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A COHEN TYPE INEQUALITY FOR LEGENDRE-SOBOLEV EXPANSIONS

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Abstract

Let introduce the Sobolev-type inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)dx + N[f'(1)g'(1) + f'(-1)g'(-1)],$$

where $N \geq 0$. In this paper we prove a Cohen type inequality for Fourier expansion in terms of the polynomials associated to the Sobolev inner product.

1 Introduction and Main Result

The purpose of this paper is to derive a lower bound for the norm associated to the Sobolev spaces of the polynomial expansions relative to Sobolev inner product. For classical orthogonal expansions such inequalities were proved by Dreseler and Soardi [5] and Markett [7].

Let us first introduce some notation. We shall say that $f \in L^p$ if f is measurable on the $[-1,1]$ and $\|f\|_{L^p} < \infty$, where

$$\|f\|_{L^p} = \begin{cases} \left(\int_{-1}^{1} |f(x)|^p dx\right)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty, \\ \underset{-1<x<1}{ess\ sup}\, |f(x)| & \text{if } p = \infty. \end{cases}$$

Now let us introduce the Sobolev spaces

$$S_p = \{f \ : \ \|f\|_{S_p}^p = \|f\|_{L^p}^p + N[|f'(1)|^p + |f'(-1)|^p] < \infty\}, \qquad 1 \leq p < \infty,$$

$$S_\infty = \{f \ : \ \|f\|_{S_\infty} = \|f\|_{L^\infty(d\mu)} < \infty\}, \qquad p = \infty.$$

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We also introduce the discrete Sobolev-type inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)dx + N[f'(1)g'(1) + f'(-1)g'(-1)] \tag{1}$$

for any functions f, g for which the right side makes sense and $N \geq 0$. We denote by $\hat{B}_n$ the orthonormal polynomials with respect to the inner product (1) (see [2], [3], [6]). We call these polynomials the Legendre-Sobolev polynomials. For $N = 0$, denoted by p_n, we have classical Legendre orthonormal polynomials.

For $f \in S_1$, the Fourier expansion in Legendre-Sobolev polynomials is

$$\sum_{k=0}^{\infty} \hat{f}(k)\hat{B}_k(x), \tag{2}$$

$$\hat{f}(k) = \langle f, \hat{B}_k\rangle.$$

The Cesàro means of order δ of the expansion (2) are defined by (see [9, p. 76-77])

$$\sigma_n^{\delta} f(x) = \sum_{k=0}^{n} \frac{A_{n-k}^{\delta}}{A_n^{\delta}} \hat{f}(k)\hat{B}_k(x),$$

where $A_k^{\delta} = \binom{k+\delta}{k}$.

For a function $f \in S_p$ and a given sequence $\{c_{k,n}\}_{k=0}^{n}$ of complex numbers with $|c_{n,n}| > 0$, we define the operator T_n^N by

$$T_n^N(f) = \sum_{k=0}^{n} c_{k,n}\hat{f}(k)\hat{B}_k.$$

Now we formulate main result

Theorem 1. *Let $1 \leq p \leq \infty$. There exists a positive constant c, independent of n, such that*

$$\|T_n^N\|_{[S_p]} \geq c|c_{n,n}| \begin{cases} n^{\frac{2}{p}-\frac{3}{2}} & \text{if } 1 \leq p < 4/3 \\ (log\ n)^{\frac{1}{4}} & \text{if } p = 4/3,\ p = 4 \\ n^{\frac{1}{2}-\frac{2}{p}} & \text{if } 4 < p \leq \infty, \end{cases}$$

where by $[S_p]$ we denote the space of all bounded, linear operators from a space S_p into itself, furnished with the usual operator norm $\|\cdot\|_{[S_p]}$.

Corollary 1. *Let $1 \leq p \leq \infty$. For $c_{k,n} = 1$, $k = 0, ..., n$, and for p outside the Pollard interval $(4/3, 4)$*

$$\|S_n\|_{[S_p]} \to \infty, \qquad n \to \infty,$$

where S_n denotes the nth partial sum of expansion (2).

