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A Combinatorial Approach to Singularities of Normal Surfaces

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Abstract. In this paper we study generic coverings of $\mathbb{C}^2$ branched over a curve s.t. the total space is a normal analytic surface, in terms of a graph representing the monodromy of the covering, called monodromy graph. A complete description of the monodromy graphs and of the local fundamental groups is found in case the branch curve is $\{x^n = y^m\}$ (with $n \leq m$) and the degree of the cover is equal to n or $n-1$.

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1. – Introduction

In this paper we introduce a combinatorial approach to study normal singularities of complex analytic surfaces. In particular, since a normal surface has only isolated singularities (see, e.g., [Na]), we restrict our attention to germs (S, s) where S is a connected normal surface, $s \in S$ and $S \setminus s$ is non-singular. We begin by setting some standard notation.

A *normal generic covering* (S, π) (ngc in the sequel) is a finite holomorphic map $\pi : S \longrightarrow \mathbb{C}^2$ from a connected normal surface S to the complex plane $\mathbb{C}^2$, which is an analytic covering branched over a curve $B \subset \mathbb{C}^2$, such that the fiber over a smooth point of B is supported on $\deg\pi - 1$ distinct points. An ngc is called *smooth* if S is non-singular.

Two ngc's (S_1, π_1), (S_2, π_2) are called (analytically) *equivalent* if there exists an isomorphism $\phi : S_1 \to S_2$ such that $\pi_1 = \pi_2 \circ \phi$. In the sequel, we will consider equivalent ngc's to be the same covering. For instance, when we say "π is unique", we mean "π is unique up to equivalence".

The main interest in ngc's comes from the well-known fact that, by the Weierstrass preparation theorem, given an analytic surface $S \subset \mathbb{C}^n$, a generic

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projection $S \xrightarrow{\pi} \mathbb{C}^2$ is (at least locally, in order to insure $\deg\pi < \infty$) an ngc branched over a curve (see [GR]). Moreover, since over a non-singular point of B π is locally (in S) equivalent to the map of the complex plane to itself which takes (x, y) to (x^a, y) with $a = 1, 2$, we can restrict to the case in which the branch locus B has only one singular point (which we may assume to be the origin O). Namely, given a germ (S, s) as above, there exists an ngc (S, π) whose branch curve B is non-singular away from $\pi(s) = O$.

For a fixed curve B there are three natural problems related to ngc's: the *existence* problem (does there exist an ngc branched over B?); the *uniqueness* problem (under which hypothesis is the covering unique?); and the *smoothness* problem (does there exist a smooth ngc branched over B?). Moreover, if we allow the branch curve to vary, we have the *classification* problem, i.e. to find all pairs (B, d) for which there exists an ngc of degree d branched over B and find which couples correspond to smooth ngc's. Notice that for $\deg\pi = 2$ the problem is trivial, since there always exists a (unique) ngc of degree 2 branched over each curve B. Namely, if B has equation $f(x, y) = 0$, it is sufficient to consider the projection on the x, y-plane of the surface in $\mathbb{C}^3$ defined by the equation $z^2 = f(x, y)$.

A standard way to study ngc's is the following: given an ngc (S, π) with branch curve B, one defines the *monodromy* homomorphism $\rho : \pi_1(\mathbb{C}^2 \setminus B) \to \mathcal{S}_{\deg \pi}$ as the action of this fundamental group on the fiber of π over a fixed regular value. The "generic" condition means that for each geometric loop (i.e. a simple loop in $\mathbb{C}^2 \setminus B$ around a smooth point of the curve B) its monodromy is a transposition. We will call such a homomorphism a *generic monodromy* for $\pi_1(\mathbb{C}^2 \setminus B)$.

It is well-known that one can reconstruct the covering from the pair (B, ρ) (cfr. [GrRe]). However, despite the explicit construction, understanding the singularity of the covering in this way is very difficult (except in specific cases). It is, for example, still an open problem to classify all the possible pairs coming from smooth ngc's. Observe that monodromies of equivalent ngc's differ only by an inner automorphism of the symmetric group, so we will say that two monodromies $\rho_1, \rho_2 : \pi_1(\mathbb{C}^2 \setminus B) \to \mathcal{S}_d$ are *equivalent* if there exists $\sigma \in \mathcal{S}_d$ such that $\rho_1(\gamma) = \sigma\rho_2(\gamma)\sigma^{-1}$ for all $\gamma \in \pi_1(\mathbb{C}^2 \setminus B)$ and we have the bijection $\{$ ngc's branched over $B\}$/equivalence $\leftrightarrow$ $\{$ generic monodromies for $\pi_1(\mathbb{C}^2 \setminus B)\}$/equivalence.

In this paper we restrict to the case where the branch curve has (up to analytic equivalence) the equation $\{x^n = y^m\}$. Let us point out that, according to the Puisieux classification (see [BK]), this class of singularities is a natural first step for a complete classification and that much is known in the non-generic case (see [T1], T2]).

Our aim is to translate the problem into a combinatorial one by representing the equivalence class of the monodromy ρ of an ngc of degree d branched on the curve $B_{n,m} = \{x^n = y^m\}$ by a connected graph $\Gamma \in Gr_{d,n}$ called the *monodromy graph*, where $Gr_{d,n}$ is the set of (isomorphism classes of) graphs with d vertices and n labeled edges: if $\gamma_1, \dots, \gamma_n$ are geometric loops that generate $\pi_1(\mathbb{C}^2 \setminus B)$, after identifying $\mathcal{S}_d$ with the set of permutations of the vertices of the graph, then in Γ the edge labeled i connects the vertices exchanged by the transposition $\rho(\gamma_i)$. Observe that the monodromy graph does not carry all the information needed to reconstruct the covering: Γ has n edges, but we have lost m. For a connected $\Gamma \in Gr_{d,n}$, we say that m is *compatible* with Γ if Γ defines an ngc branched over $B_{n,m} = \{x^n = y^m\}$. Given a connected graph $\Gamma \in Gr_{d,n}$ and a compatible integer m, we can uniquely reconstruct the monodromy homomorphism, and hence, an ngc which has Γ as monodromy graph. We then have the bijection {ngc's of degree d branched over $B_{n,m}$}/equivalence $\leftrightarrow$ $\{(\Gamma, m) \in Gr_{d,n} \times \mathbb{N}$ such that Γ is connected and m is compatible with $\Gamma\}$.

In order to translate the problem into a combinatorial one we define some actions of the free group F_n on n generators on $Gr_{d,n}$ in terms of which we can decide if a given integer m is compatible with a given graph Γ. Exploiting this action, in [MP] we were able to give a complete classification of the ngc's branched over irreducible curves of type $\{x^n = y^m\}$ in terms of the monodromy graphs:

THEOREM 1.1. *The monodromy graphs of ngc's (S, π) of degree $d \geq 3$ branched over the curve $\{x^n = y^m\}$, with $(n, m) = 1$, are the following*:

1. *"Polygons" with d vertices, valence $\frac{n}{d}$ (or $\frac{m}{d}$) and increment j, with $(j, d) = 1$, $j < \frac{d}{2}$, $j(d-j)|m$ (resp. $j(d-j)|n$). Moreover, d must divide n (resp. m).*
2. *"Double stars" of type $(j, d-j)$ and valence $\frac{n}{j(d-j)}$ (or $\frac{m}{j(d-j)}$), with $(j, d) = 1$, $j < \frac{d}{2}$, $j(d-j)|n$ (resp. $j(d-j)|m$). Moreover, d must divide m (resp. n).*

Duality induced by fiber product with the map $\psi(x, y) = (y, x)$ takes graphs of type (i) to graphs of type (ii), and vice-versa.

(For the definition of polygons and double stars see [MP].)

In this paper we change the point of view: rather than imposing restrictions on the branch curve, we fix the degree of the covering and we classify all monodromy graphs associated to ngc's branched over $B_{n,m}$ of degrees n and $n-1$ (see below for exact statements). Using monodromy graphs and the Reidemeister-Shreier method, we are also able to compute a presentation of the local fundamental group of the total space of these ngc's, thus giving a complete answer to the smoothness problem in these cases.

We now describe the content of each section. In Section 2 we give the basic definitions of the combinatorial and algebraic setting. In Section 3 we define the monodromy graph associated to an ngc, we prove that the set of integers compatible with a given connected graph Γ is always non-empty –since there is one, called *standard*, which we can compute from Γ itself– and that this set is the positive part of an ideal in $\mathbb{Z}$, (hence, there is a minimum integer compatible with Γ). This implies that among all ngc's that have Γ as monodromy graph, there is a "minimal" one, and we give a way to construct all the others out of the minimal one. In Section 4 we give a complete classification of monodromy graphs in $Gr_{n+1,n}$ which define minimal non-standard coverings. The result is the following:

THEOREM 1.2. *A tree* $\Gamma \in Gr_{n+1,n}$ *defines a minimal non-standard ngc* $\iff$ $\exists\ \alpha : \alpha | n$ *and* Γ *is a coherently labeled* α*-centered graph. In this case, the minimum integer compatible with* Γ *is* $m = \frac{n}{\alpha}(n+1)$ *with* α *maximal (for the fixed labeling of* Γ*).*

For the definition of a coherently labeled α-centered graph see Definition 4.2. In Section 5 we give a complete classification of monodromy graphs in $Gr_{n,n}$ which define minimal non-standard coverings. The result is the following:

THEOREM 1.3. *A connected graph* $\Gamma \in Gr_{n,n}$ *defines a minimal non-standard ngc* $\iff$ $\exists\ \alpha, s : \alpha | n$, $(s, \alpha) = 1$ *and* Γ *is a s-coherently labeled* α*-ring. In this case, the minimum integer compatible with* Γ *is* $m = \frac{hk}{(h,k)} \frac{n}{\alpha}$ *with* α *maximal (for the fixed labeling of* Γ*).*

For the definition of an s-coherently labeled α-ring, h and k, see Section 5. In Section 6 we compute the local fundamental group of the surfaces associated to the graphs constructed in Sections 4 and 5, obtaining:

THEOREM 1.4. *Let* (S, π) *be an ngc branched over* $B_{n,m}$ *and* Γ *be its monodromy graph. If* Γ *is a tree then* S *is smooth. If* Γ *is a coherently labeled* α*-ring then the fundamental group of* $S \setminus \pi^{-1}(O)$ *is cyclic of order* $\frac{m}{hk}$.

(Here h and k have the same meaning as in Theorem 1.3.)

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2. – Graphs and homomorphisms to symmetric groups

In this section we introduce the basic definitions and notation that we will use throughout the paper.

Let V be a set with d elements. We denote by $Gr_{V,n}$ the set of graphs with n labeled edges, labeled $1, \dots, n$, and whose set of vertices is V, and by $Gr_{d,n}$ the set of isomorphism classes of graphs with d vertices and n labeled edges, labeled $1, \dots, n$. If $\Gamma \in Gr_{V,n}$, we denote its isomorphism class by $\Gamma_* \in Gr_{d,n}$. Observe that if $\Gamma_* = \Gamma'_*$ then the isomorphism between Γ and Γ' is given by $\tau \in \mathcal{S}(V)$, a permutation of the set V.

In what follows we will make no difference between an edge and its label and we will say "the edge a" for "the edge labeled a". For instance, if the edge l with the label i has vertices p and q we will write $l = \overline{p, q} = i$ indifferently.

