A cluster expansion formula (A_n case)

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Abstract

We consider the Ptolemy cluster algebras, which are cluster algebras of finite type A (with non-trivial coefficients) that have been described by Fomin and Zelevinsky using triangulations of a regular polygon. Given any seed Σ in a Ptolemy cluster algebra, we present a formula for the expansion of an arbitrary cluster variable in terms of the cluster variables of the seed Σ. Our formula is given in a combinatorial way, using paths on a triangulation of the polygon that corresponds to the seed Σ.

0 Introduction

Cluster algebras, introduced in [FZ1], are commutative algebras equipped with a distinguished set of generators, the *cluster variables*. The cluster variables are grouped into sets of constant cardinality n, the *clusters*, and the integer n is called the *rank* of the cluster algebra. Starting with an initial seed consisting of a cluster $\mathbf{x}$ together with a skew symmetrizable integer $n \times n$ matrix $B = (b_{ij})$ and a coefficient vector $\mathbf{p}$, the set of cluster variables is obtained by repeated application of so called *mutations*.

Ptolemy cluster algebras have been introduced by Fomin and Zelevinsky in [FZ2, section 12] as examples of cluster algebras of type A. The Ptolemy algebra of rank n is described using the triangulations of a regular polygon with $n+3$ vertices. In this description, the seeds of the cluster algebra are in bijection with the triangulations of the polygon. The cluster of the seed corresponds to the diagonals, while the coefficients of the seed correspond to the boundary edges of the triangulation. The Laurent phenomenon [FZ1] states that, given an arbitrary seed Σ one can write any cluster variable of the cluster algebra as a Laurent polynomial in the cluster variables and the coefficients of the seed Σ.

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The main result of this paper is an explicit formula for these Laurent polynomials, see Theorem 1.2. Each term of the Laurent polynomial is given by a path on the triangulation corresponding to the seed Σ. As an application, we prove the positivity conjecture of Fomin and Zelevinsky [FZ1] for Ptolemy algebras, see Corollary 1.7.

There is an interesting connection between our work and that of Propp [P] who uses perfect matchings arising from the triangulation to calculate these Laurent polynomials.

The polygon model has also been used in [CCS] to construct the cluster category associated to the cluster algebra, compare [BMRRT]. In that context, the cluster algebra has trivial coefficients and our formula naturally applies to that situation, see Remark 1.5. Therefore, there is an interesting intersection with the work of Caldero and Chapoton [CC], who have obtained a formula for cluster expansions when the given seed is acyclic, meaning that each triangle in the triangulation has at least one side on the boundary of the polygon. Their formula, and its generalization by Caldero and Keller [CK], also applies to cluster algebras (with trivial coefficients) of other types, but, again, only in the case where the seed Σ is acyclic. Their description uses the representation theory of finite dimensional algebras and is very different from ours.

We have been informed that our cluster expansion formula was also known to Carroll and Price [CP] and to Fomin and Zelevinsky [FZ3].

The author thanks Hugh Thomas for interesting discussions on the subject.

1 Cluster expansions in the Ptolemy algebra

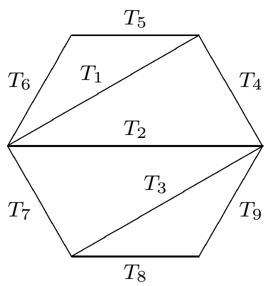
Throughout this paper, let n be a positive integer and let P be a regular polygon with $n+3$ vertices. A *diagonal* in P is a line segment connecting two non-adjacent vertices of P and two diagonals are said to be *crossing* if they intersect in the interior of the polygon. A *triangulation* T is a maximal set of non-crossing diagonals together with all boundary edges. Any triangulation T has n diagonals and $n+3$ boundary edges. Denote the boundary edges of P by $T_{n+1}, \ldots, T_{2n+3}$.