For $c_{k,n} = \frac{A^\delta_{n-k}}{A^\delta_n}$, $0 \le k \le n$, the Theorem 1 yield:

Corollary 2. *Let given numbers p and δ such that $1 \le p \le \infty$;*

$$\begin{cases} 0 < \delta < \frac{2}{p} - \frac{3}{2} & \text{if } 1 \le p < 4/3 \\ 0 < \delta < \frac{1}{2} - \frac{2}{p} & \text{if } 4 < p \le \infty. \end{cases}$$

Then, for $p \notin [4/3, 4]$

$$\|\sigma^\delta_n\|_{[S_p]} \to \infty, \qquad n \to \infty.$$

2 Preliminaries

We summarize the properties of Legendre-Sobolev polynomials we need, cf. [6] (see also [2], [3]).

Let μ be the Gegenbauer (or ultraspherical) measure, $d\mu(x) = (1-x^2)^\alpha dx$, $\alpha > -1$, and let $p^{(\alpha)}_n$ the corresponding Gegenbauer orthonormal polynomials. The representation of $\hat{B}_n$ is

$$\hat{B}_n(x) = A_n(1-x^2)^2 p^{(4)}_{n-4}(x) + B_n(1-x^2)p^{(2)}_{n-2}(x) + C_n p_n(x) \tag{3}$$

where

$$A_n \cong 1, \qquad B_n \cong -2, \qquad C_n \cong -1$$

and by $u_n \cong v_n$ we mean that the sequence u_n/v_n converges to 1.

The maximum of $\hat{B}_n$ on $[-1, 1]$ is

$$\max_{x\in[-1,1]} |\hat{B}_n(x)| \sim n^{1/2} \tag{4}$$

where by $u_n \sim v_n$ we mean that there exist some positive constants c_1 and c_2 such that $c_1 u_n \le v_n \le c_2 u_n$ for sufficiently large n.

The polynomials $\hat{B}_n$ satisfy the estimate

$$|\hat{B}_n(cos\theta)| = \begin{cases} O(\theta^{-1/2}) & \text{if } c/n \le \theta \le \pi/2 \\ O(n^{1/2}) & \text{if } 0 \le \theta \le c/n \end{cases} \tag{5}$$

and c is positive constant.

The formula of Mehler-Heine type for Gegenbauer orthonormal polynomials is (see [8, Theorem 8.1.1] and [8, (4.3.4)])

$$\lim_{n\to\infty} n^{-\alpha-1/2} p^{(\alpha)}_n(cos\frac{z}{n}) = z^{-\alpha} J_\alpha(z), \tag{6}$$

where α real number and $J_\alpha(z)$ is the Bessel function. This formula holds uniformly for $|z| \le R$, R fixed.

From (6) it can be shown that

$$\lim_{n\to\infty} n^{-\alpha-1/2} p^{(\alpha)}_n(cos\frac{z}{n+j}) = z^{-\alpha} J_\alpha(z) \tag{7}$$

holds uniformly for $|z| \le R$, R fixed, and $j \in N \cup \{0\}$.

Lemma 1. *Let $N > 0$. Then*

$$\lim_{n\to\infty} n^{-1/2}\hat{B}_n(cos\frac{z}{n}) = J_4(z) - 2J_2(z) - J_0(z)$$

which holds uniformly for $|z| \le R$, R fixed.

