

In presenting the dissertation as a partial fulfillment of the requirements for an advanced degree from the Georgia Institute of Technology, I agree that the Library of the Institute shall make it available for inspection and circulation in accordance with its regulations governing materials of this type. I agree that permission to copy from, or to publish from, this dissertation may be granted by the professor under whose direction it was written, or, in his absence, by the Dean of the Graduate Division when such copying or publication is solely for scholarly purposes and does not involve potential financial gain. It is understood that any copying from, or publication of, this dissertation which involves potential financial gain will not be allowed without written permission.

\_\_\_\_\_

7/25/68

A COLLISION AVOIDANCE WARNING CRITERION  
FOR MANEUVERING AIRCRAFT

A THESIS

Presented to

The Faculty of the Division of Graduate  
Studies and Research

by

Roscoe McClendon Hinson, Jr.

In Partial Fulfillment  
of the Requirements for the Degree  
Doctor of Philosophy  
in the School of Mechanical Engineering

Georgia Institute of Technology

March, 1972

A COLLISION AVOIDANCE WARNING CRITERION  
FOR MANEUVERING AIRCRAFT

Approved: \_\_\_\_\_

Chairman \_\_\_\_\_

Date approved by Chairman: 3/31/72

## ACKNOWLEDGEMENTS

The author is grateful to many people and organizations who have assisted in the completion of this work, especially his wife, Elizabeth, and his parents, Roscoe and Virginia Hinson.

Sincere thanks is given to Dr. S. L. Dickerson for his invaluable guidance from the initiation of the thesis to its completion. Appreciation is expressed also to Dr. J. E. Brown, Dr. R. B. Evans, Dr. J. J. Talavage, and Dr. W. M. Williams for reviewing the thesis and making suggestions for its improvement.

For their help in the conceptual phase of the research, gratitude is given to T. Walsh, and J. H. Schrader of Langley Research Center; R. M. Buck, J. Brennan and F. Brandl of the Federal Aviation Administration in Washington; F. C. White of the American Transport Association; and C. L. Britt of the Research Triangle Institute. For their help in obtaining the Atlanta Radar Data, appreciation is extended to R. J. Cannon and C. D. Jolitz of the National Aviation Facilities Experimental Center.

And finally, without the financial aid of the National Aeronautics and Space Administration, \* the research would not have been possible. The author was the recipient of a NASA System Design Traineeship.

---

\* NGT 11-002-064 & NGR 11-002-081



## TABLE OF CONTENTS

	Page
ACKNOWLEDGEMENTS . . . . .	ii
LIST OF ILLUSTRATIONS . . . . .	iv
SUMMARY . . . . .	vii
NOMENCLATURE . . . . .	ix
Chapter	
I. INTRODUCTION . . . . .	1
II. THREAT PREDICTION CRITERIA . . . . .	5
Previous Criteria	
A New Criterion	
III. THEORY OF COLLISION PROBABILITY . . . . .	10
Definition of Collision Probability	
Previous Methods of Calculation	
Marks' Method for CAS Applications	
A New Method	
IV. STATISTICAL ANALYSIS AND COMPUTATIONAL TECHNIQUES . . . . .	32
Introduction	
Position Statistics	
Mean Dwell Time	
V. RESULTS AND CONCLUSIONS . . . . .	71
Numerical Results	
Future Research Possibilities	
Application of the Methodology	
LITERATURE CITED . . . . .	83
VITA. . . . .	86

## LIST OF ILLUSTRATIONS

Figure	Page
1. Distribution Function of $\underline{x}_a$ and $\underline{x}_b$ . . . . .	13
2. The $\Sigma_{+W_x}$ Plane with an Incremental Volume . . . . .	19
3. Flight Tracks Recorded at Hartsfield Airport . . . . .	34
4(a). Density Function of $\bar{v}$ After 10 Seconds . . . . .	38
4(b). Density Function of $\bar{v}$ After 30 Seconds . . . . .	39
4(c). Density Function of $\bar{v}$ After 60 Seconds . . . . .	40
4(d). Density Function of $\bar{v}$ After 120 Seconds . . . . .	41
5. Aircraft Positions in the $x, y$ Coordinate System After 30 Seconds . . . . .	42
6. Aircraft Positions in the $x', y'$ Coordinate System After 30 Seconds . . . . .	43
7. Aircraft Positions in the $x, y$ Coordinate System After 120 Seconds . . . . .	44
8. Aircraft Positions in the $x', y'$ Coordinate System After 120 Seconds . . . . .	45
9. Constant Turn Radius Flight Path . . . . .	46
10. Aircraft Positions After 60 Seconds Superimposed on $\epsilon, k$ Coordinate System . . . . .	49
11. Joint Density Function of $\underline{\epsilon}$ and $\underline{k}$ . . . . .	50
12. Collision Volume. . . . .	54
13. Plane View of Collision Volume . . . . .	54
14. Cross Section View of Collision Volume . . . . .	54

## LIST OF ILLUSTRATIONS (CONTINUED)

Figure	Page
15. Density Function of the Heading Error After Assuming a Constant Radius Turn, with $\Phi = 120^\circ$ and $t = 60$ Seconds . . . . .	56
16. Density Function of the Heading Error After Assuming a Constant Radius Turn, with $\Phi = 60^\circ$ and $t = 60$ Seconds . . . . .	57
17. Density Function of the Heading Error After Assuming a Constant Radius Turn, with $\Phi = 20^\circ$ and $t = 60$ Seconds . . . . .	58
18. Density Function of the Heading Error After Assuming a Constant Radius Turn, with $\Phi = 0^\circ$ and $t = 60$ Seconds . . . . .	59
19. Density Function of the Heading Error After Assuming a Constant Radius Turn, with $\Phi = -20^\circ$ and $t = 60$ Seconds . . . . .	60
20. Density Function of the Heading Error After Assuming a Constant Radius Turn, with $\Phi = -60^\circ$ and $t = 60$ Seconds . . . . .	61
21. Density Function of the Heading Error After Assuming a Constant Radius Turn, with $\Phi = -120^\circ$ and $t = 60$ Seconds . . . . .	62
22. Density Function of the Heading Error After Assuming a Two Minute Turn, with $\Phi = 120^\circ$ and $t = 60$ Seconds . . . . .	64
23. Density Function of the Heading Error After Assuming a Two Minute Turn, with $\Phi = 60^\circ$ and $t = 60$ Seconds . . . . .	65
24. Density Function of the Heading Error After Assuming a Two Minute Turn, with $\Phi = 20^\circ$ and $t = 60$ Seconds . . . . .	66

## LIST OF ILLUSTRATIONS (CONTINUED)

Figure	Page
25. Density Function of the Heading Error After Assuming a Two Minute Turn, with $\Phi = 0$ , and $t = 60$ Seconds . . . . .	67
26. Density Function of the Heading Error After Assuming a Two Minute Turn, with $\Phi = -20^{\circ}$ and $t = 60$ Seconds . . . . .	68
27. Density Function of the Heading Error After Assuming a Two Minute Turn, with $\Phi = -60^{\circ}$ , and $t = 60$ Seconds . . . . .	69
28. Density Function of the Heading Error After Assuming a Two Minute Turn, with $\Phi = -120^{\circ}$ , and $t = 60$ Seconds . . . . .	70
29. Independent Variables on Which the Collision Probability Depends . . . . .	72
30. Collision Probability for $\beta = 0$ . . . . .	74
31. Collision Probability for $\beta = 15^{\circ}$ . . . . .	75
32. Collision Probability for $\beta = 30^{\circ}$ . . . . .	76
33. Comparison of Collision Avoidance Warning Criteria. . . . .	78



## SUMMARY

An effective aircraft collision avoidance system must incorporate a means to discriminate between aircraft which pose a threat of collision and aircraft which do not. This problem of discrimination becomes especially difficult in areas where the aircraft densities are high and where aircraft maneuvers occur frequently. The purpose of this research was to develop a warning criterion suitable for this environment. The aircraft flight paths were considered to be stochastic processes and the warning criterion was therefore based on the probability of a collision.

Information recorded from Atlanta's Hartsfield Airport radar was obtained from the Federal Aviation Administration. This information contained in digital form at frequent intervals the coordinates and velocities of the aircraft being tracked. It was found that the statistics concerning the movements of an aircraft in this environment could be reduced to a very concise form. In addition, a method was developed which could use these statistics to calculate the probability of a collision between two aircraft given certain initial conditions.

This collision probability was used as the basis of a warning criterion which determines if a maneuver can reduce the collision probability sufficiently enough to warrant its execution. Previous criteria have dealt primarily with one avoidance maneuver, but this criterion has the capability of incorporating any number of avoidance maneuvers. Also previous criteria have relied upon intuitive judgment rather than a consideration of actual cost and benefit.

A complete warning criterion suitable for direct implementation was not developed, but the results presented show significant differences between this and previously developed criteria.

## NOMENCLATURE

a	distance related to collision volume, feet
b	distance related to collision volume, feet
c	distance related to collision volume, feet
C	set of tracks that contain collisions
$C_o$	set of initial conditions
d	distance along arc, nautical miles
$d'$	$d/V$ , seconds
$d_c$	distance along arc, nautical miles
$d'_c$	$d_c/V$ , seconds
g	unit of acceleration, $32.2 \text{ ft/sec}^2$
$g_i$	gain associated with $i^{\text{th}}$ maneuver
h	one-half the height of the collision volume, feet
k	inverse of radius
$m_i$	$i^{\text{th}}$ maneuver
n	number of collision
$p_i$	penalty associated with $i^{\text{th}}$ maneuver
$P_{c_i}$	collision probability associated with $i^{\text{th}}$ maneuver
$q_i$	point in state space
$Q_i$	set of points in $x, y, z, \theta, V_1$ , and $V_2$ state space
r	range, nautical miles; radius, feet



$\vec{r}$	relative position vector, nautical miles
R	region in $x^*$ , $y^*$ , $z^*$ and $t$ space
$S_a$	set of flight tracks of aircraft A
$S_b$	set of flight tracks for aircraft C
$S_c$	set of flight track pairs
$s_r$	separation in $r$ direction, nautical miles
$t$	time, seconds
$t^*$	time of collision, seconds
$t_1^*$	time of first collision, seconds
$t_\mu$	time to closest approach, seconds
T	time
U	maximum aircraft acceleration, g
$\bar{v}$	normalized velocity error
$v_h$	horizontal velocity, knots
V	velocity, knots
$\vec{V}$	velocity vector, knots
$W_r$	dimension of aircraft in $r$ direction, feet
x	position coordinate, nautical miles
$x'$	$x/V$ , seconds
$x_a^*$	coordinate at collision, nautical miles
$\tilde{x}$	relative displacement coordinate, nautical miles
y	position coordinate, nautical miles
$y'$	$y/V$ , seconds

$y_a^*$	coordinate at collision, nautical miles
$\tilde{y}$	relative displacement coordinate, nautical miles
$z$	position coordinate, nautical miles
$z'$	$z/V$ , seconds
$z_a^*$	coordinate at collision, nautical miles
$\tilde{z}$	relative displacement coordinate, nautical miles

#### Greek Symbols

$\alpha$	benefit proportionality constant; angle, degrees
$\alpha(t)$	collision rate
$\beta$	relative bearing, degrees
$\beta(t)$	derivate of collision probability
$\gamma$	azimuth, degrees
$\delta$	dwelt time, seconds
$\epsilon$	$1 - \frac{2}{kt} \tan^{-1} \frac{x}{y}$ , dimensionless
$\theta$	heading, degrees
$k$	inverse of radius
$\lambda_r$	separation loss frequency in r direction
$\mu$	miss distance, feet
$\rho$	radius, nautical miles
$\tau$	$r/\dot{r}$ , seconds
$\tau_m$	minimum time to collision, seconds
$\Phi$	bearing, degrees
$\Sigma$	plane

## CHAPTER I

### INTRODUCTION

Soon after the beginning of commercial air service, the need for a method of assuring adequate separation between aircraft became apparent. Since the terminal area contained the most dense traffic, the first air traffic control (ATC) system was established in the terminal area. As densities increased, the ATC system was extended to include enroute traffic as well. Today, nearly the entire airspace over the United States is under radar surveillance. The air traffic controller can therefore direct radar traffic from the ground in such a way as to maintain safe separations between the aircraft operating in the system.

Even though this system has operated satisfactorily for many years, attention is being given to ways to improve the system. And again because of high densities, the terminal area is considered to be the most critical. One method for improvement which has been receiving considerable attention is the collision avoidance system (CAS). The term CAS refers to a class of devices that monitor the behavior of other aircraft operating in the vicinity of the protected aircraft and evaluate the danger of a midair collision between the other aircraft and the protected aircraft. This threat evaluation by the CAS must be carried out automatically and should not require any effort on the part of the pilot. When the CAS determines that the danger is great enough, a warning is given to the pilot and the pilot is instructed as to the appropriate avoidance maneuver. The

CAS is considered to be a backup system to the present ATC and is not to be considered a replacement.

In 1955 the scheduled airlines, working through the Air Transport Association (ATA), requested industry to propose or produce a CAS. Following this request several companies began development of collision avoidance systems. Some of the earlier developers were Bendix Aviation Corporation (1), Boeing Company (2), Hughes Aircraft Company, the Army Ordnance Corps (3), and Collins Radio.

Much of the initial work was directed to the development of airborne equipment to detect and determine the relative flight paths of intruding aircraft. The problem proved to be more difficult than originally thought and several of the first systems were found unacceptable (4). The primary difficulty is obtaining sufficient data accuracy at reasonable cost. Equipment development is still continuing today and many feel that the newer techniques can be successfully applied (5, 6).

In addition to the equipment development, analytical studies of the problem have been conducted. These studies were directed toward such topics as when to give a warning (7), escape maneuver required (8, 9, 10), effects of air turbulence on measured parameters (11), and the effects of data measurement errors (12). Of primary concern has been the development of a threat prediction criterion that gives a high degree of success in predicting impending collision and yet does not generate excessive false alarms.

Previous criteria have been for the large part based on deterministic



flight paths. Statistical analyses have been performed on these deterministic criteria by introducing random measurement error; also statistical analyses have been conducted to evaluate these criteria in a realistic environment. The criteria themselves, however, are deterministic in nature. The criterion investigated in the present work considers the aircraft flight paths to be stochastic processes. In this way a warning criterion which incorporates the behavior of actual aircraft can be developed. The theoretical basis for this criterion, which is based on the probability of a collision occurring, is presented. Then by using actual radar data taken by the Federal Aviation Administration and by making extensive use of a digital computer, the probability of a collision was computed.

The probability of a midair collision was considered to be dependent on six independent initial conditions. These initial conditions were the airspeeds of both aircraft; the range of the intruding aircraft; and the bearing, heading, and altitude of the intruding aircraft relative to the protected aircraft. For the collision probability results given in Chapter V, the airspeeds of both aircraft were fixed at 160 knots and the initial altitude separation was fixed at zero. The remaining three initial conditions (range, relative heading, and relative bearing) were expressed in terms of miss distance, time to closest approach, and relative bearing. The collision probability was then calculated as a function of these three variables as initial conditions.

The scope of this work did not include the development of a complete avoidance criterion which could be directly implemented. The mathematical basis was developed and the computational feasibility was established by giving

sample results. These sample results do, however, point out fairly significant differences between this criterion and previously proposed criteria.

## CHAPTER II

## THREAT PREDICTION CRITERIA

A number of threat prediction criteria have previously been developed and analyzed. The mathematical development of these criteria is usually straightforward and therefore will not be included. However, the results and underlying assumptions will be discussed.

Previous Criteria

The simplest criterion are based on the assumption of straight-line flight. With this assumption, it is possible to compute the closest approach distance, or miss distance, and the time to the closest approach by knowing certain parameters concerning the aircraft's positions and movements. These criteria differ primarily by the input information required.

Consider one aircraft to be the protected aircraft. All measurements will be made relative to this protected aircraft. If the relative position vector  $\vec{r}$  and relative velocity vector  $\vec{v}$  of an intruding aircraft are known, then the miss distance  $\mu$  is

$$\mu = \frac{|\vec{v} \times \vec{r}|}{|\vec{v}|} \quad (2.1)$$

and the time to closest approach,  $t_{\mu}$ , is



$$t_{\mu} = - \frac{\vec{r} \cdot \vec{v}}{|\vec{v}|^2} \quad (2.2)$$

If only the range  $r$  of the intruding aircraft is known and the aircraft are co-altitude, then

$$\mu = \frac{r^3 \ddot{r}}{\ddot{r} r + \dot{r}^2} \quad (2.3)$$

and

$$t_{\mu} = \frac{r \dot{r}}{\ddot{r} r + \dot{r}^2} \quad (2.4)$$

From a hardware viewpoint,  $r$  and  $\dot{r}$  are fairly easy for a CAS to obtain; however,  $\ddot{r}$  is much more difficult. Therefore, the use of equation (2.4) for finding  $t_{\mu}$  is of limited application since  $\ddot{r}$  is required. For collision courses, the ratio

$$\tau = \frac{r}{\dot{r}} \quad (2.5)$$

gives the time to a collision.

Even though  $\tau$  equals  $t_{\mu}$  only for unaccelerated collision courses, in most instances  $\tau$  gives a reasonable approximation of  $t_{\mu}$ . This approximation forms the basis for the "tau" criterion. A warning is given whenever  $\tau$  becomes less than some specified value. Britt (14) has determined, using the same radar data as used for this study, that in a realistic environment the probability of a false alarm using the tau criterion is .49 during a 13 minute flight. A warning threshold of  $\tau = 60$  sec. was used and only aircraft within an altitude range of  $\pm 500$  feet were considered. Although the threat prediction criteria using equation (2.5)

has several advantages, it seems that the high false alarm rate will detract from its acceptability.

Another criterion which has been developed allows for the consideration of accelerated flight paths. This criterion assumes some maximum aircraft acceleration and then uses this maximum acceleration to calculate the minimum time to collision. This time is then used to determine when a warning should be given. In terms of range and range-rate, the minimum time to collision,  $\tau_m$  is

$$\tau_m = \frac{-\dot{r} + (\dot{r}^2 - 2Ur)^{\frac{1}{2}}}{U} \quad (2.6)$$

where  $U$  is the maximum range acceleration considered. This has been termed the "modified tau" criterion. Using the modified tau criterion, Britt has also determined that the probability of a warning is equal to .59 during a 13 minute flight when the value  $\tau_m = 35$  sec. was used with  $U = \frac{1}{2}g$ . Again only aircraft within an altitude range of  $\pm 500$  ft. were considered. This false alarm rate again seems high.

#### A New Criterion

This present work intends to investigate a different structure for the threat prediction criteria. It considers the possibility of basing the threat prediction criterion on the probability of a collision occurring. This new criterion, then, will not only take into account the possibility of a maneuver occurring, but it will also consider the probability that this particular maneuver occurs. The criterion itself is fairly simple in that a warning is given and, consequently, a

maneuver is initiated only when the benefit gained from making the maneuver outweighs the undesirability of making the maneuver.

A more precise statement of the criterion is as follows. Consider an aircraft in flight. At any time there are  $n$  different maneuvers it may make over some time interval  $\Delta T$ . Let the  $i^{\text{th}}$  maneuver be denoted by  $M_i$ . Associated with each maneuver is a penalty  $p_i$ . Let there be one maneuver,  $M_0$ , for which the associated penalty is always zero. The maneuver,  $M_0$ , is therefore the maneuver the pilot would make if left to proceed normally. All maneuvers other than  $M_0$  are considered to be evasive maneuvers. Notice that maneuvers as used here refer to any possible actions the aircraft can take including flying straight.

For each maneuver there is also a benefit  $b_i$  defined as

$$b_i = \alpha(P_{c_0} - P_{c_i}) \quad (2.7)$$

where  $P_{c_0}$  is the probability of a collision occurring during  $\Delta T$  given that the maneuver  $M_0$  is initiated,  $P_{c_i}$  is the probability of a collision occurring during  $\Delta T$  given that the maneuver  $M_i$  is initiated, and  $\alpha$  is the proportionality constant.

The gain  $g_i$  is defined as

$$g_i = b_i - p_i \quad (2.8)$$

The warning criterion is established by maximizing the gain and initiating the maneuver when this maximum gain becomes greater than zero. Therefore, the  $i^{\text{th}}$  maneuver is initiated when

$$\max[g_i] > 0 \quad (2.9)$$



The American Transport Association (ATA), which has been very active in specifying what type of CAS would be acceptable to the commercial airlines, has already established certain maneuvers to be used. This may be viewed as determining a priori which maneuver maximizes the gain. In order to make this criterion compatible with the ATA system, the set of all possible maneuvers contains only the climb maneuver defined by the ATA. The criterion for giving the warning will then be when the gain from making this maneuver is greater than zero.

Perhaps one disturbing aspect of this criterion is that even after the avoidance maneuver is taken, a finite probability of a collision could still exist. As will be determined later this aspect of the stochastic approach can be fairly easily avoided by initiating a maneuver that will reduce the collision probability to such a small level that it is indistinguishable from zero. However, even if the probability could not be reduced to zero, any system that offers some reduction would definitely be beneficial. The benefit appears to be even greater when one considers that the CAS apparatus is a backup to an already good ATC. If a CAS could reduce the number of mid-air collisions, say by a factor of 100, then mid-air collisions would be virtually eliminated.

## CHAPTER III

### THEORY OF COLLISION PROBABILITY

Several methods have been used to determine the probability of a collision between two aircraft. These methods have, however, been developed primarily for evaluation of air traffic control (ATC) regulation rather than for collision avoidance applications. For instance, the proper separation for airways and the proper separation for parallel runways have been examined by calculating the collision probability. Even though the methods and assumptions for calculating the collision probability for ATC applications can not be applied directly to the calculation of collision probabilities for CAS applications, there are several aspects common to both applications. Therefore, the previously developed methods form a good basis for starting the development of the methods and assumptions for the present collision avoidance problem.

#### Definition of Collision Probability

The probability that two aircraft will collide during some time interval given certain initial conditions is the probability needed for collision avoidance threat prediction. The important aspect for collision avoidance application is that the probability be conditional on the initial conditions. Thus the probability of a future collision can be calculated from knowledge of the current situation.

If  $t^*$  is the time at which a collision between two aircraft occurs, and  $C_0$

represents the initial conditions, then the desired probability can be expressed as

$$P\{\text{a collision occurs on } [0, T] \text{ given } C_0\} = P\{0 \leq t^* \leq T \mid C_0\} \quad (3.1)$$

Even though only one collision between two aircraft is possible physically, multiple collisions are possible mathematically. As used in equation (3.1),  $t^*$  can be the time at some collision other than the first since it is only important to convey that a collision, not necessarily the first, has occurred prior to  $T$ . However, if any collision does occur before  $T$ , then it is also true that the first collision occurred before  $T$ . If  $t_1^*$  denotes the time at the first collision, then equation (3.1) can be written as

$$P\{\text{a collision occurs on } [0, T] \text{ given } C_0\} = P\{0 \leq t^* \leq T \mid C_0\} = P\{0 \leq t_1^* \leq T \mid C_0\} \quad (3.2)$$

This probability can also be written in the form

$$P\{0 \leq t^* \leq T \mid C_0\} = \int_0^T \beta(t) dt \quad (3.3)$$

The interpretation of  $\beta(t)$  is rather fundamental to the difficulty in calculating the collision probability. The function  $\beta(t)$  should properly be defined by the relation

$$\beta(t) dt = P\{t \leq t_1^* \leq t + dt \mid C_0\} \quad (3.4)$$



where the time at the first collision is used. Equation (3.4) indicates that the last form of the probability defined in equation (3.2) is preferable even though either definition is accurate. Unfortunately, the function  $\beta(t)$  is not easy to obtain. A function  $\alpha(t)$  defined by

$$\alpha(t)dt = P\{t \leq t^* \leq t + dt | C_0\} \quad (3.5)$$

is easier than  $\beta(t)$  to obtain. The only difference between  $\alpha(t)$  and  $\beta(t)$  is that the time of the collision is defined in  $\alpha(t)$  to be the time at a collision, not necessarily the first, and is defined in  $\beta(t)$  to be the time of the first collision. The integration of  $\alpha(t)$  gives the expected number of collisions rather than the collision probability. This distinction is made at this point to more precisely define the probability needed. These concepts are dealt with again later in this chapter and the discussion concerning the availability of  $\alpha(t)$  and  $\beta(t)$  will be deferred until then.