1.1 The Ptolemy cluster algebra

We recall some facts about the Ptolemy cluster algebra of rank n from [FZ2, section 12.2]. The cluster variables x_M of this algebra are in bijection with the diagonals M of the polygon P, and the generators of its coefficient semifield are in bijection with the boundary edges $T_{n+1}, \ldots, T_{2n+3}$ of P. To be more precise, the coefficient semifield is the tropical semifield $\mathrm{Trop}(x_{n+1}, x_{n+2}, \ldots, x_{2n+3})$, which is a free abelian group, written multiplicatively, with generators $x_{n+1}, \ldots, x_{2n+3}$, and with auxiliary addition $\oplus$ given by

$$\prod_j x_j^{a_j} \oplus \prod_j x_j^{b_j} = \prod_j x_j^{\min(a_j, b_j)}.$$

Clusters are in bijection with triangulations of the polygon P. Given a triangulation $T = \{T_1, \ldots, T_n, T_{n+1}, \ldots, T_{2n+3}\}$ let $\mathbf{x}_T = \{x_1, \ldots, x_n\}$ be the corresponding cluster,



$$
\begin{aligned}
(p_1^+,\, p_1^-) &= (x_5\,,\, x_4x_6)\\
(p_2^+,\, p_2^-) &= (x_4x_7\,,\, 1\,)\\
(p_3^+,\, p_3^-) &= (x_8\,,\, x_7x_9)
\end{aligned}
\qquad
B = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}
$$

Figure 1: Snake triangulation, initial coefficients and initial matrix in rank 3

where we use the notation $x_i = x_{T_i}$ for short. The mutation in direction k is described as follows: T_k is a diagonal in a unique quadrilateral in T. Let $T_{k'}$ be the other diagonal in that quadrilateral. Then the mutation in direction k of T is the triangulation obtained from T by replacing T_k by $T_{k'}$. The corresponding exchange relation is $x_k x_{k'} = x_a x_c + x_b x_d$, where T_a, T_c are two opposite sides of the quadrilateral and T_b, T_d are the other two opposite sides.

Let us choose the initial seed $(\mathbf{x}, \mathbf{p}, B)$ consisting of a cluster $\mathbf{x}$, a coefficient vector $\mathbf{p} = (p_1^+, p_1^-, p_2^+, p_2^-, \ldots, p_n^+, p_n^-) \in (\mathrm{Trop}(x_{n+1}, x_{n+2}, \ldots, x_{2n+3}))^{2n}$ and a $(2n+3)\times n$ matrix $B = (b_{ij})$ as follows: Taking the initial cluster to be the snake triangulation of [FZ2, section 12], the initial matrix $B = (b_{ij})$ is given by the conditions $b_{ii} = 0, b_{ij} \in \{-1, 0, 1\}$ and $b_{ij} = 1$ (respectively $b_{ij} = -1$) if and only if $i \neq j$, the edges T_i and T_j bound the same triangle and the sense of rotation from T_i to T_j is counterclockwise (respectively clockwise). The corresponding initial coefficient vector is given by

$$
p_j^+ = \prod_{i \geq n+1 \,:\, b_{ij}=1} x_j \qquad \text{and} \qquad p_j^- = \prod_{i \geq n+1 \,:\, b_{ij}=-1} x_j .
$$

An example of rank 3 is given in Figure 1.

1.2 T-paths

Let $T = \{T_1, T_2, \ldots, T_n, T_{n+1}, \ldots, T_{2n+3}\}$ be a triangulation of the polygon P, where $T_1, \ldots, T_n$ are diagonals and $T_{n+1}, \ldots, T_{2n+3}$ are boundary edges. Let a and b be two non-adjacent vertices on the boundary and let $M_{a,b}$ be the diagonal that connects a and b.

Definition 1 *A* T-path α *from* a *to* b *is a sequence*

$$
\alpha = (a_0, a_1, \ldots, a_{\ell(\alpha)} \mid i_1, i_2, \ldots, i_{\ell(\alpha)})
$$

such that

(T1) $a = a_0, a_1, \ldots, a_{\ell(\alpha)} = b$ *are vertices of* P.

(T2) *$i_k \in \{1, 2, \ldots, 2n+3\}$ such that T_{i_k} connects the vertices $a_{i_{k-1}}$ and a_{i_k} for each $k = 1, 2, \ldots, \ell(\alpha)$.*

(T3) *$i_j \neq i_k$ if $j \neq k$.*

(T4) *$\ell(\alpha)$ is odd.*

(T5) *T_{i_k} crosses $M_{a,b}$ if k is even.*

(T6) *If $j < k$ and both T_{i_j} and T_{i_k} cross $M_{a,b}$ then the crossing point of T_{i_j} and $M_{a,b}$ is closer to the vertex a than the crossing point of T_{i_k} and $M_{a,b}$.*

Thus, a T-path from a to b is a path on the edges of the triangulation T, that does not use any edge twice, whose crossing points with $M_{a,b}$ are progressing from a towards b, and, when classifying the edges into even and odd edges according to their order of appearance, then every even edge is crossing $M_{a,b}$.