Proof. From (3) we have

$$n^{-1/2}\hat{B}_n(cos\frac{z}{n}) = A_n\ sin^4(\frac{z}{n})\ n^{-1/2}p^{(4)}_{n-4}(cos\frac{z}{n}) + B_n\ sin^2(\frac{z}{n})\ n^{-1/2}p^{(2)}_{n-2}(cos\frac{z}{n}) + C_n n^{-1/2}p_n(cos\frac{z}{n}).$$

Finally, we take the limit $n \to \infty$ and use (3) and (7) to obtain

$$\lim_{n\to\infty} n^{-1/2}\hat{B}_n(cos\frac{z}{n}) = z^4 z^{-4}J_4(z) - 2\ z^2 z^{-2}J_2(z) - J_0(z) = J_4(z) - 2J_2(z) - J_0(z).$$

□

We also need to know the S_p norms for Jacobi-Sobolev polynomials

$$\|\hat{B}_n\|^p_{S_p} = \int_{-1}^{1} |\hat{B}_n(x)|^p dx + N|(\hat{B}_n)'(1)|^p + N|(\hat{B}_n)'(-1)|^p \quad (8)$$

where $1 \le p < \infty$. Hence, it is sufficient to estimate just the L^p norms for $\hat{B}_n$. For $N = 0$ the calculation of these norms is in [8, p.391. Exercise 91] (see also [7, (2.2)]).

Lemma 2. *Let $N \ge 0$. Then*

$$\int_0^1 |\hat{B}_n(x)|^p dx \sim \begin{cases} c & if\ p < 4, \\ log\ n & if\ p = 4, \\ n^{p/2-2} & if\ p > 4. \end{cases}$$

Proof. From (5), for $p \ne 4$, we have

$$\int_0^1 |\hat{B}_n(x)|^p dx \sim \int_0^{\pi/2} \theta\ |\hat{B}_n(cos\theta)|^p d\theta = O(1)\int_0^{n^{-1}} \theta\ n^{p/2} d\theta + O(1)\int_{n^{-1}}^{\pi/2} \theta\ \theta^{-p/2} d\theta = O(n^{p/2-2}) + O(1),$$

and for $p = 4$ we have

$$\int_0^1 |\hat{B}_n(x)|^p dx = O(log\ n).$$

On the other hand, according to Lemma 1, we have

$$\int_0^{\pi/2} \theta|\hat{B}_n(cos\theta)|^p d\theta > \int_0^{n^{-1}} \theta|\hat{B}_n(cos\theta)|^p d\theta$$
$$\cong c \int_0^1 (z/n) n^{p/2}\ n^{-1} |J_4(z) - 2J_2(z) - J_0(z)|^p dz \sim n^{p/2-2}.$$

Using a similar argument as above, for $p = 4$, we have

$$\int_0^{\pi/2} \theta|\hat{B}_n(cos\theta)|^4 dx > c \int_0^{n^{-1/2}} \theta|\hat{B}_n(cos\theta)|^4 dx$$
$$\cong c \int_0^{n^{1/2}} z|J_4(z) - 2J_2(z) - J_0(z)|^4 dz \sim n \geq c\ log\ n.$$

Finally, from (3) and [8, Theorem 8.21.8] we obtain

$$\int_0^{\pi/2} \theta\ |\hat{B}_n(cos\theta)|^p d\theta > \int_{\pi/4}^{\pi/2} \theta\ |\hat{B}_n(cos\theta)|^p d\theta \sim c.$$

□

3 Proof of Theorem 1

For the proof of Theorem 1 we will use the test function

$$g_n^j(x) = (1-x^2)^j p_n^{(j)}(x)$$

where $j \in N\backslash\{1\}$. From (2) and (3) the Fourier coefficients of the function $g_n^j(x)$ can be written as

$$(g_n^j)\,\hat{}\,(k) = \int_{-1}^1 (1-x^2)^j p_n^{(j)}(x) \hat{B}_k(x) dx$$
$$= A_k \int_{-1}^1 (1-x^2)^j p_n^{(j)}(x)\ (1-x^2)^2 p_{k-4}^{(4)}(x) dx$$
$$+ B_k \int_{-1}^1 (1-x^2)^j p_n^{(j)}(x)\ (1-x^2) p_{k-2}^{(2)}(x) dx$$
$$+ C_k \int_{-1}^1 (1-x^2)^j p_n^{(j)}(x) p_k(x) dx = I_1^{k,n} + I_2^{k,n} + I_3^{k,n}$$

where it is assumed $p_i^{(\alpha)}(x) = 0$, for $i = -1, -2, -3, -4$.