#### Previous Methods of Calculation

In one of the first papers to consider the probability of a collision between two aircraft, Taylor (15) examines the case of aircraft flying in the same direction along parallel flight paths at the same altitude. Let  $y$  be the distance along the track and let  $x$  be the across track distance. Taylor assumes that the two aircraft maintain their zero separation in the  $y$  direction and zero separation in the  $z$  direction, but drift randomly from the nominal flight paths in the  $x$  direction. Let  $x_a$  be the  $x$  position of one of the aircraft and  $x_b$  be the  $x$  position of the other



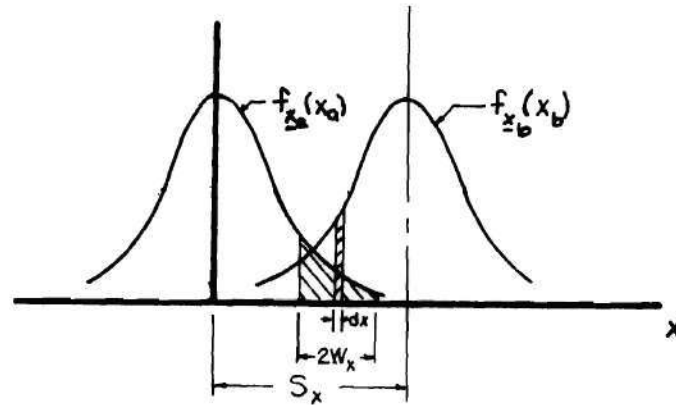


Figure 1. Distribution Function for  $\underline{x}_a$  and  $\underline{x}_b$ .

aircraft. The density functions Taylor assumed for the  $x$  positions of the two aircraft are (the bar below a variable will distinguish it as a random variable)

$$f_{\underline{x}_a}(x_a) = \frac{1}{\sigma_a \sqrt{2\pi}} e^{-\frac{x_a^2}{2\sigma_a^2}} \quad (3.6)$$

$$f_{\underline{x}_b}(x_b) = \frac{1}{\sigma_b \sqrt{2\pi}} e^{-\frac{(x_b - S_x)^2}{2\sigma_b^2}} \quad (3.7)$$

where  $S_x$  is the nominal separation between the tracks. The probability that the separation between the two aircraft is less than  $W_x$  is

$$P\{|\underline{x}_a - \underline{x}_b| < W_x\} = \int_{-\infty}^{\infty} \int_{\underline{x}_a - W_x}^{\underline{x}_a + W_x} \frac{1}{\sigma_a \sqrt{2\pi}} e^{-\frac{\underline{x}_a^2}{2\sigma_a^2}} \cdot \frac{1}{\sigma_b \sqrt{2\pi}} e^{-\frac{(\underline{x}_b - S_x)^2}{2\sigma_b^2}} d\underline{x}_b d\underline{x}_a \quad (3.8)$$

Taylor does not include time as a parameter, but a simple extension of equation (3.8) gives the more general form,

$$P\{|\underline{x}_a(t) - \underline{x}_b(t)| < W_x\} = \int_{-\infty}^{\infty} \int_{\underline{x}_a - W_x}^{\underline{x}_a + W_x} f_{\underline{x}_a}(\underline{x}_a; t) f_{\underline{x}_b}(\underline{x}_b; t) d\underline{x}_b d\underline{x}_a \quad (3.9)$$

Equation (3.9) assumes that the deviations from the nominal flight paths are random processes with  $f_{\underline{x}_a}(\underline{x}_a; t)$  and  $f_{\underline{x}_b}(\underline{x}_b; t)$  as the corresponding density functions. The idea of the aircraft being able to maintain separations less than the physical dimensions of the aircraft is possible, of course, only mathematically. The probability given in equation (3.9) is the probability at any instant that the separation is less than  $W_x$ .

The expected length of time that the aircraft would spend in contact is the integral of the probability in equation (3.9). Therefore, the expected value of the contact time over the interval  $[0, T]$  is

$$E \{\text{contact time}\} = \int_0^T P \{ |\mathbf{x}_a(t) - \mathbf{x}_b(t)| \leq W_x \} dt \quad (3.10)$$

Neither the probability in equation (3.9) nor the expected length of time in contact given by equation (3.10) is equal to the collision probability in equation (3.1) and should not be used for collision avoidance applications.

Koetsch (16), also working with parallel flight paths, performs an analysis similar to Taylor's in order to find the probability of a collision. Koetsch reasons that if the aircraft are only separated in the x direction then it would also be undesirable to have an orientation of the two aircraft such that their positions are the reverse of the orientation of the parallel tracks. This reversal of position means of course that the aircraft must have crossed through each other to have arrived on opposite sides. This analysis is true, however, for aircraft that have a separation in only one direction. Where the aircraft are not flying parallel at the same altitude this assumption would not be appropriate.

Marks (17) developed a more general procedure which first finds the rate of collision and then the expected number of collisions. This method was used by Reich (18) to find the probability of a collision over the North Atlantic and again by Steinburg (19) to find the probability of landing accidents. The collision rate  $\alpha(t)$  is the apparent frequency at which the separation between the aircraft becomes less than the dimensions of the aircraft. Integrating  $\alpha(t)$  over an interval of time gives the expected number of collisions. Thus

$$E \{n\} = \int_0^T \alpha(t) dt \quad (3.11)$$

where  $n$  is the number of collisions in the interval  $[0, T]$  and  $E\{n\}$  is the expected number of collisions.

The quantity  $\alpha(t)$  is interpreted as the apparent frequency of collision, but can also be interpreted as the collision probability per unit time. Hence

$$\alpha(t)dt = P\{t \leq t^* \leq t + dt\} \quad (3.12)$$

where  $t^*$  is the time at which the separation between the aircraft<sup>1</sup> becomes less than the physical dimensions. Equation (3.12) is therefore the same as equation (3.5) except for the conditionality on  $C_0$  which Marks does not use.

If the expected number of collisions on the interval  $[0, T]$  is small compared to one, then the expected number of collisions in equation (3.11) is approximately equal to the probability of a collision occurring on the interval.

The problem now reduces to one of finding the collision rate  $\alpha(t)$ . The situation Marks was most interested in was when aircraft tried to follow straight level flight tracks or airways. These airways have lateral separation as well as altitude separation. Several aircraft are permitted on one airway and the separation along a track is maintained by establishing a certain time separation as the aircraft enter the airway. The aircraft maintains the time separation by flying at a predetermined velocity. Because of navigational errors the aircraft will not be able to maintain their position exactly. Let a coordinate system be

---

<sup>1</sup>This apparent frequency and probability is equivalent to the apparent frequency and probability associated with the zero crossing problem described by Rice (20).



established such that the y axis is parallel to the tracks, and the x coordinate system is horizontal and perpendicular to the tracks. The z axis is vertical. Consider two aircraft, denoted by a and b, on the airways whose positions are given  $x_a, y_a, z_a, x_b, y_b,$  and  $z_b$ . The separation between the two aircraft can be written

$$s_x = |x_a - x_b| \quad (3.13a)$$

$$s_y = |y_a - y_b| \quad (3.13b)$$

$$s_z = |z_a - z_b| \quad (3.13c)$$

where  $s_x, s_y,$  and  $s_z$  are the separations along the x, y, and z axes. Marks assumed that the error components along the three axes were independent. The components of the separation would also be independent since the airways are parallel. The collision rate can be written as the sum of the collision rates along the three axes as

$$\alpha(t) = \alpha_x(t) + \alpha_y(t) + \alpha_z(t) \quad (3.14)$$

Because the separation components were assumed independent, the collision rates can be written

$$\alpha_x(t) = \lambda_x(t) P\{s_y \leq W_y\} P\{s_z \leq W_z\} \quad (3.15a)$$

$$\alpha_y(t) = \lambda_y(t) P\{s_x \leq W_x\} P\{s_z \leq W_z\} \quad (3.15b)$$

$$\alpha_z(t) = \lambda_z(t) P\{s_x \leq W_x\} P\{s_y \leq W_y\} \quad (3.15c)$$

where  $\lambda_x(t)$ ,  $\lambda_y(t)$ , and  $\lambda_z(t)$  are the frequencies at which the respective separations become less than the physical dimensions,  $W_x$ ,  $W_y$ , and  $W_z$ .

The probabilities  $P\{s_x \leq W_x\}$ ,  $P\{s_y \leq W_y\}$  and  $P\{s_z \leq W_z\}$  in equations (3.15a), (3.15b), and (3.15c) would be fairly easy to calculate if the density functions  $f_{s_x}(s_x; t)$ ,  $f_{s_y}(s_y; t)$ , and  $f_{s_z}(s_z; t)$  are known. The loss of separation frequencies are<sup>2</sup>

$$\lambda_x(t) = \int_{-\infty}^0 -s_x f_{s_x}(W_x, s_x; t) ds_x \quad (3.16a)$$

$$\lambda_y(t) = \int_{-\infty}^0 -s_y f_{s_y}(W_y, s_y; t) ds_y \quad (3.16b)$$

$$\lambda_z(t) = \int_{-\infty}^0 -s_z f_{s_z}(W_z, s_z; t) ds_z \quad (3.16c)$$

Assuming that the position errors along the x, y, and z axis are independent is fairly good perhaps when the aircraft are attempting to maintain straight line level flight, but this assumption becomes less accurate if the aircraft are involved in maneuvers. Therefore, the assumption of independence among the x, y, and z positions is not accurate for collision avoidance applications since the aircraft will in general be maneuvering.

<sup>2</sup>See Bendat (21) page 125.

### Marks' Method for CAS Applications

It would be of interest to determine what degree of complexity is introduced into the expressions for collision rates by not assuming  $x$  and  $y$  to be independent. The following is for the case where  $x$  and  $y$  are not independent and is primarily an extension of the zero-crossing problem given by Bendat (20). This development is for finding  $\alpha_x(t)$ . The collision rates  $\alpha_y(t)$  and  $\alpha_z(t)$  can be found with the same approach.

A small zero superscript will be used to denote a density function conditional on specified condition at  $t = 0$ . For example

$$\underline{f}_{uv}^0(u, v; t) = \underline{f}_{uv}(u, v | u(0) = u_0, v(0) = v_0; t) \quad (3.17)$$

The coordinate system in Figure 2 is a coordinate system moving with one of the aircraft. The coordinates are the relative displacement,  $\tilde{x}$ ,  $\tilde{y}$ , and  $\tilde{z}$ , instead of the separations used by Marks. The separations used by Marks are the absolute value of the relative displacement.

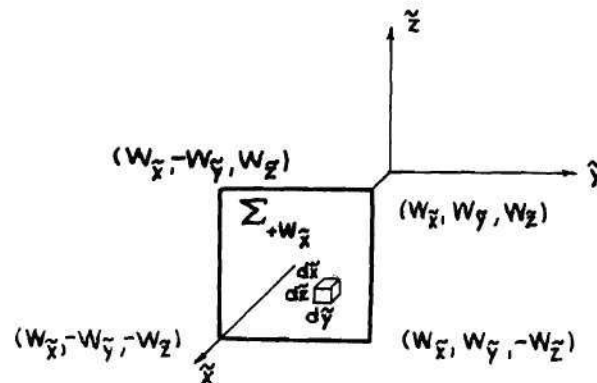


Figure 2. The  $\Sigma_{+W_{\tilde{x}}}$  Plane with an Incremental Volume.



Let the plane  $\Sigma_{+W_x}$  (Figure 2) be formed by joining the points  $(W_x, W_y, W_z)$ ,  $(W_x, -W_y, -W_z)$ ,  $(W_x, -W_y, W_z)$  and  $(W_x, W_y, -W_z)$ . The dimension  $W_i$  is the aircraft's dimensions in the  $i$  direction. Since the physical dimensions of the aircraft have been accounted for by the dimension  $W_x$ ,  $W_y$ , and  $W_z$ , the aircraft crossing this plane can be represented by a mathematical point. Consider an incremental volume centered at  $(\tilde{x}, \tilde{y}, \tilde{z})$  with the dimensions  $d\tilde{x}$ ,  $d\tilde{y}$ , and  $d\tilde{z}$ . The probability of finding the point in the incremental volume with a velocity between  $\dot{\tilde{x}}$  and  $\dot{\tilde{x}} + d\dot{\tilde{x}}$ , given the initial conditions  $C_0$ , is

$$p\{\tilde{x} < \underline{\tilde{x}} < \tilde{x} + d\tilde{x}, \tilde{y} < \underline{\tilde{y}} < \tilde{y} + d\tilde{y}, \tilde{z} < \underline{\tilde{z}} < \tilde{z} + d\tilde{z}, \dot{\tilde{x}} < \underline{\dot{\tilde{x}}} < \dot{\tilde{x}} + d\dot{\tilde{x}}; t | C_0\} = \int_{\underline{\tilde{x}}}^{\tilde{x}} \int_{\underline{\tilde{y}}}^{\tilde{y}} \int_{\underline{\tilde{z}}}^{\tilde{z}} f_{\tilde{x}\tilde{y}\tilde{z}\dot{\tilde{x}}}(\tilde{x}, \tilde{y}, \tilde{z}, \dot{\tilde{x}}; t) d\tilde{x} d\tilde{y} d\tilde{z} d\dot{\tilde{x}} \quad (3.18)$$

where the initial conditions are

$$\tilde{x}(0) = \tilde{x}_0 \quad (3.19a)$$

$$\tilde{y}(0) = \tilde{y}_0 \quad (3.19b)$$

$$\tilde{z}(0) = \tilde{z}_0 \quad (3.19c)$$

$$\dot{\tilde{x}}(0) = \dot{\tilde{x}}_0 \quad (3.19d)$$

The probability in equation (3.18) can also be interpreted as the expected time per unit time of finding the intruding aircraft in the incremental volume with an  $\tilde{x}$  velocity between  $\dot{\tilde{x}}$  and  $\dot{\tilde{x}} + d\dot{\tilde{x}}$ . For this last interpretation it can be seen that

the expected number of crossings per unit time can be found by dividing the probability in equation (3.17) by the expected time spent in crossing the volume. The  $\tilde{x}$  velocity is essentially  $\dot{\tilde{x}}$ , therefore

$$\delta = \frac{d\tilde{x}}{\dot{\tilde{x}}} \quad (3.20)$$

where  $\delta$  is the time to cross the volume. Using equation (3.18) and equation (3.20), the expected number of crossings per unit time with  $\tilde{x}$  velocity equal to  $\dot{\tilde{x}}$  is

$$\frac{f_{\tilde{x}\tilde{y}\tilde{z}\tilde{z}}^0(W_{\tilde{x}}, \tilde{y}, \tilde{z}, \dot{\tilde{x}}; t) d\tilde{x} d\tilde{y} d\tilde{z} d\dot{\tilde{x}}}{\delta} = \dot{\tilde{x}} f_{\tilde{x}\tilde{y}\tilde{z}\tilde{x}}^0(W_{\tilde{x}}, \tilde{y}, \tilde{z}, \dot{\tilde{x}}; t) d\dot{\tilde{x}} d\tilde{y} d\tilde{z} \quad (3.21)$$

Considering all velocities which result in a crossing from outside to inside (since these velocities are the only velocities which result in a collision), one obtains

$E\{\text{number of crossings per unit time } |C_o\} =$

$$\int_{-\infty}^0 -\dot{\tilde{x}} f_{\tilde{x}\tilde{y}\tilde{z}\tilde{x}}^0(W_{\tilde{x}}, \tilde{y}, \tilde{z}, \dot{\tilde{x}}; t) d\dot{\tilde{x}} d\tilde{y} d\tilde{z} \quad (3.22)$$

as the expected number of cross per unit time of the incremental volume.

Summing these expected number of crossings over the entire  $\Sigma_{+W_{\tilde{x}}}$  plane, the crossing rate  $\lambda_{+\tilde{x}}(t)$  (the  $+x$  subscript of  $\lambda$  indicates that only crossing of the  $\Sigma_{+W_{\tilde{x}}}$  plane are included) is

$$\lambda_{+\tilde{x}}(t) = \int_{-W_{\tilde{z}}}^{W_{\tilde{z}}} \int_{-W_{\tilde{y}}}^{W_{\tilde{y}}} \int_{-\infty}^0 -\dot{\tilde{x}} f_{\tilde{x}\tilde{y}\tilde{z}\tilde{x}}^0(W_{\tilde{x}}, \tilde{y}, \tilde{z}, \dot{\tilde{x}}; t) d\dot{\tilde{x}} d\tilde{y} d\tilde{z} \quad (3.23)$$

On the negative  $\tilde{x}$  axis, the separation loss rate is

$$\lambda_{-\tilde{x}}^{\sim}(t) = \int_{-W_z^{\sim}}^{W_z^{\sim}} \int_{-W_y^{\sim}}^{W_y^{\sim}} \int_0^{\infty} \tilde{x} \underline{f_{\tilde{x}\tilde{y}\tilde{z}\tilde{x}}}^{\sim}(-W_x^{\sim}, \tilde{y}, \tilde{z}, \tilde{x}; t) d\tilde{x} d\tilde{y} d\tilde{z} \quad (3.24)$$

The total separation loss rate  $\lambda_{\tilde{x}}^{\sim}(t)$  due to a separation loss in the  $\tilde{x}$  direction is

$$\lambda_{\tilde{x}}^{\sim}(t) = \lambda_{-\tilde{x}}^{\sim}(t) + \lambda_{+\tilde{x}}^{\sim}(t) \quad (3.25)$$

A similar development can be used to obtain  $\lambda_{\tilde{y}}^{\sim}(t)$  and  $\lambda_{\tilde{z}}^{\sim}(t)$ .

It is intended in this work to empirically determine the necessary density functions and in turn to calculate the probabilities from these density functions. In order to empirically determine the density function in equation (3.24) directly, it would be necessary to have a large quantity of data on aircraft pairs since the  $\tilde{x}$ ,  $\tilde{y}$  and  $\tilde{z}$  coordinates are the relative coordinates of one aircraft to another. This would be somewhat of an impossibility since the data must contain what amounts to actual collisions in order to obtain the density function at  $\tilde{x} = W_x^{\sim}$ . Another way of obtaining the density function in equation (3.24) would be to obtain the density function of the position of only one aircraft at a time with respect to some coordinate system. Then for situations involving two aircraft it will be assumed that the behavior of both the aircraft are independent of the other aircraft. The density function of the relative coordinates could then be obtained from the density functions of the individual aircraft coordinates. Of course, the behavior of



one aircraft would not normally be independent of the other aircraft if the other aircraft were nearby. However, for CAS application, it is preferable to consider the behavior of the two aircraft to be independent. This is because the CAS is to be used as a backup system to the present ATC. The warning would be given, hopefully, only in the event of a failure of the ATC. Therefore one would suspect that before the warning is given the two aircraft would be unaware of the other's presence and that the behavior of the two aircraft would be independent. Therefore the probability of a collision without a warning will be determined by assuming that the behavior of the two aircraft are independent of each other.

Let a coordinate system be established such that at  $t = 0$ , the aircraft is at the origin of the system. Let the  $z$  axis be vertical and let the  $y$  axis be aligned with the aircraft velocity vector. The joint density functions of  $x$ ,  $y$ ,  $z$ , and  $\dot{x}$  can be determined from one density function of  $x$ ,  $y$ ,  $z$ ,  $\dot{x}$ , and  $\dot{y}$  if this one density function is assumed for both aircraft. This density function of  $x$ ,  $y$ ,  $z$ ,  $\dot{x}$ , and  $\dot{y}$  contains six independents including time. Since the number of memory locations required to store an imperial function increases exponentially with the number of independent variables, it is desirable to make the numbers of independent variables as small as possible. One plausible means of doing this is to assume that the altitude coordinate is independent of the  $x$  and  $y$  coordinates. Therefore

$$f_{\underline{xyzy}}(x, y, z, \dot{x}, \dot{y}; t) = f_{\underline{xy\dot{x}\dot{y}}}(x, y, \dot{x}, \dot{y}; t) f_{\underline{z}}(z; t) \quad (3.26)$$

which gives an appreciable reduction in the memory locations required. The



density function  $f_{\underline{xyxy}}(x, y, \dot{x}, \dot{y}; t)$  still contains five independent variables. It would therefore be beneficial to find another method of finding  $\lambda(t)$  which does not require empirically determining a function of five independent variables.

### A New Method

This method, like Marks', obtains only an approximation of the collision probability. The assumptions used to justify the approximation are also similar to those made by Marks. The primary advantage of this method is that it is easier to empirically evaluate the collision probability.

Consider two aircraft, A and B, in a three dimensional space with some specified initial conditions. Let  $S_a$  be the set of all possible paths, with the specified initial conditions, that aircraft A can follow. Similarly, let the set  $S_b$  consist of all the possible flight paths that satisfy the initial conditions for aircraft B. The set  $S_c$  is defined as the cartesian product of these two sets:

$$S_c = S_a \times S_b \quad (3.27)$$

Then  $S_c$  consist of all possible pairs of one path from the set  $S_a$  and one path from the set  $S_b$ .

In order to determine when and if a collision occurs, each aircraft is viewed as being surrounded by a right circular cylinder of radius  $r$  and height  $2h$ . If these cylinders touch or overlap, then a collision is assumed to have occurred. Each outcome of  $S_c$  must either contain one or more collisions, or no collisions. For the outcomes that have a collision there is a time  $t_1^*$  at which the first

collision occurs. It is also possible to associate with each collision the coordinates  $x_a^*$ ,  $y_a^*$ , and  $z_a^*$  which are the coordinates of aircraft A at  $t_1^*$ . Let the dwell time  $\delta$  be the time the cylinder surrounding the aircraft remains in continuous contact after the first collision occurs. The set C is a subset of  $S_c$  of all the outcomes that contain a collision. By considering only the outcomes in C, a joint density and distribution function of  $\delta$ ,  $x_a^*$ ,  $y_a^*$ ,  $z_a^*$ , and  $t_1^*$  can be found. The notation for these functions are  $f_{\delta x_a^* y_a^* z_a^* t_1^*}(\delta, x_a^*, y_a^*, z_a^*, t_1^* | C)$  and  $F_{\delta x_a^* y_a^* z_a^* t_1^*}(\delta, x_a^*, y_a^*, z_a^*, t_1^* | C)$ . The conditionality on C indicates that only the outcomes of  $S_c$  with a collision were considered. Let the condition R be such that the coordinates of aircraft A at some time satisfy  $\underline{x}_a \leq x$ ,  $\underline{y}_a \leq y$ ,  $\underline{z}_a \leq z$ , and  $\underline{t}_a \leq t$ . The union of C and R gives all the outcomes of  $S_c$  that result in a collision such that  $\underline{x}_a^* \leq x$ ,  $\underline{y}_a^* \leq y$ ,  $\underline{z}_a^* \leq z$ , and  $\underline{t}_1^* \leq t$ .

This method is based on the approximate equality between the expected dwell time of the first collision and the expected time in contact. This approximation equality can then be solved for a joint density function containing the time of the first collision. Integrating this density function gives the probability of a collision occurring.

First it is desired to find the expected value of the dwell time considering all outcomes of  $S_c$ . To find this expected value, first consider finding the probability that  $\underline{\delta} \leq \delta$ ,  $\underline{x}_a^* \leq x_a^*$ ,  $\underline{y}_a^* \leq y_a^*$ ,  $\underline{z}_a^* \leq z_a^*$ , and  $\underline{t}_1^* \leq t$  given R and C. This conditional probability can be written

$$P\{\underline{\delta} \leq \delta, \underline{x}_a^* \leq x, \underline{y}_a^* \leq y, \underline{z}_a^* \leq z, \underline{t}_1^* \leq t | CR\} = \frac{P\{\underline{\delta} \leq \delta, \underline{x}_a^* \leq x, \underline{y}_a^* \leq y, \underline{z}_a^* \leq z, \underline{t}_1^* \leq t, CR\}}{P\{CR\}} \quad (3.28)$$

If the collision is an element of the set  $\{\underline{x}_a^* \leq x, \underline{y}_a^* \leq y, \underline{z}_a^* \leq z, \underline{t}_1^* \leq t\}$ , it is also a member of the set C as well as satisfying the condition R.

Therefore

$$P\{\underline{\delta} \leq \delta, \underline{x}_a^* \leq x, \underline{y}_a^* \leq y, \underline{z}_a^* \leq z, \underline{t}_1^* \leq t, CR\} =$$

$$P\{\underline{\delta} \leq \delta, \underline{x}_a^* \leq x, \underline{y}_a^* \leq y, \underline{z}_a^* \leq z, \underline{t}_1^* \leq t\} \quad (3.29)$$

By using equation (3.29) and the relation

$$P\{CR\} = P\{R|C\}P\{C\} \quad (3.30)$$

and the fact that

$$P\{\underline{\delta} \leq \delta, \underline{x}_a^* \leq x, \underline{y}_a^* \leq y, \underline{z}_a^* \leq z, \underline{t}_1^* \leq t | C\} =$$

$$\frac{P\{\underline{\delta} \leq \delta, \underline{x}_a^* \leq x, \underline{y}_a^* \leq y, \underline{z}_a^* \leq z, \underline{t}_1^* \leq t\}}{P\{C\}} \quad (3.31)$$

then equation (3.28) can be written

$$P\{\underline{\delta} \leq \delta, \underline{x}_a^* \leq x, \underline{y}_a^* \leq y, \underline{z}_a^* \leq z, \underline{t}_1^* \leq t | CR\} =$$

$$\frac{P\{\underline{\delta} \leq \delta, \underline{x}_a^* \leq x, \underline{y}_a^* \leq y, \underline{z}_a^* \leq z, \underline{t}_1^* \leq t | C\}}{P\{R|C\}} \quad (3.32)$$

By using the definition of a distribution function and differentiating, equation (3.32) can be used to obtain

$$\begin{aligned}
 \frac{f_{\delta x_a^* y_a^* z_a^* t_1^*}(\delta, x_a^*, y_a^*, z_a^*, t_1^* | RC) =}{a^3} \\
 \frac{f_{\delta x_a^* y_a^* z_a^* t_1^*}(\delta, x_a^*, y_a^*, z_a^*, t_1^* | C)}{P\{R|C\}} \quad (3.33)
 \end{aligned}$$

The expected value of  $\underline{\delta}$  given R and C is therefore

$$E\{\underline{\delta}|RC\} = \int_0^t \int_{-\infty}^z \int_{-\infty}^y \int_{-\infty}^x \int_0^\infty \frac{f_{\delta x_a^* y_a^* z_a^* t_1^*}(\delta, x_a^*, y_a^*, z_a^*, t_1^* | C)}{P\{R|C\}} d\delta dx_a^* dy_a^* dz_a^* dt_1^* \quad (3.34)$$

It can be shown that<sup>3</sup>

$$E\{\underline{x}\} = E\{\underline{x}|A_1\}P\{A_1\} + E\{\underline{x}|A_2\}P\{A_2\} \quad (3.35)$$

if

$$A_1 A_2 = 0$$

and

$$A + A_2 = S$$

where S is the certain event.

