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A closed-form equation for effective stress in unsaturated soil

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[1] We propose that the recently conceptualized suction stress characteristic curve represents the effective stress for the shear strength behavior of unsaturated soil. Mechanically, suction stress is the interparticle stress called tensile stress. The working hypothesis is that the change in the energy of soil water from its free water state is mostly consumed in suction stress. We demonstrate that the suction stress lies well within the framework of continuum mechanics where free energy is the basis for any thermodynamic formulation. Available experimental data on soil water characteristic curves and suction stress characteristic curves are used to test the hypothesis, thus validating a closed-form equation for effective stress in unsaturated soil. The proposed closed-form equation is intrinsically related to the soil water characteristic curve by two pore parameters: the air entry pressure and pore size spectrum number. Both semiquantitative and quantitative validations show that the proposed closed-form equation well represents effective stress for a variety of earth materials ranging from sands to clays. Of important practical implications are (1) the elimination of the need for any new shear strength criterion for unsaturated soil, (2) the elimination of the need for determining the Bishop's effective stress parameter χ because the new form of effective stress is solely a function of soil suction, and (3) the ready extension of all classical soil mechanics work on limit equilibrium analysis to unsaturated soil conditions.

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1. Introduction

[2] In recent years, the suction stress characteristic curve has been introduced to represent the state of stress in unsaturated soil [*Lu and Likos*, 2004, 2006]. In a broad sense, it is an expansion and extension of both Terzaghi's effective stress for saturated soil [e.g., *Terzaghi*, 1943] and Bishop's effective stress for unsaturated soil [*Bishop*, 1954, 1959]. Like previous effective stress approaches, the suction stress approach seeks a single stress variable that is responsible for the mechanical behavior of earth materials. However, suction stress differs from Terzaghi's "skeleton" stress in that forces contributing to suction stress are self-balanced at the interparticle level and thus do not pass on from one particle to another. Suction stress originates from the available interaction energy at the soil solid surface that can be conceptualized to exist in the forms of van der Waals and double-layer forces, surface tension, and solid-liquid interface forces due to pore water pressure [*Lu and Likos*, 2006]. A macroscopic continuum representation of suction stress is the tensile stress [e.g., *Mitarai and Nori*, 2006; *Lu et al.*, 2007] that can be simply determined from uniaxial tensile strength tests of unsaturated earth and other granular materials [*Lu et al.*, 2007]. The suction stress concept differs from Bishop's effective stress mainly in that it eliminates the need to define the coefficient of effective stress χ, as suction stress is solely a function of soil suction. The uniqueness of χ and its determination have been major obstacles for the wide acceptance of Bishop's effective stress. However, recent work has established a link between χ and the air entry pressure head [*Khalili and Khabbaz*, 1998; *Khalili et al.*, 2004] and the utility of Bishop's effective stress as a constitutive framework for critical state soil mechanics [*Nuth and Laloui*, 2008]. Finally, the suction stress characteristic curve, similar to the soil water characteristic curve, does not need to be restricted to a single-valued function. Hysteresis due to different wetting states could also be treated under the framework of suction stress. For example, different parameters can be used for the air entry or water entry pressure, depending on the soil's wetting state.

[3] Suction stress provides a framework for the examination of the state of stress in unsaturated soil that differs radically from the more commonly known two independent stress state variable approach [e.g., *Fredlund and Morgenstern*, 1977]. In the two independent stress state variable approach, the total stress or "net normal stress" and matric suction are hypothesized to be the necessary and sufficient stress variables for describing the mechanical behavior of unsaturated soil. Under this framework, shear strength needs to be completely modified to account for its dependence on matric suction, and this is accomplished by the introduction of an additional friction angle, ϕ^b. Despite the popularity of the two independent stress state variable approach, it is quite controversial [e.g., *Khalili and Khabbaz*, 1998; *Nuth and*

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Laloui, 2008]. Whether matric suction is a stress variable and the physical basis for the additional shear strength parameter ϕ^b are disputed at the present time [e.g., *Lu*, 2008]. The major theoretical and practical obstacle faced by the two independent stress state variable approach is that it cannot be reconciled within the context of classical mechanics for saturated soil. In classical soil mechanics, the single stress variable, effective stress, can be used for both shear strength (e.g., limit state) and deformation (e.g., consolidation) analyses. This philosophy has been widely adopted as the design basis in geotechnical practice today.

[4] The two independent stress state variable approach has been expanded to handle elastoplasticity, critical state soil mechanics, and coupled yield limits with some success [e.g., *Alonso et al.*, 1990; *Wheeler and Sivakumar*, 1995; *Gallipoli et al.*, 2003]. While these theories provide insight into important concepts such as loading collapse and suction increase and decrease curves, they add additional complexity by requiring additional material parameters that are often either variable or difficult to determine experimentally. These theories typically involve an additional variable or parameter and are most suitable for stress-strain analysis of postfailure or deformable soil. A thorough critical review and analysis of these models can be found in works by *Gens et al.* [2006] and *Nuth and Laloui* [2008].

[5] In this paper, we present our working hypothesis that the change in energy of soil water from its free water state is mostly consumed in suction stress and establish a thermodynamic justification for a closed-form equation for effective stress in variably saturated soils. We then reinterpret available experimental results from the literature to validate the closed-form equation and conclude with a discussion of theoretical and practical implications.

2. A Closed-Form Equation for Effective Stress in Variably Saturated Soil

2.1. Working Hypothesis

[6] The effective stress principle, under the framework of the suction stress characteristic curve, can be expressed as [*Lu and Likos*, 2006]

$$\sigma' = (\sigma - u_a) - \sigma^s, \tag{1}$$

where u_a is the pore air pressure, σ is the total stress, σ' is the effective stress, and σ^s is defined as the suction stress characteristic curve of the soil with a general functional form of

$$\sigma^s = -(u_a - u_w) \qquad u_a - u_w \le 0, \tag{2a}$$

$$\sigma^s = f(u_a - u_w) \qquad u_a - u_w \ge 0, \tag{2b}$$

where u_w is the pore water pressure and f is a scaling function describing the link between suction stress and matric suction. *Lu and Likos* [2004, 2006] and *Lu et al.* [2007] showed that the suction stress characteristic curve, σ^s, could be obtained by shear strength or tensile strength tests or by theoretical formulations. In this work we seek a closed-form equation for the suction stress characteristic curve as a function of either matric suction or soil saturation using a working hypothesis formed on the basis of experimental observations and thermodynamic justifications.

[7] The working hypothesis is that the change in energy of soil water from its free water state is mostly consumed in suction stress. Experiments show that for a variety of soils and granular materials, there is a relationship between the soil water characteristic curve (SWCC), plotting suction versus saturation, and the suction stress characteristic curve (SSCC), plotting the tensile stress versus saturation. Our aim here is to derive a first-order approximation for this relationship within a thermodynamic framework.

2.2. Thermodynamic Justifications

[8] The tensile stress can be calculated from the virtual work of increasing the volume of a soil system. For a partially saturated soil held at constant temperature, T, and chemical potential, μ, the work is itself stored as the grand canonical free energy, F, and so the stress is given by the derivative of the free energy with respect to volume V; that is,

$$\sigma^s = -\left.\frac{\partial F}{\partial V}\right|_{\mu,T}. \tag{3}$$

Assuming a constant density for water, the free energy will have contributions from both the free capillary water and the bound residual water layers. We analyze each contribution below.

[9] Bound residual water layers can exist because of surface hydration attraction, extending over a layer thickness of water to the surface of the soil solid. The binding results in a lower free energy for the water bound in that layer. For all N grains in a representative elementary volume (REV), the bound water layers occupy a total "residual volume," V_r, and have a total free energy, F_r. The remaining "free" capillary water thus has a volume $V_f = V_w - V_r$, where V_w is the total water volume.

[10] Since the bound residual water is taken to have significantly lower free energy density than the free water, water added to a dry granular system up to a volume $V_w \le V_r$ will accumulate in this layer first. Since our focus will be on $V_w > V_r$, we do not concern ourselves with the volume dependence of the free energy below this residual volume, other than to identify F_r as the limiting value as V_w approaches V_r. The remaining phases j (free capillary water and air) having volume V_j and interfacial (or surface) areas A_i contribute volume and surface terms to the free energy, giving

$$F = F_r - \sum_{\text{phase } j} u_j V_j + \sum_{\text{interface } i} \gamma_i A_i \quad \text{for} \quad V_w > V_r, \tag{4a}$$

with differential

$$\mathrm{d}F = -S\mathrm{d}T - \sum_j u_j \mathrm{d}V_j - \sum_j N_j \mathrm{d}\mu_j + \sum_i \gamma_i \mathrm{d}A_i \quad \text{for} \quad V_w > V_r, \tag{4b}$$

where S is the entropy of the system and N_j is the number of molecules in phase j (omitting notation specifying species type). This provides the thermodynamic definition of the surface tension for interface i,

$$\gamma_i = \left.\frac{\partial F}{\partial A_i}\right|_{T,V,\mu} = \left.\frac{\partial E}{\partial A_i}\right|_{S,V,N}, \tag{5}$$

where the second equation expressed in terms of the total energy E of the system is obtained by Legendre transformation $F = E - TS - \mu N$.

