □

THEOREM 5.2. *Let λ be a weight of M^{u^+} and γ be a weight of M. Then* $\dim M_\gamma \geqslant \dim M_\lambda$.

PROOF. Let $0 \neq v_\lambda \in M_\lambda$ and $0 \neq v_\gamma \in M_\gamma$ and assume $\dim M_\lambda > \dim M_\gamma$. We may assume further that λ and γ are related by $\gamma = \lambda + \sum k_i \alpha_i$ where $\alpha_i \in B$ and $k_i \geqslant 0$. Moreover, we may take λ and γ so that $\sum k_i$ is minimal among all such pairs.

By the irreducibility of M and the Poincaré-Birkhoff-Witt Theorem, we can redefine v_γ so that it is the image of v_λ under the action of a monomial in the $\sum k_i \alpha_i$ weight space of $U(L)$ and further that this monomial involves no elements of H and no torsion free root vectors. Lemma 5.1 implies that all roots in the root system spanned by $\{\alpha_i \mid k_i \neq 0\}$ are locally nilpotent.

The subset $B_1 = \{\alpha_i \mid k_i \neq 0\}$ of B is a base for a semisimple subalgebra L_1 of L. The weight vectors of M_λ are lowest weight vectors relative to B_1 in the L_1-module $M_1 = U(L_1)M_\lambda$. Our result then follows from the theory of finite-dimensional modules once we have shown that M_1 is finite dimensional.

Let $X_1^{e_1} \cdots X_k^{e_k}$ be a typical monomial in a Poincaré-Birkhoff-Witt basis of $U(L_1)$. (Here the X_i's denote root vectors in L_1.) Then

$$M_1 = \text{lin. span}\{X_1^{e_1} \cdots X_k^{e_k} M_\lambda \mid e_i\text{'s are nonnegative integers}\}.$$

If $S_{k-i} = \text{lin. span}\{X_{k-i}^{e_{k-i}} \cdots X_k^{e_k} M_\lambda \mid e_j\text{'s are nonnegative integers}\}$ is finite dimensional then since X_{k-i-1} is locally nilpotent there is some m such that $X_{k-i-1} m_{S_i} = 0$ and hence S_{k-i-1} is finite dimensional. It now follows, since M_λ is finite dimensional, $M_1 = S_{k-1}$ is also finite dimensional. □

COROLLARY 5.3. *If M is pointed then every weight space of M^{u^+} is 1 dimensional.*

THEOREM 5.4 [**13**, THEOREM 6]. *Let L be a simple Lie algebra with Cartan subalgebra H. Let B be a base of the root system R of L with respect to H. Let $\{B_i \mid i = 1, \ldots, k\}$ be mutually orthogonal components of B and set L_i equal to the subalgebra of L associated with the root subsystem R_i of R generated by B_i. Then the cycle subalgebra $C(L)$ can be written as a vector space direct sum*

$$C(L) = U(H)C(L_1) \cdots C(L_K) \oplus C'$$

where C' is the linear span of all PBW basis monomials of $C(L)$ involving at least one root vector not in $\bigcup R_i$. The subspace C' is an ideal in $C(L)$ and moreover if η_i: $C(L_i) \to \mathbf{C}$ are mass functions there exists a mass function η: $C(L) \to \mathbf{C}$ such that $\eta \mid C(L_i) = \eta_i$, $\eta \mid C' = 0$ and η may be defined arbitrarily on the basis elements H_μ where $\mu \notin \bigcup R_i$.

We now indicate how the construction and classification of all torsion free modules for algebras of types A and C lead to an alternate construction and classification of all pointed L modules. We first use the foregoing results to observe that every pointed L module admits (and hence by Theorem 1.8 is determined by) a mass function of a certain type. In fact, if M is a pointed L module then by Corollary 5.3 we may select a 1-dimensional weight space M_λ contained in M^{u^+}. By Theorem 1.3 there exists a base B of the root system R of L such that the symmetric part of $T(M)$ intersects B in mutually orthogonal components $\{B_i \mid i = 1, \ldots, k\}$ where each B_i is a base of the root system R_i of a simple Lie subalgebra L_i of type

A or C. Then the L_i submodule M_i of M generated by M_λ is a torsion free L_i module. Therefore the mass function η: $C(L) \to \mathbf{C}$ of the pointed L module M associated with the 1-dimensional weight space M_λ has the following properties:

(i) $\eta \mid H = \lambda$,

(ii) $\eta \mid C(L_i) =$ a mass function of a pointed torsion free L_i module, and

(iii) $\eta \mid C' = 0$.

Conversely then, using the classification of all pointed torsion free modules (and hence all mass functions of pointed torsion free modules) and applying Theorem 5.4 we can construct *all* mass functions η: $C(L) \to \mathbf{C}$ having properties (i)–(iii). The L module $U(L)/I_\eta$ associated to each such mass function η where I_η denotes the unique left ideal of $U(L)$ containing the kernel of η is then a pointed L module. Moreover the collection of all such L modules yields *all* pointed L modules.

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