To any T-path $\alpha = (a_0, a_1, \ldots, a_{\ell(\alpha)} \mid i_1, i_2, \ldots, i_{\ell(\alpha)})$, we associate an element $x(\alpha)$ in the cluster algebra by

$$x(\alpha) = \prod_{k \text{ odd}} x_{i_k} \prod_{k \text{ even}} x_{i_k}^{-1}. \tag{1}$$

Definition 2 *Let $\mathcal{P}_T(a,b)$ denote the set of T-paths from a to b.*

Lemma 1.1 *Let $\alpha, \alpha' \in \mathcal{P}_T(a,b)$ with $\alpha \neq \alpha'$. Then $x(\alpha) \neq x(\alpha')$.*

Proof. Suppose $x(\alpha) = x(\alpha')$. Then (T3) implies that the set of even edges and the set of odd edges are the same in α and α'. From conditions (T5) and (T6) it follows that the order of the even edges is the same, hence the order of all edges is the same, whence $\alpha = \alpha'$. □

1.3 Expansion formula

The following theorem is our main result.

Theorem 1.2 *Let a and b be two non-adjacent vertices of P, let $M = M_{a,b}$ be the diagonal connecting a and b and let x_M be the corresponding cluster variable. Then*

$$x_M = \sum_{\alpha \in \mathcal{P}_T(a,b)} x(\alpha). \tag{2}$$

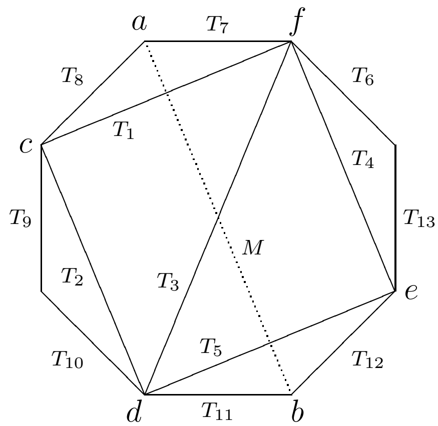
Remark 1.3 Because of conditions (T3) and (T5), each $x(\alpha)$ is a reduced fraction whose denominator is a product of cluster variables.

Remark 1.4 By Lemma 1.1, each term in the sum of equation (2) appears with multiplicity one.

Remark 1.5 Formula (2) also applies to type A cluster algebras with trivial coefficients by setting $x_t = 1$ for $t = n+1, n+2, \ldots, 2n+3$.

The proof of Theorem 1.2 will be given in section 2. To illustrate the statement, we give an example here.

Example 1.6 *The following figure shows a triangulation $T = \{T_1, \ldots, T_{13}\}$ and a (dotted) diagonal $M = M_{a,b}$. Next to it is a complete list of elements of $\mathcal{P}_T(a,b)$.*



$(a, f, d, b \mid 7, 3, 11)$
$(a, f, c, d, e, b \mid 7, 1, 2, 5, 12)$
$(a, c, f, e, d, b \mid 8, 1, 4, 5, 11)$
$(a, c, f, d, e, b \mid 8, 1, 3, 5, 12)$
$(a, f, c, d, f, e, d, b \mid 7, 1, 2, 3, 4, 5, 11)$

Theorem 1.2 thus implies that

$$x_M = \frac{x_7 x_{11}}{x_3} + \frac{x_7 x_2 x_{12}}{x_1 x_5} + \frac{x_8 x_4 x_{11}}{x_1 x_5} + \frac{x_8 x_3 x_{12}}{x_1 x_5} + \frac{x_7 x_2 x_4 x_{11}}{x_1 x_3 x_5}.$$

1.4 Positivity

In the case of the Ptolemy algebra, the following positivity conjecture of [FZ1] is a direct consequence of Theorem 1.2 and Remarks 1.3 and 1.4.

Corollary 1.7 *Let x be any cluster variable in the Ptolemy cluster algebra and let $\{x_1, \ldots, x_n\}$ be any cluster. Let*

$$x = \frac{f(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{2n+3})}{x_1^{d_1} \ldots x_n^{d_n}}$$

be the expansion of x in the cluster $\{x_1, \ldots, x_n\}$, where f is a polynomial which is not divisible by any of the $x_1, \ldots, x_n$. Then the coefficients of f are either 0 or 1, thus, in particular, they are non-negative integers.