[11] Specifically, for a free water phase with volume $V_f = V_w - V_r$ and air phase with volume $V_a = V_v - V_w = (V_v - V_r) - V_f$, where V_v is the void volume, we have

$$F = F_r - u_a V_a - u_w V_f + \sum_i \gamma_i A_i$$
$$= F_r - u_a[(V_v - V_r) - (V_w - V_r)] - u_w(V_w - V_r) + \sum_i \gamma_i A_i \quad \text{for } V_w > V_r, \qquad (6)$$

or, expressed in terms of the matric suction, $(u_a - u_w)$,

$$F = F_r - u_a(V_v - V_r) + (u_a - u_w)(V_w - V_r) + \sum_i \gamma_i A_i \quad \text{for} \quad V_w > V_r. \qquad (7)$$

We can use this expression for the free energy to derive the tensile stress or suction stress from a virtual work argument. As the REV is subject to tension, an infinitesimal extension with volume change dV will lead to an infinitesimal increase in the free energy, dF, corresponding to the input work. In particular, we consider a uniaxial strain leading to a change in the total volume, $V = V_s + V_v$, where V_s is the constant volume occupied by the solid grains. Since the volumes of the rigid grains and of the bound water layer are essentially constant for $V_w > V_r$, we have $\mathrm{d}V = \mathrm{d}V_w + \mathrm{d}V_a$. At constant T and μ, we thus have

$$\left.\frac{\partial F}{\partial V}\right|_{\mu,T} = -u_a \left.\frac{\partial V_a}{\partial V}\right|_{\mu,T} - u_w \left.\frac{\partial V_w}{\partial V}\right|_{\mu,T} + \sum_i \gamma_i \left.\frac{\partial A_i}{\partial V}\right|_{\mu,T}$$
$$= -u_a + (u_a - u_w)\left.\frac{\partial V_w}{\partial V}\right|_{\mu,T} + \sum_i \gamma_i \left.\frac{\partial A_i}{\partial V}\right|_{\mu,T} \quad \text{for } V_w > V_r, \qquad (8)$$

where we have assumed $\left.\frac{\partial F_r}{\partial V}\right|_{\mu,T} = 0$ since the tightly bound residual water is unperturbed by a small change in total volume.

[12] We thus need to specify how much the water volume, V_w, and the interfacial areas, A_i, change with V under uniaxial strain. The ratio of free water volume to the total available to it is specified by the effective saturation,

$$S_e \equiv \frac{S - S_r}{1 - S_r} \equiv \frac{\frac{V_w}{V_v} - \frac{V_r}{V_v}}{1 - \frac{V_r}{V_v}} = \frac{V_f}{V_v - V_r}, \qquad (9)$$

or $V_f = S_e\,(V_v - V_r)$, with S_r being the residual saturation. Given that $\mathrm{d}V = \mathrm{d}V_v$ and $\mathrm{d}V_w = \mathrm{d}V_f$ for $V_w > V_r$, we have

$$\mathrm{d}V_w = S_e \mathrm{d}V + (V_v - V_r)\mathrm{d}S_e \quad \text{for} \quad V_w > V_r. \qquad (10)$$

Since the arrangement of soil particles within the REV is assumed to be random, there is a distribution of local grain arrangements and strengths, and we propose that the sample will deform at a localized region. This region under deformation can draw upon both water and air in the neighboring area, which acts as a reservoir. As a first approximation, we can assume that the effective saturation, S_e, remains constant, i.e., $\mathrm{d}S_e = 0$, in which case the change in water volume is simply proportional to the change in void volume:

$$\mathrm{d}V_w = S_e \mathrm{d}V \quad \text{for} \quad V_w > V_r. \qquad (11)$$

Accounting for the pressure, u_a, provided by the surrounding atmospheric air, this leads to an expression for the tensile stress (suction stress):

$$\sigma^s = -\left.\frac{\partial F}{\partial V}\right|_{\mu,T} - u_a = -(u_a - u_w)\left.\frac{\partial V_w}{\partial V}\right|_{\mu,T} - \sum_i \gamma_i \left.\frac{\partial A_i}{\partial V}\right|_{\mu,T} \quad \text{for} \quad V_w > V_r, \qquad (12)$$

or

$$\sigma^s = -(u_a - u_w)S_e - \sum_i \gamma_i \left.\frac{\partial A_i}{\partial V}\right|_{\mu,T,\mathrm{uniaxial}} \quad \text{for} \quad V_w > V_r. \qquad (13)$$

If the second term (interfacial contribution) is ignored, we have the approximation of suction stress:

$$\sigma^s = -(u_a - u_w)S_e \quad \text{for} \quad V_w > V_r. \qquad (14)$$

Although the surface tension term has been neglected in previous work [e.g., *Houlsby*, 1997], no convincing physical justification has been reported. In the capillary (occluded air bubble state) regime, the surface tension term should be small or zero, but it could be significant in both the pendular (discontinuous menisci state) and the funicular (continuous menisci and pore air) regimes. *Houlsby* [1997] argued that the relative velocity of the interface and the soil solid is small, so that the surface tension can be ignored, but the exact circumstances where such a condition applies, as well as the quantitative role of interfacial tension in the tensile stress, have yet to be established. The work done by the tensile stress, captured by equation (13) or equation (14), is similar to that proposed by *Bishop* [1954] or derived by *Houlsby* [1997, equation (26)], except that the energy in the residual water bound by surface hydration is removed from the tensile stress. As argued above, the residual water content is the amount of water that remains primarily in the form of thin films surrounding the soil particle surfaces at very high suctions but has very little effect on the interparticle suction stress. The graphical representation of suction stress is the area under the normalized soil water characteristic curve and is shown in Figure 1a. In this work, we rely on the physical justification of the validity of neglecting the surface tension term using experimental data, shown in section 3.

2.3. A Closed-Form Equation for Effective Stress

[13] *Bishop*'s [1954, 1959] effective stress can be written as

$$\sigma' = (\sigma - u_a) + \chi(u_a - u_w). \qquad (15)$$

Bishop [1954] also suggested that for his effective stress parameter χ, $\chi = S$,

$$\sigma' = (\sigma - u_a) + S(u_a - u_w). \qquad (16)$$

Lu and Likos [2006] proposed a form of suction stress that is consistent with Terzaghi's effective stress:

$$\sigma' = (\sigma - u_a) - \sigma^s, \qquad (17)$$

where $\sigma^s = -(u_a - u_w)S$.

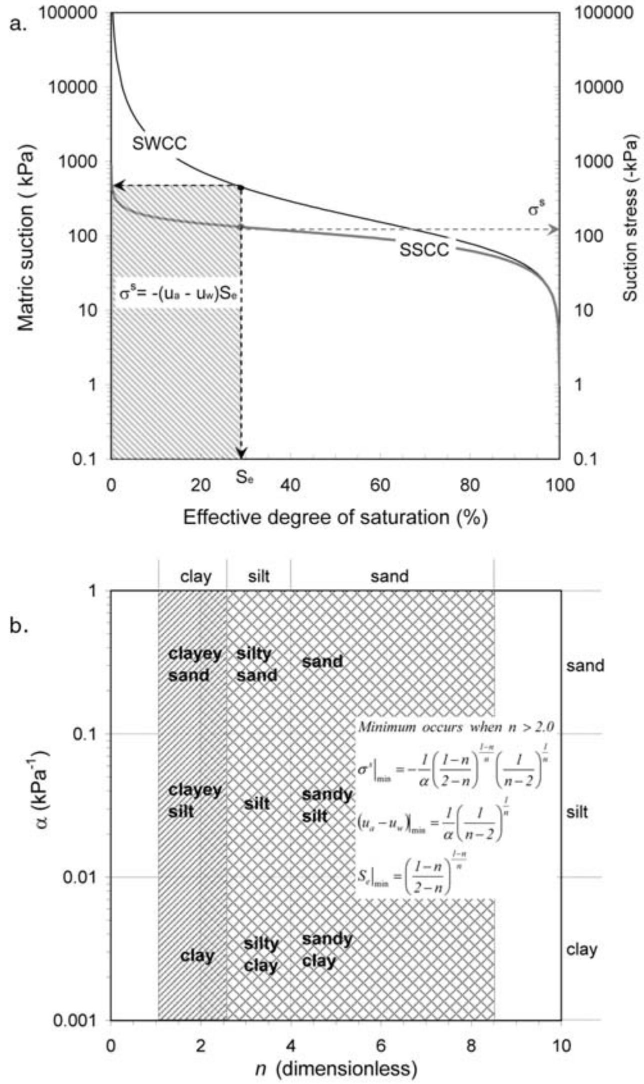

Figure 1. (a) Interrelationship between the soil water characteristic curve (SWCC) and the suction stress characteristic curve (SSCC) and (b) illustration of suction stress regimes for various soils.

