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We consider the notion of monotonic independence in a more general frame, similar to the construction of operator-valued free probability. The paper presents constructions for maps with similar properties to the *H* and *K* transforms from the literature, semi-inner-product bimodule analogues for the monotone and weakly monotone product of Hilbert spaces, an ad-hoc version of the Central Limit Theorem, an operator-valued arcsine distribution as well as a connection to operator-valued conditional freeness.

1. Introduction

An important notion in noncommutative probability is monotonic independence, introduced by P. Y. Lu and Naofumi Muraki. Since its beginning, the study of this notion of independence was done by constructions, techniques and developments similar to the theory of free probability. R. Speicher [1998] developed an operator-valued analogue of free independence. The present paper addresses problems similar to ones discussed in that work, but in the context of monotonic independence.

Other motivation is that while for the free Fock space over a Hilbert space there is a straightforward analogous semi-inner-product bimodule construction, as illustrated in [Pimsner 1997; Speicher 1998], there are no similar constructions for its various deformations, such as the q-Fock spaces [Effros and Popa 2003]. As shown in Section 4, the monotone and weakly monotone Fock-like spaces, which are strongly connected to monotonic independence, admit analogous semi-inner-product bimodules.

The paper is structured in six sections. Section 2 presents the definition of the monotonic independence over an algebra. In Section 3 there are constructed maps with similar properties to the maps H and K from the theory of monotonic independence, as introduced in [Muraki 2000; Bercovici 2005b]. Section 4 deals with semi-inner-product bimodule analogues of the monotone and weakly monotone products of Hilbert spaces and algebras of annihilation operators, as introduced in

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[Muraki 2000; 2001; Wysoczański 2005]. Section 5 presents a Central Limit Theorem in the frame of monotonic independence over a C*-algebra and a positivity result concerning it. Since in the scalar-valued case the density of the limit distribution is the arcsine function [Lu 1997; Muraki 2000], the limit in Theorem 5.3 can be regarded as an "operator-valued arcsine law". Section 6 introduces a notion of conditionally free product of conditional expectations extending the definition and positivity results from [Młotkowski 2002] and shows the connection to monotonic independence analogous to [Franz 2005, Proposition 3.1].

2. Preliminaries

Let $\mathfrak{B}$ be an algebra (not necessarily unital). Within this paper, the notation $\mathfrak{B}_+\langle \xi_1, \dots, \xi_n\rangle$ will stand for the free noncommutative algebra generated by $\mathfrak{B}$ and the symbols $\xi_1, \dots, \xi_n$. For the smaller algebra $\mathfrak{B}_+\langle \xi_1, \dots, \xi_n\rangle \ominus \mathfrak{B}$ we will use the notation $\mathfrak{B}\langle \xi_1, \dots, \xi_n\rangle$.

If $\mathfrak{B}$ is a $*$-algebra, we can consider $*$-algebra structures on $\mathfrak{B}_+\langle \xi\rangle$ and $\mathfrak{B}\langle \xi\rangle$ either by letting $(\xi)^* = \xi$ (that is, the symbol ξ is self-adjoint) or considering $\mathfrak{B}_+\langle \xi, \xi^*\rangle$ with $(\xi)^* = \xi$.

We also need to consider an extended notion of nonunital complex algebra. $\mathfrak{A}$ will be called a $\mathfrak{B}$-algebra if $\mathfrak{A}$ is an algebra such that $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$ or there exists an algebra $\widetilde{\mathfrak{A}}$ containing $\mathfrak{B}$ as a subalgebra such that $\widetilde{\mathfrak{A}} = \mathfrak{A} \sqcup \mathfrak{B}$. Note that the symbol $\sqcup$ stands for disjoint union.

A map $\Phi : \mathfrak{A} \longrightarrow \mathfrak{B}$ is said to be $\mathfrak{B}$-*linear* if

$$\Phi(b_1 x b_2 + y) = b_1 \Phi(x) b_2 + \Phi(y)$$

for all $x, y \in \mathfrak{A}$ and $b_1, b_2 \in \mathfrak{B}$.

If $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$ and $\Phi(b) = b$ for all $b \in \mathfrak{B}$, then Φ will be called a *conditional expectation*.

Definition 2.1. Suppose that $\mathfrak{A}$ is a $\mathfrak{B}$-algebra and I is a totally ordered set.

A family $\{\mathfrak{A}_j\}_{j\in I}$ of subalgebras of $\mathfrak{A}$ is said to be *monotonically independent over* $\mathfrak{B}$ if given $X_j \in \mathfrak{A}_j$ $(j \in I)$, the following conditions are satisfied:

(a) for all $i < j > k$ in I and $A, B \in \mathfrak{A}$:

$$\Phi(A X_i X_j X_k B) = \Phi(A X_i \Phi(X_j) X_k B);$$

(b) for all $i_m > \cdots > i_1 < k_1 < \cdots < k_n$ in I:

$$\begin{aligned}
\Phi(X_{i_m} \cdots X_{i_1}) &= \Phi(X_{i_m}) \cdots \Phi(X_{i_1}),\\
\Phi(X_{k_1} \cdots X_{k_n}) &= \Phi(X_{k_1}) \cdots \Phi(X_{k_n}),\\
\Phi(X_{i_m} \cdots X_{i_1} X_{k_1} \cdots X_{k_n}) &= \Phi(X_{i_m}) \cdots \Phi(X_{i_1}) \Phi(X_{k_1}) \cdots \Phi(X_{k_n}).
\end{aligned}$$

The elements X_j $(j \in I)$ from $\mathfrak{A}$ are said to be *monotonically independent over* $\mathfrak{B}$ if the subalgebras of $\mathfrak{A}$ generated by X_j and $\mathfrak{B}$ are monotonically independent over $\mathfrak{B}$.

Following [Muraki 2000] or [Muraki 2001], one may consider the stricter definition of monotonic independence by replacing the first condition with

(a′) $X_i X_j X_k = X_i \Phi(X_j) X_k$ whenever $i < j > k$.

Yet, Definition 2.1 (similar to that in [Franz 2005]) suffices for the results within this paper.

3. The maps κ, ρ and $\mathfrak{h}$

Two important instruments in monotonic probability are the maps H_X and K_X associated to a selfadjoint element X from a unital $*$-algebra $\mathcal{A}$ with a $\mathbb{C}$-linear functional φ such that $\varphi(1) = 1$. Namely H_X is reciprocal Cauchy transform

$$H_X(z) = (G_X(z))^{-1},$$

where G_X is the Cauchy transform corresponding to X

$$G_X(z) = \varphi\big((z - X)^{-1}\big)$$

and the map K_X is given by

$$K_X(z) = \frac{\psi_X(z)}{1 + \psi_X(z)},$$

where

$$\psi_X(z) = \varphi\big(zX(1 - zX)^{-1}\big).$$

Their key properties [Bercovici 2005a; Franz 2005] are that for X, Y, respectively $U - 1$, V monotonically independent with respect to φ, one has

$$H_{X+Y} = H_X \circ H_Y,$$
$$K_{UV} = K_{VU} = K_U \circ K_V.$$

In the scalar-valued case, the moment generating series of X can be recovered from H and K. For the $\mathfrak{B}$-valued setting, the n-th moment of X is the multilinear function

$$m_{X,n} : \mathfrak{B}^{n-1} \longrightarrow \mathfrak{B}$$

given by

$$m_{X,n}(b_1, \ldots, b_{n-1}) = \Phi(X b_1 X \cdots X b_{n-1} X).$$

The mathematical object replacing the moment generating series is a multilinear function series over $\mathfrak{B}$ [Dykema 2007], that cannot be recovered from a $\mathfrak{B}$-valued analytic map.

The use of analytic tools (such as the Cauchy transform) is strongly impaired by the previous considerations; hence the combinatorial approach is very convenient in the present framework. We will first construct the $\mathfrak{B}$-valued analytic functions $\mathfrak{h}$ replacing H, κ and ρ replacing K. Based on these constructions, the second part of the section will address the more general framework of multiplicative function series over an algebra.

In this section we require $\mathfrak{A}$ to be a $*$-algebra, $\mathfrak{B}$ to be a C*-algebra with norm $\|\cdot\|$, and $\Phi : \mathfrak{A} \longrightarrow \mathfrak{B}$ to be a positive conditional expectation. $\overline{\mathfrak{A}}$ will denote the closure of $\mathfrak{A}$ in the topology given by $X \mapsto \|\Phi(X^*X)\|$. For simplicity, we will denote the continuous extension of Φ to $\overline{\mathfrak{A}}$ also with Φ.

Definition 3.1. For $X \in \mathfrak{A}$, consider the $\mathfrak{B}$-valued function $\mathfrak{h}_X$:

$$\left\{z \in \mathfrak{B} : \ \|z\| < \|X\|^{-1}\right\} \ni z \mapsto \mathfrak{h}_X(z) = (1 - z\Phi(X))^{-1} z \in \mathfrak{B}.$$

Observe that $\mathfrak{h}$ is an analytic function defined in a neighborhood of $0 \in \mathfrak{B}$ and $\mathfrak{h}(0) = 0$.

Theorem 3.2. *If $X, Y \in \mathfrak{A}$ are monotonically independent, then*

$$\mathfrak{h}_{X+Y}(z) = \mathfrak{h}_X \circ \mathfrak{h}_Y(z)$$

for z in a neighborhood of $0 \in \mathfrak{B}$.