For $k \geq 4$, according to the [8, (4.3.4)] we obtain

$$I_1^{k,n} = A_k\ \{h_n^{j,j}\}^{-1/2}\ \{h_{k-4}^{4,4}\}^{-1/2} \int_{-1}^1 (1-x^2)^j P_n^{(j)}(x)\ (1-x^2)^2 P_{k-4}^{(4)}(x) dx,$$

where $h_n^{\alpha,\alpha} = 2^{2\alpha} n^{-1}$.

On the other hand, from [7, (2.8)]

$$(1-x^2)^j P_n^{(j)}(x) = \sum_{m=0}^{2j} b_{m,j}(0,0,n) P_{n+m}(x) \tag{9}$$

and

$$(1-x^2)^2 P_{k-4}^{(4)}(x) = \sum_{l=0}^{4} b_{l,2}(2,2,k-4) P_{k+l-4}^{(2)}(x),$$

where

$$b_{0,j}(\alpha,\alpha,n) = 4^j \frac{(\Gamma(n+\alpha+j+1))^2}{(\Gamma(n+\alpha+1))^2} \frac{\Gamma(2n+2\alpha+2)}{\Gamma(2n+2\alpha+2j+2)} \cong 4^j,$$

$$b_{2j,j}(\alpha,\alpha,n) = (-4)^j \frac{\Gamma(n+2j+1)}{\Gamma(n+1)} \frac{\Gamma(2n+2\alpha+2j+1)}{\Gamma(2n+2\alpha+4j+1)} \cong (-4)^j.$$

Thus

$$\begin{cases} I_1^{k,n} = 0, & 4 \le k \le n-1 \\ I_1^{n,n} = 0, & n \ge 4, \ 0 < m \le 2j. \end{cases}$$

Let $k = n \ge 4$ and $m = 0$. Then

$$I_1^{n,n} = A_n \ \{h_n^{j,j}\}^{-1/2} \{h_{n-4}^{4,4}\}^{-1/2} b_{0,j}(0,0,n) \ b_{4,2}(2,2,n-4) \int_{-1}^{1} P_n(x) \ P_n^{(2)}(x) dx$$

Since (see [1, p. 359, Theorem 7.1.4])

$$P_n^{(2)}(x) = \frac{16(n+1/2)(n+3/2)}{(n+3)(n+4)} P_n + Q_{n-1}(x),$$

we get

$$\begin{aligned} I_1^{n,n} = & \frac{16 A_n (n+1/2)(n+3/2)}{(n+3)(n+4)} \ \{h_n^{j,j}\}^{-1/2} \{h_{n-4}^{4,4}\}^{-1/2} \ h_n^{0,0} \\ & \times b_{0,j}(0,0,n) \ b_{4,2}(2,2,n-4) \ \cong 16 \cdot 2^j. \end{aligned}$$

In similar way, for $k \ge 2$, using [8, (4.3.4)] and (9)

$$I_2^{k,n} = B_k \ \{h_n^{j,j}\}^{-1/2} \ \{h_{k-2}^{2,2}\}^{-1/2} \sum_{m=0}^{2j} b_{m,j}(0,0,n) \int_{-1}^{1} P_{n+m}(x) \ (1-x^2) \ P_{k-2}^{(2)}(x) dx.$$