Given a graph Γ, p a vertex of Γ and l an edge of Γ, let the *degree* or *valence* of p be the number of edges of Γ having p as an end point, and let the *valence* of l be the number of edges of Γ with the same end points as l. Call an *end* a vertex of valence 1, and a *leaf* an edge with an end as vertex.

If Γ' is a subgraph of Γ we set $\partial\Gamma' = \{v \text{ vertex of } \Gamma' | \exists\, l \text{ edge of } \Gamma, l \not\subset \Gamma' : v \in l\}$ and $\Gamma - \Gamma' = (\Gamma \setminus \Gamma') \cup \partial\Gamma'$, that is, we delete from Γ all edges in Γ' and all vertices in $\Gamma' \setminus \partial\Gamma'$. If p is a vertex of Γ of degree i, we denote by $\Gamma - p$ the graph obtained from $\Gamma \setminus \{p\}$ by adding i new vertices, one to each edge with p as end point.

A sequence $c = (j_i)_{i=1,\dots,l}$ of distinct edges of Γ such that the edge j_i intersects the edge j_{i+1} only in one vertex is called a *chain* in Γ of length l. If $l \geq 2$, we say that the vertex of j_1 (resp. j_l) not in common with j_2 (resp. j_{l-1}) is the *starting vertex* (resp. *ending vertex*) of c. If $l = 1$, then both vertices of j_1 are considered either starting or ending vertices of c. We also consider a single vertex as a trivial chain of length $l = 0$. A *p-chain* is a chain with p as starting vertex, while a *p, q-chain* is a p-chain with q as ending vertex.

DEFINITION 2.1. For a fixed index $1 \leq i \leq n$, we define the i-th action of $\mathcal{S}(V)$ on $Gr_{V,n}$ in the following way: for $\sigma \in \mathcal{S}(V)$ and $\Gamma \in Gr_{V,n}$, $\sigma_{(i)}(\Gamma)$ is the graph obtained from Γ by deleting the edge $i = \overline{p, q}$ and adding an edge with label i between the vertices $\sigma(p)$ and $\sigma(q)$.

It is easy to see that if $\Gamma_* = \Gamma'_*$ and if the isomorphism from Γ to Γ' is given by $\tau \in \mathcal{S}(V)$, then $\sigma_{(i)}(\Gamma)$ and $\tau^{-1}\sigma\tau_{(i)}(\Gamma')$ are again isomorphic via τ.

Let F_n be the free group on n generators $\overline{\gamma}_1, \dots, \overline{\gamma}_n$, then $Gr_{V,n}$ is in one–to–one correspondence with the set of homomorphisms $\overline{\varphi} : F_n \longrightarrow \mathcal{S}(V)$ such that $\overline{\varphi}(\overline{\gamma}_i)$ is a transposition for all $i = 1, \dots, n$. To see this, let $\Gamma \in Gr_{V,n}$ and define $\mathrm{Hom}(\Gamma) : F_n \longrightarrow \mathcal{S}(V)$ by setting $\mathrm{Hom}(\Gamma)(\overline{\gamma}_i) = (p, q)$ if the edge i of Γ has vertices $i = \overline{p, q}$. Conversely, given a homomorphism $\overline{\varphi} : F_n \longrightarrow \mathcal{S}(V)$ such that $\overline{\varphi}(\overline{\gamma}_i)$ is a transposition for all i, we define $\mathrm{Graph}(\overline{\varphi}) \in Gr_{V,n}$ by taking for each $i = 1, \dots, n$ an edge labeled i of vertices $i = \overline{p, q}$ where p and q are the points of V exchanged by the transposition $\overline{\varphi}(\overline{\gamma}_i)$. It is immediate to verify that $\mathrm{Graph}(\mathrm{Hom}(\Gamma)) = \Gamma$ and $\mathrm{Hom}(\mathrm{Graph}(\overline{\varphi})) = \overline{\varphi}$. For the sake of simplicity, we will denote $\mathrm{Hom}(\Gamma)(\overline{\gamma})$ by $\overline{\gamma}_\Gamma$; so, if $p \in V$, we will write $\overline{\gamma}_\Gamma(p)$ or simply $\overline{\gamma}(p)$ if no confusion arises.

REMARK. Since the only transitive subgroup of $\mathcal{S}(V)$ generated by transpositions is $\mathcal{S}(V)$ itself, then $\overline{\varphi}$ is surjective if and only if the associated graph $\mathrm{Graph}(\overline{\varphi})$ is connected. Also, observe that $(\mathrm{Graph}(\overline{\varphi}))_*$ represents the conjugacy class of $\overline{\varphi}$ modulo inner automorphisms of $\mathcal{S}(V)$, since for $\tau \in \mathcal{S}(V)$, the graph isomorphic to $\mathrm{Graph}(\overline{\varphi})$ obtained by permuting the vertices by τ corresponds to the homomorphism $\overline{\varphi}' : F_n \to \mathcal{S}(V)$ such that $\overline{\varphi}'(\overline{\gamma}) = \tau^{-1}\overline{\varphi}(\overline{\gamma})\tau$, for every $\overline{\gamma} \in F_n$.

For a fixed $\Gamma \in Gr_{V,n}$ we can make F_n act on V via the representation given by $\mathrm{Hom}(\Gamma)$ and composing the map Hom: $Gr_{V,n} \longrightarrow \mathrm{Hom}(F_n, \mathcal{S}(V))$ with the i-th action of $\mathcal{S}(V)$ we are able to define the i-th action of F_n on $Gr_{V,n}$.

Observe that if $\Gamma_* = \Gamma'_*$ and the isomorphism from Γ to Γ' is given by $\tau \in \mathcal{S}(V)$, then $\overline{\gamma}_\Gamma = \tau\overline{\gamma}_{\Gamma'}\tau^{-1}$, so that the i-th action of $\overline{\gamma}_\Gamma$ on Γ is isomorphic via τ to the i-th action of $\tau^{-1}\overline{\gamma}_\Gamma\tau = \overline{\gamma}_{\Gamma'}$ on Γ'. Thus both actions pass to $Gr_{d,n}$: we can make F_n act on the set of vertices of a fixed graph in $Gr_{d,n}$ and have:

DEFINITION 2.2. For $\overline{\gamma} \in F_n$ and $\Gamma \in Gr_{V,n}$ the formula

$$\overline{\gamma}_{(i)}(\Gamma_*) = ((\overline{\gamma}_\Gamma)_{(i)}(\Gamma))_*$$

defines the i-th action of F_n on $Gr_{d,n}$. If in $\overline{\gamma}_{(i)}(\Gamma_*)$ the edge i has the same vertices as the edge j we say that $\overline{\gamma}$ sends the edge i of Γ_* to the edge j of Γ_* and we write $\overline{\gamma}[i] = j$. Observe that, if $i = \overline{p,q}$ and $j = \overline{p',q'}$ are edges of Γ, then $\overline{\gamma}[i] = j$ if and only if $\{\overline{\gamma}(p), \overline{\gamma}(q)\} = \{p', q'\}$.

Let G be a group which admits a presentation of the following type:

$$G =< \gamma_1, \dots, \gamma_n | \ \gamma_{k_j} = T_j\gamma_{h_j}T_j^{-1} \ \ j = 1, \dots, m > \tag{2.3}$$

and fix such a presentation. Let $g : F_n \to G$ be the (generator) map such that $g(\overline{\gamma}_i) = \gamma_i$ for $i = 1, \dots, n$ and, if $\omega = \gamma_{r_1} \dots \gamma_{r_s}$ is a word in the γ's, then set $\overline{\omega} = \overline{\gamma}_{r_1} \dots \overline{\gamma}_{r_s}$. We call $\overline{\omega}$ the *verbal lifting* of ω.

DEFINITION 2.4. A *generic monodromy* for (the fixed presentation of) G is a homomorphism $\varphi : G \longrightarrow \mathcal{S}(V)$ such that $\varphi(\gamma_i)$ is a transposition for each $i = 1, \dots, n$. We say that two homomorphisms $\varphi_1, \varphi_2 : G \to \mathcal{S}(V)$ are *equivalent* if there exists $\tau \in \mathcal{S}(V)$ such that $\varphi_1(\gamma) = \tau\varphi_2(\gamma)\tau^{-1}$ for all $\gamma \in G$. Given a generic monodromy φ for G, the *monodromy graph* associated to φ is the graph Graph($\overline{\varphi}$), where $\overline{\varphi} = \varphi \circ g$ is the lifting of φ to F_n under the map g. Observe that the monodromy graph depends on the presentation of G (more specifically on the chosen set of generators).

Conversely, given a graph $\Gamma \in Gr_{V,n}$, it defines a generic monodromy $\overline{\varphi}$ =Hom(Γ) for F_n. $\overline{\varphi}$ factors through g if and only if, setting $\sigma_j = \overline{\varphi}(\overline{T}_j)$, $\overline{\varphi}(\gamma_{k_j}) = \sigma_j\overline{\varphi}(\gamma_{h_j})\sigma_j^{-1}$ for all j, that is, σ_j sends the vertices of the edge k_j of Γ to the vertices of the edge h_j for all j. This is exactly the condition for $\overline{T}_j$, the verbal lifting of T_j, to send the edge k_j of Γ_* to the edge h_j for all j. Remark also that the monodromy graphs of two equivalent generic monodromies for G are in the same isomorphism class in $Gr_{d,n}$, thus we have proved:

PROPOSITION 2.5. *A graph $\Gamma_* \in Gr_{d,n}$ represents the equivalence class of a generic monodromy $\varphi : G \longrightarrow \mathcal{S}(V)$ for G $\iff$ $\overline{T}_j[k_j] = h_j$ for all $j = 1, \dots, m$.*

In case $T_j = T$ does not depend on j, we have that $\overline{T}$ sends the edge k_j to the edge h_j for all j and if moreover $\{k_1, \dots, k_m\} = \{h_1, \dots, h_m\}$ then we can think of T (actually $\overline{T}$) as acting on the set of edges $\{k_1, \dots, k_m\}$ of the associated graph Γ (or of its isomorphism class Γ_*). Observe that this action respects incidence relation, i.e. if the edges k_i and k_j do not intersect (resp. have one vertex in common, resp. have the same end points) then the edges h_i and h_j do the same. In particular, all edges of the same $\overline{T}$-orbit have the same valence and if l is a leaf of Γ, then all edges in the same $\overline{T}$-orbit are leaves.

We end this section with a definition which will be used later. Let $\overline{T} = \overline{\gamma}_{i_1} \dots \overline{\gamma}_{i_r} \in F_n$ and let p be a vertex of the graph $\Gamma \in Gr_{V,n}$. Let $p_j = \overline{\gamma}_{i_1} \dots \overline{\gamma}_{i_j}(p)$, $(p_0 = p)$. If $p_j \neq p_{j+1}$, then the edge $i_{j+1} = \overline{p_j, p_{j+1}}$.

Definition 2.6. We define the *motion* of p under the action of $\overline{T}$ to be the sequence of edges $(l_1, l_2, \dots)$ where $l_j = i_k$ for $k = \min\{h \mid p_h \neq p_{j-1}\}$, $(l_0 = 0)$.

We also identify the motion of p with the oriented path described by the (ordered) union of the edges in the sequence. A similar definition may be given if p is a vertex of a graph in $Gr_{d,n}$. We will mainly use this definition in case $\overline{T} = \overline{T}_{h,k} = \overline{\gamma}_h \dots \overline{\gamma}_{h+k-1} \in F_n$, where the indices in the $\overline{\gamma}_i$ are taken to be cyclical mod$n$.