Proof. First, note that, for $X_1, X_2 \in \mathfrak{A}$ of sufficiently small norm, we have

$$\sum_{n=0}^{\infty}(X_1 + X_2)^n = \sum_{p=0}^{\infty}\left(\left(\sum_{k=0}^{\infty} X_2^k\right)X_1\right)^p\left(\sum_{m=0}^{\infty} X_2^m\right). \tag{3-3}$$

Indeed,

$$\begin{aligned}\sum_{n=0}^{\infty}(X_1 + X_2)^n &= \sum_{m=0}^{\infty} \sum_{\alpha_0,\beta_m \geq 0} \sum_{\alpha_j,\beta_j \geq 1} X_1^{\alpha_0} X_2^{\beta_0} \cdots X_1^{\alpha_m} X_2^{\beta_m} \\ &= \sum_{n=0}^{\infty}\left(\sum_{\beta_j \geq 0}\left(\prod_{j=0}^{n} X_2^{\beta_j} X_1\right)\right)\left(\sum_{m=0}^{\infty} X_2^m\right) \\ &= \sum_{p=0}^{\infty}\left(\left(\sum_{k=0}^{\infty} X_2^k\right)X_1\right)^p\left(\sum_{m=0}^{\infty} X_2^m\right).\end{aligned}$$

Substituting $X_1 = zX$ and $X_2 = zY$ in (3-3) we have

$$\sum_{n=0}^{\infty}(z(X+Y))^n = \sum_{p=0}^{\infty}\left(\left(\sum_{k=0}^{\infty}(zY)^k\right)zX\right)^p\left(\sum_{m=0}^{\infty}(zY)^m\right).$$

And therefore

$$(1-z(X+Y))^{-1}z=\sum_{p=0}^{\infty}\bigg(\Big(\sum_{k=0}^{\infty}(zY)^{k}\Big)zX\bigg)^{p}\Big(\sum_{m=0}^{\infty}(zY)^{m}\Big)z.$$

We deduce that

$$\Phi\big((1-z(X+Y))^{-1}z\big)=\Phi\bigg(\sum_{p=0}^{\infty}\bigg(\Big(\sum_{k=0}^{\infty}(zY)^{k}\Big)zX\bigg)^{p}\Big(\sum_{m=0}^{\infty}(zY)^{m}\Big)z\bigg).$$

Hence,

$$\mathfrak{h}_{X+Y}(z)=\Phi\bigg(\sum_{p=0}^{\infty}\big(\big((1-zY)^{-1}z\big)X\big)^{p}(1-zY)^{-1}z\bigg).$$

Let

$$Z=(1-zY)^{-1}z\in\overline{\mathfrak{A}}.$$

Z is in the closure of the algebra generated by Y and $\mathfrak{B}$. If X, Y are monotonically independent over $\mathfrak{B}$, the continuity of Φ and Definition 2.1 (a) imply

$$\Phi((ZX)^{p}Z)=\Phi(ZXZ\cdots ZXZ)=\Phi(ZX\Phi(Z)\cdots\Phi(Z)XZ).$$

Since $X\Phi(Z)X\cdots\Phi(Z)X$ is in the algebra generated by X and $\mathfrak{B}$, Definition 2.1 (b) gives

$$\Phi((ZX)^{p}Z)=(\Phi(Z)X)^{p}\Phi(Z).$$

Therefore,

$$\mathfrak{h}_{X+Y}(z)=\Phi\bigg(\sum_{p=0}^{\infty}(\mathfrak{h}_{Y}(z)X)^{p}\mathfrak{h}_{Y}(z)\bigg)=\mathfrak{h}_{X}\circ\mathfrak{h}_{Y}(z).\qquad\square$$

Definition 3.3. For $X\in\mathfrak{A}$ and z in a neighborhood of $0\in\mathfrak{B}$, define the maps

$$\begin{aligned}\vartheta_{X}(z)&=\Phi\big((1-zX)^{-1}zX\big),\\ \kappa_{X}(z)&=(1+\vartheta_{X}(z))^{-1}\vartheta_{X}(z),\\ \varrho_{X}(z)&=\Phi\big(Xz(1-Xz)^{-1}\big),\\ \rho_{X}(z)&=\varrho_{X}(z)(1+\varrho_{X}(z))^{-1}.\end{aligned}$$

These are $\mathfrak{B}$-valued analytic maps for which 0 is a fixed point.

Theorem 3.4. *Let U, $V\in\mathfrak{A}$ be such that $U-1$ and V are monotonically independent over $\mathfrak{B}$. Then, for z in some neighborhood of $0\in\mathfrak{B}$,*

$$\begin{aligned}\kappa_{VU}(z)&=(\kappa_{U}\circ\kappa_{V})(z),\\ \rho_{UV}(z)&=(\rho_{U}\circ\rho_{V})(z).\end{aligned}$$

Proof. With the notation $U-1=X$, we obtain

$$\begin{aligned}\vartheta_{VU}(z) &= \Phi\big((1-zVU)^{-1}zVU\big) = \Phi\big((1-zVU)^{-1}zVU\big)\\ &= \Phi\bigg(\sum_{k=0}^{\infty}(zVU)^k zVU\bigg) = \Phi\bigg(\sum_{k=0}^{\infty}(zV(X+1))^k zVU\bigg)\\ &= \Phi\bigg(\sum_{k=0}^{\infty}\sum_{\substack{\alpha_1+\cdots+\alpha_p=k+1\\ \alpha_j\geq 1}}(zV)^{\alpha_1}X(zV)^{\alpha_2}X\cdots(zV)^{\alpha_p}U\bigg).\end{aligned}$$

As in the proof of Theorem 3.2, using Definition 2.1 in the above equation we have

$$\begin{aligned}\vartheta_{VU}(z) &= \Phi\bigg(\sum_{k=0}^{\infty}(\vartheta_V(z)X)^k\vartheta_V(z)U\bigg) = \Phi\big((1-\vartheta_V(z)X)^{-1}\vartheta_V(z)U\big)\\ &= \Phi\big((1+\vartheta_V(z)-\vartheta_V(z)U)^{-1}(1+\vartheta_V(z))(1+\vartheta_V(z))^{-1}\vartheta_V(z)U\big)\\ &= \Phi\Big(\big((1+\vartheta_V(z))^{-1}(1+\vartheta_V(z)-\vartheta_V(z)U)\big)^{-1}(1+\vartheta_V(z))^{-1}\vartheta_V(z)U\Big)\\ &= \Phi\Big(\big(1-(1+\vartheta_V(z))^{-1}\vartheta_V(z)U\big)^{-1}\kappa_V(z)U\Big)\\ &= \Phi\big((1-\kappa_V(z)U)^{-1}\kappa_V(z)U\big)\\ &= \vartheta_U(\kappa_V(z)).\end{aligned}$$

Therefore,

$$\kappa_{VU}(z) = (1+\vartheta_{VU}(z))^{-1}\vartheta_{VU}(z) = (1+\vartheta_U(\kappa_V(z)))^{-1}\vartheta_U(\kappa_V(z)) = \kappa_U(\kappa_V(z)).$$

The identity for ρ follows analogously. $\square$

The proofs of Theorems 3.2 and 3.4 do not use the analyticity of the maps $\mathfrak{h}$, κ, ρ, but only properties from Definition 2.1 and some combinatorial identities that are true for any formal series. This leads to an easy reformulation of the results in the more general frame, presented in [Dykema 2007], of multilinear function series over an algebra.

In the following paragraphs we will briefly remind the reader the construction and several results on multilinear function series.

Let $\mathfrak{B}$ be an algebra. We set $\widetilde{\mathfrak{B}}$ equal to $\mathfrak{B}$ if $\mathfrak{B}$ is unital and to the unitalization of $\mathfrak{B}$ otherwise. For $n\geq 1$, we denote by $\mathscr{L}_n(\mathfrak{B})$ the set of all multilinear mappings

$$\omega_n : \underbrace{\mathfrak{B}\times\cdots\times\mathfrak{B}}_{n\ \text{times}} \longrightarrow \mathfrak{B}.$$

A *formal multilinear function series* over $\mathfrak{B}$ is a sequence $\omega=(\omega_0,\omega_1,\dots)$, where $\omega_0\in\widetilde{\mathfrak{B}}$ and $\omega_n\in\mathscr{L}_n(\mathfrak{B})$ for $n\geq 1$. According to [Dykema 2007], the set of all multilinear function series over $\mathfrak{B}$ will be denoted by $Mul[[\mathfrak{B}]]$.

For $F, G \in Mul[[\mathfrak{B}]]$, the *sum* $F+G$ and the *formal product* FG are the elements from $Mul[[\mathfrak{B}]]$ defined by

$$(F+G)_n(b_1, \dots, b_n) = F_n(b_1, \dots, b_n) + G_n(b_1, \dots, b_n),$$
$$(FG)_n(b_1, \dots, b_n) = \sum_{k=0}^{n} F_k(b_1, \dots, b_k) G_{n-k}(b_{k+1}, \dots, b_n)$$

for any $b_1, \dots, b_n \in \mathfrak{B}$.

If $G_0 = 0$, then the *formal composition* $F \circ G \in Mul[[\mathfrak{B}]]$ is defined by

$$(F \circ G)_0 = F_0,$$
$$(F \circ G)_n(b_1, \dots, b_n) = \sum_{k=1}^{n} \sum_{\substack{p_1, \dots, p_k \geq 1 \\ p_1 + \dots + p_k = n}} F_k\big(G_{p_1}(b_1, \dots, b_{p_1}), \dots, G_{p_k}(b_{q_k+1}, \dots, b_{q_k+p_k})\big)$$

where $q_j = p_1 + \dots + p_{j-1}$ for $j \geq 1$.

With these operations, $Mul[[\mathfrak{B}]]$ is an algebra with additional properties similar to those of power series [Dykema 2007, Propositions 2.3, 2.6].

Proposition 3.5. *Let $E, F, G \in Mul[[\mathfrak{B}]]$. Then*

(i) $1 = (1, 0, 0, \dots) \in Mul[[\mathfrak{B}]]$ *is a multiplicative identity element*;

(ii) $F = (F_0, F_1, \dots)$ *has a multiplicative inverse if and only if* F_0 *is an invertible element of* $\widetilde{\mathfrak{B}}$;

(iii) *if* $F_0 = 0$ *and* $G_0 = 0$, *then*

$$(E \circ F) \circ G = E \circ (F \circ G);$$

(iv) *if* $G_0 = 0$, *then*

$$(E+F) \circ G = E \circ G + F \circ G, \quad \textit{and} \quad (EF) \circ G = (E \circ G)(F \circ G);$$

(v) $I = (0, id_{\mathfrak{B}}, 0, 0, \dots) \in Mul[[\mathfrak{B}]]$ *is an identity element for the formal composition*;

(vi) $F = (0, F_1, F_2, \dots) \in Mul[[\mathfrak{B}]]$ *has a compositional inverse*, *denoted* $F^{\langle -1 \rangle}$, *if and only if* F_1 *is an invertible element of* $\mathcal{L}_1(\mathfrak{B})$;

(vii) *if* $F = (0, F_1, F_2, \dots) \in Mul[[\mathfrak{B}]]$, *then*

$$(1-F)^{-1} = 1 + \sum_{k=1}^{\infty} F^k.$$

For the next definitions and results, $\mathfrak{A}$ will be a $\mathfrak{B}$-algebra ($\mathfrak{B}$ and $\mathfrak{A}$ are not necessarily $*$-algebras).

Definition 3.6. For $X \in \mathfrak{A}$ consider $\mathfrak{H}_X = (\mathfrak{H}_{X,0}, \mathfrak{H}_{X,1}, \dots) \in Mul[[\mathfrak{B}]]$, where

$$\begin{aligned}\mathfrak{H}_{X,0} &= 0,\\ \mathfrak{H}_{X,1}(b) &= b,\\ \mathfrak{H}_{X,n}(b_1, \dots, b_n) &= \Phi(b_1 X b_2 \cdots b_{n-1} X b_n),\end{aligned}$$

for all $b, b_1, \dots, b_n \in \mathfrak{B}$, $n \geq 1$.