By an extension of equation (3.35) the following results:

$$E\{\underline{\delta}|R\} = E\{\underline{\delta}|RC\}P\{C|R\} + E\{\underline{\delta}|R\bar{C}\}P\{\bar{C}|R\} \quad (3.36)$$

But since

$$E\{\underline{\delta}|\bar{C}R\} = 0 \quad (3.37)$$

then

---

<sup>3</sup>See Papoulis (22) page 144.



$$E\{\underline{\delta} | R\} = E\{\underline{\delta} | RC\} P\{C | R\} \quad (3.38)$$

Also  $f_{\underline{\delta} x_a y_a z_a t_1}^{* * * *}(\underline{\delta}, x_a^*, y_a^*, z_a^*, t_1^* | C)$  can be written as

$$\begin{aligned} & f_{\underline{\delta} x_a y_a z_a t_1}^{* * * *}(\underline{\delta}, x_a^*, y_a^*, z_a^*, t_1^* | C) = \\ & \underline{\delta} f_{\underline{\delta} | x_a^*, y_a^*, z_a^*, t_1^*} f_{x_a^*, y_a^*, z_a^*, t_1^*}^{* * * *}(x_a^*, y_a^*, z_a^*, t_1^* | C) \end{aligned} \quad (3.39)$$

Using Equation (3.34), (3.35) and (3.36) and carrying out the integration with respect to  $\underline{\delta}$ , the expected value of  $\underline{\delta}$  given R is given as

$$\begin{aligned} E\{\underline{\delta} | R\} &= \frac{P\{C | R\}}{P\{R | C\}} \int_0^t \int_{-\infty}^z \int_{-\infty}^y \int_{-\infty}^x E\{\underline{\delta} | x_a^*, y_a^*, z_a^*, t_1^*\} \\ & \cdot f_{x_a^*, y_a^*, z_a^*, t_1^*}^{* * * *}(x_a^*, y_a^*, z_a^*, t_1^*) dx_a^* dy_a^* dz_a^* dt_1^* \end{aligned} \quad (3.40)$$

The expected time in contact can be found by integrating with respect to time the probability at any instant of time that the two aircraft are in contact (see discussion related to Equation (3.5)). By noting that

$$\frac{f_{x_a^*, y_a^*, z_a^*}^{* * * *}(x_a^*, y_a^*, z_a^* | R; t_a)}{P\{R\}} = \frac{f_{x_a^*, y_a^*, z_a^*}^{* * * *}(x_a^*, y_a^*, z_a^*; t_a)}{P\{R\}} \quad (3.41)$$

then

$$E\{\text{time in contact} | R\} = \frac{1}{P\{R\}} \int_0^t \int_{-\infty}^z \int_{-\infty}^y \int_{-\infty}^x f_{x_a y_a z_a}(x_a, y_a, z_a; t) \\ \cdot \left[ \int_{x_a - r}^{x_a + r} \int_{y_a - \sqrt{r^2 - x_a^2}}^{y_a + \sqrt{r^2 - x_a^2}} \int_{z_a - h}^{z_a + h} f_{x_b y_b z_b}(x_b, y_b, z_b; t) dx_b dy_b dz_b \right] dx_a dy_a dz_a dt_a \quad (3.42)$$

If the expected number of collisions for all the outcomes of  $S_c$  is small compared to one, then the expected time in contact will be approximately equal to the expected dwell time of the first collision. The contribution to the expected contact time of the dwell time associated with collisions other than the first will be small as compared to the contribution of the dwell time associated with the first collision. Also, attention must be given to the relation between the length of time interval over which the integral is taken and the length of the dwell time. If the time interval is short as compared to the expected dwell time, the number of instances where only a position of the dwell time lies inside the time interval becomes a significant portion of the total number of dwell times. The expected time in contact would then be much smaller than the expected dwell time. Under the appropriate conditions, the expected dwell time in equation (3.40) will be approximately equal to the expected contact time in equation (3.42). By making use of the relation

$$P\{RC\} = P\{R|C\}P\{C\} = P\{C|R\}P\{R\} \quad (3.43)$$

equation (3.40) and equation (3.42) give

$$P\{C\} \int_0^t \int_{-\infty}^z \int_{-\infty}^y \int_{-\infty}^x E\{\underline{\delta} | x_a^*, y_a^*, z_a^*, t_1^*\} \cdot$$

$$\frac{f_{x_a^* y_a^* z_a^* t_1^*}(x_a^*, y_a^*, z_a^*, t_1^* | C) dx_a^* dy_a^* dz_a^* dt_1^*}{-a^* - a^* - a^* - 1} \approx$$

$$\int_0^t \int_{-\infty}^z \int_{-\infty}^y \int_{-\infty}^x f_{x_a^* y_a^* z_a^*}(x_a^*, y_a^*, z_a^*; t) \cdot$$

$$\left[ \int_{x_a^* - r}^{x_a^* + r} \int_{y_a^* - \sqrt{r^2 - x_a^2}}^{y_a^* + \sqrt{r^2 - x_a^2}} \int_{z_a^* - h}^{z_a^* + h} f_{x_b^* y_b^* z_b^*}(x_b^*, y_b^*, z_b^*; t) dx_b^* dy_b^* dz_b^* \right] dx_a^* dy_a^* dz_a^* dt \quad (3.44)$$

Taking the partial derivatives of the terms of equation (3.44) with respect to  $t, z, y,$  and  $x$  gives

$$P\{C\} \frac{f_{x_a^* y_a^* z_a^* t_1^*}(x_a^*, y_a^*, z_a^*, t_1^* | C)}{-a^* - a^* - a^* - 1} \approx \frac{f_{x_a^* y_a^* z_a^*}(x_a^*, y_a^*, z_a^*; t)}{-a^* - a^* - a^*}$$

$$\frac{\int_{x_a^* - r}^{x_a^* + r} \int_{y_a^* - \sqrt{r^2 - x_a^2}}^{y_a^* + \sqrt{r^2 - x_a^2}} \int_{z_a^* - h}^{z_a^* + h} f_{x_b^* y_b^* z_b^*}(x_b^*, y_b^*, z_b^*; t) dx_b^* dy_b^* dz_b^*}{E\{\underline{\delta} | x_a^*, y_a^*, z_a^*, t_1^*\}} \quad (3.45)$$

Integrating the left hand side of equation (3.45) over the full range of  $x_a^*, y_a^*,$  and  $z_a^*$  and integrating with respect to time on the interval  $[0, t]$ , the probability of a collision on the interval  $[0, t]$  can be found. This can therefore be approximated by integrating the right hand side equation (3.45). Since only the

flight paths of aircraft A and B that satisfy the specified initial conditions are used in  $S_c$ , and since the density functions in equation (3.45) are defined on  $S_c$ , different density functions are required for each set of initial conditions.

The density functions in equation (3.45) are simpler than the density functions required for Marks' method since they do not include the velocities.

The discussion on how to obtain  $E\{\delta | x_a^*, y_a^*, z_a^*, t_1^*\}$  is given in the next chapter.

Equation (3.45) is the general case where both aircraft paths are random processes. It could also be used where one aircraft path is random and the other is deterministic. Suppose that aircraft A's path is deterministic with

$$x_a(t) = x(t) \quad (3.46)$$

$$y_a(t) = y(t) \quad (3.47)$$

$$z_a(t) = z(t) \quad (3.48)$$

then  $f_{\underline{x}_a \underline{y}_a \underline{z}_a}(x_a, y_a, z_a; t)$  would consist of the impulse function

$$f_{\underline{x}_a \underline{y}_a \underline{z}_a}(x_a, y_a, z_a; t) = \begin{cases} 1 & \text{if } \underline{x}_a = x(t), \underline{y}_a = y(t), \underline{z}_a = z(t) \text{ at } t \\ 0 & \text{otherwise} \end{cases} \quad (3.49)$$

The density function of equation (3.49) could then be used in equation (3.45) to find the collision probability.



## CHAPTER IV

### STATISTICAL ANALYSIS AND COMPUTATIONAL TECHNIQUES

#### Introduction

The data used was collected in 1968 by The Federal Aviation Administration at Atlanta's Hartsfield Airport. Hartsfield Airport employs the ARTS radar system which has the capability of tracking each aircraft and recording in digital form on magnetic tape, the x and y position, the x and y velocities, and the identifying track number. If the aircraft has onboard a mode C transponder, the altitude of the aircraft is also available. Where the altitude of the aircraft was not available from the ARTS radar data, the FAA used voice recordings of the pilot-controller communications to manually insert altitude data at a later date. This data was taken during eleven one hour sessions. The eleven sessions were spaced over a period of four days so as to fairly represent all the traffic conditions at the terminal.

The x and y velocities on the tape received from the FAA were unrecoverable. It was necessary to reconstruct the velocity information from the position information. Since direct computation of the velocity by taking the difference of successive positions proved to have a high noise level, and  $\alpha$ - $\beta$  tracker (23) was employed to smooth the data and supply the velocity data. Fairly heavy filtering was employed which might cause the filteral position to be somewhat displaced from the position on the original tapes. Because of the use to which the data would

be used, the noise was thought less desirable than the shift of position. The data from each time scan was also sorted according to aircraft and the resulting tracks placed sequentially on a new tape. A few of the flight tracks were found with obvious errors. All of the tracks were then checked for velocity ranges, altitude rate, and turn rates. After removing several tracks, the result was 640 tracks that span 93 hours, 36 minutes and 15 seconds. Figure 3 shows a portion of the tracks used.

The statistics of the data were determined by considering only single flight tracks. All flight tracks were rotated and shifted so that the starting point corresponded to the origin of the axes and the velocity vector was aligned with the y-axis. The rotating and shifting obscures any dependence the statistics have on the absolute position of the aircraft.

Since the absolute position of the aircraft was not considered, then the starting point of the track was arbitrary. Any point along the track could have been considered the starting point. Therefore one track was used several times, each time with a new starting point.

The time span over which the statistics were taken was a matter subject to several considerations. The collision probability is the integral of a time varying function. The integration is carried out over the entire time span for which the statistics are known. The portion of the probability that is of most interest for avoiding collision is the portion at a time greater than the time required for an evasive maneuver since this is the position of the probability that can be most affected by a maneuver. If an evasive maneuver in the order

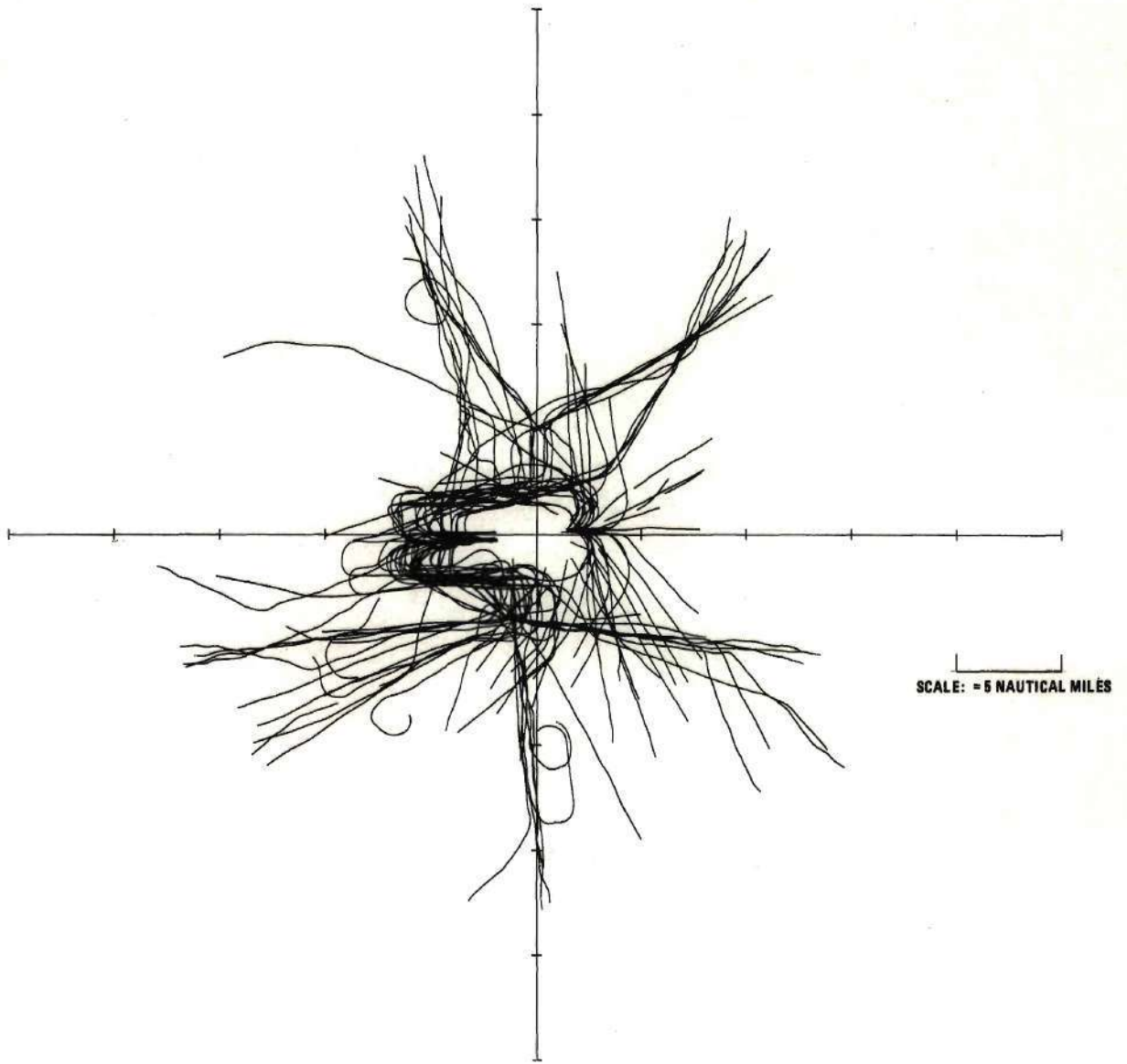


Figure 3. Flight Tracks Recorded at Hartsfield Airport.



of 20 to 30 seconds is considered, then perhaps the time span should extend one or two minutes. Another factor considered was the number of tracks available that extend the full range of the desired time span. The length of the time span together with the time intervals at which the data was collected determined the amount of computer shortage required for the statistics. With all these factors under consideration, a time span of 120 seconds was chosen with data taken at 10 second intervals.

### Position Statistics

It was of interest to find the probability of an aircraft occupying any arbitrary portion of air space at various times given the initial position and velocity. All aircraft were assumed to behave statistically identical regardless of the absolute starting coordinates. Thus only the position relative to the starting position need be dealt with. It was felt that the position statistics would be correlated to the initial velocity. The required density function therefore is  $f_{xyz}(x, y, z; \dot{y}_0, \dot{z}_0, t)$ . The velocity component in the x direction,  $\dot{x}$ , is not included since it will always be zero if the initial velocity vector is aligned with the y axis.

A simplification was gained by considering z to be independent of x and y. Thus giving

$$f_{xyz}(x, y, z; \dot{y}_0, \dot{z}_0, t) = f_{xy}(x, y; \dot{y}_0, t) f_z(z; \dot{z}_0, t) \quad (4.1)$$

If the range of each variable were divided into 10 increments, an array containing  $1 \times 10^6$  elements would be required for the density function on the left-hand side



of equation (4.1) but only  $1.1 \times 10^4$  elements would be required for the density function on the right.

The decision not to include  $\dot{z}$  was also made. It is unlikely that a CAS would have ready access to  $\dot{z}$ . It therefore would not be utilized in a threat prediction criteria and to have carried it further would have been an unnecessary burden. Also, since the altitude rate can be changed readily over the two minute span, the correlation of  $z$  to the initial altitude rate would diminish fairly rapidly with time. The desired density function was reduced to  $f_{\underline{xy}}(x, y; \dot{y}_0, t) \cdot f_{\underline{z}}(z; t)$ . The density function  $f_{\underline{z}}(z; t)$  did not present any difficulty as far as computer storage is concerned; however,  $f_{\underline{xy}}(x, y; \dot{y}_0, t)$  could still become quite large. Ways of further reducing the storage requirements were sought.

The position of an aircraft along some arbitrary flight path can be written

$$\vec{r}(t) = \int_0^t \vec{v}(\tau) d\tau \quad (4.2)$$

By considering the x and y components, then

$$x(t) = \int_0^t v(\tau) \sin \theta(\tau) d\tau \quad (4.3)$$

$$y(t) = \int_0^t v(\tau) \cos \theta(\tau) d\tau \quad (4.4)$$

where  $\theta(t)$  is the heading (measured positively clockwise from the y axis such as azimuth is) and  $v(t)$  is the absolute velocity (airspeed). If the airspeed is constant,

i. e.  $v(t) = V$ , then equation (4.3) and (4.4) becomes

$$\frac{x(t)}{V} = \int_0^t \sin \theta(\tau) d\tau \quad (4.5)$$

$$\frac{y(t)}{V} = \int_0^t \cos \theta(\tau) d\tau \quad (4.6)$$

By letting  $x'(t) = \frac{x(t)}{V}$  and  $y'(t) = \frac{y(t)}{V}$ , the required density function becomes  $f_{\underline{x}, \underline{y}}'(x', y'; t)$ . For any particular velocity the density function for  $x$  and  $y$  was obtained by the relation

$$f_{\underline{xy}}(x, y; \dot{y}_0, t) = V f_{\underline{x}, \underline{y}}'\left(\frac{x}{V}, \frac{y}{V}; t\right) \quad (4.7)$$

where  $\dot{y}_0 = V$ .

In order to see how well the constant velocity assumption fits the actual data, the density function of the normalized velocity error was plotted in figures 4(a), 4(b), 4(c), and 4(d) at 10, 30, 60, and 120 seconds. The normalized velocity error  $\bar{v}(t)$  in the figures is defined as:

$$\bar{v}(t) = \frac{v(t) - v(0)}{v(0)} \quad (4.8)$$

As can be seen from the figures, the constant velocity assumption seems fairly good. Even after 120 seconds the error is seldom greater than 20 percent.

Figures 5, 6, 7, and 8 compare the positions of aircraft in the two coordinate systems. The points were taken from the Atlanta radar data. The  $x$  and  $y$

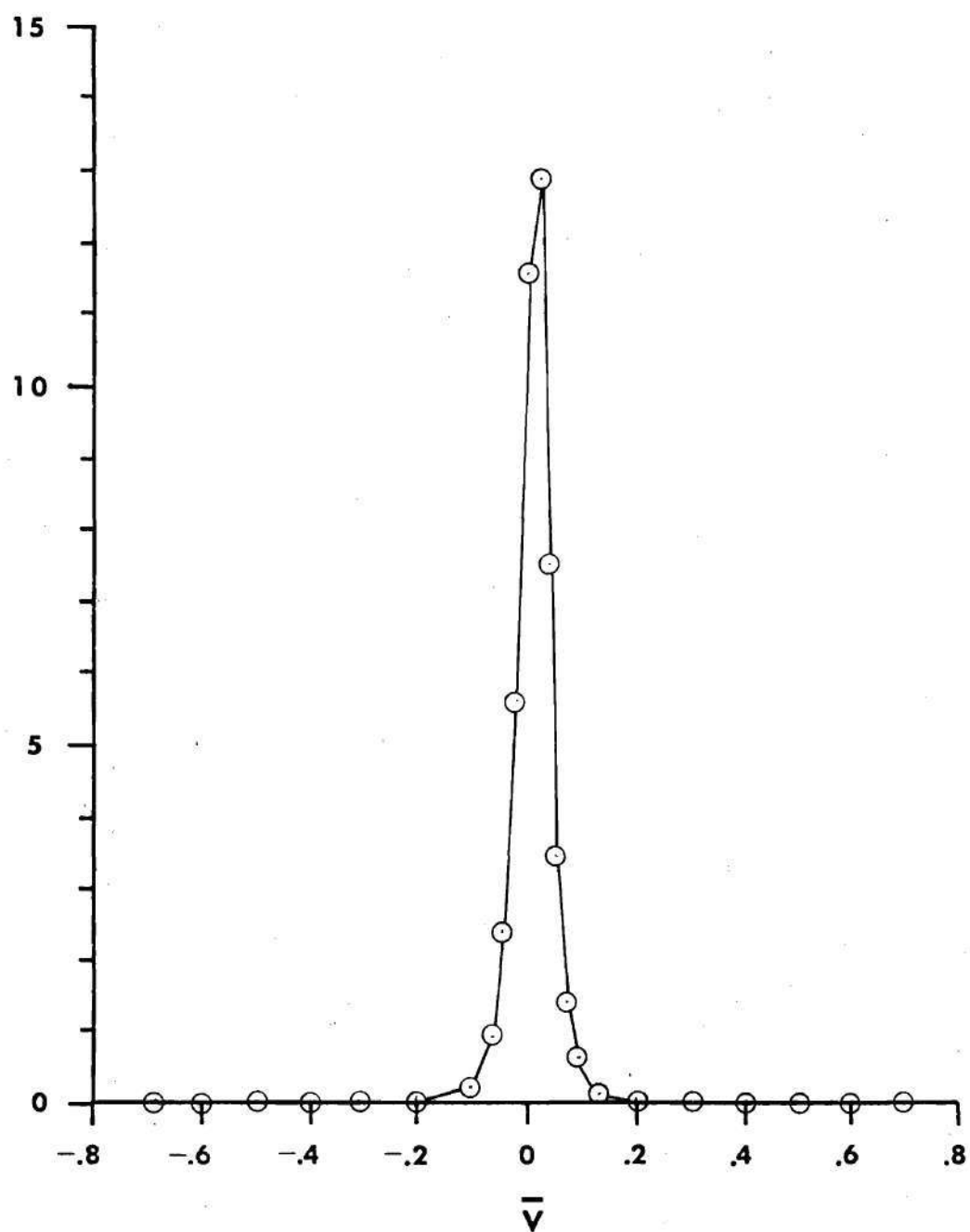


Figure 4(a). Density Function of  $\bar{v}$  After 10 Seconds.

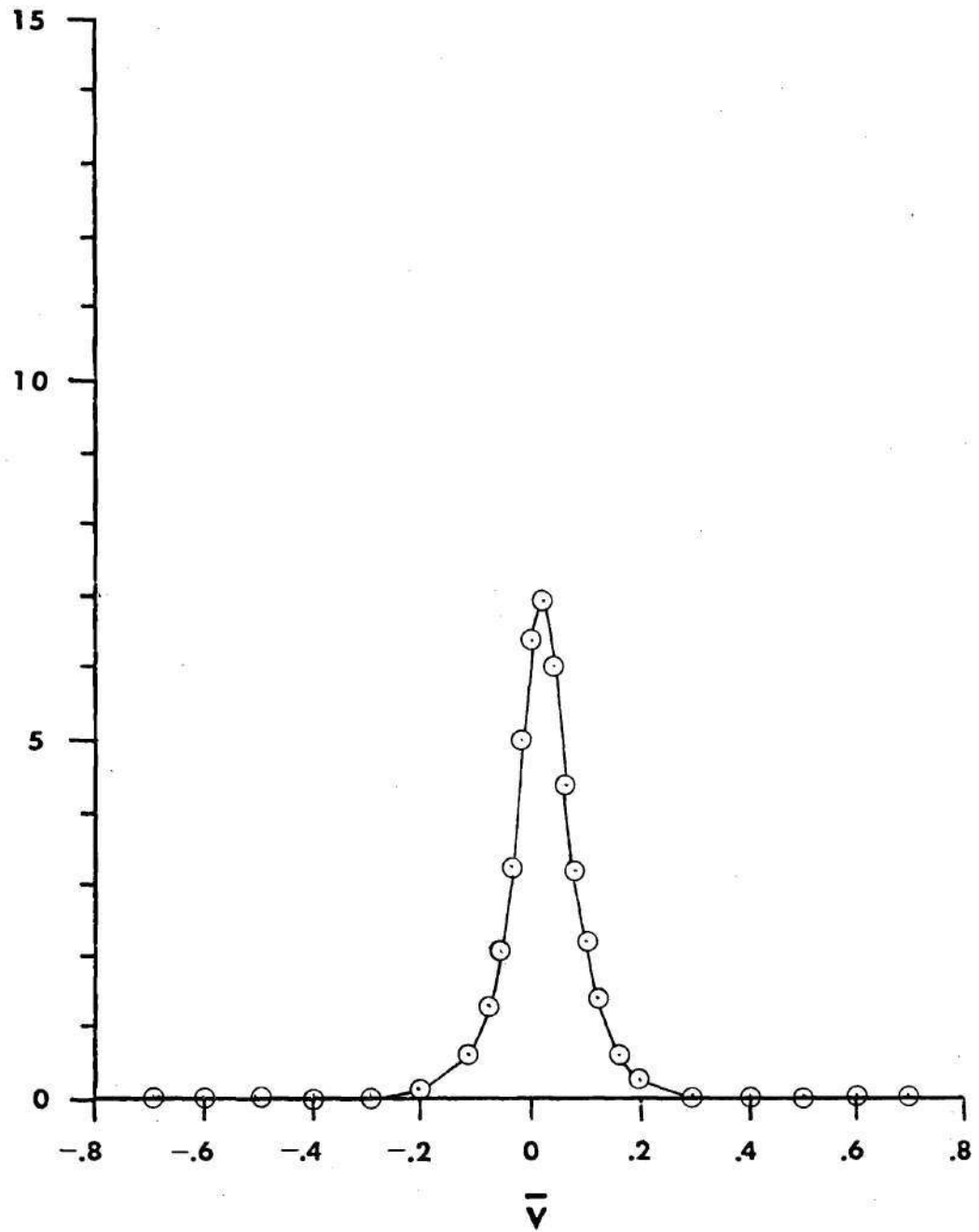


Figure 4(b). Density Function of  $\bar{v}$  After 30 Seconds.



