PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 100, Number 1, May 1987

A CLOSED SURFACE OF GENUS ONE IN E^3 CANNOT CONTAIN SEVEN CIRCLES THROUGH EACH POINT

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ABSTRACT. There exists a closed surface of genus one in E^3 which contains six cirlces through each point, but any closed surface of genus one in E^3 cannot contain seven circles through each point.

1. Introduction. A sphere in E^3 is characterized as a closed surface which contains an infinite number of circles through each point. But we do not know a surface other than a sphere or a plane, which contains many circles through each point of it.

In 1980, Richard Blum [**1**] found a closed C^∞ surface of genus one which contains six circles through each point, and he gave a conjecture: A closed C^∞ surface in E^3 which contains seven circles through each point is a sphere.

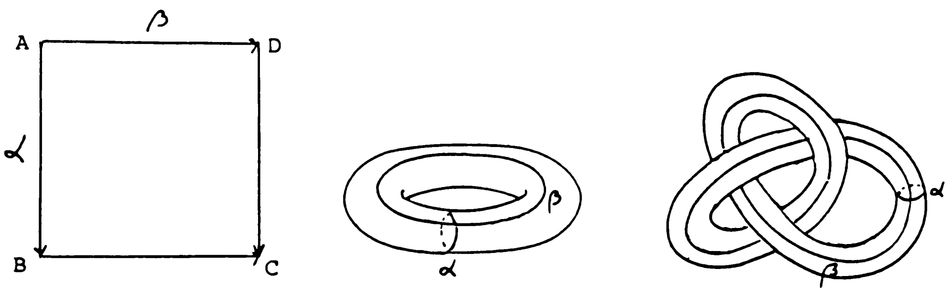
We proved in [**3**] that a closed simply connected C^∞ surface in E^3 which contains three circles through each point is a sphere.

The purpose of this paper is to obtain the following theorem for a closed C^∞ surface of genus one.

THEOREM. *A closed C^∞ surface of genus one in E^3 cannot contain seven circles through each point.*

The author wishes to express her hearty thanks to Professor K. Ogiue for his constant encouragement and valuable suggestions and Professor S. Kojima for his valuable suggestions.

2. Circles on a closed surface of genus one. Let M be a closed surface of genus one. Then M is topologically obtained from a square $ABCD$ by identifying $\overrightarrow{AB}$ with $\overrightarrow{DC}$ and $\overrightarrow{BC}$ with $\overrightarrow{AD}$.



Received by the editors September 10, 1985.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 53C45.
Key words and phrases. A closed surface of genus one, circles.

Let α and β be the closed curves on M corresponding to AB and BC, respectively. Then the homotopy classes $[\alpha]$ and $[\beta]$ defined by α and β are generators of the fundamental group $\pi_1(M)$ of M (see the diagram). Then the intersection numbers are given as follows:

$$\operatorname{Int}(\alpha,\alpha)=0, \quad \operatorname{Int}(\beta,\beta)=0,$$
$$\operatorname{Int}(\alpha,\beta)=1 \quad \text{and} \quad \operatorname{Int}(\beta,\alpha)=-1.$$

Let γ be a closed curve on M. Then the homotopy class $[\gamma]$ defined by γ can be written as $[\gamma]=m[\alpha]+n[\beta]$ for some integers m and n.

The following facts are basic (see, for example, [**4**]).

FACT 1. γ is a simple closed curve if and only if m and n are relatively prime.

FACT 2. γ is knotted if and only if $|m|\geq 2$ and $|n|\geq 2$, or M is knotted and $n\neq 0$.

We see from these facts that

($*$) if γ is a circle, then either $m=n=0$ or $|m|=1$ or $|n|=1$.

REMARK. Since we disregard the orientation of curves, we identify γ with $-\gamma$.

3. Lemmas. Let M be a closed C^∞ surface of genus one in E^3. Then we have the following lemmas for curves on M in view of ($*$):

LEMMA 1. *If two circles on M are homotopic and if they have only one point in common, then they are tangent to each other at the point.*

PROOF. Let c_1 and c_2 be two circles on M which belong to a homotopy class $m[\alpha]+n[\beta]$. Since the intersection number Int(,) is bilinear, $\operatorname{Int}(c_1,c_2)=\operatorname{Int}(m\alpha+n\beta,m\alpha+n\beta)=m^2\operatorname{Int}(\alpha,\alpha)+mn\operatorname{Int}(\alpha,\beta)+nm\operatorname{Int}(\beta,\alpha)+n^2\operatorname{Int}(\beta,\beta)=0$.