Theorem 3.7. *If $X, Y \in \mathfrak{A}$ are monotonically independent over $\mathfrak{B}$, then*

$$\mathfrak{H}_{X+Y} = \mathfrak{H}_X \circ \mathfrak{H}_Y.$$

Proof. It suffices to show that

$$\mathfrak{H}_{X+Y,n} = (\mathfrak{H}_X \circ \mathfrak{H}_Y)_n$$

for all $n \geq 0$.

For $n = 0, 1$, the assertion is trivial. For $n \geq 2$,

$$\begin{aligned}&(\mathfrak{H}_X \circ \mathfrak{H}_Y)_n(b_1, \dots, b_n)\\ &\qquad = \sum_{k=1}^{n} \sum \mathfrak{H}_{X,k}\big(\mathfrak{H}_{Y,p_1}(b_1, \dots, b_{p_1}), \dots, \mathfrak{H}_{Y,p_k}(b_{q_k+1}, \dots, b_n)\big)\\ &\qquad = \Phi\bigg(\sum_{k=1}^{n} \sum \mathfrak{H}_{Y,p_1}(b_1, \dots, b_{p_1}) X \cdots X \mathfrak{H}_{Y,p_k}(b_{q_k+1}, \dots, b_n)\bigg)\\ &\qquad = \Phi\bigg(\sum_{k=1}^{n} \sum \Phi(b_1 Y \cdots Y b_{p_1}) X \cdots X \Phi(b_{q_k+1} Y \cdots Y b_n)\bigg),\end{aligned}$$

where $q_j = p_1 + \cdots + p_{j-1}$ and the second summation is over all $p_1, \dots, p_k \geq 1$ such that $p_1 + \cdots + p_k = n$.

Using Definition 2.1, this becomes

$$\begin{aligned}(\mathfrak{H}_X \circ \mathfrak{H}_Y)_n(b_1, \dots, b_n) &= \sum_{k=1}^{n} {\sum}' \Phi(b_1 Y \cdots Y b_{p_1} X \cdots X b_{q_k+1} Y \cdots Y b_n)\\ &= \sum_{\substack{X_i \in \{X, Y\}\\ i=1,\dots,n-1}} \Phi(b_1 X_1 \cdots X_{n-1} b_n)\\ &= \Phi(b_1(X+Y) \cdots (X+Y) b_n),\end{aligned}$$

where $\sum'$ indicates the convention that if $p_j = 1$ then $b_{q_j+1} Y \cdots Y b_{q_{j+p_j}} = b_{q_j+1}$.

On the other hand,

$$\mathfrak{H}_{X+Y,n}(b_1, \dots, b_n) = \Phi(b_1(X+Y) \cdots (X+Y) b_n)$$

and hence the conclusion follows. □

If the algebra $\mathfrak{A}$ is unital, there also exist multilinear function series analogous to κ, ρ. First, for $X \in \mathfrak{A}$, define the elements β_X and γ_X of $Mul[[\mathfrak{B}]]$ by

$$\begin{aligned} \beta_{X,0} &= 0, \\ \beta_{X,n}(b_1, \ldots, b_n) &= \Phi(b_1 X b_2 \cdots b_n X), \\ \gamma_{X,0} &= 0, \\ \gamma_{X,n}(b_1, \ldots, b_n) &= \Phi(X b_1 X \cdots X b_n). \end{aligned}$$

From the property in Proposition 3.5 (ii), the multilinear function series $\mathfrak{K}_X$ and $\mathfrak{r}_X$ are well-defined, where

$$\begin{aligned} \mathfrak{K}_X &= (1+\beta_X)^{-1}\beta_X, \\ \mathfrak{r}_X &= \gamma_X(1+\gamma_X)^{-1}. \end{aligned}$$

Theorem 3.8. *Let $U, V \in \mathfrak{A}$ be such that $U-1$ and V are monotonically independent over $\mathfrak{B}$. Then*

$$\begin{aligned} \mathfrak{K}_{VU} &= \mathfrak{K}_U \circ \mathfrak{K}_V, \\ \mathfrak{r}_{UV} &= \mathfrak{r}_V \circ \mathfrak{r}_U. \end{aligned}$$

The proof is a routine (though tedious) verification, using Proposition 3.5 and the techniques from the proof of Theorems 3.4 and 3.7.

4. Semiinner product bimodules

The terminology used in this section is the one from [Lance 1995]. Let $\mathfrak{B}$ be a unital C*-algebra. A *semi-inner-product $\mathfrak{B}$-bimodule* is a linear space $\mathscr{E}$ which is a $\mathfrak{B}$-bimodule, together with a map: $\mathscr{E} \times \mathscr{E} \longrightarrow \mathfrak{B}$, $(x, y) \mapsto \langle x, y\rangle$ such that

(i) $\langle x, \alpha y + \beta z\rangle = \alpha\langle x, y\rangle + \beta\langle x, z\rangle$ for any $x, y, z \in \mathscr{E}$, $\alpha, \beta \in \mathbb{C}$;

(ii) $\langle x, ya\rangle = \langle x, y\rangle a$ for any $x, y \in \mathscr{E}$, $a \in \mathfrak{B}$;

(iii) $\langle y, x\rangle = \langle x, y\rangle^*$ for any $x, y \in \mathscr{E}$;

(iv) $\langle x, x\rangle \geq 0$ for any $x \in \mathscr{E}$.

$\mathscr{E}$ is called an *inner-product $\mathfrak{B}$-bimodule* if $\langle x, x\rangle = 0$ implies $x = 0$ and a *Hilbert $\mathfrak{B}$-bimodule* if it is complete with respect to the norm $\|x\|_0 = \|\langle x, x\rangle\|^{1/2}$, where the second norm is the C*-algebra norm of $\mathfrak{B}$. The algebra of $\mathfrak{B}$-linear (not necessarily bounded) operators on $\mathscr{E}$ will be denoted by $\mathscr{L}(\mathscr{E})$.

4.1. Given a family $\{\mathscr{E}_i\}_{i\in I}$ of semi-inner-product $\mathfrak{B}$-bimodules indexed by a totally ordered set $I \subseteq \mathbb{Z}$, we define, following [Muraki 2000; Pimsner 1997], the *monotonic product* $\mathscr{E}^m$ *of* $\{\mathscr{E}_i\}_{i\in I}$ to be the semi-inner-product $\mathfrak{B}$-bimodule

$$\mathscr{E}^m = \mathfrak{B} \oplus \Big(\bigoplus_{n\geq 1} \bigoplus_{(i_1,\dots,i_n)\in\lfloor I,n\rfloor} \mathscr{E}_{i_1} \otimes \mathscr{E}_{i_2} \otimes \cdots \otimes \mathscr{E}_{i_n} \Big)$$

where all the tensor products are with amalgamation over $\mathfrak{B}$ and

$$\lfloor I, n\rfloor = \{(i_1, \dots, i_n) :\ i_1, \dots, i_n \in I, i_1 > \cdots > i_n\},$$

with the inner-product given by

$$\langle f_1 \otimes \cdots \otimes f_n, e_1 \otimes \cdots \otimes e_m\rangle = \delta_{m,n}\langle f_n, \langle f_{n-1}, \dots, \langle f_1, e_1\rangle \dots e_{n-1}\rangle e_n\rangle.$$

Note that in general $\mathscr{E}^m$ is not an inner-product $\mathfrak{B}$-bimodule even if $\mathscr{E}_i$ are inner-product bimodules or Hilbert bimodules. For example, if $\langle f_1, f_1\rangle = b^*b > 0$ and $bf_2 = 0$, then

$$\langle f_1 \otimes f_2, f_1 \otimes f_2\rangle = \langle f_2, \langle f_1, f_1\rangle f_2\rangle = \langle f_2, b^*bf_2\rangle = 0;$$

see also [Speicher 1998].

If $i \in I$ is fixed, we have the natural identification

$$\mathscr{E}^m = \big((\mathfrak{B} \oplus \mathscr{E}_i) \otimes_{\mathfrak{B}} (\mathfrak{B} \oplus \mathscr{E}^m_{(<i)})\big) \oplus \mathscr{E}^m_{(>i)}$$

where

$$\mathscr{E}^m_{(<i)} = \bigoplus_{n\geq 1} \bigoplus_{\substack{(i_1,\dots,i_n)\in\lfloor I,n\rfloor \\ i_1<i}} \mathscr{E}_{i_1} \otimes \mathscr{E}_{i_2} \otimes \cdots \otimes \mathscr{E}_{i_n},$$

$$\mathscr{E}^m_{(>i)} = \bigoplus_{n\geq 1} \bigoplus_{\substack{(i_1,\dots,i_n)\in\lfloor I,n\rfloor \\ i_1>i}} \mathscr{E}_{i_1} \otimes \mathscr{E}_{i_2} \otimes \cdots \otimes \mathscr{E}_{i_n}.$$

Based on this decomposition, one also has the (nonunital) $*$-representation

$$\lambda_i : \mathscr{L}(\mathfrak{B} \oplus \mathscr{E}_i) \longrightarrow \mathscr{L}(\mathscr{E}^m),$$
$$\lambda_i(A) = \big(A \otimes I_{\mathfrak{B}\oplus\mathscr{E}^m_{(<i)}}\big) \oplus 0_{\mathscr{E}^m_{(>i)}}.$$

Theorem 4.1. *With the above notations*, $\lambda_i(\mathscr{L}(\mathfrak{B} \oplus \mathscr{E}_i))$ $(i \in I)$ *are monotonically independent in* $\mathscr{L}(\mathscr{E}^m)$ *with respect to the conditional expectation* $\Phi(\cdot) = \langle 1, \cdot 1\rangle$.

Proof. We need to show that the two conditions from Definition 2.1 for monotonic independence are satisfied. In fact, it will be shown that the family

$$\big\{ \lambda_i(\mathscr{L}(\mathfrak{B} \oplus \mathscr{E}_i))\big\}_{i\in I}$$

satisfies (b) and the stricter condition (a′) of Definition 2.1.

The proof is similar to the proof of [Muraki 2000, Theorem 2.1]. For $i \in I$, consider $A_i \in \mathscr{L}(\mathfrak{B} \oplus \mathscr{E}_i)$ and $X_i = \lambda_i(A_i)$.

We can write

$$X_i 1 = \alpha_i + s_i,$$

where $1 \in \mathfrak{B} \subset \mathfrak{B} \oplus \mathscr{E}_{i,j}$, $\alpha_i \in \mathfrak{B}$, and $s_i \in \mathscr{E}_i$.

If $k < l$,

$$X_k X_l 1 = X_k(\alpha_l + s_l) = X_k 1 \alpha_l = X_k \langle 1, X_l 1\rangle;$$

therefore,

$$X_j X_{k_1} \cdots X_{k_n} 1 = \langle 1, X_j 1\rangle\langle 1, X_{k_1} 1\rangle \ldots \langle 1, X_{k_n} 1\rangle$$

whenever $j < k_1 < \cdots < k_n$.