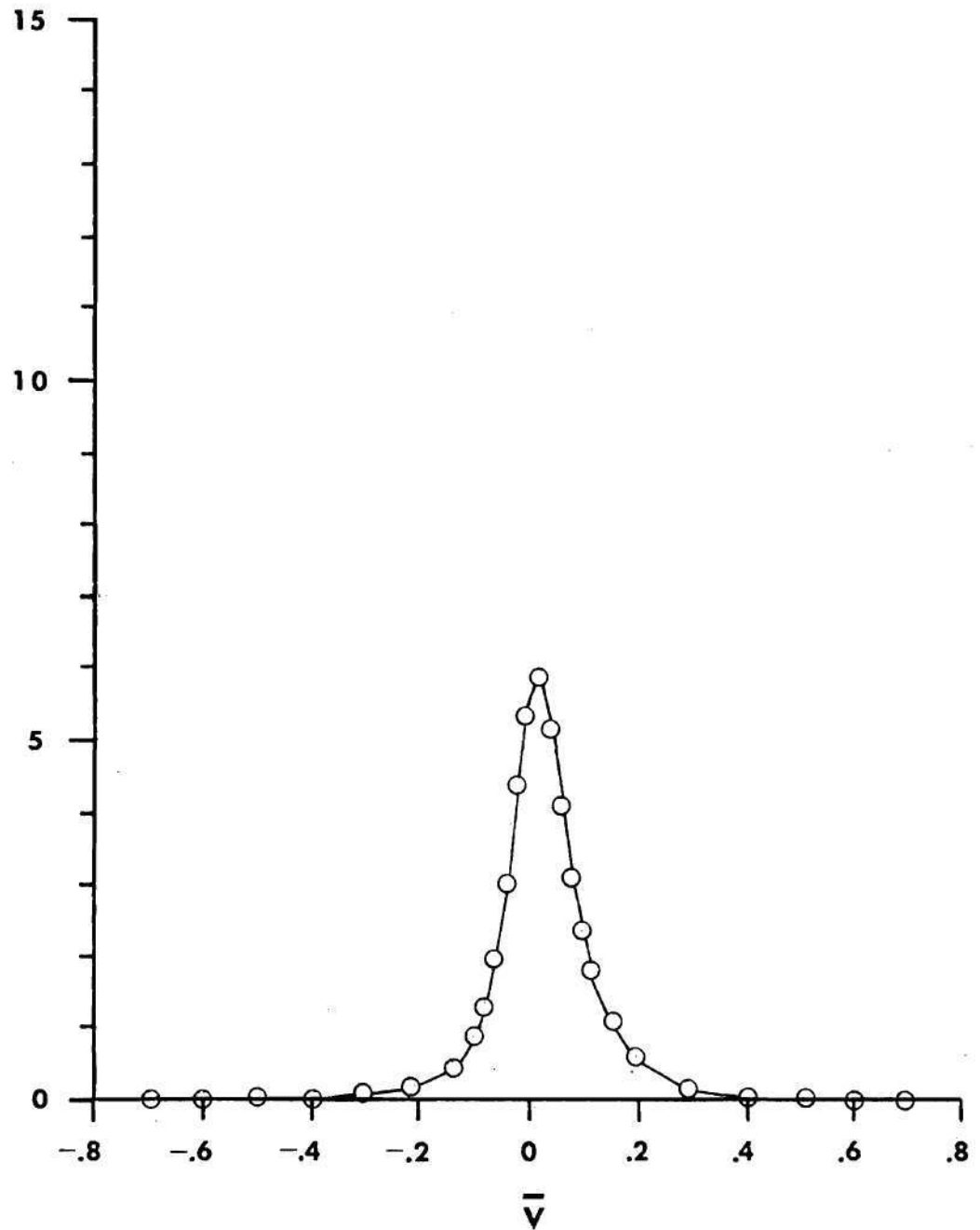


Figure 4(c). Density Function of  $\bar{v}$  After 60 Seconds.

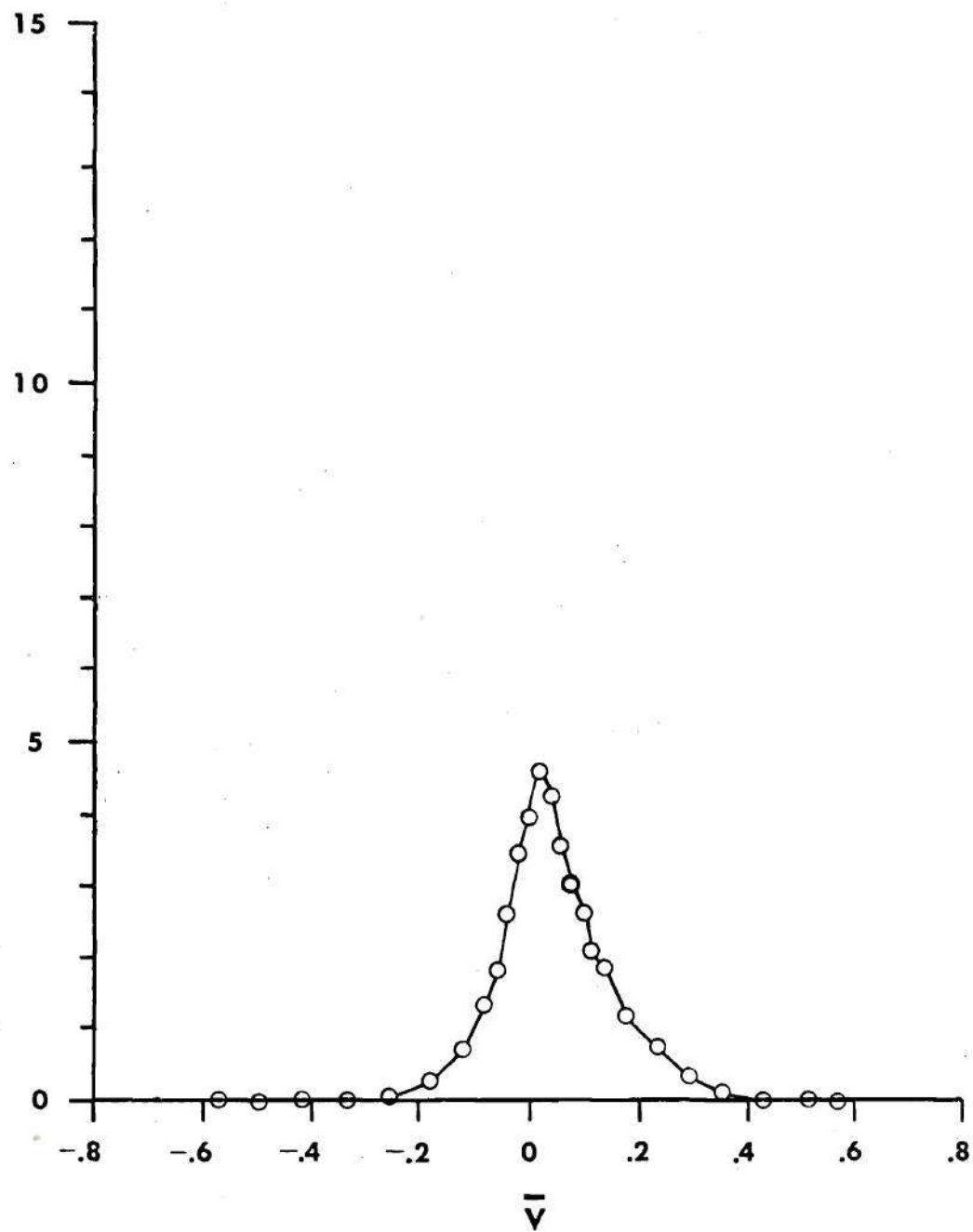


Figure 4(d). Density Function of  $\bar{v}$  After 120 Seconds.

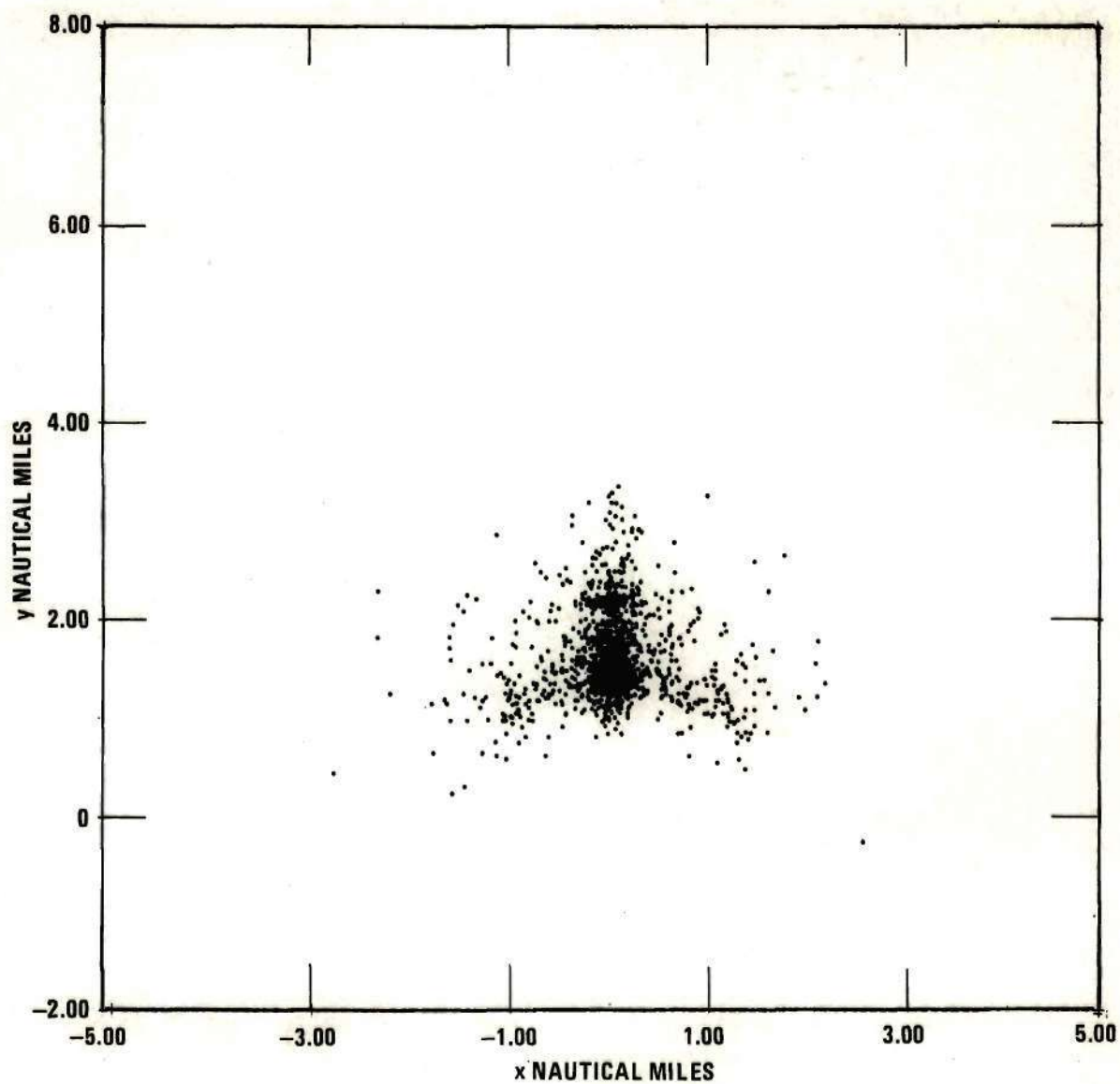


Figure 5. Aircraft Positions in the x,y Coordinate System

After 30 Seconds.

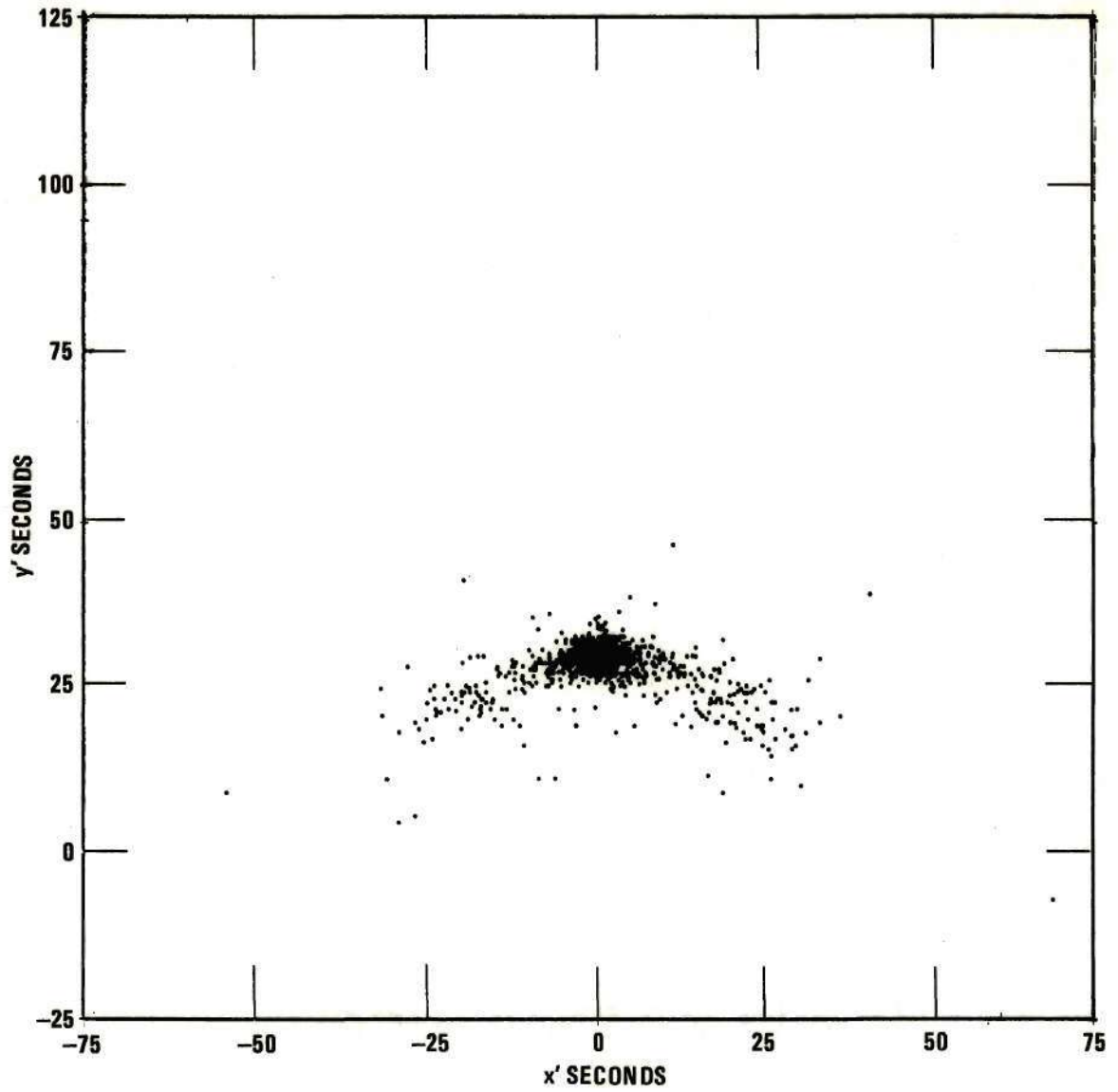


Figure 6. Aircraft Positions in the  $x'$ ,  $y'$  Coordinate System

After 30 Seconds.



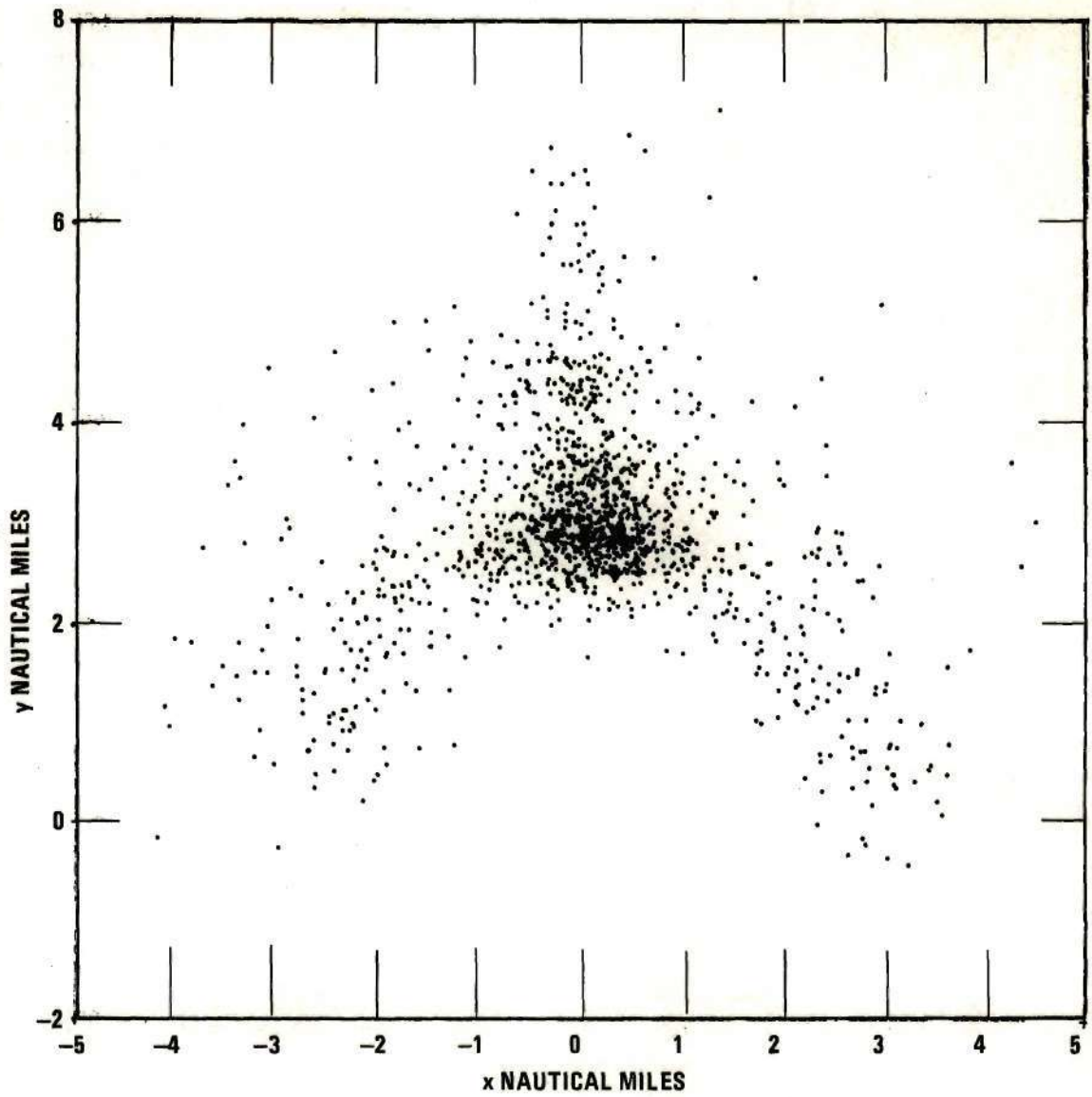


Figure 7. Aircraft Positions in the x, y Coordinate System

After 120 Seconds.

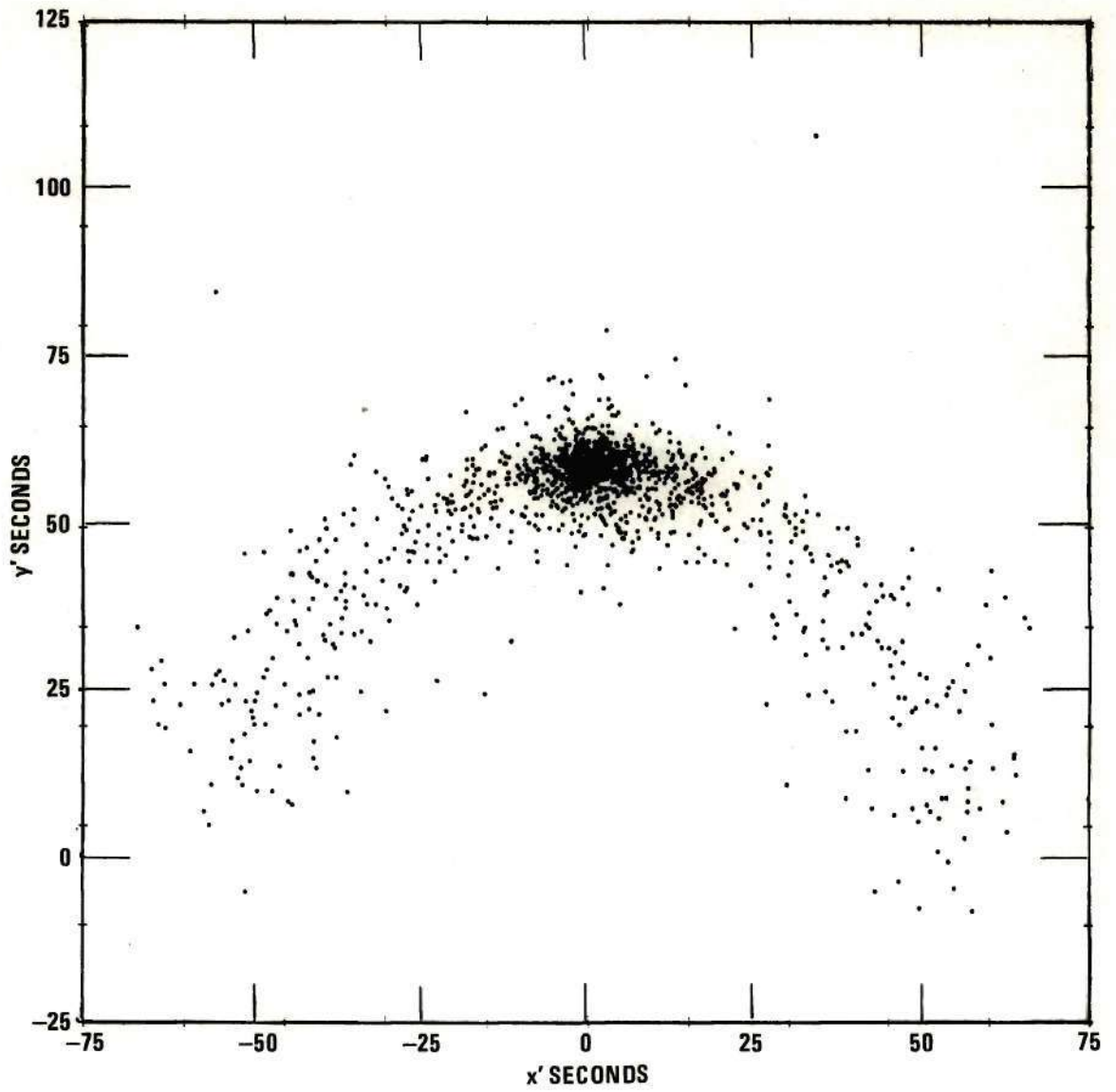


Figure 8. Aircraft Positions in the  $x'$ ,  $y'$  Coordinate System

After 120 Seconds.

coordinates are used in Figures 5 and 7 and the  $x'$  and  $y'$  coordinates are used in Figures 6 and 8. Using  $x'$  and  $y'$  results in a tighter dispersion of points. Notice the crescent shape of the highest concentrations in Figures 6 and 8. This shape suggested a method of representing the data more efficiently than the  $x'$  and  $y'$  coordinate system.