Again, as applications of [7, (2.8)] and [1, p. 359, Theorem 7.1.4] we point out the following relations

$$(1-x^2) \ P_{k-2}^{(2)}(x) = \sum_{l=0}^{2} b_{l,1}(1,1,k-2) P_{k+l-2}^{(1)}(x).$$

and

$$P_n^{(1)}(x) = \frac{4n+2}{n+2} P_n + Q_{n-1}(x).$$

Thus

$$\begin{cases} I_2^{k,n} = 0, & 2 \le k \le n-1 \\ I_2^{n,n} = \frac{(4n+2)B_n}{n+2} \{h_n^{j,j}\}^{-1/2} \{h_{n-2}^{2,2}\}^{-1/2} \ h_n^{0,0} & \\ \qquad\qquad \times \ b_{0,j}(0,0,n) \ b_{2,1}(1,1,n-2) \cong 8 \cdot 2^j & n \ge 2, \ m = 0 \\ I_2^{n,n} = 0 & n \ge 2, \ 0 < m \le 2j. \end{cases}$$

Finally, for $k \ge 0$

$$I_3^{k,n} = C_k \ \{h_n^{j,j}\}^{-1/2} \ \{h_k^{0,0}\}^{-1/2} \sum_{m=0}^{2j} b_{m,j}(0,0,n) \int_{-1}^{1} P_{n+m}(x) \ P_k(x) dx.$$

Thus

$$\begin{cases} I_3^{k,n} = 0, & 0 \le k \le n-1 \\ I_3^{n,n} = C_n \{h_n^{j,j}\}^{-1/2} \{h_n^{0,0}\}^{1/2} \ b_{0,j}(0,0,n) \cong -2^j, & n \ge 0, \ m = 0 \\ I_3^{n,n} = 0, & n \ge 0, \ 0 < m \le 2j. \end{cases}$$

As a conclusion

$$\begin{cases} (g_n^j)\hat{\ }(k) = 0, & 0 \le k \le n-1 \\ (g_n^j)\hat{\ }(n) \cong -2^j, & n = 0,1 \\ (g_n^j)\hat{\ }(n) \cong 6 \cdot 2^j, & n = 2,3 \\ (g_n^j)\hat{\ }(n) \cong 23 \cdot 2^j, & n \ge 4. \end{cases} \tag{10}$$

On the other hand, from [8, p.391. Exercise 91] (see also [7, (2.2)])

$$\begin{aligned} \|g_n^j\|_{S_p}^p = \|g_n^j\|_{L^p}^p = & \int_{-1}^{1} (1-x)^{jp}(1+x)^{jp} |p_n^{(j)}(x)|^p dx \\ & \le c_1 \int_0^1 (1-x)^{jp} |p_n^{(j)}(x)|^p dx \\ & \qquad + c_2 \int_{-1}^{0} (1+x)^{jp} |p_n^{(j)}(x)|^p dx \le c. \end{aligned} \tag{11}$$

for $j > 1/2 - 2/p$ and $4 \le p < \infty$.

It is well known (see, for example, [4, Theorem 1]) that

$$|p_n^{(j)}(x)| \le c(1-x^2)^{-j/2-1/4}$$

for $x \in (-1,1)$.

Therefore

$$\|g_n^j\|_{S_\infty} = \|g_n^j\|_{L^\infty} \le c(1-x^2)^{j/2-1/4} \le c, \tag{12}$$

for $j > 1/2$.

Now we are in position to prove our main result:

Proof of Theorem 1. By duality, it suffices to assume that $4 \le p \le \infty$. Now we apply the operator T_n^N to the test function g_n^j for some $j > 1/2 - 2/p$. Hence, from (10), (11) and (12), we have

$$\|T_n^N\|_{[S_p]} \ge [\|g_n^j\|_{S_p}]^{-1} \|T_n^N g_n^j\|_{S_p} \ge c|c_{n,n}| \, \|\hat{B}_n\|_{S_p}. \tag{13}$$

From (8) and Lemma 2 we obtain that

$$\|\hat{B}_n\|_{S_p} \ge c \begin{cases} (log\ n)^{1/p} & \text{if } p = 4, \\ n^{1/2-2/p} & \text{if } 4 < p < \infty. \end{cases}$$

On combining this and (4) with (13), the statement is seen to be true.

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