Also, write

$$\mathscr{E}^m = \mathfrak{B} \oplus \mathscr{E}^0.$$

Note that $X_l f \in \mathscr{E}^0$ for any $f \in \mathscr{E}^0$ and any $l \in I$, and that, for $k < l$,

$$\begin{aligned} X_l X_k 1 &= X_l(\alpha_k + s_k) = X_l 1 \alpha_k + X_l(1 \otimes s_k) \\ &= X_l \langle 1, X_k 1\rangle + (\alpha_l + s_l) \otimes s_k = X_l \langle 1, X_k 1\rangle + f, \end{aligned}$$

for some $f \in \mathscr{E}^0$.

Iterating the above relations for $i_m > \cdots > i_1 > j < k_1 < \cdots < k_n$, we obtain

$$\begin{aligned} &\langle 1, X_{i_m} \cdots X_{i_1} X_j X_{k_1} \cdots X_{k_n} 1\rangle \\ &\qquad = \langle 1, X_{i_m} \cdots X_{i_1} 1\rangle\langle 1, X_j 1\rangle \ldots \langle 1, X_{k_n} 1\rangle \\ &\qquad = (\langle 1, X_{i_m} 1\rangle \ldots \langle 1, X_{i_1} 1\rangle + \langle 1, f\rangle)\langle 1, X_j 1\rangle \ldots \langle 1, X_{k_n} 1\rangle \\ &\qquad = \langle 1, X_{i_m} 1\rangle \ldots \langle 1, X_{k_n} 1\rangle, \end{aligned}$$

that is, property (b).

For $i < j > k$, a direct computation gives

$$\begin{aligned} X_i X_j X_k 1 &= X_i X_j(\alpha_k + s_k) = X_i(X_j 1)\alpha_k + X_i X_j(1 \otimes s_k) \\ &= X_i(\alpha_j + s_j)\alpha_k + X_i(\alpha_j + s_j) \otimes s_k \\ &= X_i(\alpha_j \alpha_k) + X_i(\alpha_j s_k) + X_i(s_j \otimes s_k) \\ &= X_i \alpha_j(\alpha_k + s_k) = X_i \langle 1, X_j 1\rangle X_k 1, \end{aligned}$$

so it remains to show (a′) on elements of the form

$$\widetilde{h} = h_{i_1} \otimes \cdots \otimes h_{i_n}, \quad h_{i_l} \in \mathscr{E}_{i_l}.$$

If $i_1 > i$, then $X_2 h_{i_1} \otimes \cdots \otimes h_{i_n} = 0$; therefore,

$$X_1 Y X_2 \widetilde{h} = 0 = X_1 \langle 1, Y 1\rangle X_2 \widetilde{h}.$$

If $i_1 = i$, with the notations $h^0 = h_{i_2} \otimes \cdots \otimes h_{i_n}$ and $X_2 h_{i_1} = \theta \oplus u$ for some $\theta \in \mathfrak{B}$, $u \in \mathscr{E}_i$, one has

$$\begin{aligned} X_1 Y X_2 h_{i_1} \otimes \cdots \otimes h_{i_n} &= X_1 Y(\theta \oplus u) h^0 \\ &= X_1(\beta\theta \oplus t\theta \oplus (\beta \oplus t) \otimes u) \otimes h^0 \\ &= X_1(\beta\theta + \beta u) \otimes h^0 = X_1 \beta(\theta \oplus u) \otimes h^0 \\ &= X_1 \langle 1, Y1 \rangle X_2 h_{i_1} \otimes \cdots \otimes h_{i_n}. \end{aligned}$$

The case $i_1 < i$ is similar. □

4.2. The *weakly monotone product* of the bimodules $\{\mathscr{E}_i\}_{i \in I}$ is the semi-inner-product $\mathfrak{B}$-bimodule

$$\mathscr{E}^{wm} = \mathfrak{B} \oplus \bigoplus_{n=1}^{\infty} \Bigl(\bigoplus_{i_1 \geq \cdots \geq i_n} \mathscr{E}_{i_1} \otimes \cdots \otimes \mathscr{E}_{i_n} \Bigr).$$

If I has only one element i_0, then $\mathscr{E}^{wm}$ is the full Fock bimodule over $\mathscr{E}_{i_0}$, $\mathscr{F}(\mathscr{E}_{i_0})$ [Pimsner 1997; Speicher 1998].

For $j \in I$, let

$$\mathfrak{J} = \{l \in I : \ l \leq j\}$$

and let $\mathscr{E}^{wm}(j)$ be the weakly monotonic product of $\{\mathscr{E}_l\}_{l \in \mathfrak{J}}$. We will also use the notations

$$\begin{aligned} \mathscr{F}_0(\mathscr{E}) &= \mathscr{F}(\mathscr{E}) \ominus \mathfrak{B}, \\ \mathscr{E}_0^{wm}(\mathscr{E}) &= \mathscr{E}^{wm} \ominus \mathfrak{B}, \\ \mathscr{E}_0^{wm}(j) &= \mathscr{E}^{wm}(j) \ominus \mathfrak{B}. \end{aligned}$$

For $f \in \mathscr{E}_i$, define the $\mathfrak{B}$-linear creation and annihilation maps $a^*(f)$ and $a(f)$ on $\mathscr{E}^{wm}$ by

$$\begin{aligned} a^*(f)1 &= f, \\ a^*(f) f_{i_1} \otimes \cdots \otimes f_{i_n} &= \begin{cases} f \otimes f_{i_1} \otimes \cdots \otimes f_{i_n}, & \text{if } i \geq i_1, \\ 0, & \text{if } i < i_1, \end{cases} \\ a(f)1 &= 0, \\ a(f) f_{i_1} \otimes \cdots \otimes f_{i_n} &= \begin{cases} \langle f, f_{i_1} \rangle f_{i_2} \otimes \cdots \otimes f_{i_n}, & \text{if } i = i_1, \\ 0, & \text{if } i \neq i_1. \end{cases} \end{aligned}$$

Note that $a(f)$ and $a^*(f)$ are adjoint to each other. Denote by $G(f)$ their sum, $G(f) = a(f) + a^*(f)$, and by $\mathfrak{A}_i$ the algebra generated over $\mathfrak{B}$ by $\{G(f) : \ f \in \mathscr{E}_i\}$.

We will use the shorthand notation $\Phi(\cdot)$ for the $\mathfrak{B}$-valued functional $\langle 1, \cdot 1 \rangle$ on the set of all $\mathfrak{B}$-linear maps on $\mathscr{E}^{wm}$. Also, for

$$\widetilde{e} = e_1 \otimes \cdots \otimes e_n,$$

with $e_l \in \mathscr{E}_k$, $1 \le l \le n$, we will use the notations

$$A^*(\widetilde{e}) = a^*(e_1) \cdots a^*(e_n),$$
$$A(\widetilde{e}) = a(e_1) \cdots a(e_n).$$

Lemma 4.2. *For any $f_1, \dots, f_n \in \mathscr{E}_k$ there are some sequences of elements of $\mathscr{E}_0^{wm}(k)$,*

$$(\widetilde{e}_r)_{r=1}^{N_1},\ (\widetilde{g}_s)_{s=1}^{N_2},\ (\widetilde{h}_q)_{q=1}^{N_3},\ (\widetilde{k}_q)_{q=1}^{N_3},$$

such that

$$P = \prod_{l=1}^{n} G(f_l)$$

can be written as

$$P = \Phi(P) + \sum_{r=1}^{N_1} A^*(\widetilde{e}_r) + \sum_{s=1}^{N_2} A(\widetilde{g}_s) + \sum_{q=1}^{N_3} A^*(\widetilde{h}_q)A(\widetilde{k}_q). \tag{4-4}$$

Proof. Let (4-4)$'$ be the weaker form of (4-4) where $\Phi(P)$ is replaced by some element $\alpha \in \mathfrak{B}$. Note that (4-4)$'$ is in fact equivalent to (4-4), since

$$\begin{aligned}\Phi(P) &= \langle 1, P1\rangle\\ &= \Big\langle 1, \alpha 1 + \sum_{r=1}^{N_1} A^*(\widetilde{e}_r)1 + \sum_{s=1}^{N_2} A(\widetilde{g}_s)1 + \sum_{q=1}^{N_3} A^*(\widetilde{h}_q)A(\widetilde{k}_q)1\Big\rangle\\ &= \Big\langle 1, 1\alpha + \sum_{r=1}^{N_1} \widetilde{e}_r\Big\rangle = \alpha.\end{aligned}$$

It remains to prove (4-4)$'$.

Note that

$$P = \prod_{l=1}^{n} G(f_l) = \prod_{l=1}^{n}\big(a^*(f_l) + a(f_l)\big) = \sum_{\substack{(\varepsilon_1,\dots,\varepsilon_n)\\ \varepsilon_l \in \{1,2\}}} a_{\varepsilon_1}(f_1)\dots a_{\varepsilon_n}(f_n),$$

where a_1 stands for a and a_2 stands for a^*.

Also, for any $f, g, h \in \mathscr{E}_k$, $\alpha \in \mathfrak{B}$ and $\varepsilon \in \{1, 2\}$,

$$a(f)\, a^*(g) = \langle f, g\rangle I,$$
$$a_\varepsilon(h)\langle f, g\rangle = a_\varepsilon(h\langle f, g\rangle),$$
$$\alpha a(f) = a(\alpha^* f),$$
$$\alpha a^*(f) = a^*(\alpha f).$$

It follows that in the expression of $a_{\varepsilon_1}(f_1)\dots a_{\varepsilon_n}(f_n)$, any $a(f_p)\,a^*(f_{p+1})$ can be reduced to $\langle f_p, f_{p+1}\rangle$ which can be included in the expression of the previous or following factor. After finite steps of iteration, no summand will have factors of the type $a(f_q)$ in front of factors of the type $a^*(f_p)$, so (4-4)$'$ is proved. □

Lemma 4.3. *Any $X \in \mathfrak{A}_i$ is satisfying the following properties*:

(i) *$\mathscr{E}^{wm}(i)$ is X-invariant*;

(ii) *$X1 = \Phi(X) + s$, for some $s \in \mathscr{F}_0(\mathscr{E}_i)$*;

(iii) *if $u \in \mathscr{E}^{wm}(j)$, $j < i$, then*

$$Xu = \Phi(X)u + t \otimes u,$$

for some $t \in \mathscr{F}_0(\mathscr{E}_i)$;

(iv) *if $i < j$ and $v \in \mathscr{F}_0(\mathscr{E}_j) \otimes \mathscr{E}^{wm}(k)$, $k < j$, then $Xv = 0$.*

Proof. (i) It is enough to verify that the property holds for $X = G(f)$, $f \in \mathscr{E}_i$. Indeed,

$$G(f)1 = f \in \mathscr{F}(\mathscr{E}_i) \subset \mathscr{E}^{wm}(i)$$

and for any $f_1 \otimes \cdots \otimes f_n \in \mathscr{E}^{wm}(i)$,

$$G(f)f_1 \otimes \cdots \otimes f_n = f \otimes f_1 \otimes \cdots \otimes f_n + \langle f, f_1\rangle f_2 \otimes \cdots \otimes f_n \in \mathscr{E}^{wm}(i).$$

(ii) If $f, f_1, \dots, f_n \in \mathscr{F}(\mathscr{E}_i)$,

$$G(f)f_1 \otimes \cdots \otimes f_n = f \otimes f_1 \otimes \cdots \otimes f_n + \langle f, f_1\rangle f_2 \otimes \cdots \otimes f_n \in \mathscr{F}(\mathscr{E}_i).$$

It follows that $\mathscr{F}(\mathscr{E}_i)$ is invariant to $\mathfrak{A}_i$. Since $1 \in \mathscr{F}(\mathscr{E}_i)$, we have $X1 \in \mathscr{F}(\mathscr{E}_i)$, and the conclusion follows from the orthogonality of $\mathfrak{B}$ and $\mathscr{F}_0(\mathscr{E}_i)$.