3. – Normal generic coverings and monodromy graphs

Let $B \subset \mathbb{C}^2$ be an algebraic curve and let P be a point not in B. It is well-known that the fundamental group $\pi_1(\mathbb{C}^2 \setminus B, P)$ of the complement of B admits a presentation in the form (2.3) where the γ's are geometric generators, i.e. simple loops around a smooth point of B (see [Mo]). If B is the branch curve of an ngc (S, π) of degree d, then $\pi_1(\mathbb{C}^2 \setminus B, P)$ acts on the fiber $V = \pi^{-1}(P)$ giving a surjective generic monodromy φ for $\pi_1(\mathbb{C}^2 \setminus B, P)$. Changing the base point or taking an equivalent ngc produces equivalent generic monodromies, so, by the bijection {ngc's branched over B}/equivalence $\leftrightarrow$ {generic monodromies for $\pi_1(\mathbb{C}^2 \setminus B)$}/equivalence (see introduction), in order to classify ngc's (S, π) of degree d branched over a curve B, we can classify the connected graphs in $Gr_{d,n}$ associated to their monodromies. Since we are interested in equivalence classes of generic monodromies, we can get rid of the base point and fix a bijection of V with the set $D = \{1, 2, \dots, d\}$ to obtain a homomorphism $\varphi : \pi_1(\mathbb{C}^2 \setminus B) \to \mathcal{S}_d = \mathcal{S}(D)$, well defined up to conjugation. Observe that if B has equation $\{f(x, y) = 0\}$, then the projection on the x, y plane of the surface S given in $\mathbb{C}^3$ by the equation $z^2 = f(x, y)$ exhibits S as an ngc of degree 2 branched over B, which corresponds to the unique homomorphism $\varphi : \pi_1(\mathbb{C}^2 \setminus B) \longrightarrow \mathcal{S}_2$ (such that $\varphi(\gamma) = (1, 2)$ for each geometric generator γ), or to the unique graph in $Gr_{2,n}$. So, in what follows suppose $d \geq 3$.

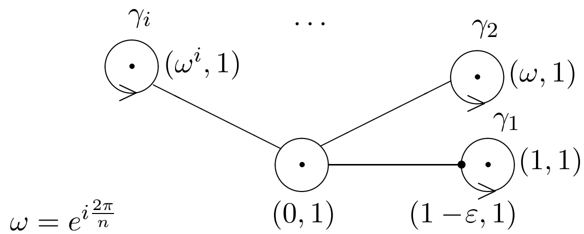
Let $G_{n,m}$ be the group presented by

$$G_{n,m} =< \gamma_1, \dots, \gamma_n | \ \gamma_j = T\gamma_{j+m}T^{-1} \ \ j = 1, \dots, n > \tag{3.1}$$

where $T = \gamma_1 \cdots \gamma_m$ and all indices are taken to be cyclical mod n. If $B = B_{n,m}$ is the curve of equation $\{x^n = y^m\}$ then (see [O], [MP])

$$\pi_1(\mathbb{C}^2 \setminus B) \cong G_{n,m}$$

where we have taken as generators the free basis of $\pi_1(\{y = 1\} \setminus B)$ shown in the picture, called *horizontal standard generators*. Indeed, we could also choose as generators the *vertical standard generators* in the plane $x = 1$ obtaining $\pi_1(\mathbb{C}^2 \setminus B) \cong G_{m,n}$. Notice that standard generators are geometric generators.



Unless explicitly stated otherwise, we will always use the presentation $\pi_1(\mathbb{C}^2 \setminus B_{n,m}) \cong G_{n,m}$ in terms of horizontal standard generators.

DEFINITION 3.2. Given an ngc (S, π) of degree d branched over the curve $B_{n,m}$, let $\varphi : \pi_1(\mathbb{C}^2 \setminus B_{n,m}) \to \mathcal{S}_d$ be its monodromy. We define the *monodromy graph* associated to π to be the isomorphism class in $Gr_{d,n}$ of the monodromy graph associated to φ considered as a generic monodromy for $G_{n,m}$.

Observe that, in $G_{n,m}$, T acts by conjugation on the set $\{\gamma_1, \dots, \gamma_n\}$ and we have (m, n) orbits, each with $\frac{n}{(m,n)}$ elements. Given a generic monodromy for $G_{n,m}$, $\varphi : G_{n,m} \to \mathcal{S}_d$, we have that (cfr. 2.5) T acts on the whole set of edges of the associated graph Γ, by sending the edge j to the edge $j + m$. Since the action of T on the edges of Γ respects incidence relations, if an edge j has end points of degree a and b, then each edge of the same orbit also has end points of degree a and b. More precisely, if the edge j intersects the edge i then the edge $j + km$ intersects the edge $i + km$ for each k.

DEFINITION 3.3. For a fixed Γ in $Gr_{d,n}$, we say that $h \in \mathbb{N}\setminus\{0\}$ is *compatible* with Γ if $\overline{T}_{1,h}[i] = i + h$ for each $i = 1, \dots, n$.

Since all geometric generators of $\pi_1(\mathbb{C}^2 \setminus B_{n,m})$ are conjugated to one of the γ_i's, we have the following:

PROPOSITION 3.4. *Given a graph $\Gamma \in Gr_{d,n}$ and an integer h compatible with Γ, then Γ defines a homomorphism $\varphi : \pi_1(\mathbb{C}^2 \setminus B_{n,h}) \to \mathcal{S}_d$ which maps geometric generators to transpositions. φ is unique up to conjugation.*

This proposition together with 2.5 gives that the pair (Γ, h), where $\Gamma \in Gr_{d,n}$ is connected and h is compatible with Γ, uniquely determines (up to equivalence) an ngc (S, π) of degree d branched over $B_{n,h}$ which has Γ as monodromy graph.

In case $n|m$ the presentation in 3.1 reduces to the unique condition that T is central. Then for each generic monodromy $\varphi : G_{n,m} \to \mathcal{S}_d$ ($d \geq 3$), we must have $\varphi(T) = 1$ and a homomorphism $\overline{\varphi} : F_n \to \mathcal{S}_d$ factors through $G_{n,m}$ if and only if $\overline{\varphi}(\overline{T}) = 1$. This allows us to remark that for every connected graph $\Gamma \in Gr_{d,n}$ the set of integers compatible with Γ is non-empty: indeed, $t = n \cdot o(\overline{\sigma})$ is compatible with Γ, where $\overline{\sigma}$ =Hom$(\Gamma)(\overline{\gamma}_1 \cdots \overline{\gamma}_n)$. We call the ngc associated to the pair (Γ, t) the *standard* ngc associated to Γ.

Consider now the map $f : \mathbb{C}^2 \to \mathbb{C}^2$, $f(x, y) = (\omega x, y)$, where $\omega = e^{i\frac{2\pi}{n}}$. Since $f(B_{n,m}) = B_{n,m}$, f induces an isomorphism

$$f_* : \pi_1(\mathbb{C}^2 \setminus B_{n,m}) \to \pi_1(\mathbb{C}^2 \setminus B_{n,m})$$

which acts on the horizontal standard generators as a cyclical permutation. Cyclically permuting the γ_i does not change the presentation of $G_{n,m}$, so, if we compose φ with f_*, we obtain another generic monodromy for $G_{n,m}$. So we will say that

DEFINITION 3.5. Two ngc's (S, π), (S', π') branched over $B_{n,m}$ are *cyclically equivalent* if (S', π') is (analytically) equivalent to the fiber product $S \times_{\mathbb{C}^2} \mathbb{C}^2$ obtained from (S, π) by base change with a power of f.

The monodromy homomorphisms $\varphi, \varphi' : \pi_1(\mathbb{C}^2 \setminus B_{n,m}) \longrightarrow \mathcal{S}_d$ of two cyclically equivalent ngc's are obtained one from the other by composing with a power of f_* (and an inner automorphism of $\mathcal{S}_d$) and the cyclical equivalence class of an ngc branched over $B_{n,m}$ contains at most (n, m) elements. Summing up we have:

PROPOSITION 3.6. *Let* (S, π), (S', π') *be ngc's with the same branch curve* $B_{n,m}$, *and* $\Gamma, \Gamma' \in Gr_{d,n}$ *their monodromy graphs.* (S, π), (S', π') *are cyclically equivalent* $\Rightarrow$ Γ, Γ' *differ for a cyclical permutation of the labels of the edges. Conversely, given two graphs* $\Gamma, \Gamma' \in Gr_{d,n}$ *which differ for a cyclical permutation of the labels of the edges* $\Rightarrow$ *they have the same set of compatible integers and the ngc's defined by* (Γ, m) *and* (Γ', m) (*with the same compatible integer*) *are cyclically equivalent.*

Since we know how to produce all the ngc's of a cyclical equivalence class provided we know one of them, given a graph $\Gamma \in Gr_{d,n}$, we can always suppose that a given edge of Γ has the label 1.

Let (S, π) be an ngc branched over $B_{n,m}$. In [MP] we proved that base change via the map $f_{a,b} : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ such that $f_{a,b}(x, y) = (x^a, y^b)$, gives $\pi' : S' \longrightarrow \mathbb{C}^2$ which is an ngc branched over the curve $\{x^{an} = y^{bm}\}$ and this defines a partial order on the set of ngc's branched over curves of type $\{x^n = y^m\}$. Call an ngc *minimal* if it cannot be induced by other coverings via one of these base changes. Moreover, if $\Gamma \in Gr_{d,n}$ is the monodromy graph associated to π, the graph $\Gamma' \in Gr_{d,an}$ associated to π' is obtained from Γ by adding $a-1$ edges labeled $i+n, \dots, i+(a-1)n$ with the same end points as the edge i, for all $i=1,\dots,n$. We call Γ' the *a-pullback* of Γ. Summing up we have:

PROPOSITION 3.7. *Let (S,π), (S',π') be ngc's as above and $\Gamma, \Gamma' \in Gr_{d,n}$ their monodromy graphs. S' is obtained from S by base change via the map $f_{a,b} \Rightarrow \Gamma'$ is the a-pullback of Γ. Conversely, Γ' is the a-pullback of $\Gamma \Rightarrow$ they have the same set of compatible integers and each ngc defined by Γ' is obtained from a ngc defined by Γ by base change with the map $f_{a,1}$.*

Notice that using base changes with the maps $f_{1,b}$ we immediately get that if m is compatible with Γ, then also bm is compatible with Γ for each $b \geq 1$.

PROPOSITION 3.8. *The set of integers compatible with a graph is the positive part of an ideal in $\mathbb{Z}$.*

PROOF. Let suppose that m, m' are compatible with the graph $\Gamma_* \in Gr_{d,n}$ with $m \leq m'$. Let $\varphi : G_{n,m} \to \mathcal{S}(V)$ and $\varphi' : G_{n,m'} \to \mathcal{S}(V)$ be the generic monodromies associated to the couples $(\Gamma, m), (\Gamma, m')$ respectively. Then, if $\gamma_1, \dots, \gamma_n$ and $\gamma'_1, \dots, \gamma'_n$ are the (horizontal standard) generators for the groups $G_{n,m}$ and $G_{n,m'}$ respectively, we have $\varphi(\gamma_j) = \varphi'(\gamma'_j) = \sigma_j$ for all j.