2 Proof of Theorem 1.2

This section is devoted to the proof of our main result. Let T be a triangulation of the polygon P, let a and b be two vertices of P and $M = M_{a,b}$ the diagonal connecting a

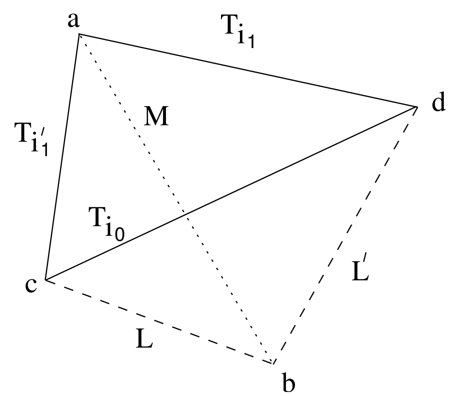


Figure 2: Proof of Theorem 1.2, solid edges are in the triangulation, dashed edges are not necessarily in the triangulation and dotted edges are not in the triangulation

and b. Suppose that $M \notin T$. Among all diagonals of T that cross M, there is a unique one, T_{i_0}, such that its intersection point with M is the closest possible to the vertex a. Then there is a unique triangle in T having T_{i_0} as one side and the vertex a as third point. Denote the other two sides of this triangle by T_{i_1} and $T_{i'_1}$ and let c be the common endpoint of $T_{i'_1}$ and T_{i_0}, and d the common endpoint of T_{i_1} and T_{i_0} (see Figure 2). Note that $T_{i_1}, T_{i'_1}$ may be boundary edges. Now consider the unique quadrilateral in which M and T_{i_0} are the diagonals. Two of its sides are T_{i_1} and $T_{i'_1}$. Denote the other two sides by L and L' in such a way that L is the side opposite to T_{i_1} (see Figure 2). The symmetry of this configuration will be used in the sequel. We will keep this setup for the rest of this section.

Lemma 2.1 *(a) If $T_i \in T$ crosses L (respectively L'), then T_i crosses M.*

(b) If $T_i \in T$ is adjacent to a, then T_i does not cross L nor L'.

(c) If $T_i \in T$ crosses M and does not cross L (respectively L'), then T_i is adjacent to c (respectively d).

(d) If $T_i, T_j \in T$ both cross M and L (respectively L'), then the order of the crossing points is the same.

Proof. This follows directly from the construction. □

Let $\mathcal{P}_T(a,b)_j$ denote the subset of $\mathcal{P}_T(a,b)$ of all T-paths α that start with the edge T_j and let $\mathcal{P}_T(a,b)_{-j}$ be the subset of $\mathcal{P}_T(a,b)$ of all T-paths α that do not contain the edge T_j. Similarly, let $\mathcal{P}_T(a,b)_{jk}$ denote the subset of $\mathcal{P}_T(a,b)_j$ of all T-paths α that start with the edges T_jT_k and let $\mathcal{P}_T(a,b)_{j,-k}$ be the subset of $\mathcal{P}_T(a,b)_j$ of all T-paths α that start with the edge T_j and do not contain the edge T_k.

Lemma 2.2 *We have $\mathcal{P}_T(a,b) = \mathcal{P}_T(a,b)_{i_1} \sqcup \mathcal{P}_T(a,b)_{i'_1}$.*

Proof. Let $\alpha = (a_0, a_1, \ldots, a_{\ell(\alpha)} \mid j_1, j_2, \ldots, j_{\ell(\alpha)})$ be an arbitrary element of $\mathcal{P}_T(a,b)$. We know that $T_{j_1} \in T$ and that one of its endpoints is the vertex a. Moreover, T_{j_2} crosses M, by property (T5) of T-paths. Let P_a and P_b denote the two pieces of the polygon obtained by cutting along T_{i_0}, where the vertex a lies in the piece P_a, while the vertex b lies in P_b. The piece P_a also contains the diagonal T_{j_1} whereas the piece P_b contains all the diagonals of T that cross M and, therefore, P_b contains T_{j_2}. Consequently, the diagonals T_{j_1}, T_{j_2} and T_{i_0} share one endpoint, which is either c or d. There are therefore two possibilities: either $T_{j_1} = T_{i_1}$ or $T_{j_1} = T_{i'_1}$. □