(iii) It is enough to prove the relation for $X = G(f_1)\cdots G(f_n)$, $f_l \in \mathscr{E}_i$. First note that for any $\tilde{f} \in \mathscr{F}_0(\mathscr{E}_i)$,

$$A^*(\tilde{f})u = \tilde{f} \otimes u,$$
$$A(\tilde{f})u = 0.$$

From Lemma 4.2, there are some sequences

$$(\widetilde{e}_r)_{r=1}^{N_1},\ (\widetilde{g}_s)_{s=1}^{N_2},\ (\widetilde{h}_q)_{q=1}^{N_3},\ (\widetilde{k}_q)_{q=1}^{N_3}$$

of elements of $\mathcal{F}(\mathcal{E}_i)$ such that

$$\begin{aligned} Xu &= \Phi(X)u + \sum_{r=1}^{N_1} A^*(\widetilde{e}_r)u + \sum_{s=1}^{N_2} A(\widetilde{g}_s)u + \sum_{q=1}^{N_3} A^*(\widetilde{h}_q)A(\widetilde{k}_q)u \\ &= \Phi(X)u + \sum_{r=1}^{N_1} \widetilde{e}_r \otimes u \\ &= \Phi(X)u + \left(\sum_{r=1}^{N_1} \widetilde{e}_r\right) \otimes u. \end{aligned}$$

(iv) Similarly, it is enough to prove the relation for $X = G(f_1)\cdots G(f_n)$, $f_l \in \mathcal{E}_i$. First note that for any $\tilde{f} \in \mathcal{F}_0(\mathcal{E}_i)$

$$A^*(\tilde{f})v = A(\tilde{f})v = 0.$$

From Lemma 4.2, there are some sequences

$$(\widetilde{e}_r)_{r=1}^{N_1},\ (\widetilde{g}_s)_{s=1}^{N_2},\ (\widetilde{h}_q)_{q=1}^{N_3},\ (\widetilde{k}_q)_{q=1}^{N_3}$$

of elements of $\mathcal{F}(\mathcal{E}_i)$ such that

$$\begin{aligned} Xv &= \Phi(X)v + \sum_{r=1}^{N_1} A^*(\widetilde{e}_r)u + \sum_{s=1}^{N_2} A(\widetilde{g}_s)v + \sum_{q=1}^{N_3} A^*(\widetilde{h}_q)A(\widetilde{k}_q)v \\ &= \Phi(X)v. \end{aligned}$$

□

Theorem 4.4. *The algebras $\mathfrak{A}_i$ $(i \in I)$ are monotonically independent with respect to the $\mathfrak{B}$-valued functional $\Phi(\cdot) = \langle 1, \cdot 1\rangle$.*

Proof. Let $X_i \in \mathfrak{A}_i$, $i \in I$. We will prove that they satisfy the relations (b) and (a′) from the definition of the monotonic independence. If $k < l$, from Lemma 4.3, we have

$$\begin{aligned} X_k X_l 1 &= X_k(\Phi(X_l) + t) \quad (\text{for some } t \in \mathcal{F}_0(\mathcal{E}_l)) \\ &= X_k 1 \Phi(X_l), \end{aligned}$$

since $Xt = 0$; therefore,

$$X_j X_{i_1} \cdots X_{k_n} 1 = X_j 1 \Phi(X_{k_1}) \cdots \Phi(X_{k_n})$$

whenever $j < k_1 < \cdots < k_n$. Similarly, for $k < l$,

$$\begin{aligned} X_l X_k 1 &= X_l(\Phi(X_k) + t_k) \quad (\text{for some } t_k \in \mathcal{F}_0(\mathcal{E}_k)) \\ &= \Phi(X_l)\Phi(X_k) + \Phi(X_l)t_k + t_l \otimes (\Phi(X_k) + t_k) \quad (\text{for some } t_l \in \mathcal{F}_0(\mathcal{E}_l)) \\ &= \Phi(X_l)\Phi(X_k) + t \quad (\text{for some } t \in \mathcal{E}_0^{wm}(l)). \end{aligned}$$

Using the above relations, we obtain

$$\begin{aligned}\Phi(X_{i_1}\cdots X_{1_n}X_jX_{k_1}\cdots X_{k_m}) &= \Phi(X_{i_1}\cdots X_{1_n}X_j)\Phi(X_{k_1})\cdots\Phi(X_{k_m})\\ &= \Phi\big(\Phi(X_{i_1})\cdots\Phi(X_j)+s\big)\Phi(X_{k_1})\ldots\Phi(X_{k_m})\\ &= \Phi\big(\Phi(X_{i_1})\cdots\Phi(X_j)+s\big)\Phi(X_{k_1})\cdots\Phi(X_{k_m})\end{aligned}$$

for some $s \in \mathscr{E}_0^{wm}(k_m)$, that is, property (b).

Also, for $i < j > k$,

$$\begin{aligned}&X_iX_jX_k1\\ &\quad= X_iX_j(\Phi(X_k)+s) \quad (\text{for some } s \in \mathscr{F}_0(\mathscr{E}_k))\\ &\quad= (X_iX_j1)\Phi(X_k)+X_iX_js\\ &\quad= X_i(\Phi(X_j){+}t_1)\Phi(X_k)+X_i(\Phi(X_j)s+t_2\otimes s) \quad (\text{for some } t_1, t_2 \in \mathscr{F}_0(\mathscr{E}_j))\\ &\quad= X_i\Phi(X_j)\Phi(X_k)+X_i\Phi(X_j)s\\ &\quad= X_i\Phi(X_j)(\Phi(X_k)+s)\\ &\quad= X_i\Phi(X_j)X_k1.\end{aligned}$$

It remains to prove that $X_iX_jX_k$ and $X_i\Phi(X_j)X_k$ coincide on vectors of the form

$$\tilde{f} = f_{i_1}\otimes\cdots\otimes f_{i_m},$$

where $f_{i_k} \in \mathscr{E}_{i_k}$ for $i_1 \geq \cdots \geq i_m$.

If $i_1 > k$, then $X_k\tilde{f} = 0$, so the equality is trivial.

If $i_1 \leq k$, then Lemma 4.3 implies that $X_k\tilde{f} \in \mathscr{E}^{wm}(k)$; therefore,

$$\begin{aligned}X_iX_jX_k\tilde{f} &= X_i\big(\Phi(X_j)X_k\tilde{f}+t\otimes X_k\tilde{f}\big) \quad (\text{for some } t \in \mathscr{F}_0(\mathscr{E}_j))\\ &= X_i\Phi(X_j)X_k\tilde{f}.\end{aligned}$$

□

Remark 4.5. An analogous construction can be done for creation and annihilation maps on $\mathscr{E}^m$, and similar computations will lead to the monotonic independence of the correspondent algebras [Muraki 2001].

5. Central Limit Theorem

In this section $\mathfrak{A}$ will be a $*$-algebra, $\mathfrak{B}$ a subalgebra of $\mathfrak{A}$ which is also a C*-algebra and $\Phi : \mathfrak{A} \longrightarrow \mathfrak{B}$ a conditional expectation. $\mathfrak{B}_+\langle\xi\rangle$ will be the $*$-algebra generated by $\mathfrak{B}$ and the selfadjoint symbol ξ, as described in Section 1.

As discussed in Section 2, given X as a selfadjoint element of $\mathfrak{A}$, the n-th moment of X is the multilinear function

$$m_{X,n} : \mathfrak{B}^{n-1} \longrightarrow \mathfrak{B}$$

given by

$$m_{X,n}(b_1,\ldots,b_{n-1}) = \Phi(Xb_1X\cdots Xb_{n-1}X).$$

We define *the moment function* of X as

$$\mu_X = \bigoplus_{m=1}^{\infty} \mu_X^{(m)}.$$

Before stating the main theorem of this section, we will begin with some combinatorial considerations on the joint moments of the family of selfadjoint elements $\{X_n\}_{n\geq 1}$ of $\mathfrak{A}$ with the properties:

(1) for any $i < j$, X_i and X_j are monotonically independent over $\mathfrak{B}$;

(2) all X_k have the same moment function, denoted by μ.

Let $NC(m)$ be the set of all noncrossing partitions of the ordered set

$$\{1, 2, \ldots, m\}.$$

For $\gamma \in NC(m)$, let

$$B = \{b_1, b_2, \ldots, b_p\}, \quad \text{and} \quad C = \{c_1, c_2, \ldots, c_q\}$$

be two blocks of γ. We say that C is *interior to* B if there is an index

$$k \in \{1, \ldots, p-1\}$$

such that $b_k < c_1, c_2, \ldots, c_q < b_{k+1}$. B and C will be called *adjacent* if $c_1 = b_p + 1$ or $b_1 = c_q + 1$. The block B will be called *outer* if it is not interior to any other block of γ.

To each m-tuple $(i_1, \ldots, i_m)$ of indices from $\{1, 2, , \ldots\}$, we associate a unique noncrossing partition $nc\,[i_1, \ldots, i_m] \in NC(m)$ as follows:

(1) if $m = 1$, then $nc\,[i_1] = (1)$;

(2) if $m > 1$, put

$$B = \{\, k : i_k = \min\{i_1, \ldots, i_m\}\} = \{k_1, \ldots, k_p\}$$

and define $nc\,[i_1, \ldots, i_m]$ to be

$$B \sqcup nc\,[i_1, \ldots, i_{k_1} - 1] \sqcup nc\,[i_{k_1} + 1, \ldots, i_{k_2} - 1] \sqcup \cdots \sqcup nc\,[i_{k_p} + 1, \ldots, i_m].$$

Reciprocally, the m-tuple $(i_1, \ldots, i_m)$ will be called an *admissible configuration for* $\gamma \in NC(m)$ if $nc\,[i_1, \ldots, i_m] = \gamma$.