Suppose it is assumed that each position an aircraft occupies is reached by way of a constant radius turn. Then each point would have associated with it a radius  $\rho$  (see Figure 9). The radius  $\rho'$  is  $\rho/V$ , or

$$\rho' = \frac{x'^2 + y'^2}{2x'} \quad (4.9)$$

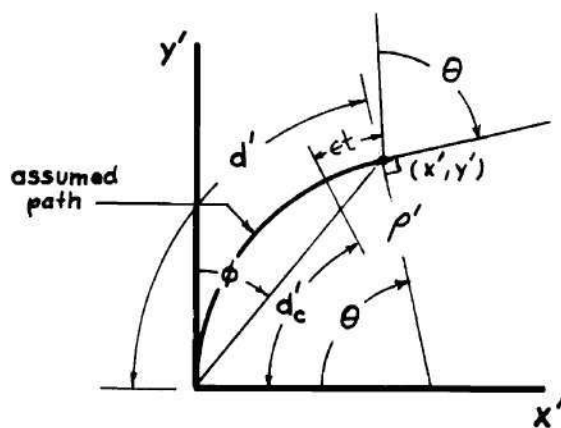


Figure 9. Constant Turn Radius Flight Path.

When the aircraft flies straight, the radius is infinite. There will also be some minimum radius,  $\rho_{\min}$ , dictated by the maximum acceleration of the aircraft. The radius varies between this minimum radius and infinity. Another unique characteristic of a constant radius turn is the curvature which

is defined as the reciprocal of the radius. The curvature has the desirable property of varying from 0 to  $1/\rho_{\min}$  instead of from  $\rho_{\min}$  to  $\infty$  as does the radius. The curvature  $k$ , in terms of  $x$  and  $y$  is

$$k = \frac{1}{\rho'} = \frac{2x'}{x'^2 + y'^2} \quad (4.10)$$

The angle  $\theta$  is the heading the aircraft would have had it followed the constant radius path. This angle in terms of  $x$  and  $y$  is

$$\theta = 2 \tan^{-1} \frac{x'}{y'} \quad (4.11)$$

The distance along the arc in Figure 9 is

$$d' = \rho' \theta \quad (4.12)$$

Had the aircraft maintained its initial velocity it would have covered a distance  $d_c$  where

$$d_c = Vt \quad (4.13)$$

For  $d_c$  to be plotted in the  $x'$  and  $y'$  coordinate system it must be divided by the initial velocity giving

$$d'_c = t \quad (4.14)$$

The difference between the distance  $d'$  in equation (4.12) and  $d'_c$  in



equation (4.14) is the error in position caused by a nonconstant velocity and a nonconstant radius turn. Since this error should have a tendency to increase with time, it can be normalized with time giving

$$\epsilon = \frac{d'_c - d'}{t} = 1 - \frac{\theta}{kt} \quad (4.15)$$

The lines of constant  $\epsilon$  and constant  $k$  could be viewed as a coordinate system for the position of the aircraft. The line for  $\epsilon = 0$  would of course be a function of time. Figure 10 shows the positions of a number of aircraft superimposed on this coordinate system for the purpose of seeing how the data actually fits the coordinate system.

It may be noted that other than the constant velocity assumptions inherent in the  $x'$  and  $y'$  coordinate system, there is no information loss in transforming the  $x'$ ,  $y'$  coordinates to the  $\epsilon$ ,  $k$  coordinates. There is a one-to-one correspondence of points. The advantage that the  $\epsilon$  and  $k$  coordinate system has over the  $x'$  and  $y'$  coordinate system is that the data has a smaller dispersion. This smaller dispersion would mean less wasted computer storage space. Figure 11 shows a three dimensional drawing of the joint density function,  $f_{\epsilon, k}(\epsilon, k; t)$  for  $t = 60$  sec. The concentration about  $\epsilon = 0$  and  $k = 0$  is obvious.

To find the collision probability it is necessary to integrate the right-hand side of equation (3.43). One of the quantities in the right-hand side of equation (3.43) is the integral of  $f_{x_b, y_b, z_b}(x_b, y_b, z_b; t)$  over the collision volume. If this density function is assumed to be constant over the area and  $z_b$  is independent of  $x_b$  and  $y_b$ , then this integral can be approximated by

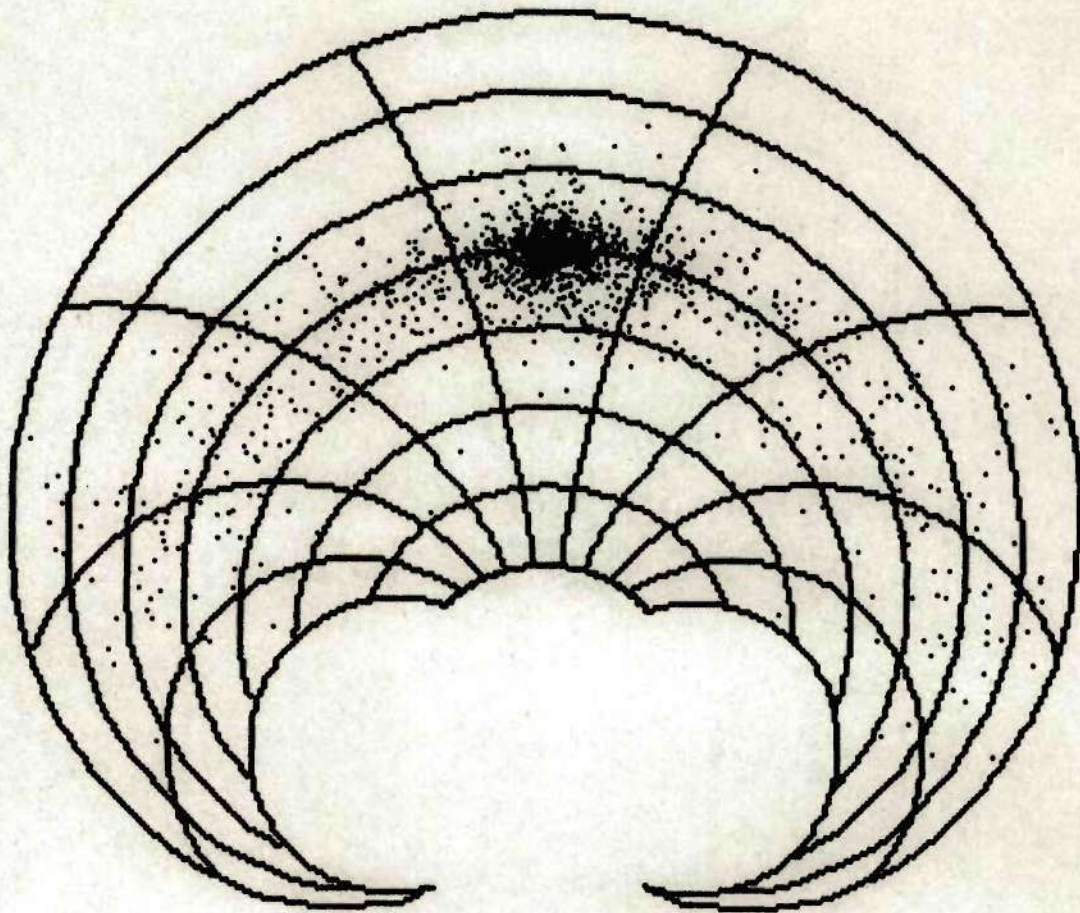


Figure 10. Aircraft Positions After 60 Seconds Superimposed  
on  $\epsilon$ ,  $k$  Coordinate System.



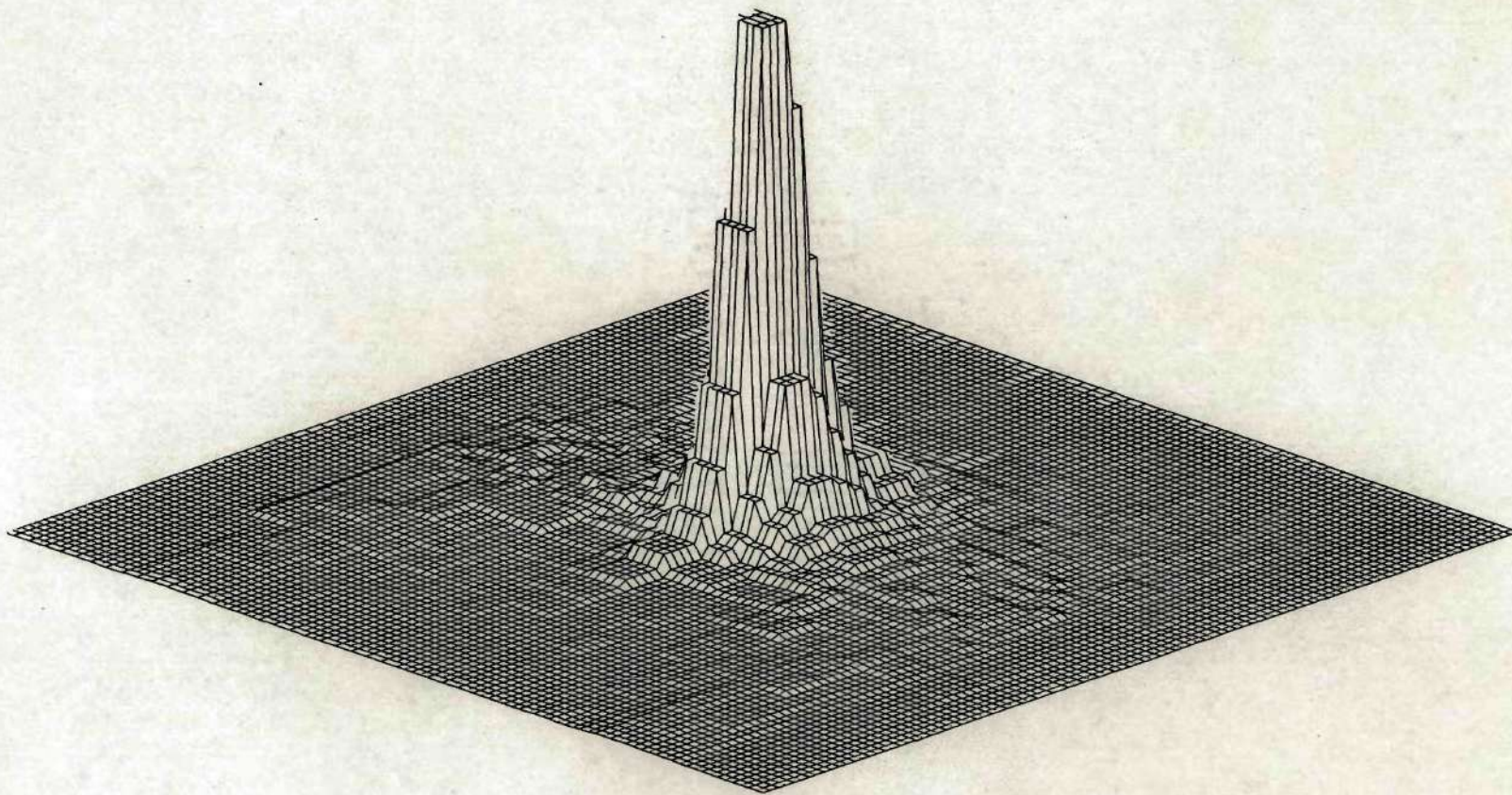


Figure 11. Joint Density Function of  $\underline{\epsilon}$  and  $\underline{k}$ .

$$f_{z_a}(z_a; t) 2hf_{x_b y_b}(x_a, y_a; t) 4\pi r^2 \quad (4.16)$$

It is necessary to find  $f_{x_b y_b}(x_b, y_b)$  from  $f_{\epsilon_b k_b}(\epsilon_b, k_b)$ .

This is

$$f_{x_b y_b}(x_b, y_b; t) = \frac{f_{\epsilon_b k_b}(\epsilon_b, k_b; t)}{\frac{\partial(x, y)}{\partial(\epsilon, k)}} \quad (4.17)$$

where  $\partial(x, y)/\partial(\epsilon, k)$  is the Jacobian of  $x$  and  $y$  with respect to  $\epsilon$  and  $k$ . This Jacobian is

$$\frac{\partial(x_b, y_b)}{\partial(k_b, \epsilon_b)} = \frac{V_b^2 t}{k_b^2} [1 - \cos k_b t (1 - \epsilon_b)] \quad (4.18)$$

where

$$k_b = \frac{2V_b x_b}{x_b^2 + y_b^2} \quad (4.19)$$

and

$$\epsilon_b = 1 - \frac{2}{k_b t} \tan^{-1} \frac{x_b}{y_b} \quad (4.20)$$

The integration could have been carried out with respect to  $x_a$ ,  $y_a$ , and  $z_a$ , but since

$$f_{x_a y_a}(x_a, y_a; t) dx_a dy_a = f_{\epsilon_a k_a}(\epsilon_a, k_a; t) d\epsilon_a dk_a \quad (4.21)$$

the integration was carried out with respect to  $\epsilon_a$  and  $k_a$ . Also the function



$f_{\epsilon, k}(\epsilon; k; t)$  was used instead of  $f_{x, y}(x_a, y_a; t)$ .

### Mean Dwell Time

The mean dwell time is needed in equation (3.45) in order to compute the collision probability.

Each aircraft is considered to be enclosed in a right circular cylinder with radius  $r$  and height  $2h$ . A collision is assumed to have occurred if these two volumes overlap. The dwell time is the length of the time the volumes remain in contact.

When the centers of the two volumes have a horizontal separation of less than  $2r$  and a height separation of less than  $2h$ , then the two volumes are in contact. Let a coordinate system be fixed on aircraft A. In this coordinate system the two aircraft are in contact when the center of aircraft B is within the region shown in Figure 12.

Figure 13 shows the plan view of the collision volume. For the purpose of finding the dwell time, it will be assumed that the relative path of aircraft B to aircraft A is essentially a straight line. This line is characterized by the angle  $\gamma$  and the distance  $b$ . The velocity component in the  $xy$  plane is  $v_h$  and the component parallel to the  $z$  axis is  $v_z$ .

The distance  $a$  is the portion of the flight path that lies over the region. The portion of the path may or may not all be within the volume depending on the  $z$  component and the point of entry. Since the cross-section is symmetric, all the relative flight tracks could be rotated to any angle without affecting the crossing time. Therefore, the crossing time is independent of the angle  $\gamma$ . Consider

all tracks to be parallel to the x axis. Figure 14 shows the cross section of the collision volume that is parallel to the z axis and contains the track.

The portion of the track that lies within the volume depends on the slope  $\alpha$  and the point c where the track crosses the z axis. Therefore the dwell time depends on b, c,  $v_h$ , and  $v_z$ . If the joint density function of the four variables as a function of x, y, z, and t were known, then the mean dwell time could be found.

Several assumptions were made to simplify these calculations. First, it was assumed that the vertical velocities were very small compared to the horizontal velocities. The angle  $\alpha$  would then be approximately 90 degrees. This is equivalent to assuming that all entrances and exits of the collision volume by the relative flight track would occur through the sides of the volume. This eliminated the dependence of the dwell time on the parameter c and the vertical velocity  $v_z$ .

For some given b and  $v_h$ , the dwell time is

$$\delta = \frac{2(4r^2 - b)^{\frac{1}{2}}}{v_h} = \frac{a}{v_h} \quad (4.22)$$

It was also assumed that the random variable  $\underline{b}$  was independent of the position and the velocity. The random variable  $\underline{a}$  would also be independent of  $v_h$ . From equation (4.22) the expected value of the dwell time can be seen to be

$$E\{\underline{\delta}\} = E\left\{\frac{1}{v_h}\right\} E\{\underline{a}\} \quad (4.23)$$

By assuming  $\underline{b}$  to be uniformly distributed between 0 and  $2r$ , the expected

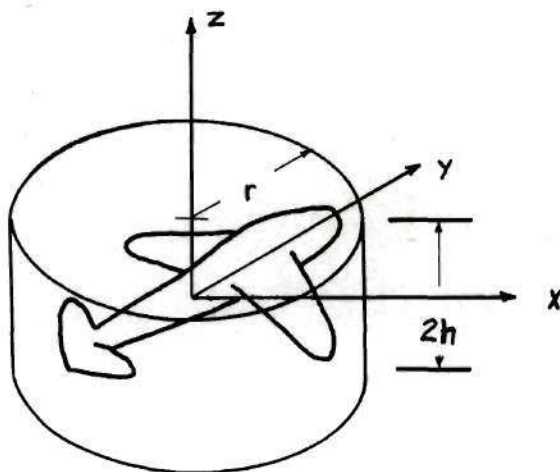


Figure 12. Collision Volume

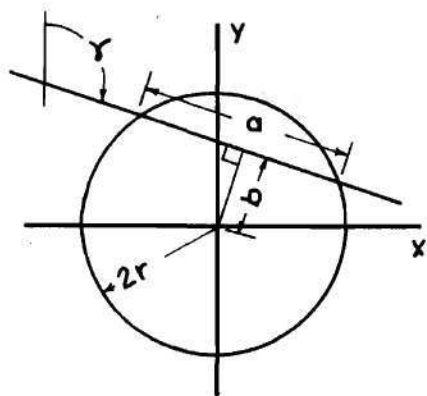


Figure 13. Plane View of Collision Volume.

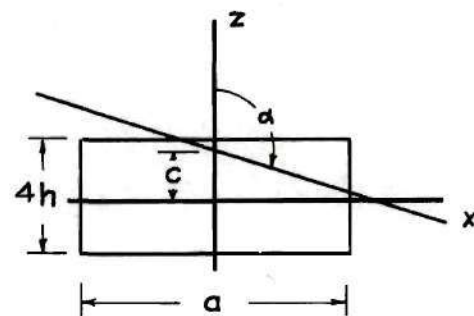


Figure 14. Cross Section View of Collision Volume.

value of  $\underline{a}$  is

$$E\{a\} = \pi r \quad (4.24)$$

It has been assumed that the aircraft maintains essentially constant airspeeds. Let  $V_1$  and  $V_2$  be the velocities for two aircraft. The relative airspeed of the two aircraft is

$$V_h = (V_1^2 + V_2^2 - 2V_1V_2 \cos \theta_r)^{\frac{1}{2}} \quad (4.25)$$

where  $\theta_r$  is the smallest angle between the two velocity vectors.

In relation to the position of an aircraft it was found that the aircraft tends to follow constant radius turns. It would be reasonable then to calculate the heading of the aircraft using this same assumption. For a given  $x$  and  $y$  the heading  $\theta$  is equal to  $2\Phi$  (Figure 9). The radar data was used to check this assumption. Figures 15 through 21 show the density function of the error between the actual heading and the heading equal to  $2\Phi$ . This density function has  $\Phi$  as well as  $t$  as a parameter. For small values of  $\Phi$  the heading calculated from the constant turn rate assumption is fairly good. As  $\Phi$  moves away from zero, the error is no longer distributed evenly about zero.

Aircraft in the terminal area normally execute a standard turn. This turn is a 3 degree/sec. turn which is equivalent to making a complete turn in two minutes. In an attempt to improve the estimate of the heading, it was assumed that an aircraft would reach any point  $x$  and  $y$  by first executing a 3 degree/sec turn followed by a straight line segment. The heading of the straight line segment



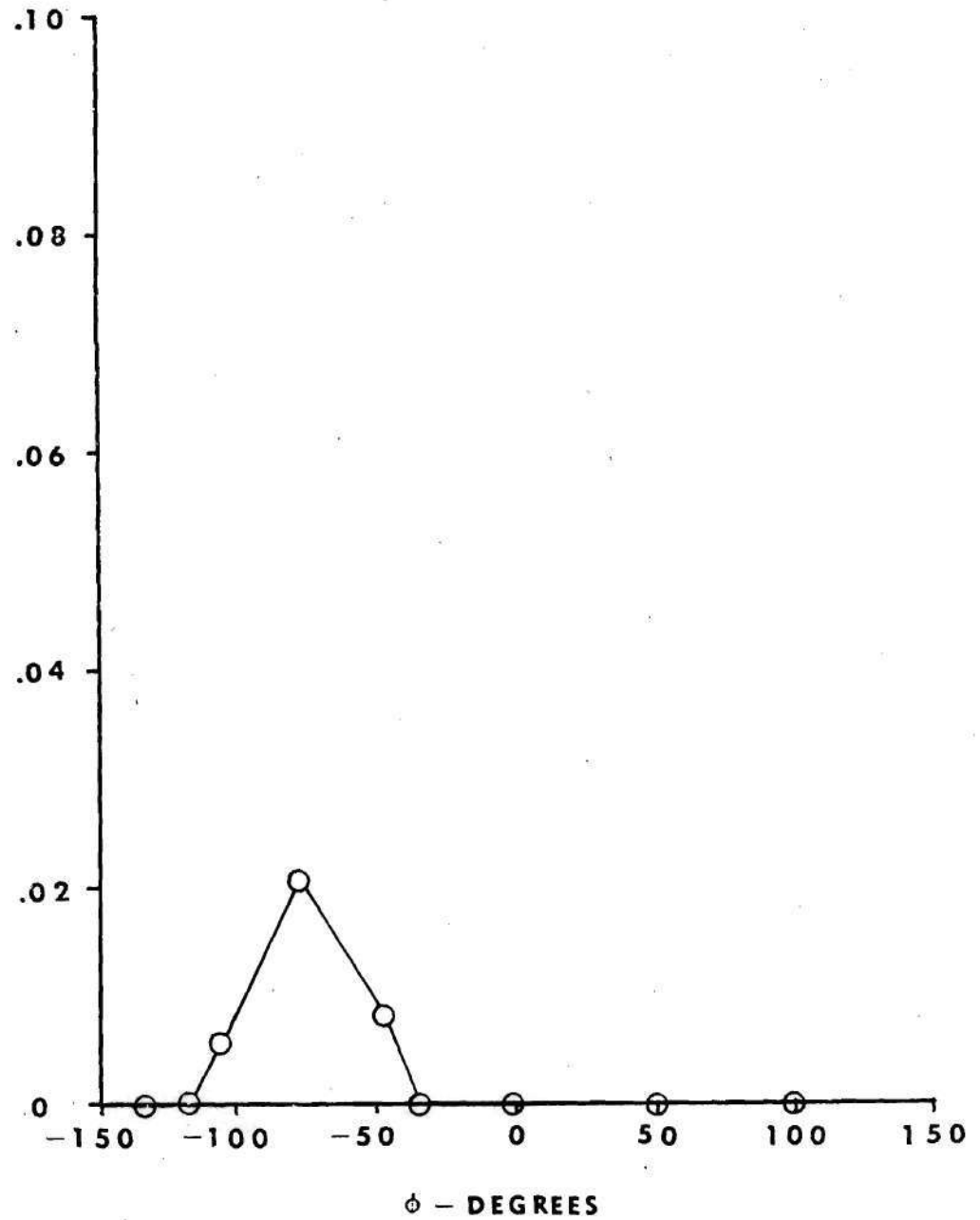


Figure 15. Density Function of the Heading Error After Assuming a Constant Radius Turn, with  $\Phi = 120^\circ$  and  $t = 60$  Seconds.

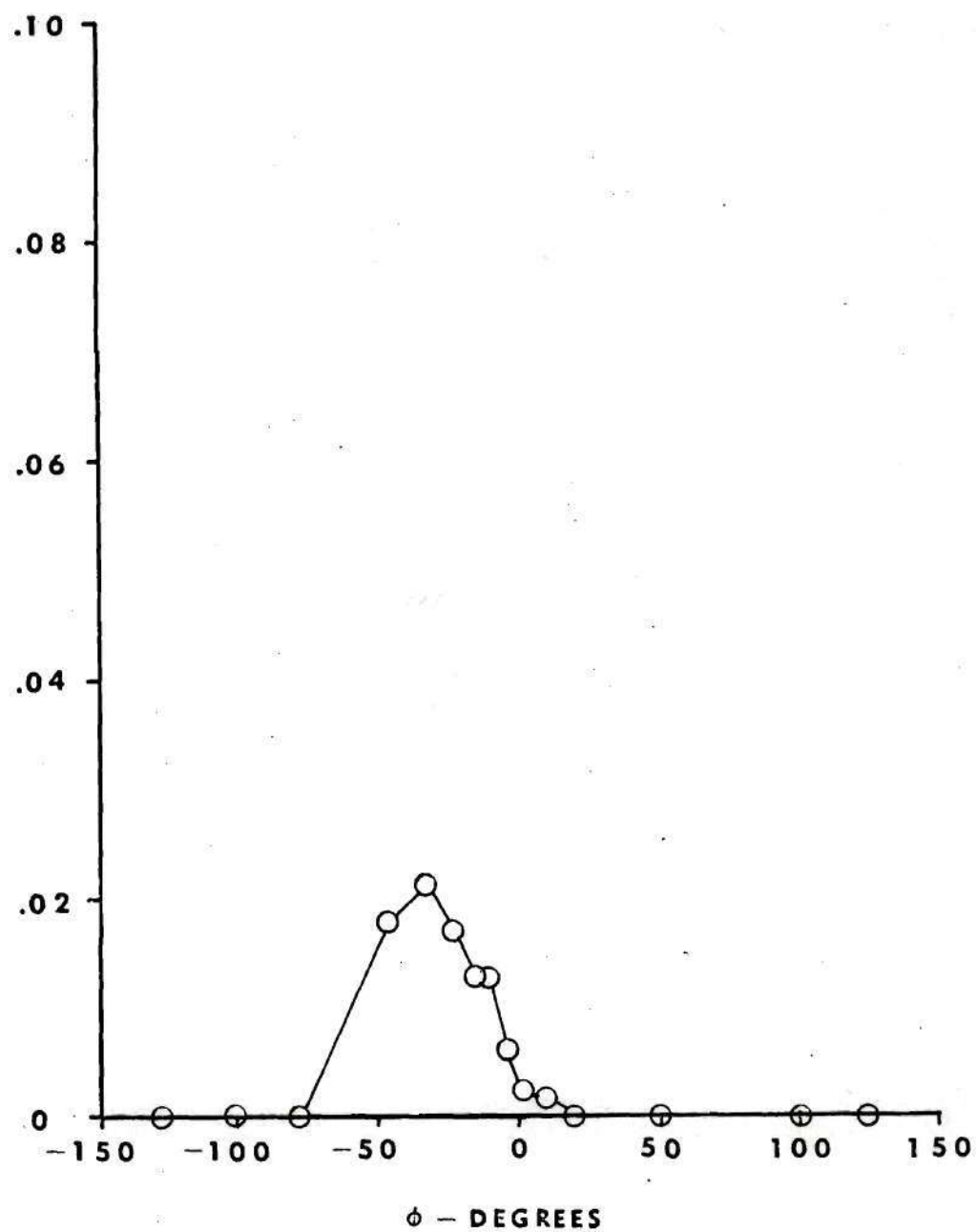


Figure 16. Density Function of the Heading Error After Assuming a Constant Radius Turn, with  $\Phi = 60^\circ$  and  $t = 60$  Seconds.