Alternative defining relations for $G_{n,m}$ are the following (see [MP])

$$\gamma_i \dots \gamma_{i+m-1} = \gamma_{i+1} \dots \gamma_{i+m}$$

and so, the transpositions σ_j satisfy

$$\sigma_i \dots \sigma_{i+m-1} = \sigma_{i+1} \dots \sigma_{i+m}$$

and

$$\sigma_i \dots \sigma_{i+m'-1} = \sigma_{i+1} \dots \sigma_{i+m'}$$

for all i. We have that

$$\sigma_i \dots \sigma_{i+m'+m-1} = \sigma_i \dots \sigma_{i+m'-1}(\sigma_{i+m'} \dots \sigma_{i+m'+m-1}) = \sigma_{i+1} \dots \sigma_{i+m'+m}$$

and

$$\sigma_i \dots \sigma_{i+m'-m-1} = \sigma_i \dots \sigma_{i+m'-1}(\sigma_{i+m'-m} \dots \sigma_{i+m'-1})^{-1} = \sigma_{i+1} \dots \sigma_{i+m'-m}$$

for all i. So, both $m'+m$ and $m'-m$ are compatible with Γ_* and hence, it is easy to see that (n,m) is compatible with Γ_*. □

Combining this with Proposition 3.6 we have

COROLLARIY 3.9. *Up to cyclical equivalence, each graph $\Gamma \in Gr_{d,n}$ defines a unique minimal ngc among all coverings that have Γ as monodromy graph, corresponding to the minimum integer compatible with Γ. All other ngc's defined by Γ are obtained from the minimal one by base change with the maps $f_{1,b}$ (again, up to cyclical equivalence).*

When in the sequel we will refer to "the" minimal ngc defined by a graph, we will refer to one of the elements in its cyclical equivalence class. Notice that if $(n,m) = 1$ then the minimal covering will actually be unique. In this case, a complete classification of monodromy graphs corresponding to ngc's branched over irreducible curves of type $\{x^n = y^m\}$ was achieved in [MP] (see the introduction for the statement of the results).

4. – Normal generic coverings branched over $x^n = y^m$ of degree $d = n + 1$

The aim of this section is to classify all graphs in $Gr_{d,n}$ associated to an ngc (S, π) branched over $B_{n,m}$ of degree $n + 1$ for $n \geq 2$. Such a graph Γ is a tree and it is easy to see that, in this case (cfr. 6.2), the surface S is smooth. Observe that, since Γ is a tree, the edges of the same $\overline{T} = \overline{T}_{1,m}$-orbit have at most one common point. Also, it is well-known that the permutation $\mathrm{Hom}(\Gamma)(\overline{\gamma}_1 \cdots \overline{\gamma}_n)$ is a $n + 1$-cycle, and we have:

Proposition 4.1. *The branch curve of the standard ngc defined by a tree* $\Gamma \in Gr_{n+1,n}$ *is* $B_{n,n(n+1)} = \{x^n = y^{n(n+1)}\}$.

Definition 4.2. Given $\alpha \geq 2$, a vertex p of a connected graph Γ is called an *α-center* if $\Gamma - p$ is given by α (possibly disconnected) forests $\Gamma_1, \ldots, \Gamma_\alpha$ each isomorphic to the others via isomorphisms which respect the distance of the vertices from p (where the distance is calculated in Γ and the new vertices have distance 0 from p).

The α-center of a graph, if it exists, is obviously unique and if p is the α-center of γ and $\beta|\alpha$, $\beta \geq 2$, then p is also the β-center of Γ. If Γ has a α-center, it is said to be *α-centered*. An α-centered graph $\Gamma \in Gr_{d,n}$ (notice that $\alpha|n$) is called *coherently labeled* if for each edge j in Γ_1 the corresponding edge in Γ_i is $j + \frac{n}{\alpha}(i - 1)$.

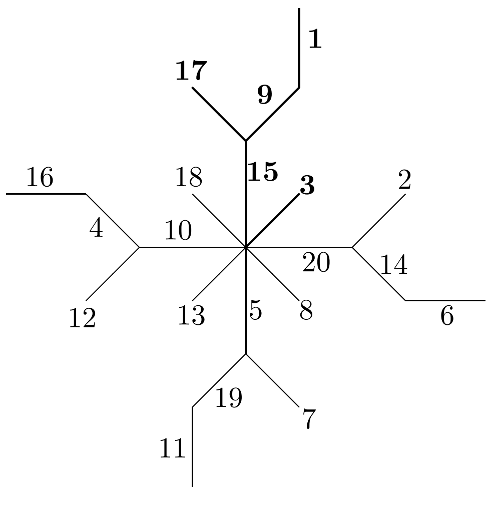


Fig. 1. A coherently labeled 4-centered tree (edges labeled in boldface form Γ_1).

Theorem 4.3. *Let* Γ *be a tree in* $Gr_{n+1,n}$ *and let m be the minimum integer compatible with* Γ. (Γ, m) *defines a non-standard minimal ngc branched over* $B_{n,m} \iff \exists\ \alpha|n$ *such that* Γ *is a coherently labeled* α*-centered graph. In this case*, $m = \frac{n}{\alpha}(n + 1)$ *with* α *maximal* (*for the fixed labeling of* Γ).

Proof. Suppose $\Gamma \in Gr_{n+1,n}$ is a tree associated to a non-standard minimal ngc branched over $B_{n,m}$ with monodromy $\varphi : G_{n,m} \to \mathcal{S}_{n+1}$. Since the covering is non-standard, then, considering the action of $\overline{T}$ on the edges of Γ, there are $\beta \neq n$ orbits, each containing $\alpha = \frac{n}{\beta} \neq 1$ elements.

Let a be a leaf of Γ. $\overline{T}[a]$ is another leaf of Γ and, since Γ is connected, there exists a unique chain $c = (j_i)_{i=1,\dots,l}$ which connects a to $\overline{T}[a]$, i.e., $j_1 = a$ and $j_l = \overline{T}[a]$. We claim that $\overline{T}[j_i] = j_{l-i+1}$ for $1 \leq i \leq \frac{l}{2}$, which implies that c has even length $l = 2k$, since no edge is fixed under the action of $\overline{T}$ ($\alpha \neq 1$). Indeed, if we assume by induction that $\overline{T}[j_i] = j_{l-i+1}$ for $1 \leq i \leq h-1$, if it were $\overline{T}[j_h] \neq j_{l-h+1}$, then $\overline{T}[j_h]$ and j_{l-h+1} would have only one common vertex, But then, if c_h is the subchain of c starting by j_h and ending by j_{l-h+1}, the chain $c_h \cup \overline{T}[c_h] \cup \ldots \cup \overline{T}^{\alpha}[c_h]$ would be a non-trivial loop in Γ (see figure below).

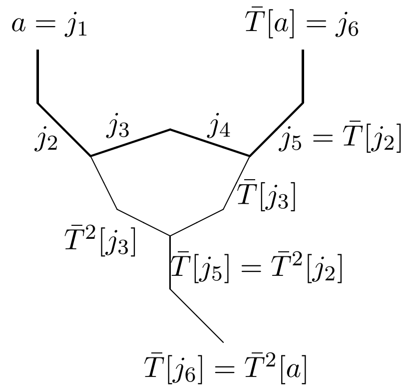


Fig. 2. $\bar{T}[j_3] \neq j_4$ is not possible (boldface edges form the chain c).

So, considering the edge j_k, we have that there exists an edge a of Γ such that a and $\overline{T}[a]$ have a common vertex p, hence p is a fixed point for the action of $\overline{T}$, that is, if an edge has p as vertex, then all edges in the same $\overline{T}$-orbit have p as vertex. Let A be a subset of all edges with p as vertex made up of one edge for each $\overline{T}$-orbit. Let Γ_1 be the sub-graph of Γ given by the union of all p-chains c such that the first edge of c is an element of A. Then, $\Gamma_i = \overline{T}^i[\Gamma_1]$, for $i = 1, \dots, \alpha - 1$, intersects Γ_1 in p and is isomorphic to Γ_1 via $\overline{T}^i$. Hence, p is a α-center of Γ and Γ is coherently labeled.

Suppose now that Γ is a coherently labeled α-centered graph in $Gr_{n+1,n}$ with α maximal for the fixed labeling. Write $n = \alpha\beta$ and let p be the α-center of Γ. Let $\Gamma_1, \dots, \Gamma_\alpha$ be the graphs in $\Gamma - p$ as in Definition 4.2, and let $\Gamma_{i,j}$ $1 \leq j \leq l$ be the connected components of Γ_i, where the enumeration is such that $\Gamma_{i,j}$ is isomorphic to $\Gamma_{i',j}$; let n_j be the number of edges of $\Gamma_{i,j}$ ($\sum_{j=1}^{l} n_j = \beta$). Let m be the minimum integer compatible with Γ. Then $\overline{T}_{1,m}$ acts on the edges of Γ sending the edge i to the edge $i+m$. Observe that p must be a fixed point for the action of $\overline{T}_{1,m}$, and that, since α is maximal, $\overline{T}_{1,m}$ must take $\Gamma_{i,j}$ to an isomorphic $\Gamma_{i',j}$. Moreover, $m \equiv k\beta \mod n$, i.e. $\overline{T}_{1,m}[i] = i + k\beta$. Since we can cyclically permute the edges, suppose 1 is the edge of $\Gamma_{1,1}$ having the center p as end point and let q be its other end point. We can also suppose that the edges of the other components $\Gamma_{1,j}$ of Γ_1 having p as end point have labels h_j such that $1 < h_2 < \ldots < h_l < 1+\beta$ and that $\Gamma_{2,1}$ has the edge $1+\beta$.

Now, the motion of p under the action of $\overline{T}_{1,N}$ for $N \gg 1$ is of the following type

$$(1, \ldots, 1, h_2, \ldots, h_2, h_3, \ldots, h_3, h_4, \ldots,)$$

i.e. p enters first $\Gamma_{1,1}$, then enters $\Gamma_{1,2}$ and so on, and its motion contains all edges of $\Gamma_{1,j}$ twice. This is because each $\Gamma_{1,j}$ is a tree and the permutation given by the product of its edges is a n_j+1 cycle. Moreover, $\overline{T}_{1,j}$ fixes p if $n_1 n + 1 \leq j < n_1 n + h_2$, $(n_1+n_2)n + h_2 \leq j < (n_1+n_2)n + h_3$, and so on.

The motion of q is similar: it leaves $\Gamma_{1,1}$ and enters $\Gamma_{1,2}$, $\Gamma_{1,3}$ and so on, eventually entering $\Gamma_{2,1}$; so, if q is sent by $\overline{T}_{1,j}$ to its corresponding point in $\Gamma_{2,1}$ then $j < \beta n + \beta + 1$ (recall that we want to find the minimum integer compatible with Γ).

Thus, for the edge 1 to be sent by $\overline{T}_{1,j}$ to the edge $1+\beta$ it must be $\beta n + h_l \leq j \leq \beta(n+1)$. We claim that, since $\beta | m$, $m = \beta(n+1)$.