[14] Following equation (14), we propose an effective stress as an extension of Bishop's and an expansion of Terzaghi's for all saturations by modifying the saturation contribution to effective stress:

$$\begin{aligned}\sigma' &= (\sigma - u_a) - [-S_e(u_a - u_w)] \\ &= (\sigma - u_a) - \left[-\frac{S - S_r}{1 - S_r}(u_a - u_w)\right] = (\sigma - u_a) - \sigma^s, \end{aligned} \tag{18}$$

where $\sigma^s = -(u_a - u_w)S_e$. Note that equation (18) is different than that of Bishop in the degree of saturations and can recover Terzaghi's effective stress $\sigma' = \sigma - u_w$ when a soil is saturated. We make an additional extension of equation (18) by applying the relationship between the normalized volumetric water content or degree of saturation and matric suction. If *van Genuchten*'s [1980] SWCC model is used, the normalized degree of saturation can be expressed as

$$S_e = \left\{\frac{1}{1 + [\alpha(u_a - u_w)]^n}\right\}^{1-1/n}, \tag{19}$$

where n and α are empirical fitting parameters of unsaturated soil properties, with α being the inverse of air entry pressure for water saturated soil and n being the pore size distribution parameter. Figure 1b shows the range of values of the α and n parameters for various soil types.

[15] A closed-form expression for suction stress for the full range of saturation can be arrived at by substituting equation (19) into equation (14) and eliminating matric suction:

$$\sigma^s = -\frac{S_e}{\alpha}\left(S_e^{\frac{n}{1-n}} - 1\right)^{\frac{1}{n}} \qquad 0 \le S_e \le 1. \tag{20}$$

Similarly, a closed-form expression for suction stress for the full range of matric suction also can be arrived at by substituting equation (19) into equation (14) and eliminating the degree of saturation:

$$\sigma^s = -(u_a - u_w) \qquad u_a - u_w \le 0, \tag{21a}$$

$$\sigma^s = -\frac{(u_a - u_w)}{\left(1 + [\alpha(u_a - u_w)]^n\right)^{(n-1)/n}} \qquad u_a - u_w \ge 0. \tag{21b}$$

In what follows, we focus on the characteristics and validation of equation (21), as most of the data available in the literature on shear strength and soil water characteristic curves are exclusively expressed in terms of matric suction. Substituting equation (21) into equation (18), the closed-form equation for effective stress in the entire pore water pressure range (all saturations) is

$$\sigma' = \sigma - u_a + (u_a - u_w) \qquad u_a - u_w \le 0, \tag{22a}$$

$$\sigma' = \sigma - u_a + \frac{(u_a - u_w)}{\left(1 + [\alpha(u_a - u_w)]^n\right)^{(n-1)/n}} \qquad u_a - u_w \ge 0. \tag{22b}$$

Equation (22a) is for saturated conditions, which is Terzaghi's equation, and equation (22b) is for unsaturated conditions. In Appendix A we show that equation (22b) continuously and smoothly approaches equation (22a) as matric suction approaches zero or from an unsaturated state to a saturated state.

[16] The general patterns of the SSCC defined by equation (21) and its interrelationship with the SWCC for different types of soil are illustrated here. Figure 2a shows SWCCs for typical sandy, silty, and clayey soils, and Figure 2b shows the corresponding SSCCs predicted using equation (21b). Each of these three "typical" soils has unique characteristics of suction stress. For sandy soil, suction stress is zero at zero matric suction (saturated condition) and at some large value of matric suction (110 kPa for the example here). Suction stress reaches a minimum value at a given matric suction (suction stress of −2 kPa at matric suction of 3 kPa or 70% saturation). The down-and-up characteristic of suction stress can be illustrated by plotting equation

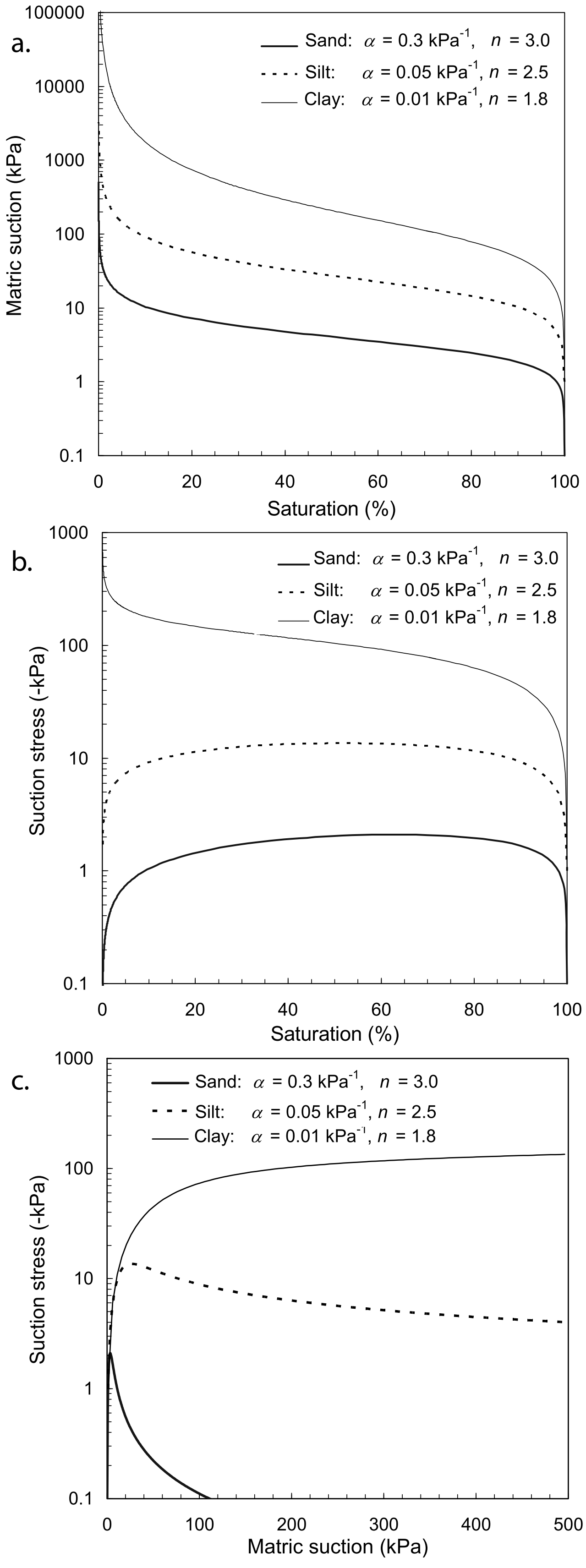

Figure 2. (a) SWCC for some typical soils, (b) SSCC for the typical soils in terms of the effective degree of saturation, and (c) SSCC for the typical soils in terms of matric suction.

(17b), as shown in Figure 2c. This behavior of suction stress is well known for sand-sized granular media [e.g., *Schubert*, 1975; *Kim*, 2001; *Lu et al.*, 2007]. It is important to note that this behavior cannot be effectively described using Bishop's effective stress approach because the effective stress was treated as a function of both matric suction and the degree of saturation (equation (16)).

[17] For a typical silty soil, suction stress follows a similar pattern to that of sandy soil (Figures 2b and 2c), except the minimum value is now on the order of 10 kPa. However, considerable suction stress remains beyond the minimum value. The range of matric suction for which the magnitude of suction stress remains considerable could be on the order of several hundreds to several thousands of kilopascals. A practical illustration of this behavior is evidenced by loess cut slopes. In these cut slopes, suction stress of several tens of kilopascals in the loess is capable of maintaining near-vertical cuts of several meters in height in dry environments even in the absence of clay films or cement.

[18] For a typical clayey soil (Figures 2b and 2c), the pattern of variation of suction stress shows some distinct characteristics. Suction stress is zero when matric suction is zero but decreases monotonically as matric suction increases. The minimum suction stress for clayey soil could be on the order of several hundreds of kilopascals in magnitude. A practical example is that when clay is dry or under high matric suction conditions, the bonding force or suction stress is very high, up to several hundreds of kilopascals, making it difficult to break. Under very moist conditions, clay is plastic, and under moderately moist conditions, clay is brittle like rock.

[19] Under the proposed framework, effective stress or suction stress is intrinsically related to the SWCC. The form of these curves is fundamentally controlled by the pore geometry and pore fluid parameters α and n. The parameter α is a direct indicator of the matric suction at which pore fluid begins to leave a drying soil water system, whereas parameter n reflects the pore size distribution of the soil. It can be shown (see Appendix B) that the SSCC can be in two distinct regimes dependent on the value of n: it monotonically decreases if $n \leq 2.0$ and decreases and then increases if $n > 2.0$. Figure 1b illustrates these two regimes in the $\alpha - n$ space. When $n > 2.0$, the minimum suction stress and its corresponding value of the equivalent degree of saturation and matric suction are

$$\sigma^s|_{\min} = -\frac{1}{\alpha}\left(\frac{1-n}{2-n}\right)^{\frac{1-n}{n}}\left(\frac{1}{n-2}\right)^{\frac{1}{n}}, \tag{23a}$$

$$(u_a - u_w)|_{\min} = \frac{1}{\alpha}\left(\frac{1}{n-2}\right)^{\frac{1}{n}}, \tag{23b}$$

$$S_e|_{\min} = \left(\frac{1-n}{2-n}\right)^{\frac{1-n}{n}}, \tag{23c}$$

respectively. It can be shown from analyzing the mathematical properties of equations (23a) and (23b) that the α parameter dominantly controls the minimum value of suction stress and the matric suction value corresponding to that minimum, whereas the parameter n solely controls the

Table 1. Soil Descriptions and Properties Used to Validate Closed-Form Equation for Effective Stress[a]