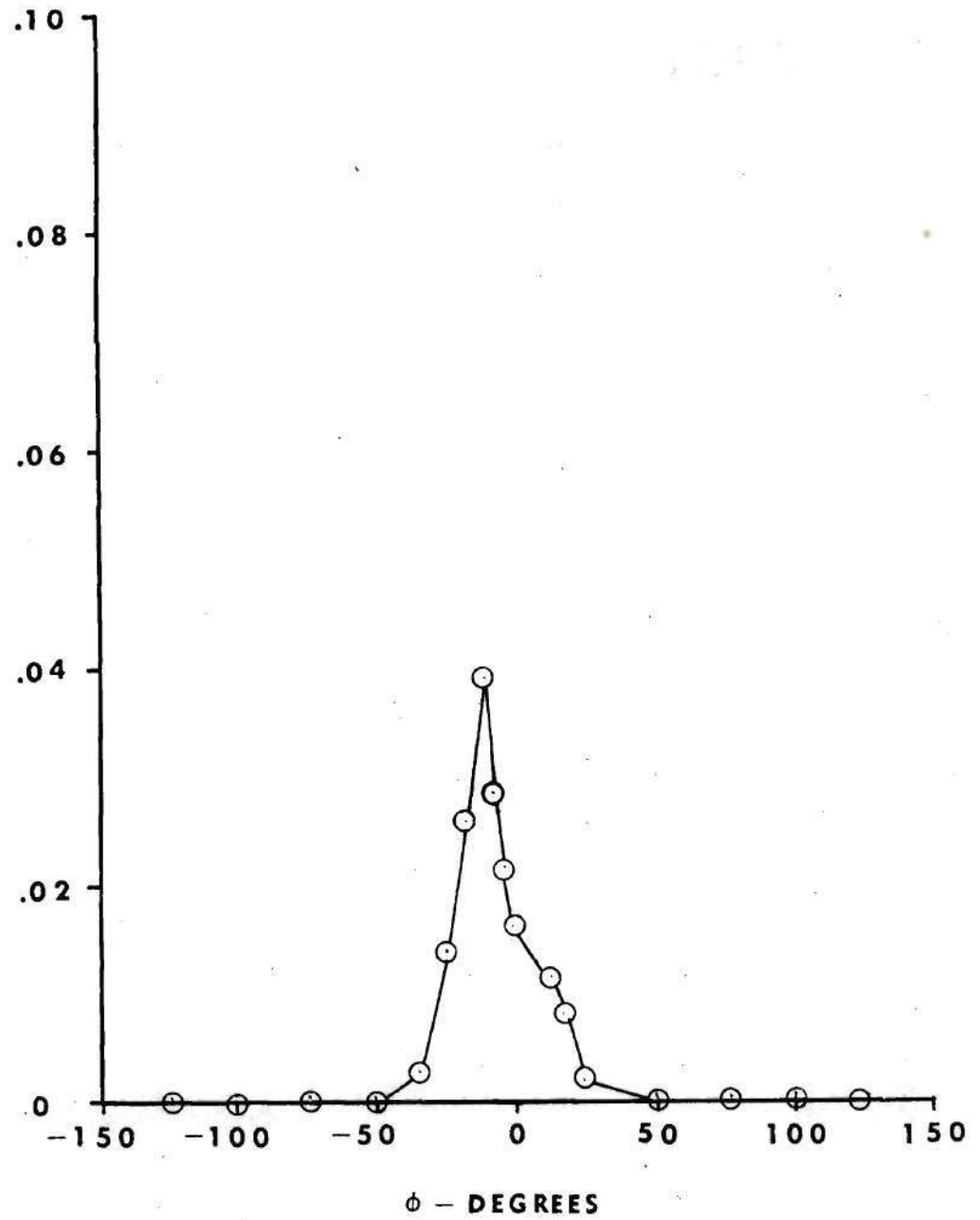


Figure 17. Density Function of the Heading Error After Assuming

a Constant Radius Turn, with  $\Phi = 20^\circ$  and  $t = 60$  Seconds.

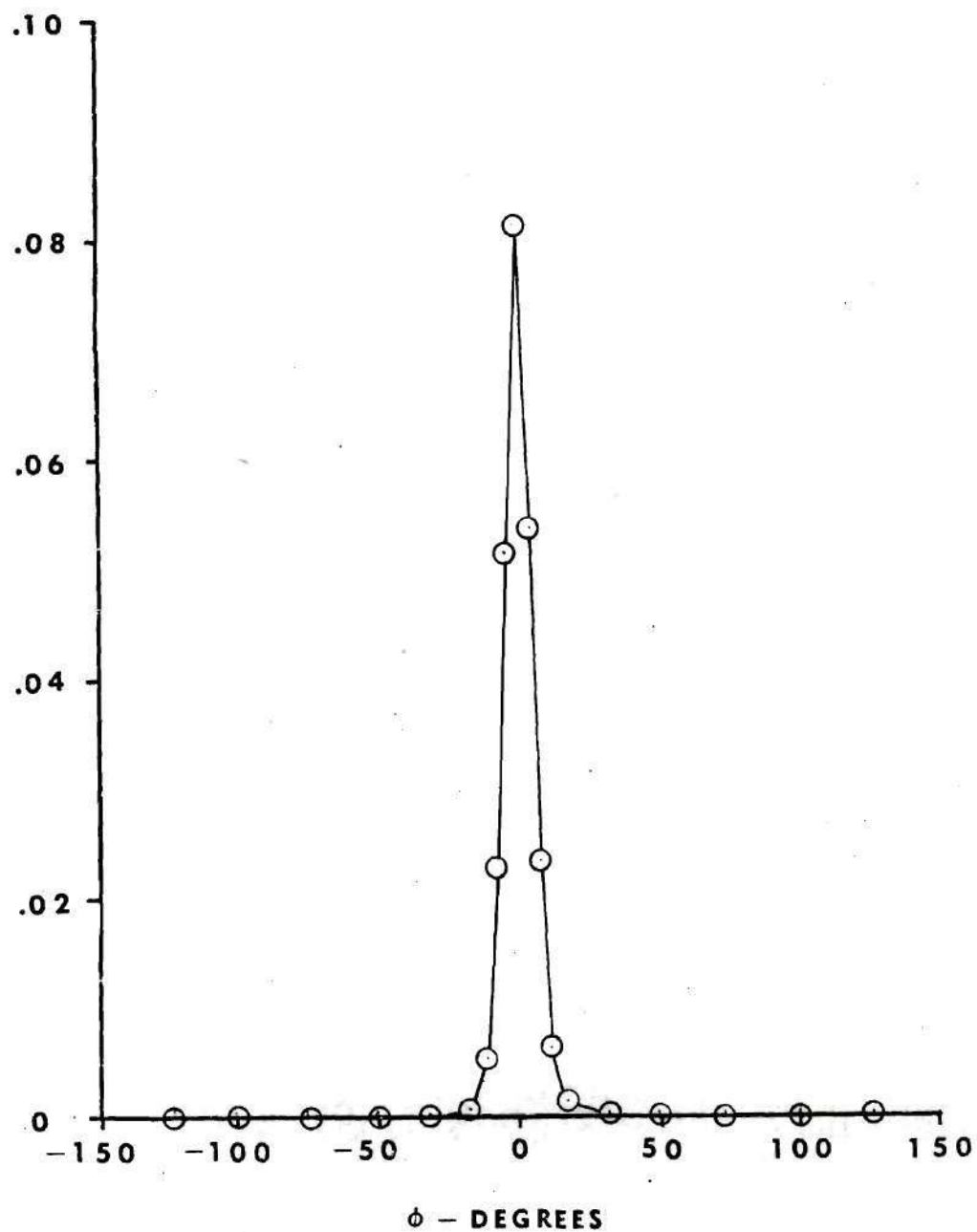


Figure 18. Density Function of the Heading Error After Assuming a Constant Radius Turn, with  $\phi = 0^{\circ}$  and  $t = 60$  Seconds.



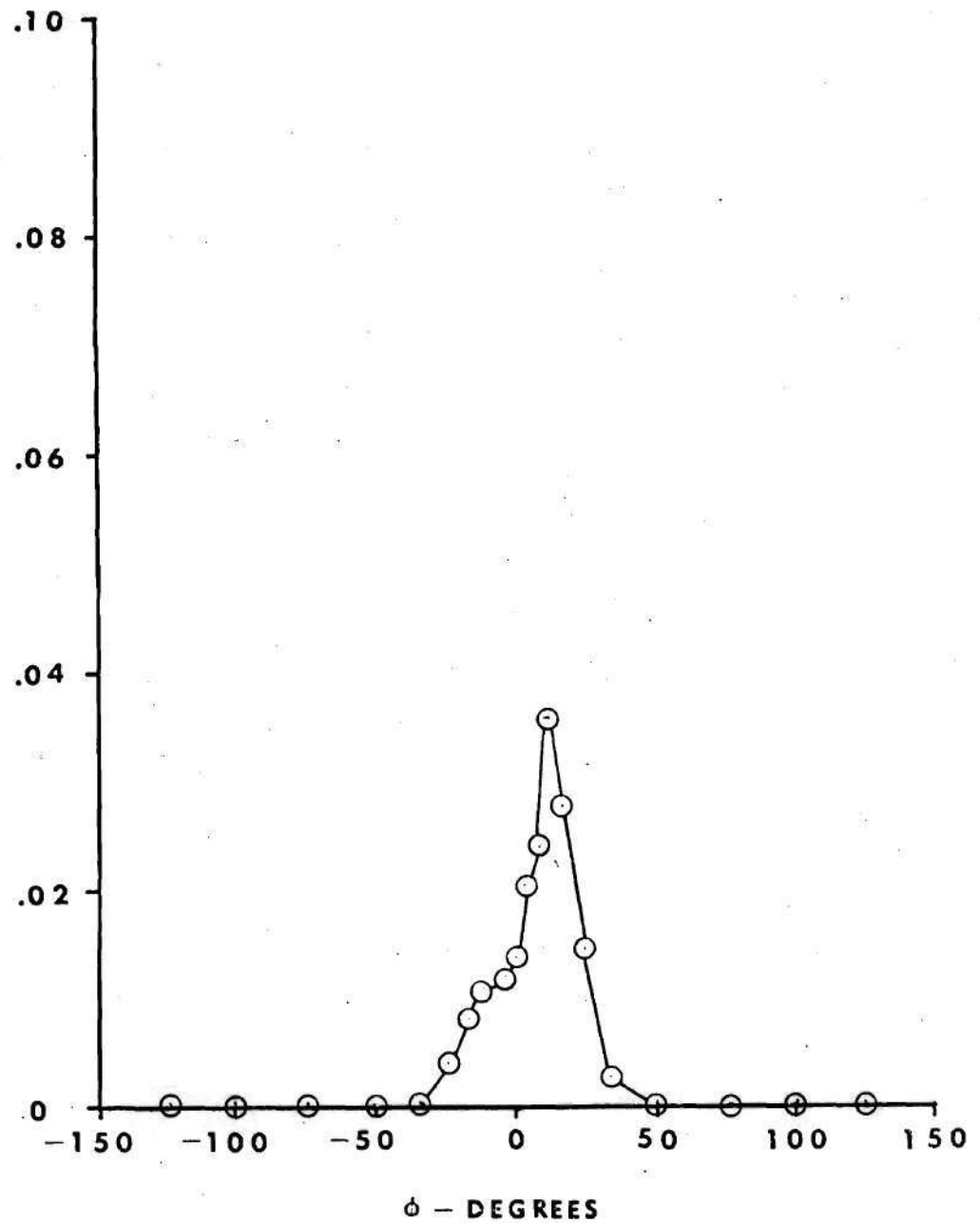


Figure 19. Density Function of the Heading Error After Assuming a Constant Radius Turn, with  $\Phi = -20^{\circ}$  and  $t = 60$  Seconds.

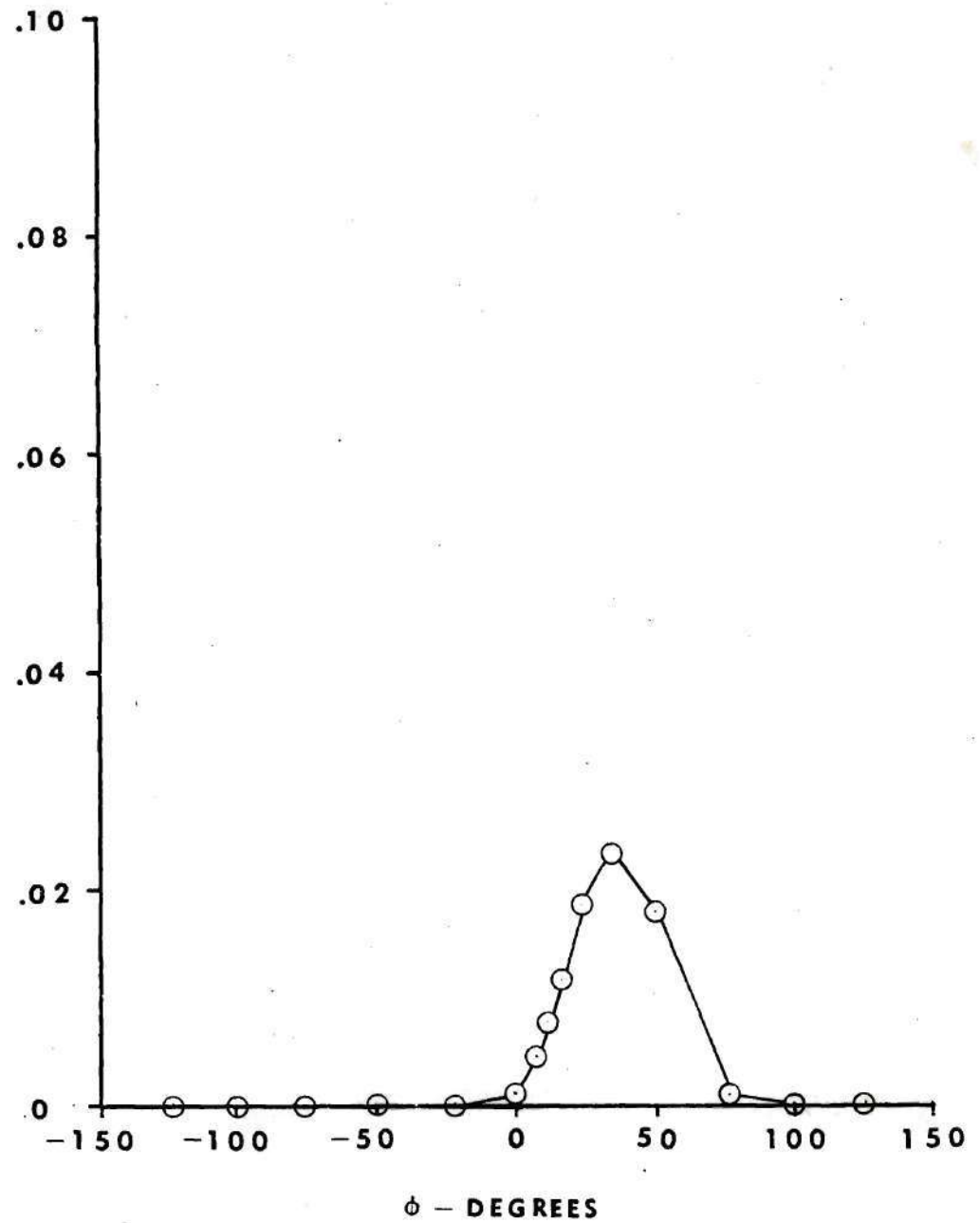


Figure 20. Density Function of the Heading Error After Assuming a Constant Radius Turn, with  $\Phi = -60^{\circ}$  and  $t = 60$  Seconds.

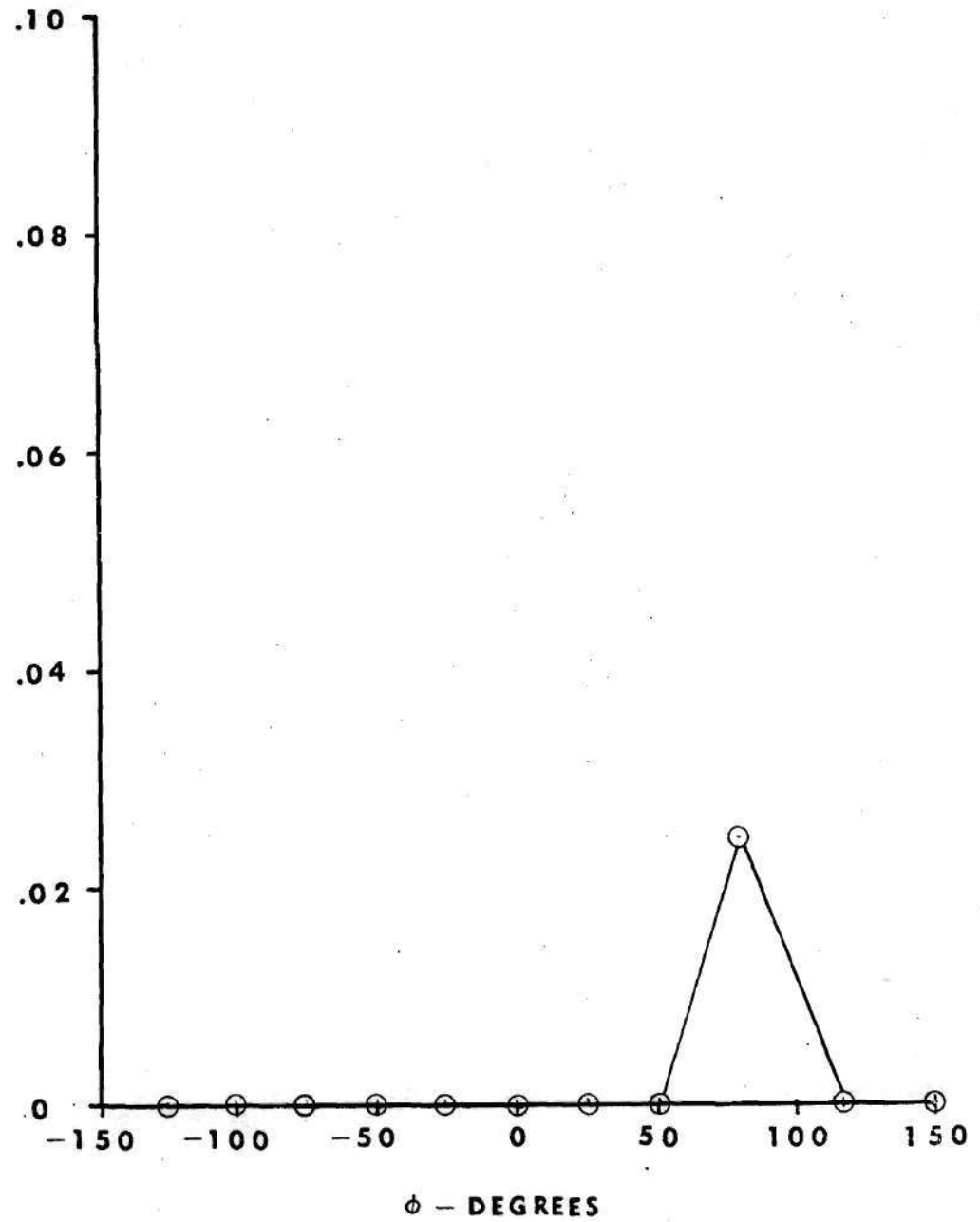


Figure 21. Density Function of the Heading Error After Assuming  
a Constant Radius Turn, with  $\Phi = -120^\circ$  and  $t = 60$  Seconds.

was assumed to be the heading at the point. If the point could not be reached by this assumed flight path, then a constant radius turn was assumed. Figures 22 through 28 show the distribution of the error in making this assumption. It can be seen that assuming a 3 degree/sec. turn is better than assuming a constant turn rate since in all cases the mean error is very near zero.

The expected dwell time at some point x and y is therefore

$$E\{\delta\} = \frac{\pi r}{(V_1^2 + V_2^2 - 2V_1 V_2 \cos \theta_r)^{\frac{1}{2}}} \quad (4.26)$$

where  $\theta_r$  is calculated by assuming that each aircraft reached the point by making a 3 degree/sec. turn followed by a straight line.



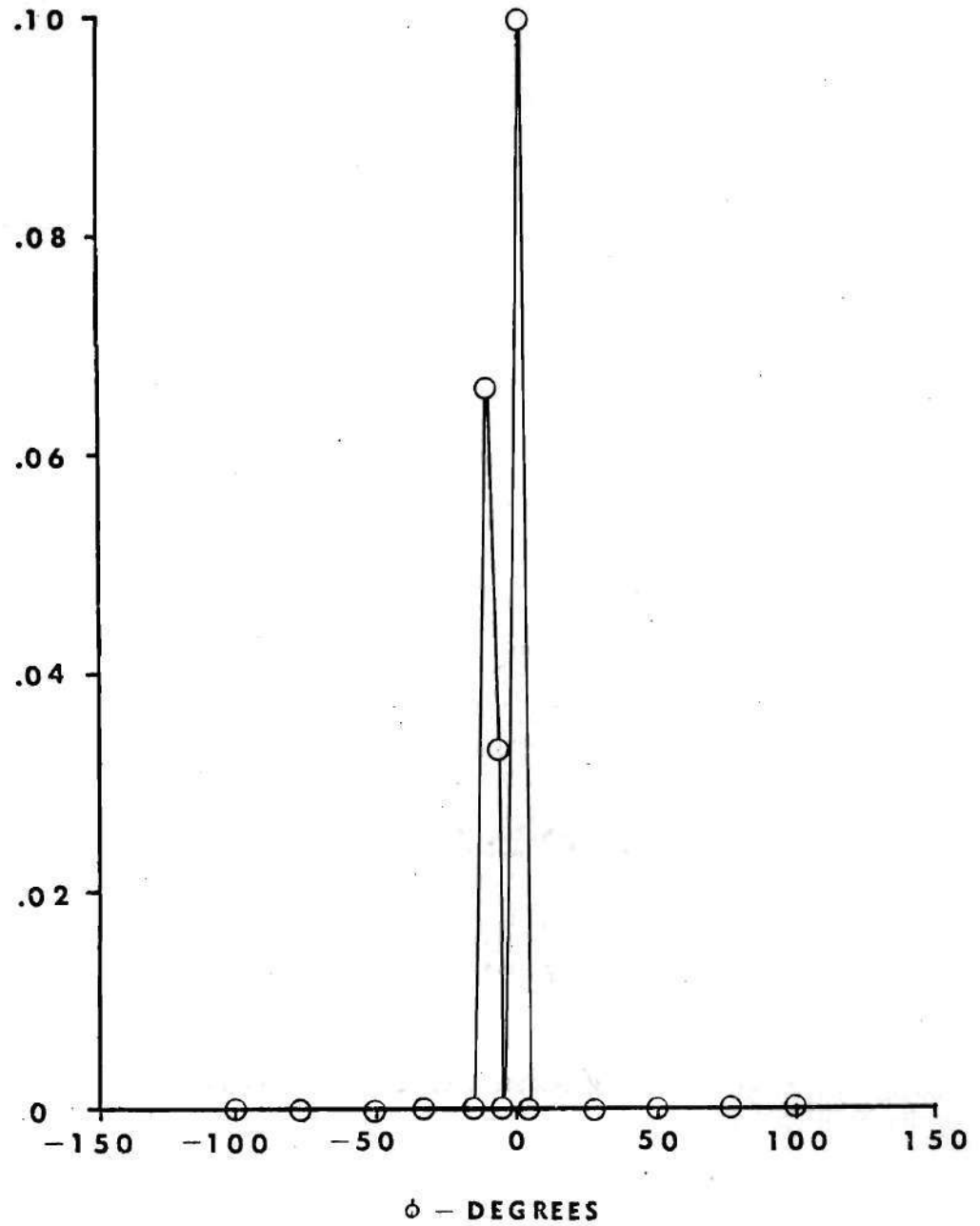


Figure 22. Density Function of the Heading Error After Assuming  
a Two Minute Turn, with  $\Phi = 120^\circ$  and  $t = 60$  Seconds.