Therefore c_1 and c_2 must be tangent to each other (cf. [**4**]).

LEMMA 2. *Let $c_1\in m[\alpha]+[\beta]$ and $c_2\in[\alpha]+n[\beta]$. If $mn\geq 4$ or $mn\leq -2$, then at least one of c_1 and c_2 cannot be a circle.*

PROOF. By the assumption, $\operatorname{Int}(c_1,c_2)=mn-1\geq 3$ or ≤ -3, so that c_1 and c_2 must have more than two points in common.

LEMMA 3. (1) *If $|n'-n|=2$, then two circles $c_1\in[\alpha]+n[\beta]$ and $c_2\in[\alpha]+n'[\beta]$ have two points in common.*

(2) *If $|n'-n|\geq 3$, then two curves $c_1\in[\alpha]+n[\beta]$ and $c_2\in[\alpha]+n'[\beta]$ must have more than two points in common, so that at least one of them cannot be a circle.*

(3) *Similar results hold for two curves $c_1\in m[\alpha]+[\beta]$ and $c_2\in m'[\alpha]+[\beta]$.*

PROOF. (1) By the assumption, $\operatorname{Int}(c_1,c_2)=n'-n=\pm 2$, so that c_1 and c_2 have two points in common.

(2) By the assumption, $\operatorname{Int}(c_1,c_2)=n'-n\geq 3$ or ≤ -3, so that c_1 and c_2 must have more than two points in common.

LEMMA 4. *If two circles $c_1\in 0\cdot[\alpha]+0\cdot[\beta]$ and c_2 have only one point in common, then they must be tangent to each other at the point.*

PROOF. $\operatorname{Int}(c_1,c_2)=0$, so that c_1 and c_2 must be tangent to each other at the point.

4. Proof of the theorem. Now we will prove our theorem by using the above lemmas and the following result:

PROPOSITION [**3**]. *Let M be a C^∞ surface in E^3. Suppose that, through each point of M, there exist three circles of E^3 contained in M, any two of which are tangent to each other or have two points in common. Then M is (a part of) a sphere or a plane.*

Let p be an arbitrary point of M. Then, from Lemmas 2 and 3, we see that the maximal sets of homotopy classes which may contain circles through p simultaneously are

$$\{@, [\alpha], m[\alpha] + [\beta], (m+1)[\alpha] + [\beta], (m+2)[\alpha] + [\beta]\}$$

for some integer m, or

$$\{@, [\beta], [\alpha] + n[\beta], [\alpha] + (n+1)[\beta], [\alpha] + (n+2)[\beta]\}$$

for some integer n, where @ stands for $0 \cdot [\alpha] + 0 \cdot [\beta]$.

It follows from Lemmas 1–4 that each of the above sets is divided into three subsets with respect to the property that any two circles through p which belong to homotopy classes in one subset are tangent to each other or have two points in common:

$$\begin{aligned} &\{@, [\alpha], m[\alpha] + [\beta], (m+1)[\alpha] + [\beta], (m+2)[\alpha] + [\beta]\} \\ &\qquad = \{@, [\alpha]\} \cup \{m[\alpha] + [\beta], (m+2)[\alpha] + [\beta]\} \cup \{(m+1)[\alpha] + [\beta]\}, \\ &\{@, [\beta], [\alpha] + n[\beta], [\alpha] + (n+1)[\beta], [\alpha] + (n+2)[\beta]\} \\ &\qquad = \{@, [\beta]\} \cup \{[\alpha] + n[\beta], [\alpha] + (n+2)[\beta]\} \cup \{[\alpha] + (n+1)[\beta]\}. \end{aligned}$$

Suppose there exist seven circles through p. Then at least one subset must contain three circles and any two of them are tangent to each other or have two points in common. Since p is arbitrary, M must be a sphere or a plane by the Proposition. This contradicts the fact that M is of genus one. Q.E.D.

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