Name	Soil Properties	$ub = 1/\alpha$ (kPa)	n	ϕ' (°)	c' (kPa)	Reference	Apparatus
		Group 1					
Kaolin	w_L = 63%, I_p = 33%, percent finer than 3 μm = 70%, γ_{dMAX} = 1.4 g/cm^3	395[b]	1.20[c]	22	24	*Khalili and Khabbaz* [1998]	Modified triaxial
Jossigny silt (low-plasticity clay)	w_L = 37%, I_p = 18%, clay-size fraction = 34%, γ_{dMAX} = 1.7 g/cm^3	182[b]	1.54[c]	22[b]	25[b]	*Cui and Delage* [1993]	Triaxial apparatus with osmotic suction control
Madrid clayey sand	w_L = 32%, I_p = 15%, fine fraction = 17%, γ_{dMAX} = 1.91 g/cm^3	127[b]	1.63[c]	38	0	*Escario and Sáez* [1986]	Modified direct shear
Sandy clay 1	N.D.	35[b]	1.59[c]	37[d]	0[d]	*Blight* [1967]	Modified triaxial with water content control
		Group 2					
Compacted glacial till	w_L = 35.5%, I_p = 18.7%, clay-size fraction = 30%, γ_{dMAX} = 1.815 g/cm^3	153[b]	1.50[c]	25.5	10	*Gan et al.* [1988]	Modified direct shear
Tappen Notch Hill silt	w_L = 57%, I_p = 32% (for clay fraction), clay-size fraction = 10%, silt = 85%,	94[b]	1.39[c]	35	0	*Krahn et al.* [1989]	Modified multistage triaxial
Sandy clay 2[d]	N.D.	70[b]	1.58[c]	30	0	*Maswaswe* [1985]	Modified triaxial
Dhanauri clay compacted to low density[d]	γ_d = 1.48 g/cm^3	62[b]	1.43[c]	28.5	7.8	*Satija* [1978]	Modified triaxial
Mature residual soil (Vista Chinesa, Brazil)	w_L = 50.7%, I_p = 18.4%, clay-size fraction = 24.4%, sand = 60%	38[b]	1.63[c]	28.7	13.7	*de Campos and Carrillo* [1995]	Modified direct shear
		Group 3					
Dhanauri clay compacted to high density[d]	γ_d = 1.58 g/cm^3	127[b]	1.30[c]	29	7.8	*Satija* [1978]	Modified triaxial
Sand-clay mixture	Clay-size fraction = 25%, sand = 75%, γ_{dMAX} = 1.92 g/cm^3	115[b]	1.40[c]	33.5	30	*Khalili and Khabbaz* [1998]	Modified triaxial
Speswhite kaolin	γ_{dMAX} = 1.2 g/cm^3	86[b]	1.42[c]	25	0	*Wheeler and Sivakumar* [1995]	Triaxial
Yellow colluvium (Vista Chinesa, Brazil)	w_L = 45.7%, I_p = 22.7%, clay-size fraction = 42.5%, sand = 50.3%	54[b]	1.62[c]	26.4	0	*de Campos and Carrillo* [1995]	Modified direct shear
Compacted nonplastic silty sand[d]	γ_{dMAX} = 1.89 g/cm^3	43	1.50[c]	38.7	11.5	*Drumright* [1989]	Modified triaxial
		Group 4					
Hume Dam clay (Southeastern Australia)	w_L = 33%, I_p = 12%, fine fraction = 74%, sand fraction = 26%, γ_d = 1.69 g/cm^3	77.5[e]	1.37[e]	29	19	*Khalili et al.* [2004]	Modified triaxial
Barcelona silt	w_L = 32%, I_p = 16%, clay-size fraction = 20%, silt = 43%, sand = 37%,	14.9[e]	1.13[e]	28	0	*Vaunat et al.* [2002]	Modified direct shear
		Group 5					
Glacial till compacted dry of optimum water content	w_L = 35.5%, I_p = 18.7%, clay-size fraction = 30%, silt = 42%, sand = 28%, γ_d = 1.73 g/cm^3	40.9[e]	1.46[e]	23	0	*Vanapalli et al.* [1996]	Modified multistage direct shear
Decomposed tuff (Hong Kong)	N.D.	46.9[e]	1.89[e]	39	0	*Fredlund et al.* [1995]	Modified multistage direct shear
		Group 6					
Ottawa sand	N.D.	2.20[e]	2.50[e]	N.D.	N.D	*Kim* [2001]	Tensile strength tests
Limestone agglomerates	N.D.	1.12[e]	12.68[e]	N.D.	N.D.	*Schubert* [1984]	Tensile strength tests

[a]N.D., no data; w_L, liquid limit; I_p, plasticity index; γ_d, dry density; γ_{dMAX}, maximum dry density.
[b]Estimates reported by *Khalili and Khabbaz* [1998].
[c]Obtained by fitting suction stress characteristic curve to shear strength test data.
[d]Data taken from *Khalili and Khabbaz* [1998].
[e]Obtained using RETC v.6.0 code [*van Genuchten et al.*, 1991].

equivalent degree of saturation corresponding to the minimum suction stress.

3. Experimental Validation

[20] Thanks to the increasing experimental effort in the past 3 decades, experimental validation of the closed-form equation for effective stress (equation (22)) can be achieved using data from shear strength or tensile strength tests. Experimental validation of the proposed effective stress equation (equation (21b)) under the tensile total stress regime has been demonstrated by *Lu et al.* [2009] for various sandy materials. Because most results reported in the literature are from shear strength tests performed in the

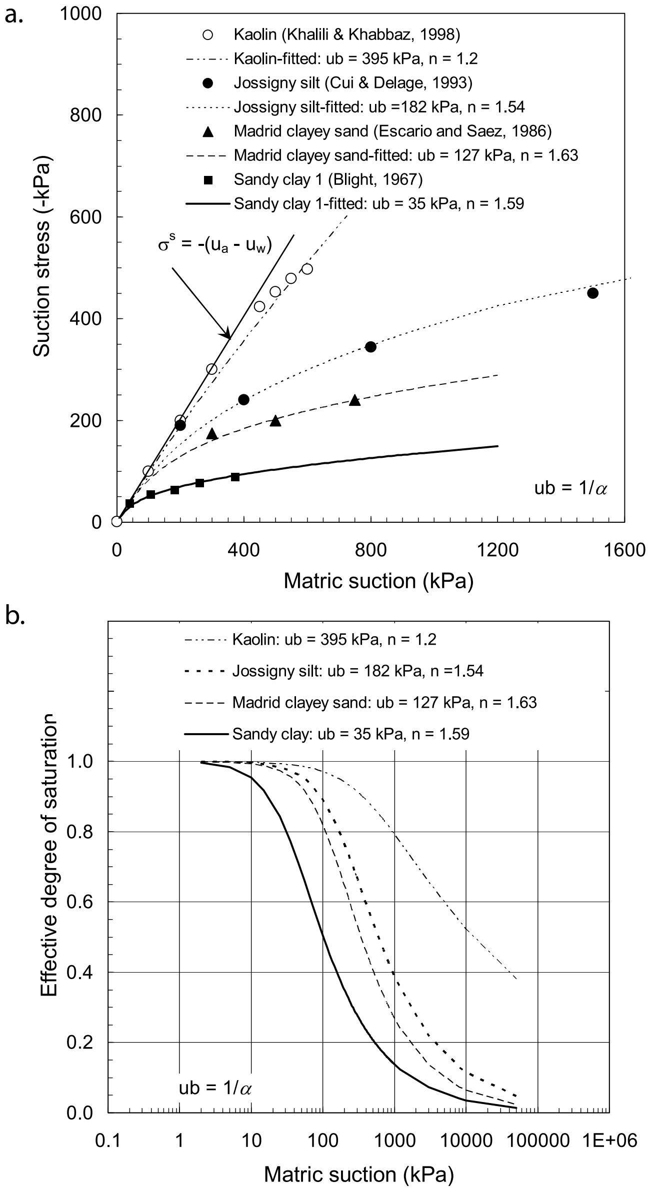

Figure 3. Semiquantitative validation of the closed-form equation for effective stress for Group 1 soils: (a) measured and fitted SSCCs for kaolin, Jossigny silt, Madrid clayey sand, and sandy clay 1 and (b) predicted SWCCs for these soils.

compressive regime, we conduct our validation using shear failure data for which the total normal stresses are compressive. These tests were typically conducted using either a direct shear or triaxial shear apparatus modified so that matric suction can be controlled. A total of 20 different soils covering the range of sand, silt, and clay was used for validation purposes (see Table 1).

[21] A true validation for a given soil requires both a SWCC and a SSCC. Suction stress characteristic curves can be experimentally obtained from a series of shear strength tests under different matric suctions. However, because of the paucity of both SWCC and SSCC for the same soil, the following strategy for validation is adopted. For those soils for which SWCC and SSCC are available, we directly compare test data with the theoretical equations (equation (19) for the SWCC and equation (21b) for SSCC). If equation (21b) is valid, a unique pair of parameters α and n is identified. For those soils for which only suction stress data are available, we perform a "semiquantitative" analysis. We identify the parameters α and n from those data and use them to predict the corresponding SWCCs. We compare these curves with the typical range of curves for different soils to see whether they follow the expected trend, as described below.