Lemma 2.3 *(a)* $\mathcal{P}_T(a,b)_{i_1} = \mathcal{P}_T(a,b)_{i_1 i_0} \sqcup \mathcal{P}_T(a,b)_{i_1,-i_0}$

(b) $\mathcal{P}_T(a,b)_{i'_1} = \mathcal{P}_T(a,b)_{i'_1 i_0} \sqcup \mathcal{P}_T(a,b)_{i'_1,-i_0}$

Proof. By construction, T_{i_0} is the edge of the triangulation T such that its crossing point with M is the closest possible to the vertex a. Hence, the result follows from condition (T6). □

Now let $\gamma = (c_0, \ldots, c_{\ell(\gamma)} \mid j_1, \ldots, j_{\ell(\gamma)})$ be any T-path from c to b. Suppose first that $j_1 = i_0$, thus $c_1 = d$. In this case, let $f(\gamma)$ be the path obtained from γ by replacing the first edge i_0 by i_1, that is

$$f(\gamma) = (a, c_1, \ldots, c_{\ell(\gamma)} \mid i_1, j_2, \ldots, j_{\ell(\gamma)}).$$

Suppose now that $j_k \neq i_0$ for all k. In this case, let $g(\gamma)$ be the composition of the paths $(a, d, c \mid i_1, i_0)$ and γ, that is

$$g(\gamma) = (a, d, c_0, c_1, \ldots, c_{\ell(\gamma)} \mid i_1, i_0, j_1, j_2, \ldots, j_{\ell(\gamma)}).$$

Let us check that $f(\gamma)$ and $g(\gamma)$ are elements of $\mathcal{P}_T(a,b)$. Indeed, the properties (T1),(T2) and (T4) are immediate and (T5) follows from Lemma 2.1(a). In order to show (T3), we need to prove that $i_1 \neq j_k$ for all k; but this follows from Lemma 2.1(b) and the fact that T_{i_1} is adjacent to a. Finally, (T6) holds since the crossing point of T_{i_0} and $M_{a,b}$ is the closest possible to a. We have the following lemma.

Lemma 2.4 *The maps f and g induce bijections*

$$f : \mathcal{P}_T(c,b)_{i_0} \to \mathcal{P}_T(a,b)_{i_1,-i_0} \qquad \textit{and} \qquad g : \mathcal{P}_T(c,b)_{-i_0} \to \mathcal{P}_T(a,b)_{i_1 i_0},$$

and

$$x(f(\gamma)) = \frac{x_{i_1}}{x_{i_0}}\, x(\gamma) \quad \textit{and} \quad x(g(\gamma)) = \frac{x_{i_1}}{x_{i_0}}\, x(\gamma) \tag{3}$$

Proof. The formulas (3) follow directly from the definitions of f and g. These formulas together with Lemma 1.1 imply the injectivity of f and g. To show the surjectivity of f, suppose that $\mathcal{P}_T(a,b)_{i_1,-i_0}$ is not empty and let $\alpha \in \mathcal{P}_T(a,b)_{i_1,-i_0}$ be an arbitrary element. Say

$$\alpha = (a, d, a_2, a_3, \ldots, a_{\ell(\alpha)} \mid i_1, j_2, j_3, j_4, \ldots, j_{\ell(\alpha)}).$$

We need to show that the path

$$\gamma = (c, d, a_2, a_3, \ldots, a_{\ell(\alpha)} \mid i_0, j_2, j_3, j_4, \ldots, j_{\ell(\alpha)})$$

is an element of $\mathcal{P}_T(c,b)_{i_0}$. Conditions (T1),(T2),(T4) hold since $\alpha \in \mathcal{P}_T(a,b)$, condition (T3) holds because the path α does not contain the edge T_{i_0} and condition (T6) holds because of Lemma 2.1(a). Moreover, since $\mathcal{P}_T(a,b)_{i_1,-i_0}$ is not empty, there exists a diagonal in $T \setminus \{T_{i_0}\}$ which is adjacent to d and crosses M. Since T is a triangulation, it follows that any diagonal in $T \setminus \{T_{i_0}\}$ that crosses M also crosses L. Thus γ satisfies condition (T5), because $\alpha \in \mathcal{P}_T(a,b)$. This shows that $\gamma \in \mathcal{P}_T(c,b)$, and since γ starts with the edge i_0, we have $\gamma \in \mathcal{P}_T(c,b)_{i_0}$. Hence f is surjective.