Lemma 5.1. *Suppose* $(i_1, \ldots, i_m)$ *is an admissible configuration for* $\gamma \in NC(m)$ *and* $B = \{k_1, \ldots, k_p\}$ *is an outer block of* γ. *Then, for any* $b_1, \ldots, b_{m-1} \in \mathfrak{B}$, *with* μ *the common moment function of* $\{X_n\}_n$, *we have*

$$\begin{aligned}\Phi(X_{i_1} b_1 X_{i_2} \cdots b_{m-1} X_{i_m}) \\ = \mu\big(\Phi(X_{i_1} b_1 \cdots X_{i_{k_1-1}} b_{k_1-1}), \Phi(b_{k_1} X_{i_{k_1+1}} \cdots b_{k_2}), \ldots, \Phi(b_{k_p} \cdots X_{i_m})\big).\end{aligned}$$

Proof. If γ has only one block, then the result is trivial. If γ has more than one block, but only one outer block B, then

$$B = \{ k : i_k = \min\{i_1, \dots, i_m\}\} = \{k_1, \dots, k_p\}$$

since the last set always forms an outer block. Also this block must contain 1 and m. The monotonic independence of $\{X_n\}_{n\geq 1}$ over $\mathfrak{B}$ implies

$$\begin{aligned}\Phi(X_{i_1}b_1X_{i_2}\cdots b_{m-1}X_{i_m})&\\ &= \Phi\big(X_{i_1}\Phi(b_1X_{i_2}b_2\cdots X_{i_{k_2-1}}b_{k_2-1})X_{i_{k_2}}\Phi(b_{k_2}X_{i_{k_2+1}}\cdots)\cdots X_{i_m}\big)\\ &= \mu\big(\Phi(b_1X_{i_2}\cdots b_{k_2-1}),\dots,\Phi(b_{k_{p-1}}X_{i_{k_{p-1}+1}}\cdots b_{m-1})\big).\end{aligned}$$

If γ has more than one outer block, the result comes by induction on the number of blocks of γ. Suppose the result is true for less than r blocks and that γ has exactly r blocks. Consider again $B_0 = \{ k : i_k = \min\{i_1, \dots, i_m\}\} = \{k_1, \dots, k_p\}$. Using again Definition 2.1, we obtain

$$\begin{aligned}\Phi(X_{i_1}b_1X_{i_2}\cdots b_{m-1}X_{i_m})&\\ &= \Phi\big(\Phi(X_{i_1}b_1\cdots b_{k_1-1})X_{i_{k_1}}\cdots X_{i_{k_m}}\Phi(b_{k_m}X_{i_{k_m+1}}\cdots X_{i_m})\big)\\ &= \mu\big(\Phi(X_{i_1}b_1\cdots b_{k_1-1}),\dots,\Phi(b_{k_m}X_{i_{k_m+1}}\cdots X_{i_m})\big).\end{aligned}$$

If $B = B_0$, then the result is proved above. If $B \neq B_0$, then without losing generality we can suppose that B is at the right of B_0, and hence

$$\begin{aligned}\Phi(X_{i_1}b_1X_{i_2}\cdots b_{m-1}X_{i_m})&\\ &= \Phi\big(\Phi(X_{i_1}b_1\cdots b_{k_1-1})X_{i_{k_1}}\cdots X_{i_{k_m}}\Phi(b_{k_m}X_{i_{k_m+1}}\cdots X_{i_m}))\\ &= \Phi(X_{i_1}b_1\cdots X_{i_{k_1-1}})\Phi(b_{k_1-1}X_{i_{k_1}}\cdots X_{i_m}).\end{aligned}$$

And the result follows by applying the induction hypothesis to $\Phi(X_{i_1}b_1\cdots X_{i_{k_1-1}})$. □

Lemma 5.2. *If $(i_1, \dots, i_m)$ and $(l_1, \dots, l_m)$ are two admissible configurations for $\gamma \in NC(m)$, then for any $b_1, \dots, b_{m-1} \in \mathfrak{B}$, one has*

$$\Phi(X_{i_1}b_1X_{i_2}\cdots b_{m-1}X_{i_m}) = \Phi(X_{l_1}b_1X_{l_2}\cdots b_{m-1}X_{l_m}).$$

Proof. Again, if γ is the partition with a single block, then $i_1 = \cdots = i_m$, $l_1 = \cdots = l_m$ and

$$\begin{aligned}\Phi(X_{i_1}b_1X_{i_2}\cdots b_{m-1}X_{i_m}) &= \Phi(X_{i_1}b_1X_{i_1}\cdots b_{m-1}X_{i_1})\\ &= \mu(b_1,\dots,b_{m-1}) = \Phi(X_{l_1}b_1X_{l_2}\cdots b_{m-1}X_{l_m}).\end{aligned}$$

The conclusion follows by induction on m.

If $m = 1$, then $\Phi(X_{i_1}) = \Phi(X_{l_1})$.

Suppose the result is true for $m \leq N-1$ and that $\gamma \in NC(N)$ has more than one block. Then let $B = \{k_1, \ldots, k_p\}$ be an outer block of γ. From Lemma 5.1,

$$\begin{aligned}
&\Phi(X_{i_1}b_1X_{i_2}\cdots b_{m-1}X_{i_m})\\
&\qquad = \mu\big(\Phi(X_{i_1}b_1\cdots X_{i_{k_1-1}}b_{k_1-1}), \Phi(b_{k_1}X_{i_{k_1+1}}\cdots b_{k_2}), \ldots, \Phi(b_{k_p}\cdots X_{i_m})\big)\\
&\qquad = \mu\big(\Phi(X_{l_1}b_1\cdots X_{l_{k_1-1}}b_{k_1-1}), \Phi(b_{k_1}X_{l_{k_1+1}}\cdots b_{k_2}), \ldots, \Phi(b_{k_p}\cdots X_{l_m})\big)\\
&\qquad = \Phi(X_{l_1}b_1X_{l_2}\cdots b_{m-1}X_{l_m}). \qquad \square
\end{aligned}$$

Since the value $\Phi(X_{i_1}b_1X_{i_2}\cdots b_{m-1}X_{i_m})$ is the same for all the admissible configurations $(i_1, \ldots, i_m)$, we will denote it by $V(\gamma, b_1, \ldots, b_{m-1})$.

Theorem 5.3. *Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of selfadjoint elements from $\mathfrak{A}$ such that*

(1) *$\{X_n\}_n$ is a monotonically independent family;*

(2) *$\mu_{X_i} = \mu_{X_j}$ for any $i, j \geq 1, n \geq 0$;*

(3) *$\Phi(X_k) = 0$.*

Then there exists a conditional expectation $\nu : \mathfrak{B}_+\langle \xi \rangle \longrightarrow \mathfrak{B}$ with the property

$$\lim_{N\to\infty} \Phi\left(f\Big(\frac{X_1 + \cdots + X_N}{\sqrt{N}}\Big)\right) = \nu(f) \tag{5-5}$$

for any $f \in \mathfrak{B}_+\langle \xi \rangle$. Moreover $\nu(f)$ depends only on the second order moments of X_i.

Proof. For convenience, we will use the notations μ for μ_{X_i} $(i \geq 1)$, $\mathfrak{a}(\gamma)$ for the set of all admissible configurations of γ, $\mathfrak{a}(\gamma, N)$ for the set of all admissible configurations of γ with indices from $\{1, 2, \ldots, N\}$, and $PP(m)$ for the set of all noncrossing pair partitions (partitions where each block has exactly two elements) of $\{1, \ldots, m\}$.

It is enough to show the property for some arbitrary $b_1, \ldots, b_{m-1} \in \mathfrak{B}$ and

$$f = \xi b_1 \xi \cdots b_{m-1}\xi.$$

From Lemma 5.2, one has

$$\begin{aligned}
&\Phi\left(\Big(\frac{X_1 + \cdots + X_N}{\sqrt{N}}\Big)b_1\cdots b_{m-1}\Big(\frac{X_1 + \cdots + X_N}{\sqrt{N}}\Big)\right)\\
&\qquad = \frac{1}{N^{m/2}} \sum_{(i_1,\ldots,i_m)} \Phi(X_{i_1}b_1\cdots b_{m-1}X_{i_m})\\
&\qquad = \frac{1}{N^{m/2}} \sum_{\gamma\in NC(m)} V(\gamma, b_1, \ldots, b_{m-1})\,\mathrm{card}(\mathfrak{a}(\gamma, N)).
\end{aligned}$$

If γ contains blocks with only one element, the condition $\Phi(X_i) = 0$ $(i \geq 1)$ and Lemma 5.1 imply that $V(\gamma, b_1, \ldots, b_{m-1}) = 0$.

Also, if γ has less than $m/2$ blocks, since

$$\operatorname{card}(\mathfrak{a}(\gamma, N)) < N^{\operatorname{card}(\gamma)} < N^{m/2},$$

we have that

$$\lim_{N\to\infty} \frac{1}{N^{m/2}} V(\gamma, b_1, \dots, b_{m-1}) \operatorname{card}(\mathfrak{a}(\gamma, N)) = 0.$$

It follows that only the pair partitions contribute to the limit, that is,

$$\lim_{N\to\infty} \Phi\Big(\Big(\frac{X_1+\cdots+X_N}{\sqrt{N}}\Big) b_1 \cdots b_{m-1} \Big(\frac{X_1+\cdots+X_N}{\sqrt{N}}\Big)\Big)$$
$$= \lim_{N\to\infty} \frac{1}{N^{m/2}} \sum_{\gamma\in PP(m)} V(\gamma, b_1, \dots, b_{m-1}) \operatorname{card}(\mathfrak{a}(\gamma, N)).$$

In particular, for m odd, the limit exists and it is equal to zero.

If m is even, note first that

$$\mathfrak{a}(\gamma, N) = \bigsqcup_{k=1}^{m/2} \mathfrak{a}(\gamma, N, k)$$

and that

$$\operatorname{card}(\mathfrak{a}(\gamma, N, k)) = \binom{N}{k} \operatorname{card}(\mathfrak{a}(\gamma, k, k)).$$

Therefore,

$$\lim_{N\to\infty} \Phi\Big(\Big(\frac{X_1+\cdots+X_N}{\sqrt{N}}\Big) b_1 \dots b_{m-1} \Big(\frac{X_1+\cdots+X_N}{\sqrt{N}}\Big)\Big)$$
$$= \lim_{N\to\infty} \frac{1}{N^{m/2}} \sum_{\gamma\in PP(m)} V(\gamma, b_1, \dots, b_{m-1}) \sum_{k=1}^{m/2} \binom{N}{k} \operatorname{card}(\mathfrak{a}(\gamma, k, k))$$
$$= \lim_{N\to\infty} \frac{1}{N^{m/2}} \sum_{\gamma\in PP(m)} V(\gamma, b_1, \dots, b_{m-1}) \binom{N}{m/2} \operatorname{card}\Big(\mathfrak{a}\Big(\gamma, m/2, m/2\Big)\Big)$$
$$= \frac{1}{(m/2)!} \sum_{\gamma\in PP(m)} V(\gamma, b_1, \dots, b_{m-1}) \operatorname{card}\Big(\mathfrak{a}\Big(\gamma, m/2, m/2\Big)\Big),$$

since

$$\lim_{N\to\infty} \frac{1}{N^{m/2}} \binom{N}{k} = \begin{cases} 0 & \text{if } k < m/2, \\ \dfrac{1}{(m/2)!} & \text{if } k = m/2. \end{cases}$$

For the last part, note that $V(\gamma, b_1, \dots, b_{m-1})$ is computed iterating the result from Lemma 5.1, so for $\gamma \in PP(m)$ it depends only on the moments of order 2 of X_i $(i \geq 1)$. $\square$

In the following paragraph we will suppose, without loss of generality, that $\mathfrak{B}$ is unital.