3.1. Semiquantitative Validation

[22] We have compiled suction stress data from 14 different soils for the semiquantitative validation. These soils cover a wide spectrum of soil types, from sandy to clayey, as listed in Table 1. Shear strength data were reduced to yield suction stress data as follows. For direct shear testing, the Mohr-Coulomb failure criterion can be written as

$$\tau_f = c' + [(\sigma - u_a) - \sigma^s] \tan\phi', \tag{24}$$

where τ_f is shear stress at failure at a given matric suction, c' is the drained cohesion, and ϕ' is drained friction angle at saturated state. Suction stress at a given matric suction is then reduced from equation (24):

$$\sigma^s = -\frac{\tau_f - c' - (\sigma - u_a)\tan\phi'}{\tan\phi'}. \tag{25}$$

For triaxial testing, suction stress at a given matric suction can be obtained as follows:

$$\sigma^s = -\frac{(\sigma_1 - u_a) - (\sigma_3 - u_a)\tan^2\left(\frac{\pi}{4} + \frac{\phi'}{2}\right) - 2c'\tan\left(\frac{\pi}{4} + \frac{\phi'}{2}\right)}{2\tan\left(\frac{\pi}{4} + \frac{\phi'}{2}\right)\tan\phi'}, \tag{26}$$

where σ_1 and σ_3 are the total principle stresses in the vertical and horizontal directions, respectively.

[23] For clarity of presentation, we have divided the soils for which the semiquantitative validation was performed into three groups (Table 1). Figure 3a shows the measured suction stress characteristic data from the four soils in Group 1. This group consists of kaolin, Jossigny silt, Madrid clayey sand, and sandy clay 1. In general, the SSCCs for these soils follow a trend of monotonic decrease in suction stress with increasing matric suction. At matric suctions less than the air entry value of the soil, suction stress strictly follows the line $\sigma^s = -(u_a - u_w)$, where Terzaghi's effective stress principle is valid. Suction stress begins to deviate from this line in the air entry regime and continues to do so as matric suction increases. The air entry pressures, *ub*, for these soils were estimated by *Khalili and Khabbaz* [1998] and are reported in Figure 3 and in Table 1. The air entry pressure, *ub*, can also be inferred from the SSCC test data as the point at which the SSCC deviates from the saturated line, although real soils do not have a clear deviating point or air entry value but rather a range of pressures in which the largest pores begin to drain. For example, the air entry pressure from Figure 3a for the kaolin is between 300 and 400 kPa (versus the reported value of 395 kPa), and that for the sandy clay 1 is around 30 kPa (versus the reported

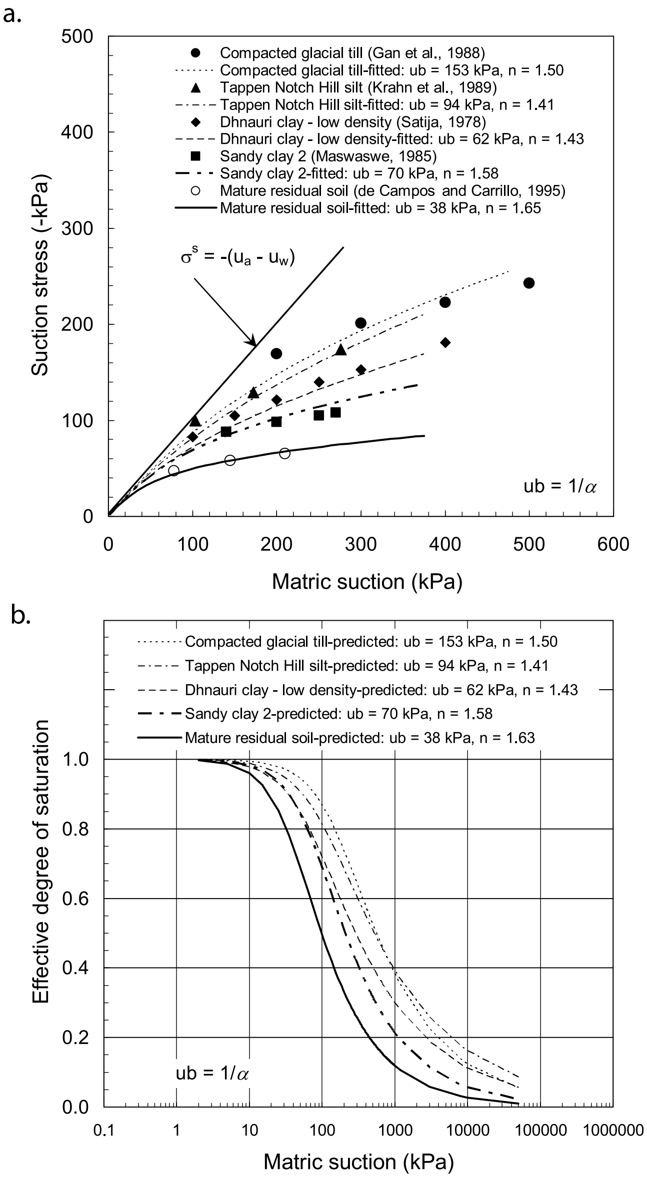

Figure 4. Semiquantitative validation of the closed-form equation for effective stress for Group 2 soils: (a) measured and fitted SSCCs for compacted glacial till, Tappen Notch Hill silt, Dhanauri clay compacted to low density, sandy clay 2, and mature residual soil and (b) predicted SWCCs for these soils.

value of 35 kPa). Using these air entry pressures from the work of *Khalili and Khabbaz* [1998], we attempt to fit the experimental data for each soil in order to obtain a value for the pore size parameter *n*, which is reported in Figure 3. The SWCCs are then calculated (using equation (19)) using the air entry parameter, α (α being the inverse of air entry pressure *ub*), and pore size parameter, *n*, and plotted in Figure 3b.

[24] In general, the finer-grained soils have larger air entry pressures and support higher matric suctions at a given degree of saturation compared with the coarser-grained soils that have smaller air entry pressures. Thus, the soil water characteristic curves should shift to the right in Figure 3b with increasing fine-grained fraction and larger air entry pressures. Figure 3b shows that equation (21) captures this expected "right shifting" trend for the Group 1 soils. The magnitude of suction stress generally increases as the soils become finer (Figure 3a and Table 1). For example, suction stress varies from slightly more than about 250 kPa at a matric suction value of 800 kPa for the Madrid clayey sand (fine fraction is 17%) to about 350 kPa for the Jossigny silt (clay-size fraction is 34%). For clayey soils (e.g., kaolin), suction stress could be as much as several hundreds of kilopascals. The predicted SWCCs for the soils in Group 1 are qualitatively within the ranges that would be expected for soils with similar air entry pressures and grain size distributions.

[25] Figure 4 shows the measured suction stress data from five additional soils, defined here as Group 2. This group consists of compacted glacial till, Tappen Notch Hill silt, sandy clay 2, Dhanauri clay compacted to low density, and a mature residual soil. The best fit SSCCs and the air entry pressures (*ub*) from the work of *Khalili and Khabbaz* [1998] and fit *n* parameters are shown in Figure 4a and Table 1. Suction stress at a matric suction of 500 kPa varies from about 100 to 250 kPa moving from the relatively coarse residual soil to the finer-grained glacial till. The *ub* and *n* parameters are used in equation (19) to predict the corresponding SWCCs, as shown in Figure 4b. For example, the leftmost SWCC is that for the mature residual soil, and it reflects its relatively large sand-size fraction (60%) (Table 1). The rightmost SWCC is that for the relatively fine grained (clay-size fraction is 30%) (Table 1) and dense compacted glacial till. Again, these estimated SWCCs are qualitatively within the ranges that would be expected for soils with similar air entry pressures and grain size distributions.

[26] The final group for this semiquantitative validation consists of five additional soils: Dhanauri clay compacted to high density, a sand-clay mixture, speswhite kaolin, yellow colluvium, and compacted nonplastic silty sand. The measured and fitted suction stress data are shown in Figure 5a and listed in Table 1, and the corresponding SWCCs from the best fit parameters are plotted in Figure 5b. Suction stress at a matric suction of 200 kPa varies from about 85 to 150 kPa moving from the relatively coarse colluvial soil to the dense Dhanauri clay. The predicted SWCCs for the soils in Group 3 are qualitatively within the range expected for soils with similar grain size distributions and bulk densities. The rightward shifting trend of the SWCCs for the finer and denser soils with greater air entry pressures is evident and qualitatively supports equation (21) for the suction stress characteristic curve.