We have to show that each vertex $q_i \in \Gamma_i$ is sent by $\overline{T}_{1,\beta(n+1)}$ to its corresponding vertex $q_{i+1} \in \Gamma_{i+1}$ (we already know that this happens for q). By the symmetry of Γ we have that $\overline{T}_{1,n}$ acts on Γ_1 in the same way as $\overline{T}_{1+(i-1)\beta,n}$ acts on Γ_i and in particular that $\overline{T}_{1,s\beta(n+1)}(p) = p$ for each s. Let k be the minimum integer such that $\overline{T}_{1,kn}(p) = q_i$. We have that

$$\overline{T}_{1+(i-1)\beta,kn-(i-1)\beta(n+1)}(p) = q_i$$

and

$$\overline{T}_{1,i\beta(n+1)-kn}(q_i) = p\,.$$

But then,

$$\overline{T}_{1+i\beta,kn-(i-1)\beta(n+1)}(p) = q_{i+1}$$

and $\overline{T}_{1,l}(q_i) = q_{i+1}$ for $l = kn - (i-1)\beta(n+1) + (i\beta(n+1) - kn) = \beta(n+1)$ as we claimed. □

Hence, the branch locus of the minimal ngc defined by a coherently labeled α-centered graph $\Gamma \in Gr_{n+1,n}$ is the curve $x^n = y^{\frac{n}{\alpha}(n+1)}$. Observe that in this case, the uniqueness problem has a negative solution since we can arbitrarily label one of the forests in the definition of an α-centered graph Γ (using labels which form a complete set of representatives for $\mathbb{Z}_\beta$) and label the other forests in such a way that Γ is coherently labeled, to obtain different ngc's branched over the same curve (even in different cyclical equivalent classes).

5. – Normal generic coverings branched over $x^n = y^m$ of degree $d = n$

The aim of this section is to classify all graphs in $Gr_{d,n}$ associated to an ngc (S, π) branched over $B_{n,m}$ of degree n for $n \geq 2$. Such a graph $\Gamma \in Gr_{n,n}$ is connected and has a unique simple loop c. If p is a vertex of c, we denote by Γ_p the (possibly empty) tree consisting of all p-chains in Γ having no edges in common with c.

LEMMA 5.1. *Given $\Gamma_* \in Gr_{n,n}$, let c be its only simple loop and let p be a vertex of c. The action of $\overline{T}_{1,n} = \overline{\gamma}_1 \cdots \overline{\gamma}_n$ on the vertices of Γ_* has two orbits. All vertices of Γ_* which belong to a fixed component of $\Gamma_p - p$ are in the same orbit.*

PROOF. Since for every $\sigma, \tau \in \mathcal{S}_d$, $\sigma\tau$ and $\tau\sigma$ have the same cyclical decomposition, a cyclical permutation of the edges of Γ will not affect the number of elements in the orbits of $\mathrm{Hom}(\Gamma)(\overline{T}_{1,n})$. Then, by a cyclical permutation of the edges, we can assume that 1 is an edge of the cycle c of Γ. In this case, $\Gamma - \{1\}$ is a maximal sub-tree of Γ and so, $\mathrm{Hom}(\Gamma)(\overline{\gamma}_2 \ldots \overline{\gamma}_n)$ is a n-cycle. Thus, the permutation $\mathrm{Hom}(\Gamma)(\overline{\gamma}_1 \cdots \overline{\gamma}_n)$, is the product of two cycles of lengths, say, h and k with $h + k = n$.

Notice that acting on $\Gamma_p - p$ by $\overline{T}_{1,n}$ is the same as acting on it by the ordered product of its edges. So, the second assertion again follows from the fact that the permutation associated to the ordered product of the edges of a tree is a cyclical permutation of all its vertices. □

Observe that, looking at the motion of a vertex a of Γ under the action of $\overline{T}_{1,N}$, a goes around c in a fixed direction (independent on N), which is shared with all h vertices in its $\overline{T}_{1,n}$-orbit, and that it makes a complete loop under the action of $\overline{T}_{1,j}$ only if $(h-1)n < j < (h+1)n$.

If c is oriented, we call the two $\overline{T}_{1,n}$-orbits the *positive* and *negative* orbits: a vertex of Γ belongs to the positive orbit if and only if the two orientations induced on some (and so all) edges of the motion of a under the action of $\overline{T}_{1,N}$ for $N \gg 1$ which belong also to c, are the same. We immediately have:

PROPOSITION 5.2. *Let h and k be the cardinality of the two $\overline{T}_{1,n}$-orbits of a graph $\Gamma \in Gr_{n,n}$, then $n = h + k$ and the standard covering defined by Γ has for branch locus the curve $B_{n,m}$ with $m = \frac{hk}{(h,k)} n$.*

DEFINITION 5.3. For three indices i, j, k we will say that $i < j < k$ *cyclically* if either $i < j < k$, or $j < k < i$, or $k < i < j$.

DEFINITION 5.4. In the notation of Lemma 5.1, fix an orientation of c and let a (resp. b) be a vertex of Γ in the positive (resp. negative) orbit. We say that a (resp. b) *enters* the component Γ' of $\Gamma_p - p$ if, cyclically $i < l < j$ (resp. cyclically $i < j < l$), where l is the edge of Γ' with p as vertex and i and j are the edges of c with p as vertex such that j follows i (resp. i follows j) in the orientation of c. Alternatively, a (resp. b) enters Γ' if the motion of a (resp. b) under the action of $T_{1,N}$ for $N \gg 1$ contains all the edges of Γ'.

We first fix our attention on a particular class of graphs.

DEFINITION 5.5. A graph $\Gamma \in Gr_{d,2\bar{n}}$ is said to be a *symmetric* graph if it is the union of two isomorphic sub-trees Γ_1 and Γ_2 which intersect (in Γ) in vertices which correspond under the isomorphism. Moreover, if the edge i is in Γ_1, then the corresponding edge in Γ_2 is the edge $i+\bar{n}$. Γ_1 and Γ_2 are called the two *halves* of Γ.

Notice that if there are t vertices of Γ belonging both to Γ_1 and Γ_2, then $\pi_1(\Gamma)$ is free of rank $t-1$, and that a 2-centered graph is symmetric.

As we will see, the minimal ngc defined by a symmetric graph is the standard one, so this class of graphs is not interesting for the purpose of this section. We start with a lemma.

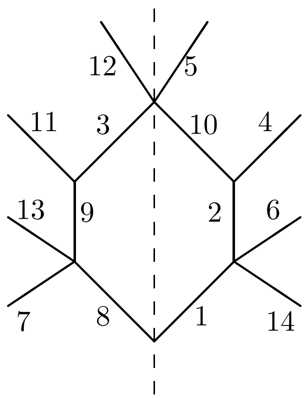


Fig. 3. A symmetric graph with 14 edges and 14 vertices.

LEMMA 5.6. *Each $\overline{T}_{1,2\bar{n}}$-orbit of a symmetric graph $\Gamma \in Gr_{2\bar{n},2\bar{n}}$ contains $\bar{n}$ elements.*

PROOF. Let Γ_1 and Γ_2 be the two halves of Γ as in Definition 5.5. Since Γ has the same number of edges and vertices, then there are exactly two vertices of Γ, p and q, which belong to both Γ_1 and Γ_2 and which divide the only simple loop c in Γ into two isomorphic parts. Give c the orientation that makes p belong to the positive $\overline{T}_{1,2\bar{n}}$-orbit.

Since if the edge i has p (resp. q) as vertex, then the edge $i+\bar{n}$ also has p (resp. q) as vertex, then if p enters a component Γ' of $\Gamma_p - p$ (resp. of $\Gamma_q - q$) it will not enter the corresponding component in the other half of Γ.

Let $a \neq p, q$ be a vertex of $\Gamma_1 \cap c$ (if any), i and j the edges of c which have a as vertex with j that follows i in the orientation of c, and $\alpha_1, \dots, \alpha_{h_a}$, $\beta_1, \dots, \beta_{k_a}$, $\delta_1, \dots, \delta_{l_a}$ the other edges of Γ which have p as vertex, such that $\alpha_1 < \dots < \alpha_{h_a} < \min\{i,j\} < \beta_1 < \dots < \beta_{k_a} < \max\{i,j\} < \delta_1 < \dots < \delta_{l_a}$. Number the components Γ'_τ of $\Gamma_a - a$ by the edge which has a as vertex. If $i < j$, then p will enter the component Γ'_τ of $\Gamma_a - a \iff \tau = \beta_1, \dots, \beta_{k_a}$, while if $j < i$, then p will enter the component Γ'_τ of $\Gamma_a - a \iff \tau = \alpha_1, \dots, \alpha_{h_a}$ or $\tau = \delta_1, \dots, \delta_{l_a}$.

Let b be the vertex of Γ_2 which corresponds to a, then the edges of c which have b as vertex are $i+\bar{n}$ and $j+\bar{n}$ and the other edges of Γ which have

b as vertex are $\alpha_1+\bar{n},\dots,\alpha_{h_a}+\bar{n}$, $\beta_1+\bar{n},\dots,\beta_{k_a}+\bar{n}$, $\delta_1+\bar{n},\dots,\delta_{l_a}+\bar{n}$. Then, if $i<j$, p will enter the component of Γ_b-b corresponding to Γ_τ if and only if $\tau=\alpha_1,\dots,\alpha_{h_a}$ or $\tau=\delta_1,\dots,\delta_{l_a}$, while if $j<i$, p will enter the component of Γ_b-b corresponding to Γ_τ if and only if $\tau=\beta_1,\dots,\beta_{k_a}$. By Lemma 5.11, p belongs to the same $\overline{T}_{1,2\bar{n}}$-orbit as exactly half of the vertices of Γ not in c.

Notice that a belongs to the same orbit as p if and only if $i>j$ and that b belongs to the same orbit as p if and only if $[j+\bar{n}]>[i+\bar{n}]$ where $[t]\equiv t \bmod 2\bar{n}$ and $0<[t]\le 2\bar{n}$. Thus, if each edge of $\Gamma_1\cap c$ has a lesser label than the corresponding edge in Γ_2, q is not in the same orbit as p and if $a\neq p,q$ is (resp. is not) in the same orbit as p, then b is not (resp. is) in the same orbit as p, and the result follows.

To conclude the proof, we will show that exchanging the labels of two corresponding edges of c yields a new symmetric graph with the same cardinalities of the $\overline{T}_{1,2\bar{n}}$-orbits. If c has only two edges, then there is nothing to prove, otherwise, let $\tilde{\Gamma}$ be the graph obtained from Γ exchanging the labels of the edges a and $a+\bar{n}$ of c. Notice that a cyclical permutation of the edges will leave the graph symmetric and will not change the cyclical decomposition of $\mathrm{Hom}(\Gamma)(T_{1,2\bar{n}})$ and hence the number of elements in the $\overline{T}_{1,2\bar{n}}$-orbits. Then, cyclically permuting the edges of Γ, we can suppose that $l=1$ is an edge of $\Gamma_1\cap c$, in which case the two vertices of l belong to distinct orbits. Suppose first that p or q is a vertex of 1. Then the two cases in figure below, where $1<i<\bar{n}$ and in which we have marked the vertices of the positive orbit by a solid dot, show that exchanging 1 with $\bar{n}+1$ does not alter the cardinality of the $\overline{T}_{1,2\bar{n}}$-orbits.

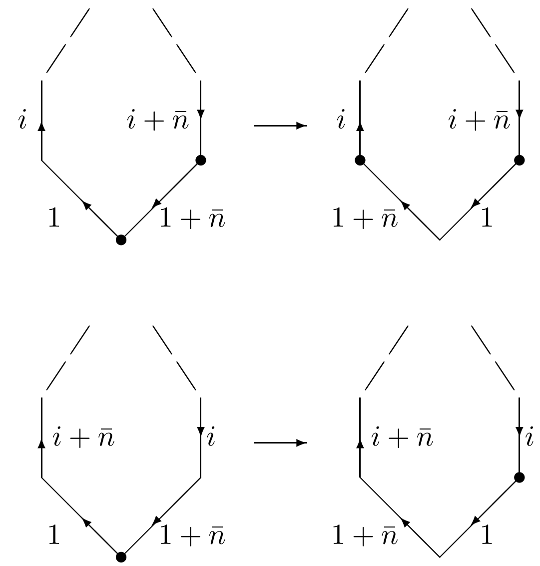


Fig. 4.