Corollary 5.4. *The functional ν is positive if and only $\Phi(X_k b^* b X_k) \geq 0$ for any $b \in \mathfrak{B}$ $(k \geq 1)$.*

Proof. One implication is trivial: if $\nu \geq 0$, then

$$\Phi(X_k b^* b X_k) = \nu((b\xi)^* b\xi) \geq 0.$$

For the other implication, consider the set of symbols $\{\zeta_i\}_{i \geq 1}$ and the linear spaces

$$\mathfrak{B}\zeta_i\mathfrak{B} = \{b_1\zeta_i b_2 : \ b_1, b_2 \in \mathfrak{B}\}$$

with the $\mathfrak{B}$-bimodule structure given by

$$a_1(b_1\zeta_i b_2)a_2 = (a_1 b_1)\zeta_i(b_2 a_2)$$

for any $a_1, a_2, b_1, b_2 \in \mathfrak{B}$, and with the $\mathfrak{B}$-valued pairing $\langle \cdot, \cdot \rangle$ given by

$$\langle a\zeta_i, b\zeta_i \rangle = \nu(\xi a^* b \xi)$$

for any $a, b \in \mathfrak{B}$.

The pairing $\langle \cdot, \cdot \rangle$ is positive, since $\nu(\xi b^* b \xi) \geq 0$ for any $b \in \mathfrak{B}$.

Let $\mathscr{E}$ be the weakly monotone product of $\{\mathfrak{B}\zeta_i\mathfrak{B}\}_{i \geq 1}$. As shown in Section 4.2, the mappings $G(\zeta_i)$ form a monotonic independent family in $\mathscr{L}(\mathscr{E})$; therefore, from Theorem 5.3, one has that

$$\nu(p^*(\xi)p(\xi)) = \lim_{N \to \infty} \Big\langle 1, p\Big(\frac{G(\zeta_1) + \cdots + G(\zeta_N)}{\sqrt{N}}\Big)^* p\Big(\frac{G(\zeta_1) + \cdots + G(\zeta_N)}{\sqrt{N}}\Big) 1 \Big\rangle$$

for any $p(\xi) \in \mathfrak{B}\langle \xi \rangle$.

The conclusion follows from the positivity of the functional $\langle 1, \cdot 1 \rangle$. $\square$

6. Positivity results and connection to operator-valued conditionally free products

Definition 6.1. Let $\mathfrak{A}_1, \mathfrak{A}_2$ be two algebras containing the subalgebra $\mathfrak{B}$ such that $\mathfrak{A}_1$ has the decomposition $\mathfrak{A}_1 = \mathfrak{B} \oplus \mathfrak{A}_1^0$ for $\mathfrak{A}_1^0$ a subalgebra of $\mathfrak{A}$ which is also a $\mathfrak{B}$-algebra. If Φ_1, Φ_2 are conditional expectations, $\Phi_j : \mathfrak{A}_j \longrightarrow \mathfrak{B}$, $j = 1, 2$, we define

$$\Phi = \Phi_1 \rhd \Phi_2,$$

the monotonic product of Φ_1 and Φ_2 to be the conditional expectation on the algebraic free product with amalgamation over $\mathfrak{B}$,

$$\mathfrak{A} = \mathfrak{A}_1 *_{\mathfrak{B}} \mathfrak{A}_2$$

given by

$$\Phi(\alpha a_1 b a_2 \beta) = \Phi(\alpha a_1 \Phi_2(b) a_2 \beta),$$
$$\Phi(b a_2 \beta) = \Phi_2(b)\Phi(a_2\beta),$$
$$\Phi(\alpha a_1 b) = \Phi(\alpha a_1)\Phi_2(b),$$

for all $a_1, a_2 \in \mathfrak{A}_1^0$, $b \in \mathfrak{A}_2$ and $\alpha, \beta \in \mathfrak{A}$.

The map Φ is well-defined since any element of $\mathfrak{A}$ can be written as a sum of finite products in which the elements from $\mathfrak{A}_1^0$ and $\mathfrak{A}_2$ alternate. The conditions above imply that

$$\Phi(b_0 a_1 b_1 \cdots a_n b_n) = \Phi_1(\Phi_2(b_0) a_1 \Phi_2(b_1) \cdots a_n \Phi_2(b_n))$$

for all $b_0, b_1, \dots, b_n \in \mathfrak{B}_2$, $a_1, \dots, a_n \in \mathfrak{A}_1^0$, and all the analogues for the other types of such products.

Proposition 6.2. *If, in the above setting, $\mathfrak{A}_1$, $\mathfrak{A}_2$ are $*$-algebras, $\mathfrak{B}$ is a C^*-algebra, and Φ_1, Φ_2 are positive (that is, $\Phi_j(a^*a) \geq 0$, for all $a \in \mathfrak{A}_j$, $j = 1, 2$), then $\Phi_1 \rhd \Phi_2$ is also positive.*

Proof. First, remember that the positivity of the conditional expectations Φ_j implies that $\Phi_j(x^*) = (\Phi_j(x))^*$, for all $x \in \mathfrak{A}_j$.

Also, the map Φ_2 is completely positive, and for any $b_1, \dots b_n \in \mathfrak{A}_2$, the element

$$\big(\Phi_2(b_i^* b_j)\big)_{i,j=1}^n \in M_n(\mathfrak{B})$$

is positive [Speicher 1998, Section 3.5].

We have to show that

$$\Phi(a^*a) \geq 0$$

for all $a \in \mathfrak{A}_1 *_{\mathfrak{B}} \mathfrak{A}_2$. Any such a can be written as a finite sum of elements of the form $b_0 a_1 b_1 \cdots a_n b_n$ with $b_0, \dots, b_n \in \mathfrak{A}_2$, $a_1, \dots, a_n \in \mathfrak{A}_1^0$. Hence

$$\begin{aligned}
&\Phi(a^*a)\\
&= \Phi\Bigg(\bigg(\sum_{i=1}^N b_{0,i} a_{1,i} b_{1,i} \cdots a_{n(i),i} b_{n(i),i}\bigg)^* \bigg(\sum_{i=1}^N b_{0,i} a_{1,i} b_{1,i} \cdots a_{n(i),i} b_{n(i),i}\bigg)\Bigg)\\
&= \Phi\bigg(\sum_{i,j=1}^N b_{n(i),i}^* a_{n(i),i}^* \cdots a_{1,i}^* b_{0,i}^* b_{0,j} a_{1,j} b_{1,j} \cdots a_{n(j),j} b_{n(j),j}\bigg)\\
&= \sum_{i,j=1}^N \Phi_1\big(\Phi_2(b_{n(i),i}^*) a_{n(i),i}^* \cdots a_{1,i}^* \Phi_2(b_{0,i}^* b_{0,j}) a_{1,j} \Phi_2(b_{1,j}) \cdots a_{n(j),j} \Phi_2(b_{n(j),j})\big).
\end{aligned}$$

Since

$$\big(\Phi_2(b_{0,i}^* b_{0,j})\big)_{i,j=1}^N \in M_N(\mathfrak{B}) \subset M_N(\mathfrak{A}_1)$$

is positive, there exists a matrix $T \in M_N(\mathfrak{A}_1)$ such that

$$\big(\Phi_2(b_{0,i}^* b_{0,j})\big)_{i,j=1}^N = T^*T.$$

With the notation

$$a_i = a_{1,j}\Phi_2(b_{1,j})\cdots a_{n(j),j}\Phi_2(b_{n(j),j}) \in \mathfrak{A}_1,$$

we obtain

$$\Phi(a^*a) = \Phi_1\big((a_1 \dots a_N)^* T^*T(a_1 \dots a_N)\big) \geq 0. \qquad \square$$

Let $\mathfrak{B}\langle \xi, \xi^* \rangle$ be the $*$-algebra of polynomials in ξ and ξ^* described in Section 2. For $X \in \mathfrak{A}$, consider $\mathfrak{A}_X$ the $*$-subalgebra of $\mathfrak{A}$ generated by X and $\mathfrak{B}$. Define the mapping $\tau_X : \mathfrak{B}\langle \xi, \xi^* \rangle \longrightarrow \mathfrak{A}_X$ to be the algebra $*$-homomorphism given by $\tau_X(\xi) = X$ and the $\mathfrak{B}$-functional $\nu_X : \mathfrak{B}\langle \xi, \xi^* \rangle \longrightarrow \mathfrak{B}$ to be given by $\nu_X = \Phi \circ \tau_X$.

Corollary 6.3. *If $X, Y \in \mathfrak{A}$ are monotonically independent over $\mathfrak{B}$ and ν_X, ν_Y are positive, then ν_Z is also positive for any element Z in the $*$-algebra generated by X and Y. In particular ν_{X+Y} and ν_{XY} are positive.*

Proof. Consider $Z = Z(X, Y)$ a polynomial in X and Y. Since the maps

$$\begin{aligned} \nu_X &: \mathfrak{B}\langle \xi_1, \xi_1^* \rangle \longrightarrow \mathfrak{B}, \\ \nu_Y &: \mathfrak{B}\langle \xi_2, \xi_2^* \rangle \longrightarrow \mathfrak{B} \end{aligned}$$

are positive, from Proposition 6.2 so is

$$\nu_x \rhd \nu_Y : \mathfrak{B}\langle \xi_1, \xi_1^* \rangle *_{\mathfrak{B}} \mathfrak{B}\langle \xi_2, \xi_2^* \rangle = \mathfrak{B}\langle \xi_1, \xi_1^*, \xi_2, \xi_2^* \rangle \longrightarrow \mathfrak{B}.$$

Remark also that

$$i_Z : \mathfrak{B}\langle \xi, \xi^* \rangle \ni f(\xi) \mapsto f(Z(\xi_1, \xi_2)) \in \mathfrak{B}\langle \xi_1, \xi_1^* \rangle *_{\mathfrak{B}} \mathfrak{B}\langle \xi_2, \xi_2^* \rangle$$

is a positive $\mathfrak{B}$-functional.