3.2. Quantitative Validation

[27] We identified six soils for which data on both the SWCC and SSCC are available from the literature. For clarity we have divided the soils into three groups. The first, Group 4, includes Hume Dam clay [*Khalili et al.*, 2004] and Barcelona silt [*Vaunat et al.*, 2002]. Test results and the best fit SWCCs are shown in Figure 6a. The curves were fit using a least squares regression and the RETC code [*van Genuchten et al.*, 1991]. The parameters α and n were then used to predict the SSCCs (Figure 6b). Comparisons of the predicted curves with the suction stress data show that the closed-form equations (19) and (21) predict the measured stresses within a few tens of kilopascals for both the Hume Dam clay and the Barcelona silt. This comparison

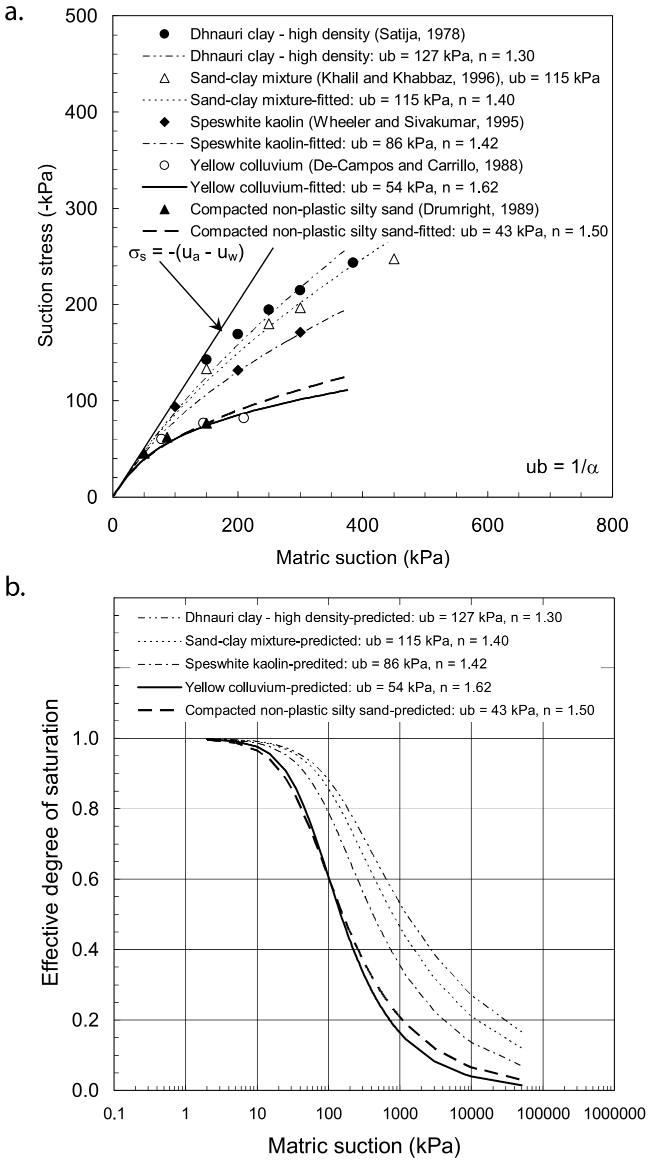

Figure 5. Semiquantitative validation of the closed-form equation for effective stress for Group 3 soils: (a) measured and fitted SSCCs for Dhanauri clay compacted to high density, sand-clay mixture, speswhite kaolin, yellow colluvium, and compacted nonplastic silty sand and (b) predicted SWCCs for these soils.

indicates that the closed-form equation (22) for effective stress is valid for the two soils.

[28] Group 5 consists of glacial till compacted dry of optimum water content [*Vanapalli et al.*, 1996] and decomposed tuff [*Fredlund et al.*, 1995]. The soil water characteristic and suction stress characteristic curve data are shown in Figures 7a and 7b, respectively. As for Group 4, the RETC model was used to identify α and n from the soil water characteristic data (Figure 7a), and the best fit values of α and n are used in equation (21) to predict the SSCCs (Figure 7b). Again, equation (21) provides an accurate prediction (within a few percent) of the measured suction stress data for both the glacial till and the decomposed tuff.

[29] Group 6 contains two relatively coarse-grained materials: fine (Ottawa) sand [*Kim*, 2001] and limestone agglomerates [*Schubert*, 1984]. The test data and the best fit for the SWCCs are shown in Figure 8a. As with the other two groups, the best fit α and n parameters were then used to predict the corresponding SSCCs (Figure 8b). Close matches between the measured and predicted SSCCs for both soils support the validity of the proposed closed-form equation (22) for effective stress in unsaturated soils. Worth noting is the "peak" behavior when parameter $n > 2.0$ (see Appendix B), an important characteristic of suction stress in coarse-grained materials; suction stress is at a minimum of −1.6 kPa at about 3 kPa of matric suction for the Ottawa

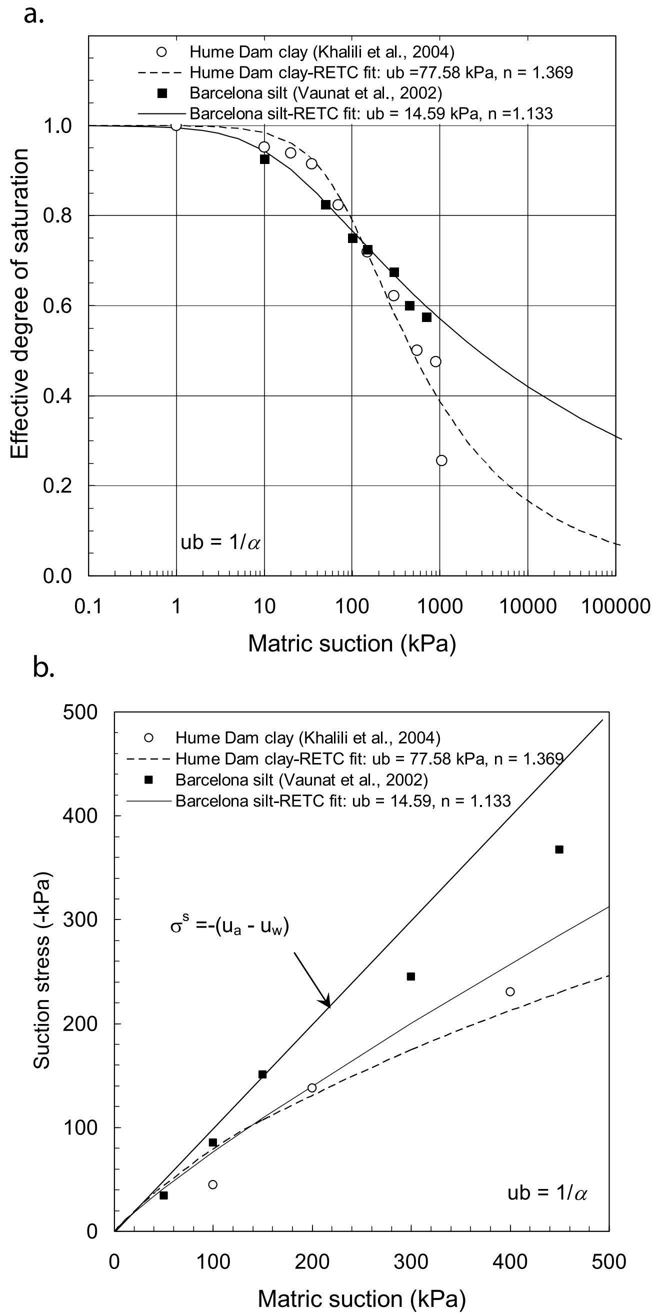

Figure 6. Quantitative validation of the closed-form equation for effective stress for Group 4 soils: (a) measured and fitted SWCCs for Barcelona silt and Hume Dam clay and (b) measured and predicted SSCCs for these soils.

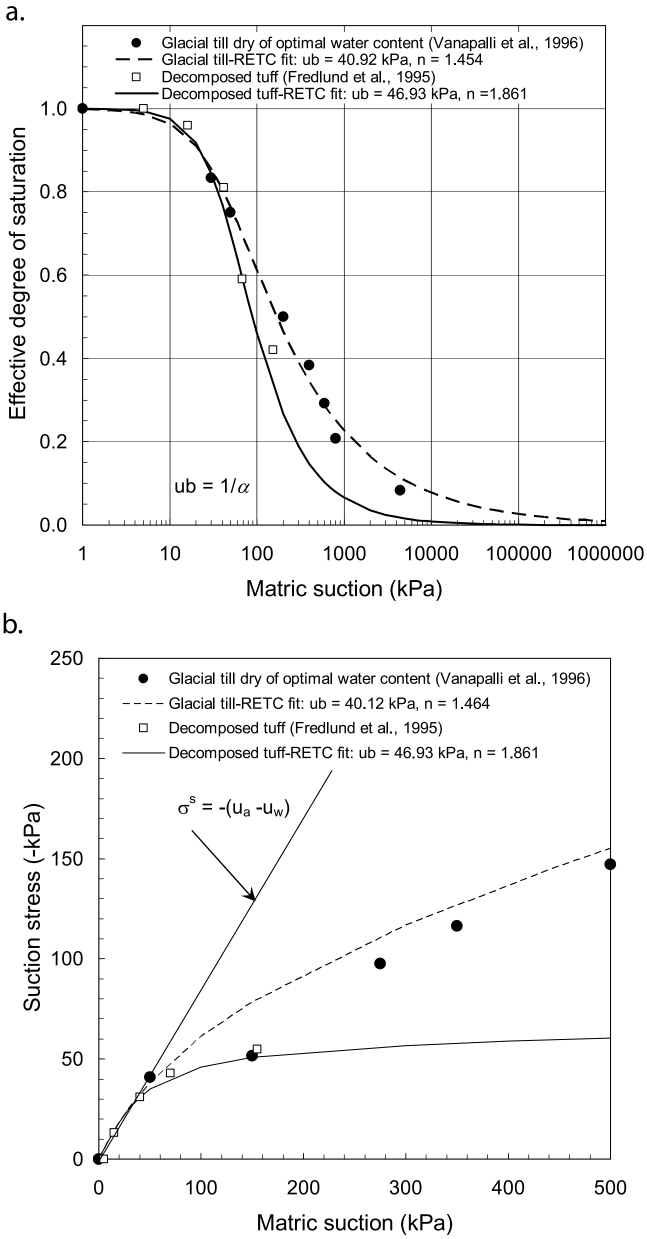

Figure 7. Quantitative validation of the closed-form equation for effective stress for Group 5 soils: (a) measured and fitted SWCCs for decomposed tuff and glacial till and (b) measured and predicted SSCCs for these soils.

sand and a minimum of about −0.85 kPa at 1 kPa of matric suction for the limestone agglomerates. This behavior cannot be predicted using either the classical effective stress approach of Bishop (equation (16)) or the extended Mohr-Coulomb criterion under the framework of two independent stress state variables, as they are not defined as a sole function of matric suction or saturation, but is well described by the proposed closed-form effective stress equation (22) when parameter $n > 2.0$.