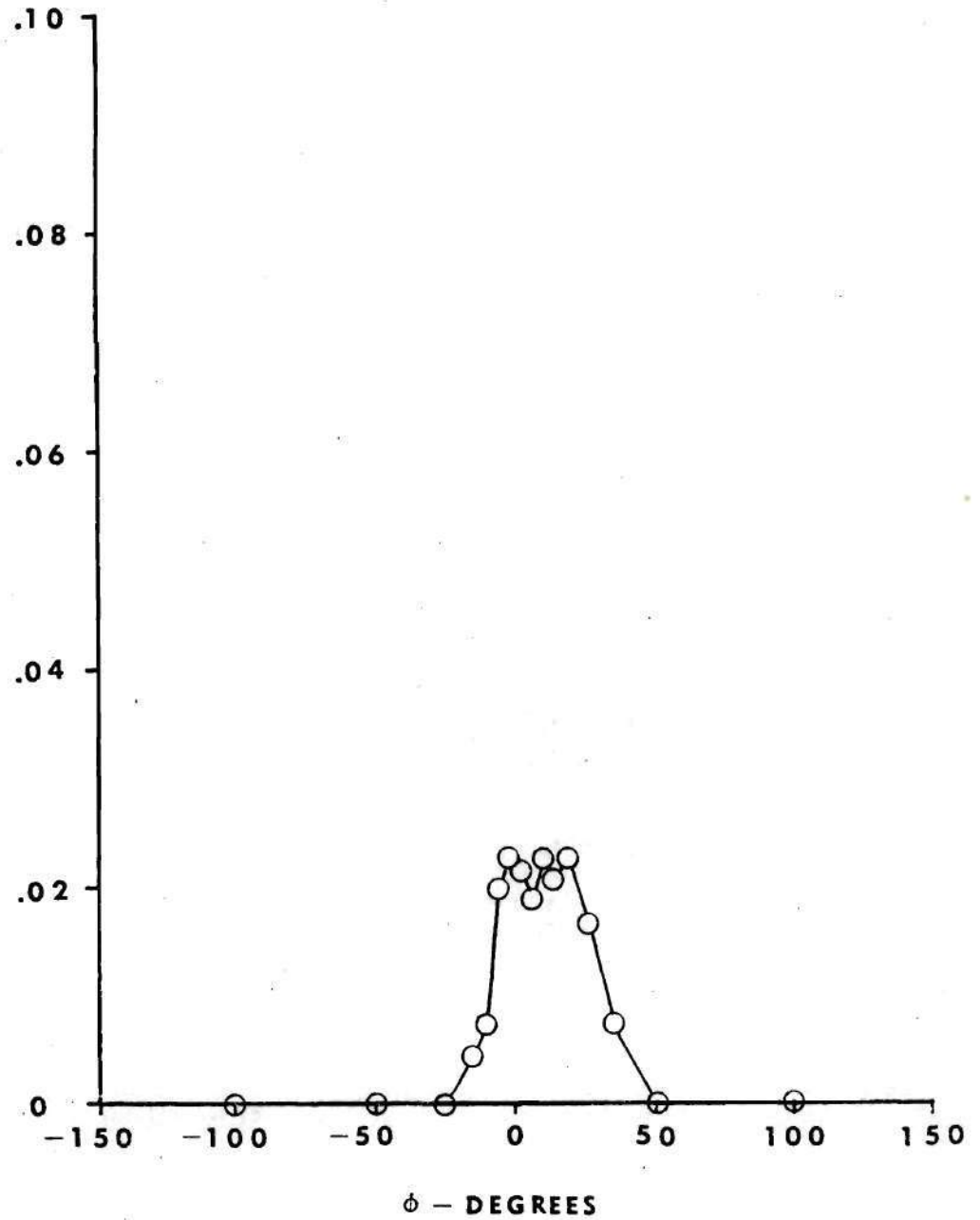


Figure 23. Density Function of the Heading Error After Assuming  
 a Two Minute Turn, with  $\Phi = 60^{\circ}$  and  $t = 60$  Seconds.

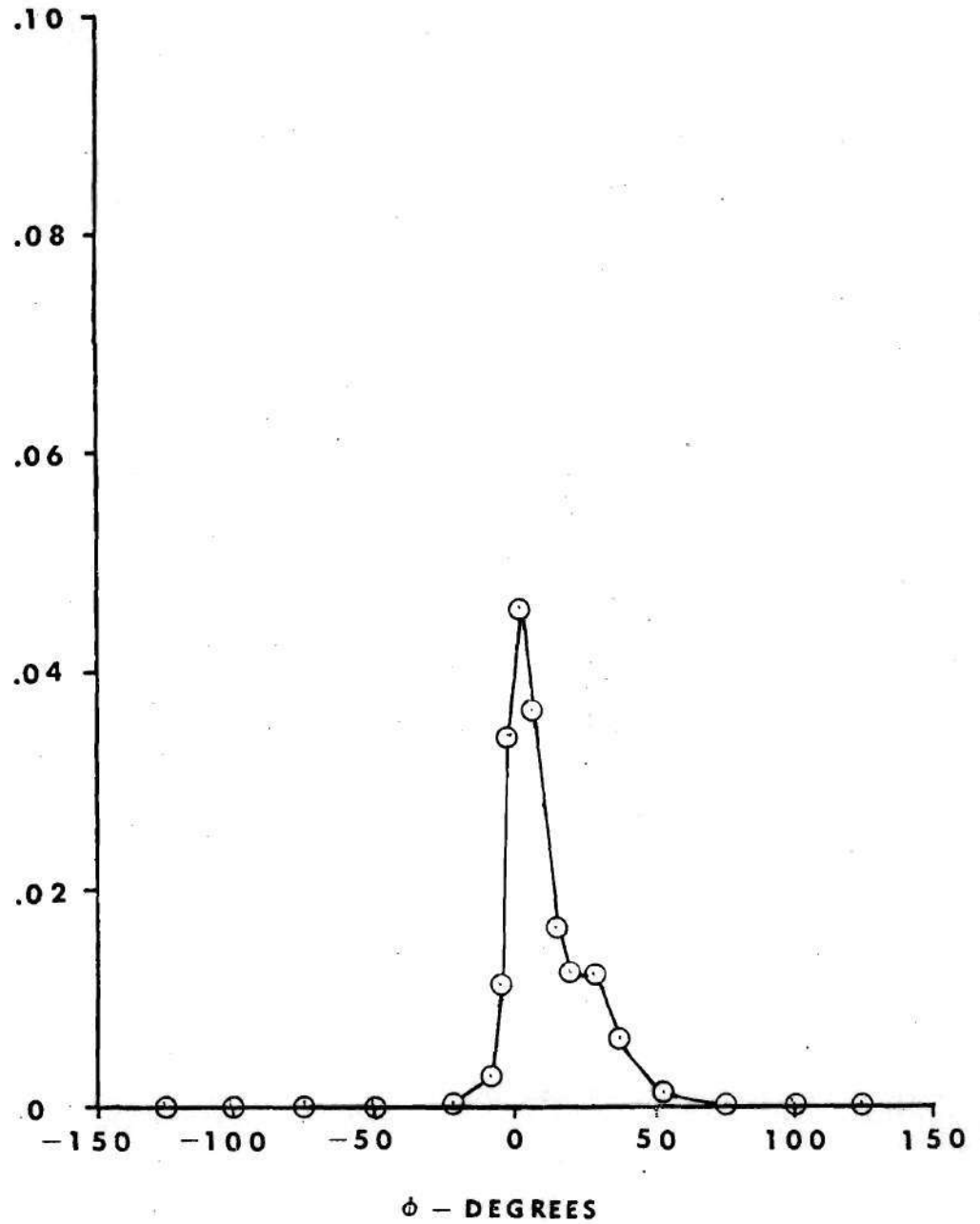


Figure 24. Density Function of the Heading Error After Assuming  
a Two Minute Turn, with  $\phi = 20^{\circ}$  and  $t = 60$  Seconds.

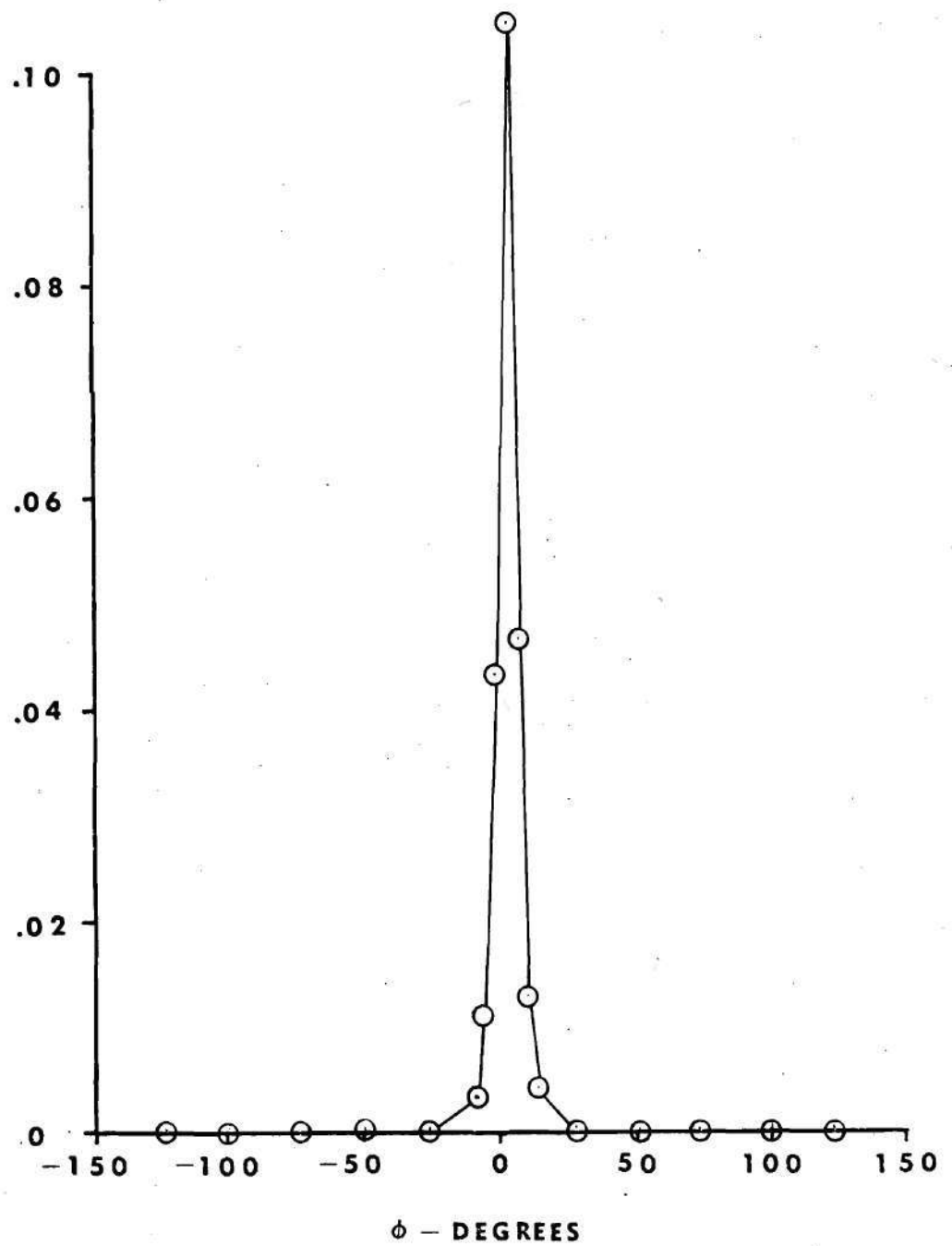


Figure 25. Density Function of the Heading Error After Assuming a Two Minute Turn, with  $\bar{\phi} = 0$  and  $t = 60$  Seconds.



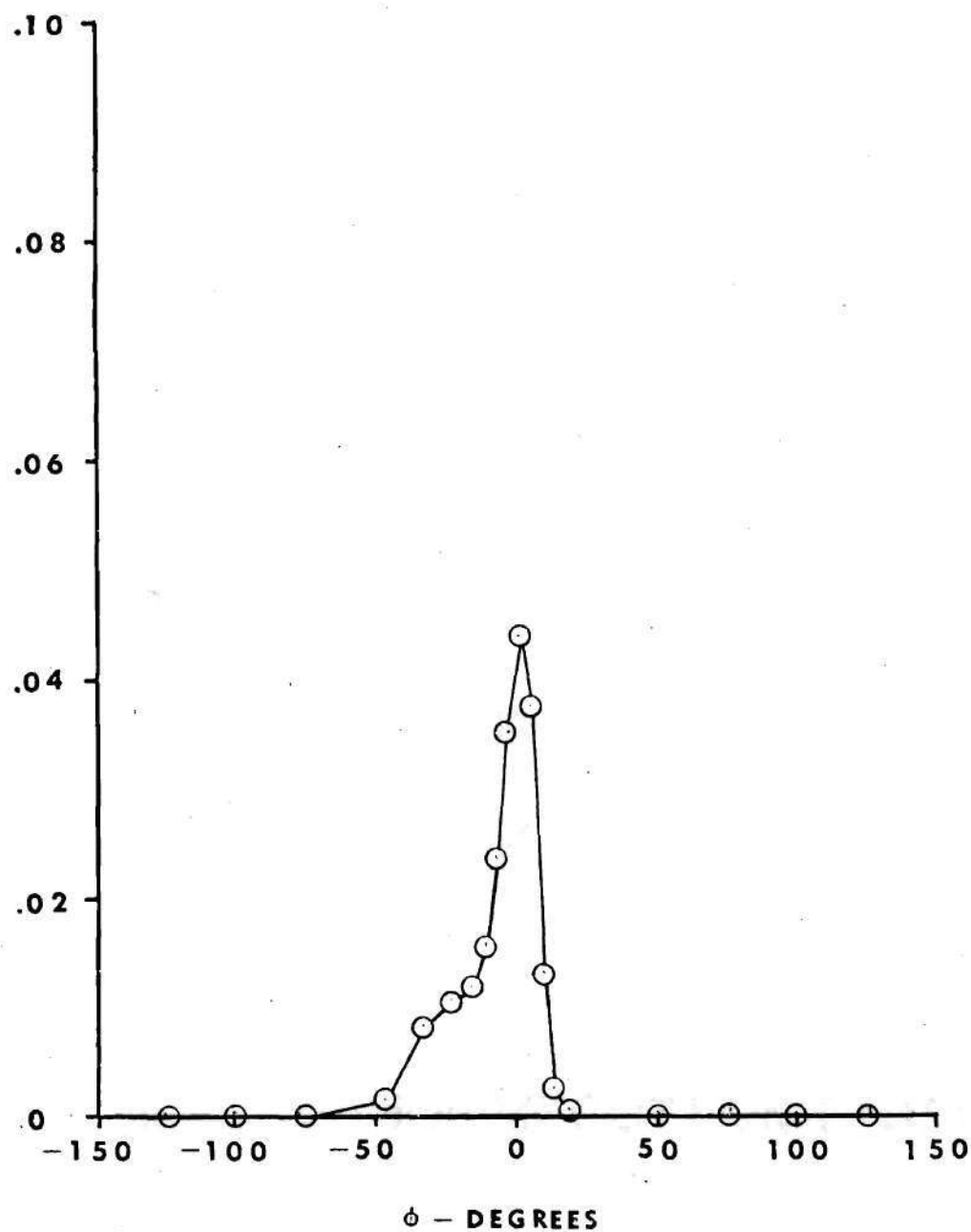


Figure 26. Density Function of the Heading Error After Assuming  
a Two Minute Turn, with  $\Phi = -20^\circ$ , and  $t = 60$  Seconds.

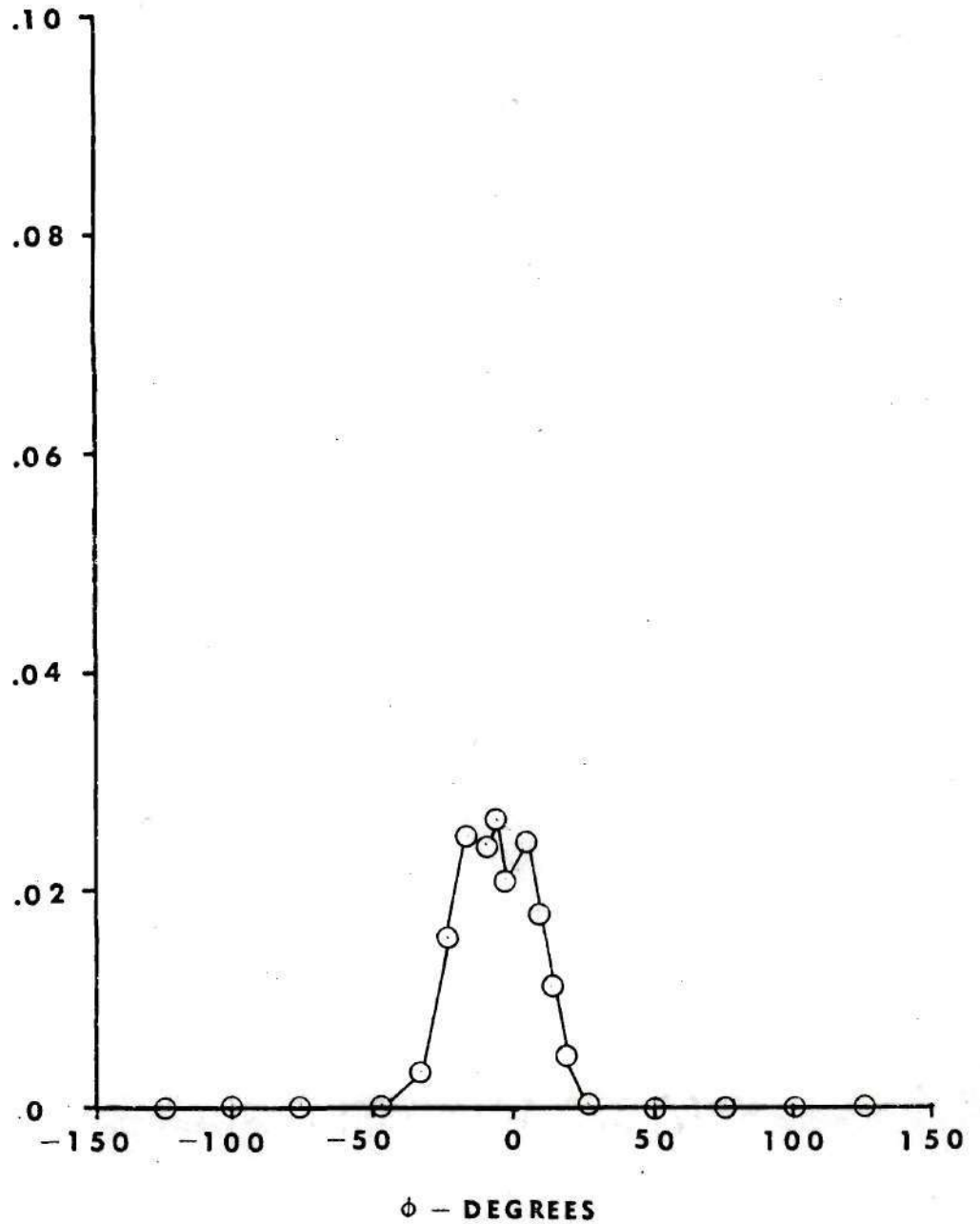


Figure 27. Density Function of the Heading Error After Assuming  
 a Two Minute Turn, with  $\Phi = -60^\circ$ , and  $t = 60$  Seconds.

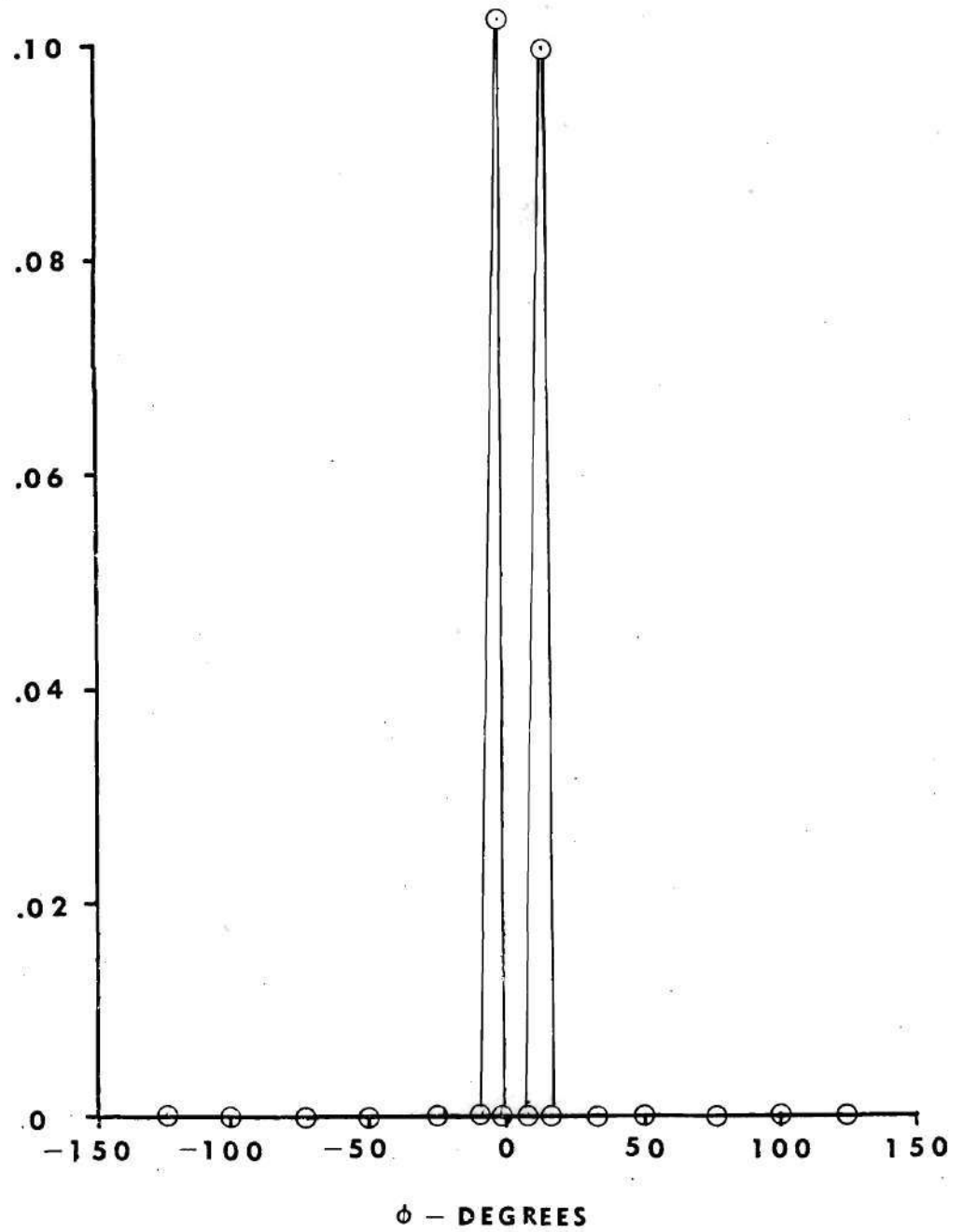


Figure 28. Density Function of the Heading Error After Assuming  
a Two Minute Turn, with  $\Phi = -120^{\circ}$ , and  $t = 60$  Seconds.

## CHAPTER V

### RESULTS AND CONCLUSIONS

#### Numerical Results

The last chapter derived the equation to be used to calculate the collision probability and this probability was used to establish the threat prediction criterion given by equation (2.9). To determine the benefit in equation (2.7) it was necessary to find the difference in the collision probability with and without an avoidance maneuver.

The avoidance maneuver considered for this work is the climb maneuver specified by the ATA(24). This maneuver consists of maintaining a vertical acceleration between  $1/8$  g and  $1/4$  g until a climb rate of 2000 ft/min is reached. The 2000 ft/min climb rate is maintained until a 650 ft. separation is reached and the aircraft then levels off. Since the termination of the maneuver depends on the vertical separation of the two aircraft, the maneuver is actually a stochastic process. The determination of the collision probability is much more difficult to calculate than the case where the avoidance maneuver is deterministic. Therefore the maneuver used was a .1875 g vertical acceleration until a climb rate of 2000 ft/min was reached. The 2000 ft/min climb rate was continued and no leveling off took place. This maneuver is deterministic since the termination of the maneuver does not depend on the vertical separation of the two aircraft. The collision probability resulting from the maneuver with leveling off should be almost identical

to the collision probability resulting from the maneuver without leveling off. This is because the collision probability after a 650 ft. climb is very small. It would therefore make little difference in the collision probability if the aircraft levels off or keeps climbing.

The independent parameters on which the collision probability depends are (Figure 29):

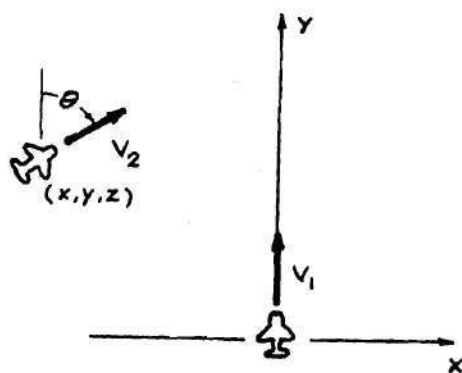


Figure 29. Independent Variables on Which the Collision Probability Depends.