If neither p nor q is a vertex of 1, the four cases in the figure below, where $1 < i, j < \bar{n}$ and in which we again have marked the vertices of the positive orbit by a solid dot, conclude the proof. □

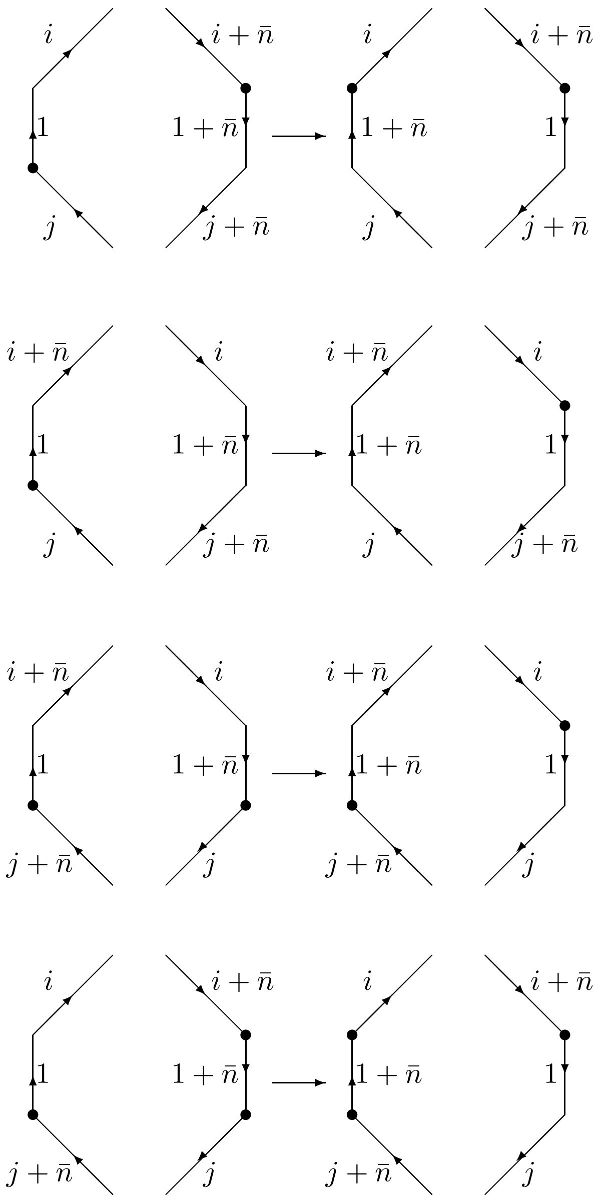


Fig. 5.

COROLLARY 5.7. *The standard ngc defined by a symmetric graph $\Gamma \in Gr_{n,n}$ has for branch locus the curve $x^n = y^m$ with $m = \frac{n^2}{2}$.*

We now state the main result concerning symmetric graphs.

LEMMA 5.8. *For a symmetric graph $\Gamma \in Gr_{n,n}$ with $n > 2$, the standard ngc is minimal.*

PROOF. Let p and q be the two vertices of Γ which belong to both halves of Γ and let c be the only simple loop in Γ. Let $x^n = y^m$ be the branch curve of the minimal ngc defined by Γ, i.e. m is the minimum integer compatible with Γ and suppose $m < \frac{n^2}{2}$. Then $\overline{T}_{1,m}$ acts on the edges of Γ sending the edge i to the edge $i+m$ and fixing c (as a set).

Since p and q are the only vertices of Γ with the property that the edge i has p (resp. q) as vertex if and only if the edge $i+\frac{n}{2}$ has p (resp. q) as vertex and the action of $\overline{T}_{1,m}$ preserves incidence relation, the set $\{p,q\}$ is fixed for the action of $\overline{T}_{1,m}$ on the vertices of Γ.

Suppose p is a fixed point for the action of $\overline{T}_{1,m}$. Then, the action of $\overline{T}_{1,m}$ must exchange the two edges of c which have p as vertex and this implies that $m \equiv \frac{n}{2} \mod n$. Cyclically permuting the edges of Γ so that the edge 1 is in c and has p as vertex, we have that if $\overline{T}_{1,j}$ fixes p then $j = an+i$ with $1+\frac{n}{2} \leq i \leq n$, so that it cannot be that $\overline{T}_{1,m}[1] = 1+\frac{n}{2}$.

On the other hand, if $\overline{T}_{1,m}(p) = q$, then $\overline{T}_{1,m}(q) = p$ and there are only two possibilities: c has only two edges, i.e., after a cyclical permutation of the edges, the edges 1 and $1+\frac{n}{2}$ have p and q as vertices and $m \equiv \frac{n}{2} \mod n$; or $n \equiv 0 \mod 4$, the two edges of c which have q as vertex are the edges $1+\frac{n}{4}$ and $1+\frac{3n}{4}$, and $m \equiv \frac{n}{4} \mod n$ or $m \equiv \frac{3n}{4} \mod n$.

In the first case, since $n > 2$, there is another edge a which has p as vertex, but then, since Γ is symmetric, the edge $a+\frac{n}{2}$ must have p as vertex, while, since $\overline{T}_{1,m}(p) = q$, the edge $a+\frac{n}{2}$ should have q as end point, thus this case cannot happen.

In the second case, consider the p,q-chain $c' = (l_i)_{i=1,\dots,s}$ in c such that $1 = l_1 \neq l_s$. We have that $\overline{T}^t_{1,m}[l_1] = l_s$ for some t, and then $\overline{T}^t_{1,m}[l_i] = l_{s-i+1}$ if $1 \leq i \leq \frac{s}{2}$. This implies that s is even, since no edge is fixed under the action of $\overline{T}^t_{1,m}$, and that the middle vertex of c' is fixed for the action of $\overline{T}^t_{1,m}$. This cannot happen since then the edges $l_{\frac{s}{2}}$, $l_{\frac{s}{2}}+\frac{n}{4}$, $l_{\frac{s}{2}}+\frac{n}{2}$, $l_{\frac{s}{2}}+\frac{3n}{4}$ will intersect in the middle vertex of c' which is different from p and q. □

We now turn to another class of graphs (which includes polygons, see the introduction).

DEFINITION 5.9. Given $\alpha \geq 2$, an *α-ring* is a graph $\Gamma \in Gr_{n,n}$ together with an orientation of its only simple loop c with the property that the set of r vertices of c may be partitioned into α sets each consisting of $\frac{r}{\alpha}$ consecutive vertices, $V_j = \{p_{1,j}, \dots, p_{\frac{r}{\alpha},j}\}$ $j = 1, \dots, \alpha$ (where the enumeration is coherent with the orientation of c), in such a way that $\Gamma_{p_{i,h}}$ is isomorphic to $\Gamma_{p_{i,k}}$ for all h and k, via an isomorphism which respects the distance of the vertices from c. If $\beta|\alpha$, $\beta \geq 2$, then Γ is also a β-ring.

A *block* of a α-ring is the union of a chain c' of length $\frac{r}{\alpha}$ in c, whose orientation is the same as that of c, and all Γ_p as p varies among the non-starting vertices of c'. The starting and ending vertices of c' are called the *starting* and *ending vertices* of the block.

A *block decomposition* of a α-ring is made by α isomorphic blocks which are disjoint if we remove from each block its starting vertex. Observe that each edge belongs to exactly one block in a block decomposition, thus, if a block contains β edges, $n = \alpha\beta$. Notice also that each vertex of c may be the starting vertex of a block.

Definition 5.10. A α-ring Γ is said to be *s-coherently labeled* if, given a block decomposition $B_1, \dots, B_\alpha$ of Γ, where the enumeration is coherent with the orientation of c, for each edge in B_1, say i, the corresponding edge in B_k is $i + (k-1)\beta s$ (cyclical indices mod n). Notice that it must be $(s, \alpha) = 1$ and if we reverse the orientation of c, then Γ is $(\alpha - 1)s$-coherently labeled.

We call a h-coherently labeled n-ring a polygon with n edges, valence 1 and increment h (see [MP] for a detailed definition of polygon).

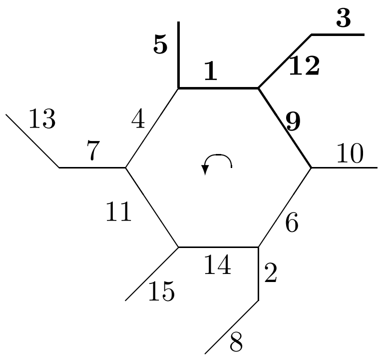


Fig. 6. A 2-coherently labeled 3-ring whit a block of 5 edges.

From Lemma 5.1 we immediately get:

Corollary 5.11. *Given a s-coherently labeled α-ring $\Gamma \in Gr_{n,n}$, let c be its only loop and p a vertex of c, then all vertices of a component Γ' of $\Gamma_p \setminus c$ and all the corresponding vertices in the other blocks* (*for any block decomposition of* Γ) *are in the same $\overline{T}_{1,n}$-orbit.*

Proof. Fix a block decomposition $B_1, \dots, B_\alpha$ of Γ such that Γ' is contained in B_1 and suppose there exists a vertex a in c in the positive $\overline{T}_{1,n}$-orbit which enters Γ'. Then, $i < l < j$ cyclically, where l is the edge of Γ' with p as vertex and i and j are the edges of c with p as vertex such that j follows i in the orientation of c. Since Γ is s-coherently labeled, the edges corresponding to i, j and l in the block B_k are $i' = i + (k-1)\beta s$, $j' = j + (k-1)\beta s$ and $l' = l + (k-1)\beta s$, so again $i' < l' < j'$ cyclically. Then a enters also all the trees corresponding to Γ' but in different blocks.

The same argument, with the appropriate modifications, may be applied in case there exists a vertex b in c in the negative orbit which enters Γ'. $\square$

Before stating the main theorem of this section, we prove two lemmas about s-coherently labeled α-rings. The first one gives some relations between α, s and the numbers h and k of elements in the positive and negative orbits. The second one will be used to compute the minimum compatible integer for such graphs.

LEMMA 5.12. *For a s-coherently labeled α-ring $\Gamma \in Gr_{n,n}$ we have*:

(i) $n = \alpha\beta = h + k$;
(ii) $h \equiv s \mod \alpha$ *(and* $k \equiv -s \mod \alpha$*) or* $k \equiv s \mod \alpha$ *(and* $h \equiv -s \mod \alpha$);
(iii) $(\alpha, h) = (\alpha, k) = 1$;
(iv) $(h, k) = (h, \beta) = (k, \beta) = (h, n) = (k, n)$.

PROOF. Let $n = \alpha\beta = h + k$ be as in the statement of Lemma 5.11. We prove (ii) by induction on β. If $\beta = 1$ then Γ is a polygon with valence 1 and increment s, and (ii) is trivial, since, in this case, $\alpha = n$ and $h = s$ or $k = s$.