4. Theoretical and Practical Implications

[30] A goal of theoretical inquiry is to describe physical behavior using mathematics. For such inquiry to be widely useful for science and engineering problems it must provide fundamental insight and also be simple and provide results of sufficient accuracy. Darcy's law for fluid flow and Terzaghi's equation for effective stress in saturated soils are examples of physical observations and mathematical abstraction that have had a profound impact on science and engineering practice. This general philosophy provides guidance in the current search for an effective stress equation for variably saturated soil. In what follows, we point out a few theoretical and practical implications of this work.

[31] We have shown that suction stress equation (21) is generally valid for a wide range of soils: from clay to

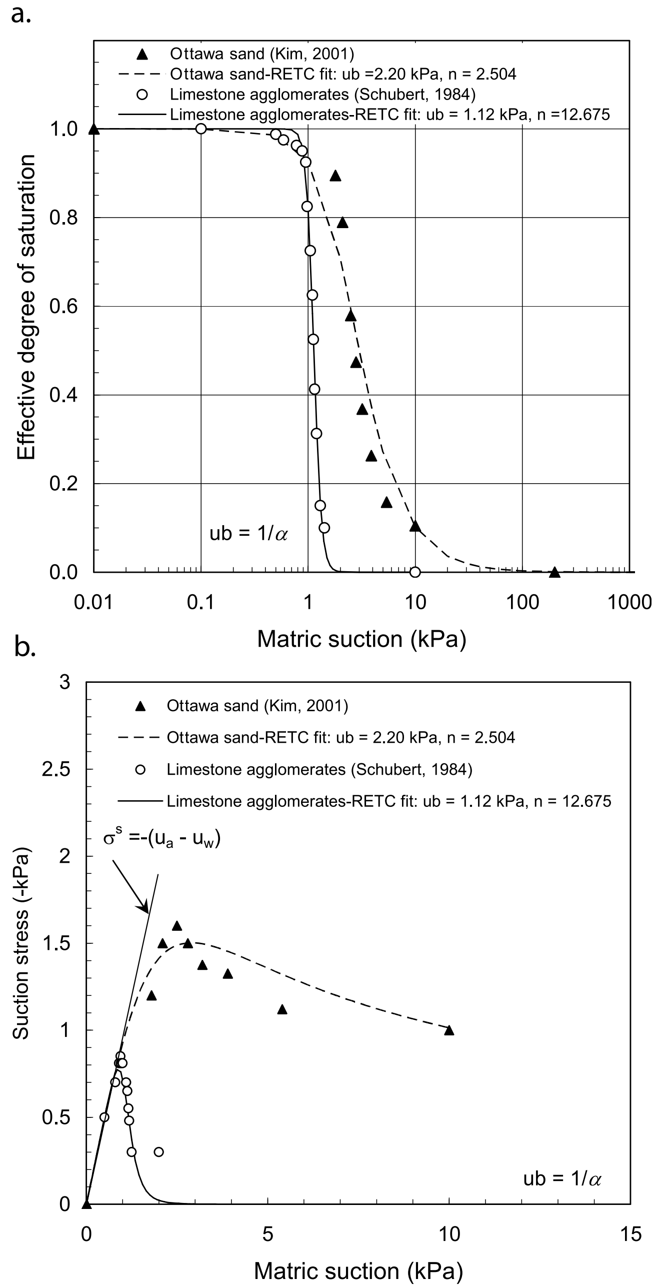

Figure 8. Quantitative validation of the closed-form equation for effective stress for Group 6 soils: (a) measured and fitted SWCCs for limestone agglomerates and Ottawa sand and (b) measured and predicted SSCCs for these soils.

limestone agglomerates (Table 1). If equation (21) for suction stress in unsaturated soil is valid in a manner similar to that of Terzaghi's effective stress for saturated soil, all limit equilibrium theories, such as those for lateral earth pressures, bearing capacity, and slope stability, in use today can be readily expanded for design and analysis under unsaturated soil conditions by simply replacing pore water pressure with the suction stress characteristic curve.

[32] Theoretically, the closed-form equation for suction stress (equation (21)) has several potentially far-reaching implications. First, it avoids the common theoretical and practical impediment embedded in Bishop's effective stress equation since there is no need to determine the coefficient of effective stress χ. Effective stress can be determined simply by reducing suction stress from shear strength test results (equation (26)) or by measuring soil water characteristic curves to identify parameters α and n. Equation (21) also captures the highly nonlinear and peak behavior of effective stress in sandy and silty soils. This behavior has been widely observed in the results of unsaturated shear strength experiments [e.g., *Vanapalli et al.*, 1996; *Kim*, 2001] that show a nonlinear increase-then-decrease dependence of shear strength with increasing matric suction (as shown in Figure 8b). This behavior is also well known from field examples such as the observation of a sudden collapse of unsaturated loess slopes upon wetting [e.g., *Higgins and Modeer*, 1996]. This work, for the first time, reconciles such behavior under the framework of effective stress. It also bridges the gap between the two independent stress variable framework and the effective stress framework by upscaling matric suction to suction stress for which matric suction is conceptualized as the controlling stress state variable. Matric suction, as demonstrated by *Lu* [2008], is not a stress at a typical REV level in soils and thus should not be defined as a stress variable. Distinctions between stress variables and stress state variables as well as their implications in theoretical soil mechanics are discussed by *Lu* [2008]. A thermodynamic distinction between the two independent stress variable frameworks and the effective stress frameworks is highlighted below.

[33] Under the two independent stress variable frameworks [*Fredlund and Morgenstern*, 1977], shear strength is conceptualized as a nonlinear function of both soil suction and saturation. For example, *Vanapalli et al.* [1996, equation (18)] proposed the following form for a modified Mohr-Coulomb criterion:

$$\tau_f = c' + (\sigma - u_a)\tan\phi' + (u_a - u_w)\tan\phi' \frac{S - S_r}{1 - S_r}. \qquad (27a)$$

In light of the effective stress equation (equation (18)), equation (27a) can be rearranged as

$$\tau_f = c' + \left[(\sigma - u_a) + (u_a - u_w)\frac{S - S_r}{1 - S_r}\right]\tan\phi'. \qquad (27b)$$

[34] Equation (27a) bears two fundamental differences with the proposed closed-form effective stress equation (equation (22)). First is a simple mathematical difference; equation (27a) or equation (27b) is a function of suction and saturation, whereas equation (22) is solely a function of suction. The second difference rests on a thermodynamic consideration. In equation (22), we state that effective stress varies with suction and that changes in suction stress lead to changes in energy in soils. In equation (27a), shear strength is conceptualized as the product of suction and saturation. From a continuum mechanics perspective, stress, such as suction stress, is part of energy (i.e., specific energy equals half stress times strain). Material strength, such as shear strength, is unambiguously not a stress and thus does not contribute to energy stored, released, or dissipated in the medium. This distinction places our effective stress equation (equation (22)) well within the framework of continuum mechanics where free energy is the basis for any thermodynamic formulation.

[35] We also seek a common basis for both fluid flow and effective stress in unsaturated soil. The SWCC (e.g., equation (19)) is commonly used in solutions to the governing equation (Richards equation) for variably saturated fluid flow in porous media. Fields of matric suction calculated in such a manner can be used directly in equations (21) and (22) for stress field and stability analysis under the same classical soil mechanics framework as saturated soils. Analyses of many practical problems, such as the state of stress in steep soil-mantled hillslopes during infiltration, require such a coupled hydromechanical model. As shown in equations (19) and (21), soil moisture and suction stress are fundamentally governed by a single variable: matric suction through the same pore fluid and solid and size distribution parameters, namely, α and n [*van Genuchten*, 1980].

[36] Beyond the well-established utility of Terzaghi's effective stress principle for strength behavior, it is also considered to be "effective" for some deformation problems such as the consolidation settlement. Strength and deformation behaviors of both saturated and unsaturated soil have been theoretically examined within the framework of critical state soil mechanics. Because the proposed effective stress equation (equation (18)) is governed by matric suction and is consistent with Terzaghi's effective stress, it can also be incorporated into elastic-plastic constitutive relations and the critical state soil mechanics frameworks.