$V_1, V_2$  -- airspeeds of the aircraft

$\theta$  -- heading of the intruding aircraft relative to the protected aircraft

$x, y, z$  -- position coordinates of intruder relative to the protected aircraft

For each point in this six-dimensional space, it is possible to find the benefit associated with the avoidance maneuver. The alarm criterion is established by locating the surface in this six-dimensional space where the benefit is equal to the penalty. Since the avoidance maneuver is always the same, the penalty is the



same. The alarm surface would therefore be a surface of constant benefit and, since the benefit is proportional to the collision probability, this surface would be one of constant probability.

It was beyond the scope of this work to include a thorough investigation of the constant benefit surface. It was thought desirable however to include some results so that this could be compared at least in part with previous criteria. The six-dimensional space was reduced to a three-dimensional space by holding three of the independent variables fixed. The two airspeeds were fixed at 160 knots and the relative altitude was held at zero. The three remaining variables  $x$ ,  $y$ , and  $\theta$  were expressed in terms of three new variables:

- $\mu$  -- separation at closest approach for unaccelerated flights (miss distance)
- $t_{\mu}$  -- time to closest approach
- $\beta$  -- relative bearing of intruder with respect to the protected aircraft

These three variables were then varied to obtain the collision probabilities in Figures 30 through 32. Each figure is for one value of the relative bearing,  $\beta$ . The vertical axes are the collision probability with no avoidance maneuver minus the collision probability with an avoidance maneuver. The range was used for the horizontal axes instead of  $t_{\mu}$  so that the comparison of this criterion with other criteria would be easier. Since it was found that the collision probability was zero when an avoidance was initiated, the vertical axes are also equal to the probability of a collision with no avoidance maneuver. Lines of constant  $\mu$  and lines of

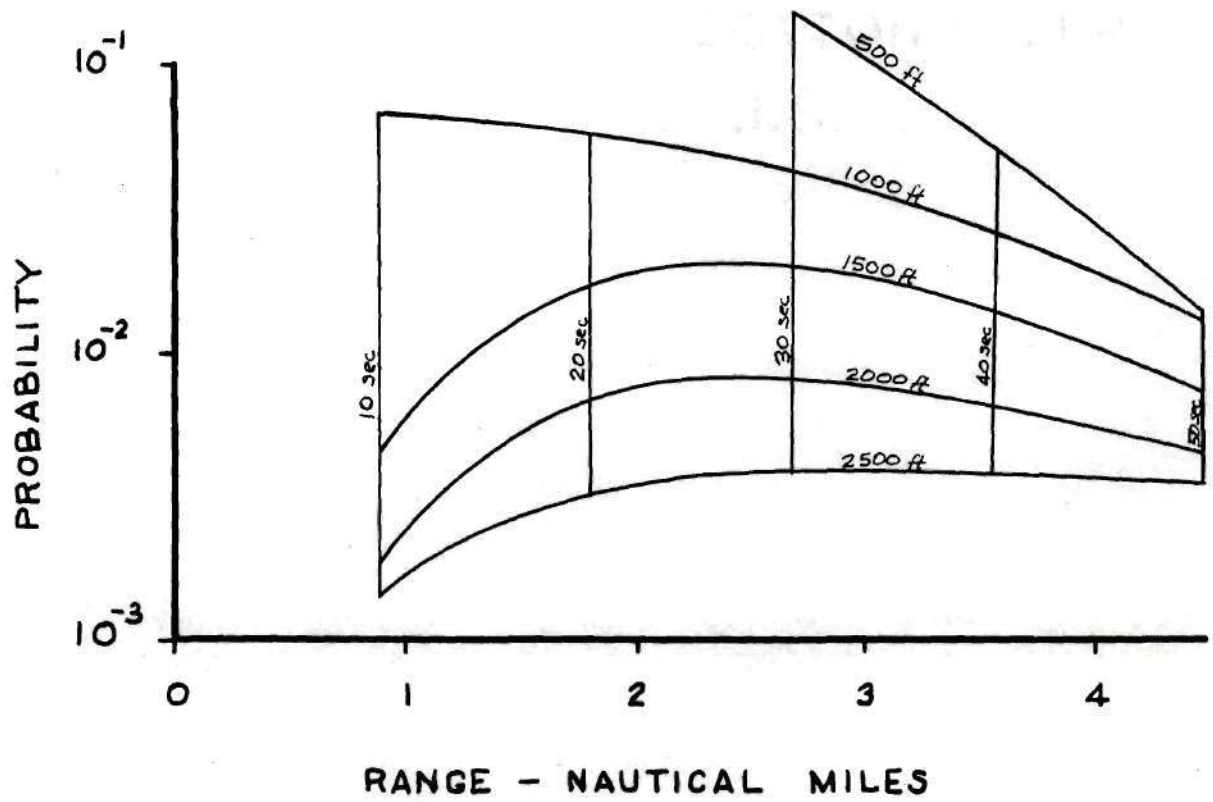


Figure 30. Collision Probability for  $\beta = 0$ .

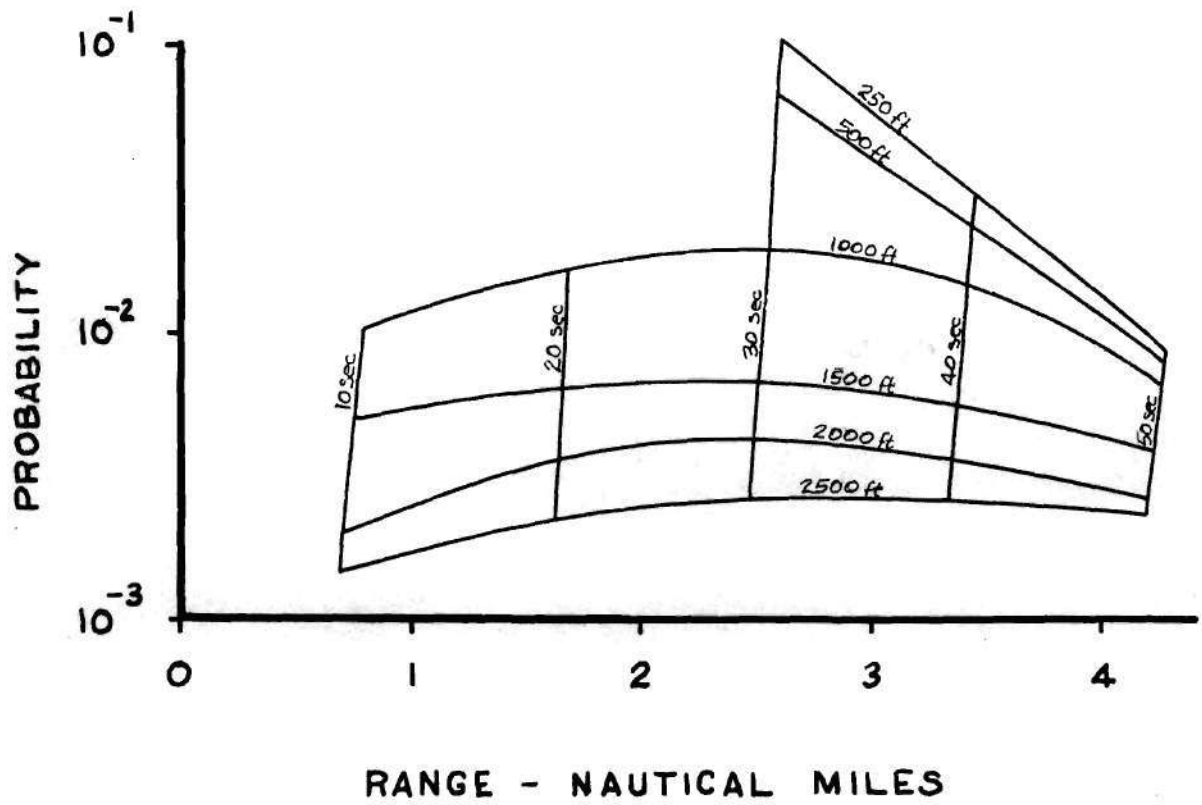


Figure 31. Collision Probability for  $\beta = 15^\circ$ .

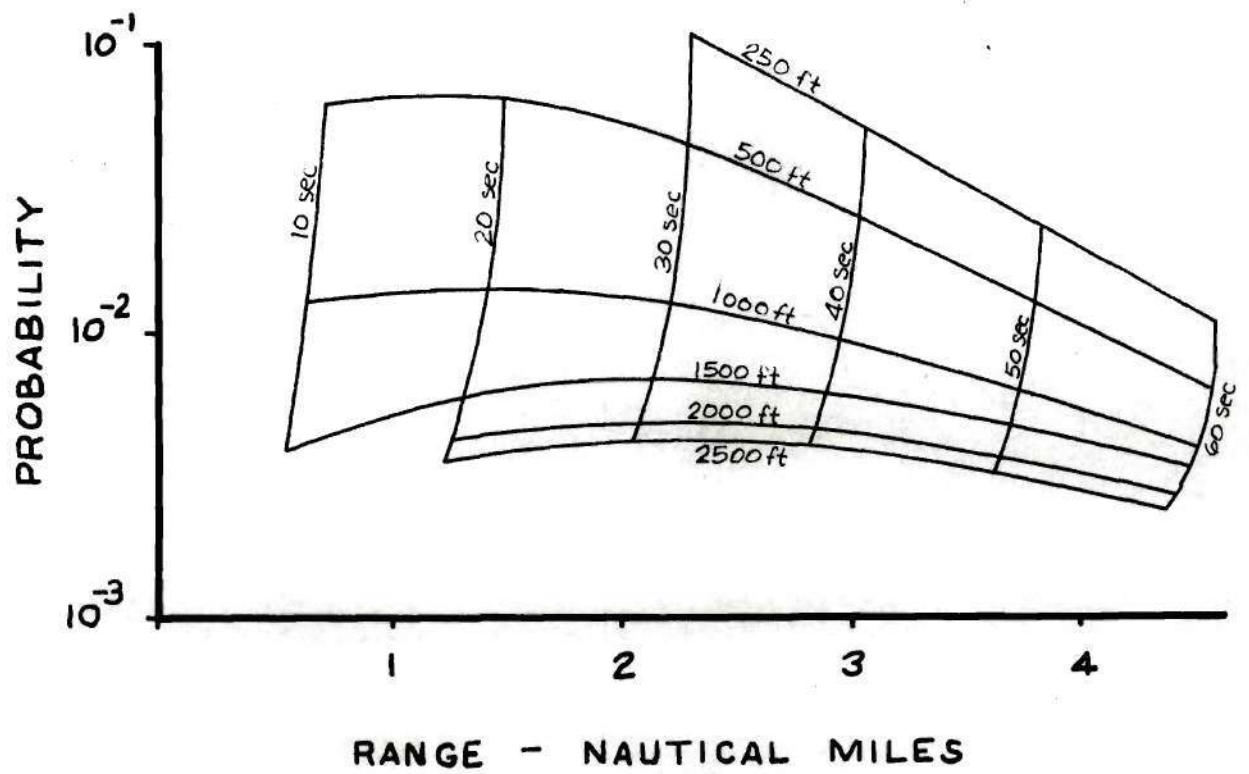


Figure 32. Collision Probability for  $\beta = 30^\circ$ .

constant  $t_{\mu}$  are also shown.

As can be seen in each figure, for a constant miss distance, the probability increases as range increases until a maximum probability is reached. The probability then decreases as range increases further. It can also be seen that the probability decreases as the miss distance increases. This corresponds to the general results one would intuitively expect.

In order to determine the warning criterion, it is necessary to select some probability to be used as a threshold value. In choosing this value it should be kept in mind that the collision probability here is not the actual collision probability when the aircrafts are under radar control but instead is the collision probability that would result in the event of a system failure. A value for the probability threshold of .01 was chosen and the resulting alarm region was plotted in Figure 33. Also present in the figure are the curves which show the alarm region defined by the tau criterion and the modified tau criterion. The lines for  $\tau$  and  $\tau_m$  were for a constant relative bearing of  $\beta = 0$ . All the criteria have the same general shape but the region covered is quite different. The tau and modified tau criteria with  $\tau$  and  $\tau_m$  equal to 30 sec cover a much larger miss distance than does the constant probability criterion. It would therefore be likely that the alarm frequency would be higher.

The constant benefit criterion as shown in Figure 33 utilizes the relative bearing in determining when to give the warning. However, the curve for  $\beta = 0$  appears to enclose the curves where  $\beta \neq 0$ . It can also be seen that the curve for  $\beta = 0$  still covers a much smaller region as do the curves for  $\tau$  and  $\tau_m$ . It



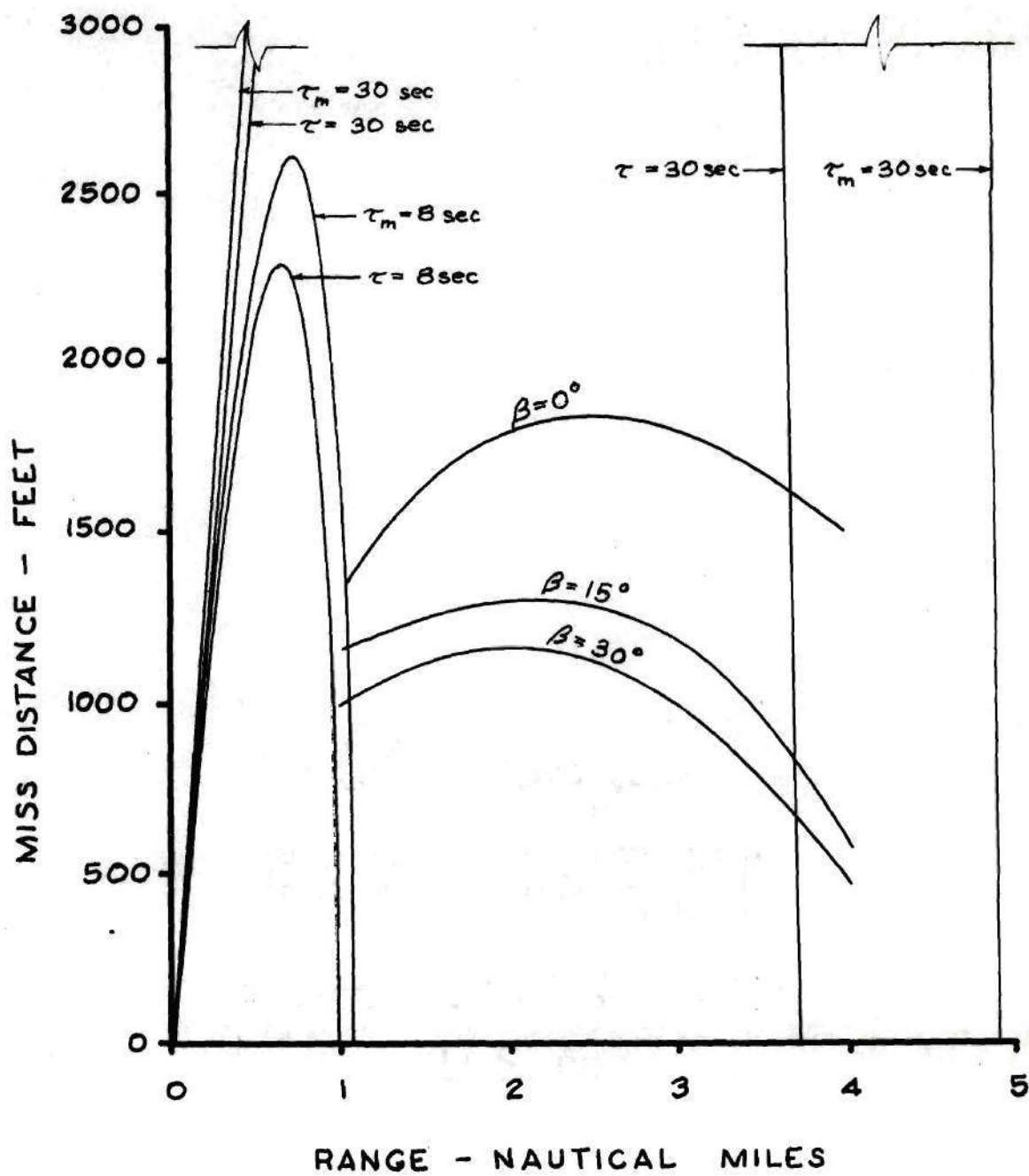


Figure 33. Comparison of Collision Avoidance Warning Criteria.

would be possible to obtain a significant false alarm reduction by using the curve for  $\beta = 0$  and the relative bearing would not be required for the criterion.

#### Future Research Possibilities

Collision avoidance systems and warning criteria can be compared by determining for each system its cost and benefit. The cost associated with each of the systems are twofold. One part of the cost is the monetary cost of the system, and another is the cost associated with the frequency of warnings and severity of maneuvers. The monetary cost is perhaps the easiest to obtain and can be found by estimating the equipment cost. The alarm cost would have to be established by assigning to each maneuver, if more than one maneuver is considered, a relative cost indicative of the disruption the maneuver would cause. The alarm frequency could be determined by considering the CAS operation in a realistic environment. The maneuver severity cost, weighted with its frequency of occurrence, would then be summed to give a single cost which when combined with the monetary cost would represent the total cost of that particular CAS.

The single benefit of a CAS is the reduction in the number of midair collisions. To determine this benefit it is necessary to know what situations have in the past and what situations will in the future contribute to collisions. It would then be possible by way of a simulation to determine how much the CAS reduces the number of collisions. By using these procedures for calculating the cost and benefit of a system, the different CAS, incorporating various warning criteria, can be compared and the appropriate system chosen. This type of analysis has

been conducted in the past to compare previous systems and criteria, but they need to be extended and refined so that a rational evaluation of the systems can be made.

Before the criterion developed in this research can be evaluated, the criterion must be developed in its entirety. One reason the criterion was not completely developed was because the number of independent variables on which the collision probability was based will not necessarily be the same as the number of independent variables available from the hardware. If the six variables related to position and velocity as used in this work are known, the criterion can be determined using these six state inputs as has been described. If the information for all six states is not known, then the criterion must be based on the information that is known. There are two ways to determine the criterion in a reduced dimensional space.

Let  $q_i$  be a point in any state space with less than six states where all of the states can be determined if the six states are known, and let  $Q_i$  be the set of points in the six dimensional space that correspond to  $q_i$ . To determine the collision probability for  $q_i$ , it is necessary to determine the collision probability for every point in  $Q_i$  and to combine these collision probabilities with the probability that that point in  $Q_i$  will occur in a realistic environment. Therefore it is necessary to have some statistical information in addition to the statistical information necessary to calculate the collision probability in six dimensional space. If these additional statistics are not available, then the appropriate procedure (see for example Tribus (25) Chapter 4) would be to use the best guess for the unknown



distribution of points in  $Q_i$ . A third and somewhat less desirable alternative would be to use the highest collision probability from all the points in  $Q_i$  for the collision probability of  $q_i$ . This last method would of course lead to a higher alarm rate.

The density function of the aircraft position and velocity were used in their numerical form for this research. It would be advantageous to obtain an analytical curve fit for this data for reasons of easier accessibility and flexibility in its use. Also an analytical curve fit would allow extension of the actual data where no data points were observed in the small sample.

The method of determining the collision probability as described in this research could also be applied to other problems. For example, this collision probability could be used as a basis for evaluating avoidance maneuvers even if the criterion based on this probability is not used.

#### Application of the Methodology

This research has applied a rational framework for a particular design problem. This is one of many similar problems where, historically, intuitive judgement has been favored over analytical methods to make design decisions. Many of these decisions, like this one, require a tradeoff be made between benefits and costs. If the benefits and costs do not both have obvious dollar equivalents, the tendency is to rely on judgement. In this case, benefits are reduced loss of human life and limb and the costs are both dollar cost and inconvenience. When life and limb are involved, the tendency to rely on judgement is even greater because one hesitates to put a dollar value on such cost.

Consider an example of two collision avoidance systems with the same dollar

cost and the same inconvenience cost. Suppose one system reduces the probability of a collision by a greater amount than the other. It is clear that failure to apply analytical methods might tragically result in the intuitive choice causing the larger loss of life and limb.



## LITERATURE CITED

1. Bendix Aviation Corporation, "Research and Development on Aircraft Proximity Warning and Collision Avoidance Techniques," Bendix Aviation Corporation, Baltimore, Maryland, 1958, Summary Report on Contract AF-33 (616)-5192.
2. Eduardo I. Pina and Henry J. Landu, "Operational Requirements for a Collision Warning System," *Operations Research*, Vol. 5, pp. 794-814, December 1957.
3. R. T. Fitzgerald, H. C. Brown, and M. D. Reed, "Radio Collision Avoidance System for Aircraft," 1959, Diamond Ordnance Fuze Laboratory, Washington.
4. Cassius Mullen, Ed., "Interurban Air Transportation System," a report of a graduate project in complex system design, December 1969, Georgia Institute of Technology, Atlanta, pp. 353-359.
5. Martin R. Bates, Harold K. Fletcher, Lewis Michnik, and Johannes W. Prast, "History of Time-Frequency Technology," *IEEE Transaction of Aerospace and Electronic Systems*, Vol. AES-4, no. 2, pp. 238-255, March 1968.
6. Air Transport Associations, "Flight Test and Evaluation of Airborne Collision Avoidance System," 1970, Washington.
7. John M. Holt, and Ronald M. Anderson, "Analysis of Warning Times for Collision Avoidance Systems," *IEEE Transactions on Aerospace and Electronic Systems*, Vol. AES-4, no. 2, pp. 305-314, March 1968.
8. Rufus Isaacs, "The Theory of Collisions Between Moving Craft," Systems Development Laboratories, SRSM-1-189 April, 1959, Hughes Aircraft Company, Culver City, California.
9. J. S. Morrel, "Physical Aspects of Collision Avoidance," *IRE Transactions on Aeronautical and Navigational Electronics*, Vol. ANE-4, no. 2, pp. 75-81, June 1957.
10. S. H. Hollingdale, "The Mathematics of Collision Avoidance in Two Dimension," *Journal of the Institute of Navigation*, Vol. 14, pp. 243-261, 1961.
11. Collins Radio Company, *Fundamentals of Collision Avoidance*, Cedar Rapids, Iowa, 1958.

12. J. Holt, L. Belden, and W. Jamerson, "Computer Simulation Study of Air-Derived Separation Assurance System in Multiple Aircraft Environments, Third Interim Report," 1968, Collins Radio Company, Cedar Rapids, Iowa, Contract FA-WA-4598.
13. Gordon Jolitz, "Status of the ATC/CAS Simulation Program," Collision Prevention Advisory Group, Minutes of the Thirty-Second Meeting- November 19, 1968. Federal Aviation Administration, Washington, D. C.
14. Charles L. Britt, "An Evaluation of Aircraft Collision-Hazard Warning Radar Techniques in the Terminal Area," TR-55, December 1969, Research Triangle Institute, Research Triangle Park, North Carolina.
15. Warren Taylor, "A Mathematical Model of the Collision Probability of Airborne Objects," RTCA Paper 188-59/AS-200. Joint RTCA-IRA Symposium, July 16, 1959.
16. J. F. Koetsch, "A Three Dimensional Mathematical Model of the Air Traffic Control Separation Problem," International Aviation Research and Development Symposium, Report of the Proceedings, April 10-14, pp. 13-35, Federal Aviation Administration, Washington, D. C.
17. B. L. Marks, "Air Traffic Control Separation Standards and Collision Risk," Technical Note Math 91, February 1963, Royal Aircraft Establishment, Farnborough, England, AD 407 312.
18. P. G. Reich, "A Theory of Safe Separation Standards for Air Traffic Control," Technical Report 64041, November 1964, Royal Aircraft Establishment, Farnborough, England, AD 455 173.
19. Herbert A. Steinberg, "Collisions and Missed Approach Risks in High-Capacity Airports Operations," Proceedings of the IEEE, Vol. 58, No. 3, pp. 314-321, March 1970.
20. S. O. Rice, "Mathematical Analysis of Random Noise," Bell System Technical Journal, Vol. 23, p. 282-332, 1944, Vol. 24, p. 46-156, 1945.
21. Julius S. Bendat, Principles and Applications of Random Noise Theory, John Wiley and Sons, Inc., New York, 1958.
22. Athanasias Papoulis, Probability, Random Variables, and Stochastic Processes, McGraw-Hill, Inc., New York, 1965.
23. Benedict and Bordner, "Synthesis of Optimum Set of Radar Track While Scan Smoothing Equations," IRE Transactions on Automatic Controls, July 1962.

24. Air Transport Associations of America, "Airborne Collision Avoidance System," ANTC Report No. 117, Revision No. 7, Washington, D. C., August 30, 1968.
25. Myron Tribus, Rational Descriptions, Decisions and Designs, Pergamon Press, New York, 1969.



## VITA

Roscoe McClendon Hinson, Jr. was born in Kingstree, South Carolina, on May 23, 1944. He attended the public schools there and graduated from Kingstree High School in June 1962. He entered Georgia Institute of Technology on September 1962 and graduated with a Bachelor of Mechanical Engineering Degree in June 1967. As an undergraduate he worked as a cooperative student with Sonoco Products Company. After entering graduate school in June 1967 at Georgia Institute of Technology, he received a Master of Science in Mechanical Engineering in December 1970.

Mr. Hinson is married to the former Elizabeth Tyler and they have one daughter, Leigh.