Suppose $\beta \geq 2$. If we contract an edge of Γ together with all the other corresponding edges in the other blocks, namely the edges $a + j\beta s$ with $j = 1, \ldots, \alpha - 1$ for a fixed $1 \leq a \leq \beta$, we obtain an α-ring with $n' = n - \alpha = \alpha(\beta - 1)$ edges. If, moreover, we relabel the edges in such a way that the original order (of the edges) is preserved, we obtain a s-coherently labeled α-ring Γ'. Indeed, the edge $i \not\equiv a \mod \beta$ in Γ becomes the edge labeled $i - [\frac{i-a}{\beta}] + 1$ in Γ'.

If a is a leaf of Γ and the $\overline{T}_{1,n}$-orbit of its end has h elements, then by the Lemma 5.11 for Γ' we have that $k' = k$ and $h' = h - \alpha$. If Γ has no ends and a is an edge of the loop c, consider the two edges, i and j, which have a vertex in common with a such that i precedes a in the orientation of c. Contract the edge a: in case $i < a < j$ cyclically, the cardinality of the positive orbit does not change, while in case $j < a < i$ cyclically, the cardinality of the negative orbit does not change (see figure on the next page in which solid dots represent vertices in the positive orbit).

Since Γ is s-coherently labeled, only one of the two cases above will occur when contracting the edge a and all the other corresponding edges in the other blocks, so, one $\overline{T}_{1,n}$-orbit will not change, while the other will have α elements less. By the inductive hypothesis, we have $h - \alpha \equiv s \mod \alpha$ (and $k \equiv -s \mod \alpha$) or $k \equiv s \mod \alpha$ (and $h - \alpha \equiv -s \mod \alpha$) as we wanted.

Since $(s, \alpha) = 1$, (iii) follows from (ii) and since $(h, k) = (h, h + k) = (h, \alpha\beta)$, (iv) follows from (iii). $\square$

LEMMA 5.13. *Let $\Gamma \in Gr_{n,n}$ be an s-coherently labeled α-ring for which the edge* 1 *belongs to c and fix a block decomposition $B_1, \ldots, B_\alpha$ of Γ, where the*

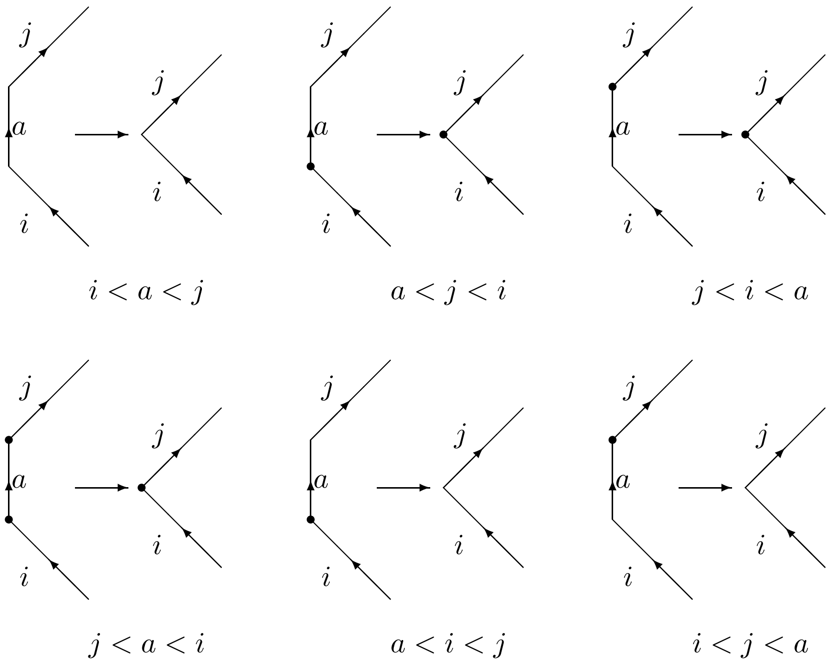


Fig. 7.

enumeration is coherent with the orientation of c *and* 1 *is the starting edge of* B_1. *Let* $\beta = \frac{n}{\alpha}$ *and let the positive orbit of vertices of* Γ (*containing the starting vertex of* B_1) *have* h *elements. Then, the action of* $\overline{T}_{1,ih\beta}$ *takes the starting vertex of* B_1 *to the starting vertex of* B_{i+1}.

PROOF. First notice that, since Γ is s-coherently labeled, it is sufficient to prove the lemma for $i = 1$, i.e. that $\overline{T}_{1,h\beta}$ takes the starting vertex of B_1 to the starting vertex of B_2. We prove this by induction on β.

If $\beta = 1$ then Γ is a polygon with valence 1 and increment s, and the result is trivial.

If $\beta \geq 2$, let a be a leaf of Γ with $1 < a \leq \beta$. Contracting a together with all its corresponding edges in the other blocks and renumbering the edges respecting the original order, we get a s-coherently labeled α-ring Γ' with $n' = \alpha(\beta - 1)$ edges.

If the end of a belongs to the positive orbit, then $h > \alpha$ and by induction, we have that the starting vertex of B_1' is sent by $\overline{T}_{1,(\beta-1)(h-\alpha)}$ to the starting vertex of B_2'. Comparing the motion of the two points, we notice that the action of $\overline{T}_{1,j}$ on the starting vertex of B_1' is the same as the action of $\overline{T}_{1,j+[\frac{j-a}{\beta}]+1}$ on the starting vertex of B_1 for $j = 1, \ldots, \bar{j}-1$, where $\bar{j}$ is such that $\overline{T}_{1,\bar{j}+[\frac{\bar{j}-a}{\beta}]+1}$ takes the starting vertex of B_1 to the end of a ($\bar{j} + [\frac{\bar{j}-a}{\beta}] + 1 \equiv a \bmod \alpha\beta$).

Observe that $\overline{T}_{1,\bar{j}+[\frac{\bar{j}-a}{\beta}]+1+i}$ will again take the starting vertex of B_1 to the end of a if $1 \le i < \alpha\beta$ and that the action of $\overline{T}_{1,j}$ on the starting vertex of B_1' for $\bar{j} > j \ge (\beta-1)(h-\alpha)$ is the same as the action of $\overline{T}_{1,j+[\frac{j-a}{\beta}]+1+\alpha\beta}$ on the starting vertex of B_1. In particular if $j = (\beta-1)(h-\alpha)$ we have

$$j + \left[\frac{j-a}{\beta}\right] + 1 + \alpha\beta = h\beta + \left[\frac{\alpha-h-a}{\beta}\right] + 1 \le h\beta\,,$$

thus, each $\overline{T}_{1,h\beta-i}$ with $[\frac{\alpha-h-a}{\beta}]+1 \le i \le 0$ takes the starting edge of B_1 to the starting edge of B_2.

If the end of a belongs to the negative orbit in Γ, then the negative orbit in Γ' contains $k-\alpha = n-h-\alpha$ elements and by induction, we have that the starting vertex of B_1' is sent by $\overline{T}_{1,(\beta-1)h}$ to the starting vertex of B_2'. Comparing the motion of the two points, we notice that the action of $\overline{T}_{1,j}$ on the starting vertex of B_1' is the same as the action of $\overline{T}_{1,j+[\frac{j-a}{\beta}]+1}$ on the starting vertex of B_1 for $j = 1, \dots, (\beta-1)h$. In particular if $j = (\beta-1)h$ we have

$$j + \left[\frac{j-a}{\beta}\right] + 1 = h\beta + \left[\frac{-h-a}{\beta}\right] + 1 \le h\beta\,,$$

thus, each $\overline{T}_{1,h\beta-i}$ with $[\frac{-h-a}{\beta}]+1 \le i \le 0$ takes the starting edge of B_1 to the starting edge of B_2.

In case Γ has no leaves, we can repeat the same argument taking an edge a of the loop c and examining all cases as in the proof of the previous lemma. □

We are ready to state the main theorem of this section.

THEOREM 5.14. *Let Γ be a graph in $Gr_{n,n}$ and let m be the minimum integer compatible with Γ. (Γ, m) defines a non-standard minimal ngc branched over $B_{n,m}$ $\iff$ $\exists\ \alpha, s : \alpha|n$ and $(s,\alpha) = 1$ such that Γ is a s-coherently labeled α-ring. In this case, $m = \frac{hk}{(h,k)}\frac{n}{\alpha}$ with α maximal (for the fixed labeling).*

PROOF. Suppose $\Gamma \in Gr_{n,n}$ is a graph associated to a non-standard minimal ngc branched over $B_{n,m}$ with monodromy $\varphi : G_{n,m} \longrightarrow \mathcal{S}_n$ corresponding to a non-standard ngc. Since the covering is non-standard, then, considering the action of $T = \overline{T}_{1,m}$ on the edges of Γ, there are $\beta \ne n$ orbits, each containing $\alpha = \frac{n}{\beta} \ne 1$ elements.

Fix an orientation for the simple loop c in Γ and let a be an edge of c. Since $T[c] = c$, $T^j[a]$ is again an edge of c for all j: let $r = T^{\bar{j}}[a]$ be the one that immediately follows a in the orientation of c. Let $c' = (l_i')_{i=1,\dots,q'-1}$ be the chain in c, oriented as c, such that $l_1' = a$ and having only one vertex in common with r. Let $l_{q'}' = r$.

Suppose $T^{\bar{j}}[c']$ intersects c'. Then it must be $T^{\bar{j}}[l_i'] = l_{q'-i+1}'$ and, since no edge is fixed under the action of T, c' must have odd length $2b-1$ and the common vertex p' of l'_b and l'_{b+1} must be fixed for the action of T. This in turn implies that $\frac{n}{(n,m)} = 2$, because no edge with p' as vertex other then l'_{b+1} can be in the same T-orbit as l'_b since there is only one simple loop in Γ. Write $n = 2\bar{n}$. The same will be true for the chain $c'' = (l''_i)_{i=1,\dots,q''-1}$ in c, oriented as c, such that $l''_1 = r$ and having only one vertex in common with a, so c contains two points p' and p'' which are fixed for the action of T and is the union of two p', p''-chains $C' = (r'_i)_{i=1,\dots,L}$ and $C'' = (r''_i)_{i=1,\dots,L}$ such that $r''_i = r'_i + \bar{n}$ and $T[C'] = C''$, $T[C''] = C'$. Moreover, if $p'_1, \dots, p'_t$ (resp. $p''_1, \dots, p''_t$) are the vertices of C' (resp. C'') numbered following the orientation of C' (resp. C''), then the trees $\Gamma_{p'_i}$ and $\Gamma_{p''_i}$ are isomorphic for $2 \geq i \geq t-1$ via an isomorphism that respects the distance of the vertices from c and if the edge a is in $\Gamma_{p'_i}$, then the corresponding edge in $\Gamma_{p''_i}$ is $a + \bar{n}$. Also, the tree $\Gamma_{p'}$ (resp. $\Gamma_{p''}$) is the union of two isomorphic trees $\Gamma^1_{p'}$ and $\Gamma^2_{p'}$ (resp. $\Gamma^1_{p''}$ and $\Gamma^2_{p''}$) via an isomorphism that respects the distance of the vertices from c and if the edge a is in $\Gamma^1_{p'}$ (resp. in $\Gamma^1_{p''}$) then the corresponding edge in $\Gamma^2_{p'}$ (resp. $\Gamma^2_{p''}$) is $a+\bar{n}$. Summing up, Γ is a symmetric graph in which one half is given by the union of C', $\Gamma^1_{p'}$, $\Gamma^1_{p''}$ and all Γ_p for p a vertex of C' other than p' and p'', and by Lemma 5.8 the standard ngc is the minimal one.