5. Summary and Conclusions

[37] A closed-form equation (equation (22)) for effective stress in unsaturated soil is proposed and validated. The closed-form equation requires only two controlling parameters: the inverse of the air entry pressure α and the pore size spectrum number n. With them, effective stress in unsaturated soils ranging from sand to silt to clay can be accurately described. These two parameters are identical to those commonly used in the soil water characteristic curve equations proposed by *van Genuchten* [1980]. Therefore, the proposed closed-form equation for effective stress can be considered to be a unified description for phenomena of flow and stress in porous granular materials.

[38] The proposed closed-form equation for the suction stress characteristic curve is an expansion of Terzaghi's effective stress principle into unsaturated conditions and a unification of Bishop's unsaturated effective stress with Terzaghi's effective stress. The proposed closed-form equation (equation (22)) is an extension of Bishop's effec-

tive stress, as the effective stress is solely a function of either matric suction or the equivalent degree of saturation, whereas Bishop's effective stress demands knowledge of the coefficient of effective stress χ and concurrent knowledge of soil suction and the degree of saturation. Under the proposed equation, the transition from saturated to unsaturated states is continuous and smooth, ensuring mathematical consistency between Terzaghi's effective stress and the effective stress equation (equation (22)).

[39] Suction stress is the tensile stress. We show that the conception of it as an effective stress is thermodynamically justifiable when the surface tension contribution to the stress can be neglected, which apparently is the case in the funicular and capillary regimes. This justification leads to the statement that suction stress is an effective stress under the conditions of no external stress and is the energy consumed by capillary pore water. The proposed closed-form equation for effective stress and the justification for neglecting the surface tension term are validated against published testing data for a wide range of soils. We conclude that the proposed effective stress or effective stress equation is valid within the inherent errors of current experimental techniques.

[40] An important practical implication of using the proposed effective stress equation is that there is no need for any new shear strength criterion for unsaturated soil and all classical soil mechanics work on limit analysis can be readily extended to unsaturated soil conditions. Of equal practical importance is the development of a thermodynamically consistent framework for analyzing coupled hydromechanical problems such as the state of stress in hillslopes during infiltration.

Appendix A: Proof of a Smooth Transition for Suction Stress at Zero Matric Suction

[41] This section shows the mathematical consistency between Terzaghi's effective stress for saturated state and the proposed effective stress equation (equation (22)) for unsaturated state. We show here that suction stress and its first derivative in the closed-form expression are continuous as matric suction varies between the partially saturated case to the fully saturated case. Thus, taking the derivative of equation (17a) with respect to matric suction, we have

$$\frac{d\sigma^s}{d(u_a-u_w)} = -1 \qquad u_a-u_w \le 0. \tag{A1}$$

Taking the derivative of equation (17b) with respect to matric suction, we have

$$\begin{aligned}\frac{d\sigma^s}{d(u_a-u_w)} =& -\frac{1}{\{1+[\alpha(u_a-u_w)]^n\}^{\frac{n-1}{n}}} \\ &+\frac{(u_a-u_w)}{\{1+[\alpha(u_a-u_w)]^n\}^{2\frac{n-1}{n}}}\frac{n-1}{n} \\ &\cdot\{1+[\alpha(u_a-u_w)]^n\}^{\frac{n-1}{n}-1}n[\alpha(u_a-u_w)]^{n-1}\alpha\cdot \\ \frac{d\sigma^s}{d(u_a-u_w)} =& -\frac{1}{\{1+[\alpha(u_a-u_w)]^n\}^{\frac{n-1}{n}}} \\ &+\frac{(n-1)[\alpha(u_a-u_w)]^n}{\{1+[\alpha(u_a-u_w)]^n\}^{\frac{2n-1}{n}}} \qquad u_a-u_w \ge 0. \end{aligned}\tag{A2}$$

Taking the limit of $(u_a - u_w) \to 0$ for both the derivatives shown in equations (A1) and (A2), we have

$$\begin{aligned}&\mathrm{Lim}\frac{d\sigma^s}{d(u_a-u_w)}\bigg|_{(u_a-u_w)\to-0} = -1 \\ &\mathrm{Lim}\frac{d\sigma^s}{d(u_a-u_w)}\bigg|_{(u_a-u_w)\to+0} = \\ &\mathrm{Lim}\left(-\frac{1}{\{1+[\alpha(u_a-u_w)]^n\}^{\frac{n-1}{n}}}+\frac{(n-1)[\alpha(u_a-u_w)]^n}{\{1+[\alpha(u_a-u_w)]^n\}^{\frac{2n-1}{n}}}\right)_{(u_a-uu_a)\to-0} \\ &=\left(-\frac{1}{\{1+[0]^n\}^{\frac{n-1}{n}}}+\frac{(n-1)[0]^n}{\{1+[0]^n\}^{\frac{2n-1}{n}}}\right)=-1.\end{aligned}$$

Taking the limit of $(u_a - u_w) \to 0$ for both the suction stress shown in equations (21a) and (21b), we have

$$\begin{aligned}&\mathrm{Lim}\sigma^s|_{(u_a-u_w)\to-0} = -(u_a-u_w) = 0 \\ &\mathrm{Lim}\sigma^s|_{(u_a-u_w)\to+0} = -\frac{(u_a-u_w)}{(1+[\alpha(u_a-u_w)]^n)^{(n-1)/n}} \\ &\qquad = -\frac{(u_a-u_w)}{(1+[0]^n)^{(n-1)/n}} = 0.\end{aligned}$$

Since at zero matric suction, both suction stress and its derivatives have unique values, the closed-form equation (21) is smooth at the point where matric suction is zero.

Appendix B: Solution Regimes for the Suction Stress Characteristic Curve

[42] This section shows the theoretical possibility of the "peak" behavior of effective stress in unsaturated soils. To find out the maxima and minima of the closed-form equation for suction stress, we can take the derivative of equation (20) with respect to the equivalent degree of saturation and set the resulting equation to zero:

$$\frac{d\sigma^s}{dS_e} = -\frac{1}{\alpha}\left(S_e^{\frac{n}{1-n}}-1\right)^{\frac{1}{n}} - \frac{S_e}{\alpha}\frac{1}{n}\left(S_e^{\frac{n}{1-n}}-1\right)^{\frac{1}{n}-1}\frac{n}{1-n}S_e^{\frac{n}{1-n}-1} = 0$$

$$-\frac{1}{\alpha}\left(S_e^{\frac{n}{1-n}}-1\right)^{\frac{1}{n}} - \frac{S_e}{\alpha}\frac{1}{n}\frac{\left(S_e^{\frac{n}{1-n}}-1\right)^{\frac{1}{n}}}{\left(S_e^{\frac{n}{1-n}}-1\right)}\frac{n}{1-n}\frac{S_e^{\frac{n}{1-n}}}{S_e} = 0$$

$$-1-\frac{S_e^{\frac{n}{1-n}}}{\left(S_e^{\frac{n}{1-n}}-1\right)}\frac{1}{1-n} = 0$$

$$-(1-n)\left(S_e^{\frac{n}{1-n}}-1\right) - S_e^{\frac{n}{1-n}} = 0$$

$$-(2-n)S_e^{\frac{n}{1-n}} + (1-n) = 0$$

$$S_e|_{\min} = \left(\frac{1-n}{2-n}\right)^{\frac{1-n}{n}}. \tag{23c'}$$

It can be deduced that for equation (23c) to have a real solution, n must be >2.0.

[43] Substituting equation (23c) into (21), we have the minimum suction stress value:

$$\sigma^s|_{\min} = -\frac{1}{\alpha}\left(\frac{1-n}{2-n}\right)^{\frac{1-n}{n}}\left(\frac{1}{n-2}\right)^{\frac{1}{n}}. \tag{23a$'$}$$

The matric suction where the minimum suction stress is attained can also be found by substituting equation (23c) into the soil water characteristic curve (equation (19)):

$$S_e|_{\min} = \left\{\frac{1}{1+[\alpha(u_a-u_w)]^n}\right\}^{\frac{n-1}{n}}$$

$$\left(S_e|_{\min}\right)^{\frac{n}{n-1}} = \left\{\frac{1}{1+[\alpha(u_a-u_w)]^n}\right\}$$

$$1+[\alpha(u_a-u_w)]^n = \left(S_e|_{\min}\right)^{\frac{n}{1-n}}$$

$$(u_a-u_w)|_{\min} = \frac{1}{\alpha}\left[\left(S_e|_{\min}\right)^{\frac{n}{1-n}}-1\right]^{\frac{1}{n}}$$

$$(u_a-u_w)|_{\min} = \frac{1}{\alpha}\left[\left\{\left(\frac{1-n}{2-n}\right)^{\frac{1-n}{n}}\right\}^{\frac{n}{1-n}}-1\right]^{\frac{1}{n}}$$

$$(u_a-u_w)|_{\min} = \frac{1}{\alpha}\left[\left(\frac{1-n}{2-n}\right)-1\right]^{\frac{1}{n}}$$

$$(u_a-u_w)|_{\min} = \frac{1}{\alpha}\left(\frac{1}{n-2}\right)^{\frac{1}{n}} \tag{23b$'$}$$

Thus, we show that when $n \leq 2.0$, the closed-form equation (21) for suction stress is a monotonically decreasing function. When $n > 2.0$, the closed-form equation (21) for suction stress has a minimum suction stress value described by equation (23a), and it occurs when the equivalent degree of saturation is at the value given by equation (23c) and matric suction is at the value given by equation (23b).

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