The conclusion follows from the fact that the monotonic independence over $\mathfrak{B}$ of X and Y is equivalent to

$$\nu_Z = (\nu_X \rhd \nu_Y) \circ i_Z. \qquad \square$$

Lemma 6.4. *Let $\mathfrak{A}_1$, $\mathfrak{A}_2$ be two $*$-algebras containing the C^*-algebra $\mathfrak{B}$, and $\Phi_j : \mathfrak{A}_j \longrightarrow \mathfrak{B}$, $j = 1, 2$, positive conditional expectations. Let $a_1, \dots, a_n \in \mathfrak{A}_1$, $a_{n+1}, \dots, a_{n+m} \in \mathfrak{A}_2$ and*

$$A = \big(A_{i,j}\big) \in M_{n+m}(\mathfrak{B})$$

be the matrix with the entries

$$A_{i,j} = \begin{cases} \Phi_1(a_i^* a_j) & \text{if } i, j \le n, \\ \Phi_1(a_i^*)\Phi_2(a_j) & \text{if } i \le n,\ j > n, \\ \Phi_2(a_i^*)\Phi_1(a_j) & \text{if } i > n,\ j \le n, \\ \Phi_2(a_i^* a_j) & \text{if } i, j > n. \end{cases}$$

Then A is positive.

Proof. As shown in [Speicher 1998, Theorem 3.5.6], the $\mathfrak{B}$-functional $\Phi_1 *_{\mathfrak{B}} \Phi_2$ is completely positive on $\mathfrak{A}_1 *_{\mathfrak{B}} \mathfrak{A}_2$. Also, note that

$$A_{i,j} = (\Phi_1 *_{\mathfrak{B}} \Phi_2)(a_i^* a_j)$$

for all $1 \le i,\ j = 1 \le n+m$, and the conclusion follows from [Speicher 1998, Lemma 3.5.2]. □

Consider now $\mathfrak{A}_1, \mathfrak{A}_2$ two $*$-algebras over the C*-algebra $\mathfrak{B}$, each endowed with two positive conditional expectations

$$\Phi_j, \Psi_j : \mathfrak{A}_j \longrightarrow \mathfrak{B}, \quad j = 1, 2.$$

We define $(\mathfrak{A}, \Phi, \Psi)$, the *conditionally free product with amalgamation over* $\mathfrak{B}$ of the triples $(\mathfrak{A}_1, \Phi_1, \Psi_1)$ and $(\mathfrak{A}_2, \Phi_2, \Psi_2)$ by

(1) $\mathfrak{A} = \mathfrak{A}_1 *_{\mathfrak{B}} \mathfrak{A}_2$;

(2) $\Psi = \Psi_1 *_{\mathfrak{B}} \Psi_2$ and $\Phi = \Phi_1 *_{(\Psi_1, \Psi_2)} \Phi_2$, that is, the functionals Ψ and Φ are determined by the relations

$$\Psi(a_1 a_2 \cdots a_n) = 0,$$
$$\Phi(a_1 a_2 \cdots a_n) = \Phi(a_1)\Phi(a_2)\cdots\Phi(a_n),$$

for any $a_i \in \mathfrak{A}_{\varepsilon(i)}$, $\varepsilon(i) \in \{1, 2\}$, such that $\varepsilon(1) \ne \varepsilon(2) \ne \cdots \ne \varepsilon(n)$ and $\Psi_{\varepsilon(i)}(a_i) = 0$.

Theorem 6.5. *In the above setting*, Φ *and* Ψ *are positive* $\mathfrak{B}$*-functionals.*

Proof. The positivity of Ψ is proved in [Speicher 1998, Theorem 3.5.6].

For the positivity of Φ we have to show that $\Phi(a^*a) \ge 0$ for any $a \in \mathfrak{A}$. Since any element of $\mathfrak{A}$ can be written as

$$a = \sum_{k=1}^{N} s_{1,k} \ldots s_{n(k),k} = \sum_{k=1}^{N} \prod_{j=1}^{n(k)} \bigl(s_{(j,k)} - \Psi(s_{(j,k)}) + \Psi(s_{(j,k)})\bigr),$$

where $s_{j,k} \in \mathfrak{A}_{\varepsilon(j,k)}$, $\varepsilon(1,k) \ne \varepsilon(2,k) \ne \cdots \ne \varepsilon(n(k),k)$, we can consider a of the form

$$a = \alpha + \sum_{k=1}^{N} a_{1,k} \ldots a_{n(k),k}$$

such that $\alpha \in \mathfrak{B}$, $a_{j,k} \in \mathfrak{A}_{\varepsilon(j,k)}$ if $\varepsilon(1,k) \neq \varepsilon(2,k) \neq \cdots \neq \varepsilon(n(k),k)$, and $\Psi_{\varepsilon(j,k)}(a_{j,k}) = 0$. Therefore

$$\begin{aligned}
\Phi(a^*a) &= \Phi\Big(\Big(\alpha + \sum_{k=1}^{N} a_{1,k}\cdots a_{n(k),k}\Big)^*\Big(\alpha + \sum_{k=1}^{N} a_{1,k}\cdots a_{n(k),k}\Big)\Big)\\
&= \Phi\left(\alpha^*\alpha + \alpha^*\Big(\sum_{k=1}^{N} a_{1,k}\cdots a_{n(k),k}\Big) + \Big(\sum_{k=1}^{N} a_{1,k}\cdots a_{n(k),k}\Big)^*\alpha\right.\\
&\qquad\qquad \left. + \Big(\sum_{k=1}^{N} a_{1,k}\cdots a_{n(k),k}\Big)^*\Big(\sum_{k=1}^{N} a_{1,k}\cdots a_{n(k),k}\Big)\right)\\
&= \Phi(\alpha^*\alpha) + \sum_{k=1}^{N}\Phi\big(\alpha^* a_{1,k}\cdots a_{n(k),k}\big) + \sum_{k=1}^{N}\Phi\big(a^*_{n(k),k}\cdots a^*_{1,k}\alpha\big)\\
&\qquad\qquad + \sum_{k,l=1}^{N}\Phi\big(a^*_{n(k),k}\cdots a^*_{1,k}a_{1,l}\cdots a_{n(l),l}\big).
\end{aligned}$$

Using the definition of the conditionally free product with amalgamation over $\mathfrak{B}$ and that $\Psi_{\varepsilon(j,k)}(a_{j,k}) = 0$ for all j, k, the above relation becomes

$$\begin{aligned}
\Phi(a^*a) = \Phi(\alpha^*\alpha) &+ \sum_{k=1}^{N}\Phi(\alpha^* a_{1,k})\Phi(a_{2,k})\cdots\Phi(a_{n(k),k})\\
&+ \sum_{k=1}^{N}\Phi(a_{n(k),k})^*\cdots\Phi(a^*_{2,k})\Phi(a^*_{1,k}\alpha)\\
&+ \sum_{k,l=1}^{N}\big(\Phi(a_{n(k),k})^*\cdots\Phi(a^*_{2,k})\big)\Phi(a^*_{1,k}a_{1,l})\Phi(a_{2,l})\cdots\Phi(a_{n(l),l}),
\end{aligned}$$

that is,

$$\begin{aligned}
\Phi(a^*a) = \Phi(\alpha^*\alpha) &+ \sum_{k=1}^{N}\Phi(\alpha^* a_{1,k})\big(\Phi(a_{2,k})\cdots\Phi(a_{n(k),k})\big)\\
&+ \sum_{k=1}^{N}\big(\Phi(a_{2,k})\cdots\Phi(a_{n(k),k})\big)^*\Phi(a^*_{1,k}\alpha)\\
&+ \sum_{k,l=1}^{N}\big(\Phi(a_{2,k})\cdots\Phi(a_{n(k),k})\big)^*\Phi(a^*_{1,k}a_{1,l})\big(\Phi(a_{2,l})\cdots\Phi(a_{n(l),l})\big).
\end{aligned}$$

Now set $a_{1,N+1} = \alpha$ and $\beta_k = \Phi(a_{2,k})\cdots\Phi(a_{n(k),k})$.

From Lemma 6.4, the matrix

$$S = \Big(\Phi\big(a_{1,i}^* a_{1,j}\big)\Big)_{i,j=1}^{N+1}$$

is positive in $M_{N+1}(\mathfrak{B})$; therefore, $S = T^*T$, for some $T \in M_{N+1}(\mathfrak{B})$.

The identity for $\Phi(a^*a)$ becomes

$$\Phi(a^*a) = (\beta_1, \dots, \beta_N, 1)^* T^*T(\beta_1, \dots, \beta_N, 1) \geq 0,$$

as claimed. □

Suppose now that the $*$-algebra $\mathfrak{A}_1$ has the decomposition $\mathfrak{A}_1 = \mathfrak{B} \oplus \mathfrak{A}_1^0$, such that $\mathfrak{A}_1^0$ is a $*$-subalgebra of $\mathfrak{A}_1$ which is also a $\mathfrak{B}$-algebra. Define the $\mathfrak{B}$-valued conditional expectation

$$\delta : \mathfrak{A}_1 \ni (\lambda + a_0) \mapsto \lambda \in \mathfrak{B} \tag{6-6}$$

for all $a_0 \in \mathfrak{A}_1^0$.

Theorem 6.6. *With the notations above*,

$$\Phi_1 \rhd \Phi_2 = \Phi_1 *_{(\delta,\Phi_2)} \Phi_2.$$

Proof. First remark that $\delta(a) = 0$ implies $a \in \mathfrak{A}_1^0$, from the definition of δ in (6-6).

For $\varepsilon(1), \dots, \varepsilon(n) \in \{1, 2\}$ such that $\varepsilon(1) \neq \cdots \neq \varepsilon(n)$ and $a_j \in \mathfrak{A}_{\varepsilon(j)}$ such that $\delta(a_j) = 0$ if $\varepsilon(j) = 1$ and $\Phi_2(a_j) = 0$ if $\varepsilon(j) = 2$, one has

$$\begin{aligned}(\Phi_1 \rhd \Phi_2)(a_1 \dots a_n) &= \Phi_1\bigg(\prod_{j=1}^{n} \big(\chi_{\mathfrak{A}_1}(a_j) + \Phi_2(\chi_{\mathfrak{A}_2}(a_j))\big)\bigg) \\ &= \Phi_1\bigg(\prod_{j=1}^{n} \chi_{\mathfrak{A}_1}(a_j)\bigg) = 0,\end{aligned}$$

where $\chi_{\mathfrak{A}_j}$ denotes the characteristic function of $\mathfrak{A}_j$, since there is at least one $a_j \in \mathfrak{A}_2$.

The conclusion follows from the above equality, since the conditional expectation $\Phi_1 *_{\delta,\Phi_2} \Phi_2$ is generated by $(\Phi_1 *_{(\delta,\Phi_2)} \Phi_2)(a_1 \dots a_n) = 0$. □

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