It remains to show that g is surjective. Let $\alpha \in \mathcal{P}_T(a,b)_{i_1 i_0}$ be arbitrary. Say

$$\alpha = (a, d, c, a_3, \ldots, a_{\ell(\alpha)} \mid i_1, i_0, j_3, j_4, \ldots, j_{\ell(\alpha)}).$$

We have to show that

$$\gamma = (c, a_3, \ldots, a_{\ell(\alpha)} \mid j_3, j_4, \ldots, j_{\ell(\alpha)}) \in \mathcal{P}_T(c,b).$$

Conditions (T1)-(T4) hold for γ because $\alpha \in \mathcal{P}_T(a,b)$ and condition (T6) holds because of Lemma 2.1(a). Let us show (T5). We need to show that any even edge of γ crosses L. Since $\alpha \in \mathcal{P}_T(a,b)$, we know that every even edge of γ crosses M. Thus by Lemma 2.1(c), if there is an even edge of γ that does not cross L then this edge has to be adjacent to c. Since γ starts at c, its first even edge T_{j_4} cannot be adjacent to c, and thus T_{j_4} crosses both M and L. Then, since α satisfies (T6), every even edge of γ crosses M and L. This shows (T5). Hence $\gamma \in \mathcal{P}_T(c,b)$ and g is surjective. $\square$

Lemma 2.5 *We have*

$$(a) \qquad \sum_{\gamma \in \mathcal{P}_T(c,b)} x(\gamma)\, \frac{x_{i_1}}{x_{i_0}} = \sum_{\alpha \in \mathcal{P}_T(a,b)_{i_1}} x(\alpha)$$

$$(b) \qquad \sum_{\gamma \in \mathcal{P}_T(d,b)} x(\gamma)\, \frac{x_{i'_1}}{x_{i_0}} = \sum_{\alpha \in \mathcal{P}_T(a,b)_{i'_1}} x(\alpha).$$

Proof. The first statement follows from Lemma 2.3(a), Lemma 2.4 and the fact that $\mathcal{P}_T(c,b) = \mathcal{P}_T(c,b)_{i_0} \sqcup \mathcal{P}_T(c,b)_{-i_0}$. The second statement follows by symmetry. $\square$

Proof of Theorem 1.2. For any $T_i \in T$, let $e(T_i, M) \in \{0,1\}$ be the number of crossings of the diagonals T_i and M. Then the total number of crossings between M and T is $e(T,M) = \sum_{T_i \in T} e(T_i, M)$.

We prove the theorem by induction on $e(T,M)$. If $e(T,M) = 0$, then $M = T_i \in T$ for some $i \in \{1, \ldots, n\}$. In this case, no element of T crosses M and, by condition (T5), the set $\mathcal{P}_T(a,b)$ contains exactly one element: $(a, b \mid i)$. Thus

$$\sum_{\alpha \in \mathcal{P}_T(a,b)} x(\alpha) = x(a, b \mid i) = x_i = x_M.$$

Suppose now that $e(T, M) \geq 1$. As before, consider the unique quadrilateral in T in which M and T_{i_0} are the diagonals (see Figure 2). Thus, in the cluster algebra, we have the following exchange relation

$$x_M \, x_{i_0} = x_{i_1} \, x_L + x_{i'_1} \, x_{L'}. \tag{4}$$

Moreover, any diagonal in T that crosses L (respectively L') also crosses M, by Lemma 2.1(a), and, moreover, T_{i_0} crosses M but crosses neither L nor L'. Thus $e(T, L) < e(T, M)$ and $e(T, L') < e(T, M)$, and by induction hypothesis

$$x_L = \sum_{\gamma \in \mathcal{P}_T(c,b)} x(\gamma) \qquad \text{and} \qquad x_{L'} = \sum_{\gamma \in \mathcal{P}_T(d,b)} x(\gamma).$$

Therefore, we can write the exchange relation (4) as

$$x_M \;\; = \sum_{\gamma \in \mathcal{P}_T(c,b)} x(\gamma) \, \frac{x_{i_1}}{x_{i_0}} \;\; + \sum_{\gamma \in \mathcal{P}_T(d,b)} x(\gamma) \, \frac{x_{i'_1}}{x_{i_0}}.$$

The theorem now follows from Lemma 2.5 and Lemma 2.2. □

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