So we have that $T^{\bar{j}}[c']$ does not intersect c', $c = \cup_{i=0}^{\alpha-1} T^i[c']$, and if we define B to be the union of c' and all chains having no edges in common with c and starting from a vertex of c' except the starting vertex, then $\Gamma = \cup_{i=0}^{\alpha-1} T^i[B]$ and this is a block decomposition of a α-ring. Moreover, the powers of T give us the isomorphisms between the blocks as required in the Definition 5.9 and, checking the labels of the edges of different blocks, we have that Γ is s-coherently labeled for $s = \frac{r-a}{\beta}$.

Suppose now that Γ is a s-coherently labeled α-ring in $Gr_{n,n}$ with α maximal for the fixed labeling. Let c be the simple loop of Γ, $\beta = \frac{n}{\alpha}$, h be the number of vertices of Γ in the positive orbit, $k = n - h$ be the number of vertices of Γ in the negative orbit and fix a block decomposition $B_1, \dots, B_\alpha$ of Γ. After a cyclical permutation of the edges, we may suppose that the edge 1 is the starting edge of B_1. Let $1 = \overline{p, q}$, where p is the starting vertex of the block B_1. Let $x^n = y^m$ be the branch curve of the minimal ngc defined by Γ, i.e. m is the minimum integer compatible with Γ. Then $\overline{T}_{1,m}$ acts on the edges of Γ sending the edge i to the edge $i+m$ and fixing c (as a set). Write $m = t\beta + r$ with $0 \leq r < \beta$. If $r \neq 0$, then let r' be the edge of B_1 corresponding to the edge r, c' be the chain in c, oriented as c, starting from the edge 1 and having only one point in common with r', and B'_1 be the union of c' together with all Γ_p for p a non-starting edge of c'. Then, the sets $T^j_{1,m}[B'_1]$ give a block decomposition of Γ as a α'-ring in which each block has r elements, so that $\alpha' > \alpha$. Since α is maximal, we have that $m \equiv 0 \bmod \beta$. By the Lemma 5.13, m must be a multiple of $h\beta$, and exchanging h with k, m must be a multiple of $k\beta$ too, so we have that $\frac{hk}{(h,k)}\beta | m$.

We claim that $m = \frac{hk}{(h,k)}\beta$. Indeed, by Lemma 5.13, p is sent by $\overline{T}_{1,lh\beta}$ to the first vertex of the block B_{l+1}, while exchanging h with k, q is sent by $\overline{T}_{1,l'k\beta}$ to its corresponding vertex in the block $B_{-l'}$, where the indices of the blocks are taken to be cyclical (modα). Then the edge 1 will be sent to the first edge of a block by $\overline{T}_{1,j}$ for $j = lh\beta = l'k\beta$ such that $l + 1 \equiv -l' + 1 \mod \alpha$. Now, since $h + k = \alpha\beta$ and from Lemma 5.12 $(h,k) = (h,\beta)$, we have that $\alpha | \frac{h+k}{(h,k)}$, and so $\alpha\frac{\beta}{(h,\beta)} = \frac{h+k}{(h,k)}$. Thus, $\overline{T}_{1,\frac{hk}{(h,k)}\beta}$ sends the edge 1 to the first edge of the block $B_{\frac{k}{(h,k)}+1}$ (recall that by Lemma 5.12 $h \equiv s \mod \alpha$), which is the edge $1 + \frac{hk}{(h,k)}\beta$.

Let p' be a vertex of the block B_f in the positive orbit and write $\frac{hk}{(h,k)}\beta = ln + r$ with $0 < r < n$ (recall that by Lemma 5.12 $(hk,\alpha) = 1$). Let $\bar{j}$ be the minimum index among all indices j for which $\overline{T}_{1,jn}$ sends p to p'. Then $\overline{T}^{-1}_{1,\bar{j}n}\overline{T}_{1,\frac{hk}{(h,k)}\beta} = \overline{T}_{1,(l-\bar{j})n+r}$ sends p to the first vertex of the block $B_{\frac{k}{(h,k)}+1}$. Notice that, since Γ is an s-coherently labeled α-ring, acting on the first vertex of the block B_j by $\overline{T}_{1+js\beta,i}$ we get the vertex in the block B_{t+j-1} corresponding to $\overline{T}_{1,i}(p) \in B_t$, thus, $\overline{T}_{1,(l-\bar{j})n+r}\overline{T}_{1+\frac{hk}{(h,k)}\beta,\bar{j}n} = \overline{T}_{1,ln+r}$ will take p' to its corresponding vertex in the block $B_{\frac{k}{(h,k)}+f}$. Since the same will be true if p' is a vertex in the block B_f in the negative orbit (reverse the orientation of c, choose a block decomposition in which 1 is the starting edge of a block and exchange h with k before applying the same argument) then $m = \frac{hk}{(h,k)}\beta$ as we wanted. □

Hence, the branch locus of the minimal ngc defined by a coherently labeled α-ring $\Gamma \in Gr_{n,n}$ is the curve $\{x^n = y^{\frac{hk}{(h,k)}\frac{n}{\alpha}}\}$. Observe that, as in the case of α-centered graphs, the uniqueness problem has a negative solution since we can arbitrarily label one block of an α-ring Γ (using labels which form a complete set of representatives for $\mathbb{Z}_\beta$) and label the other blocks in such a way that Γ is coherently labeled to obtain different ngc's branched over the same curve (even in different cyclical equivalent classes).

6. – Local fundamental groups

Let $\pi : S \to \mathbb{C}^2$ be an ngc branched over $B = B_{n,m}$ and let $\varphi : \pi_1(\mathbb{C}^2 \backslash B) \to \mathcal{S}_d$ be its monodromy and $\Gamma \in Gr_{d,n}$ the associated monodromy graph. Notice that away from $P = \pi^{-1}((0,0))$ S is smooth, thus, in order to see if S is singular we must check whether $\pi_1(S \setminus \{P\})$ is trivial or not (see [Mu]). Consider $\pi_{|_{S\setminus D}} : S \setminus D \longrightarrow \mathbb{C}^2 \setminus B$, where $D = \pi^{-1}(B)$ and $D = 2R + C$, as a divisor in $\mathrm{Pic}(S)$ (R is the ramification locus of π). This is an unramified covering, and we can identify $\pi_1(S \setminus D)$ with the subgroup of $\pi_1(\mathbb{C}^2 \setminus B)$ given by those elements γ such that $\varphi(\gamma)(j) = j$ for j a fixed index, or (which is the same) by the subgroup of those γ such that the verbal lifting $\overline{\gamma}$ stabilizes a fixed vertex of Γ.

Let Γ' be a maximal sub-tree in Γ and let p be a vertex of Γ. If $c = (k_j)_{j=1,\dots,l}$ is a p-chain in Γ', then set $\gamma_c = \gamma_{k_1} \dots \gamma_{k_l}$, and if c is the trivial p-chain, then set $\gamma_c = id$. Notice that there are exactly d p-chains in Γ'. Then, the set of all γ_c for c a p-chain in Γ' is a complete set of representatives for left cosets of the stabilizer of the vertex p: indeed, if c is a p, q-chain in Γ' then $\overline{\gamma}_c(p) = q$. Thus, to calculate $\pi_1(S \setminus D)$, we can apply the Reidemeister-Shreier method (see [MKS]) to the Shreier set $RS = \{\gamma_c |\ c \text{ is a p} - \text{chain in } \Gamma'\}$ to obtain:

Proposition 6.1. *$\pi_1(S \setminus \{P\})$ is generated by $\eta_{c,k} = \gamma_c \gamma_k \gamma_{c'}^{-1}$ for c a p-chain in Γ', k an edge of $\Gamma - \Gamma'$ which intersects c in its ending vertex and is not the last edge of c, and where $\gamma_{c'} \in RS$ is in the same left coset as $\gamma_c \gamma_k$. Moreover we have $\eta_{c,k} = \eta_{c',k}^{-1}$.*

Proof. $\pi_1(S \setminus D)$ is generated by $\eta_{c,k}$ for c a p-chain in Γ', k an edge of Γ such that $c \cup \{k\}$ is not a 1-chain in Γ' and is defined by the relators $\gamma_c R \gamma_c^{-1}$ (rewritten in terms of the η's) where c is a p-chain in Γ' and R is a relator of $\pi_1(\mathbb{C}^2 \setminus B)$. Now, to obtain a presentation for $\pi_1(S \setminus \{P\})$, we must quotient by the normal subgroup generated by all loops around the components of $D = 2R + C$. The loops around the components of C are those $\eta_{c,k}$ with k an edge not through the ending vertex of c, while the loops around the components of R are those $\eta_{c,k}$ with k equal to the last edge of c and the loops $\eta_{c,k}\eta_{c',k}$ in case k is an edge of Γ such that $c \cup \{k\}$ is not a p-chain in Γ'. □

Notice that $\pi_1(S \setminus \{P\})$ has $n - d + 1$ generators, so we have:

Corollary 6.2. *If Γ is a tree, S is smooth.*

By a result in [MP] an ngc branched over $B_{n,bn}$ is smooth if and only if the monodromy graph is a tree, so we have:

Proposition 6.3. *Let $\Gamma \in Gr_{d,n}$. The standard ngc defined by Γ is smooth if and only if Γ is a tree.*

We want now to compute the local fundamental group of the surface of the ngc defined by a s-coherently labeled α-ring. The case $\alpha = n$ of a polygon was analyzed (partially) in [MP] using an algebraic approach. In this section we will treat the general case using an argument based on the monodromy graph.

We recall the results in [MP]. Let $\pi : S \longrightarrow \mathbb{C}^2$ be the ngc branched over $B_{n,m}$ constructed from the pair ("polygon with n vertices, valence 1 and increment α", $b\alpha(n - \alpha)$).

Theorem 6.4. *With the above notation, if $\alpha > 1$ $\pi_1(S \setminus \{P\}) = \mathbb{Z}/b\mathbb{Z}$.*

Consider now the case in which $\alpha = 1$. In order to compute the local fundamental group of the surface S, it is more convenient to express the monodromy graph in terms of the standard vertical generators, or to look at the ngc

S' obtained from S by base change through the map $\psi(x, y) = (y, x)$ (which is therefore isomorphic to S). It is easy to see (cfr. [MP]) that the monodromy graph Γ' associated to S' is the b-pullback of a $\beta = n - 1$-centered tree in $Gr_{n,n-1}$. This is what we called in [MP] a double star of type $(1, \beta)$ and valence b. Since if $b = 1$, Γ' is a tree, this immediately gives us:

PROPOSITION 6.5. *If $\alpha = b = 1$ then S is smooth.*

If $b > 1$ then we may take the maximal subtree formed by the edges $1, \dots, \beta$ and their common vertex p to compute a set of generators for the stabilizer of p. Namely, using the Shreier set for left cosets $L_i = \gamma_i$ for $i = 0, \dots, \beta$ $(L_0 = 1)$, we get that $\pi_1(S \setminus D)$ is generated by

$$A_{i,j} = \gamma_i \gamma_{i+j\beta} \quad B_{i,j} = \gamma_{i+j\beta}\gamma_i^{-1}$$

for $i = 1, \dots, \beta$ and $j = 0, \dots, b - 1$ and

$$C_{i,j} = \gamma_i \gamma_j \gamma_i^{-1}$$

for $i = 1, \dots, \beta$ and $1 \le j \le (b-1)\beta$, $j \not\equiv i (\text{mod } \beta)$. In $\pi_1(S \setminus \{P\})$ we have the relations

$$C_{i,j} = A_{i,0} = B_{i,0} = B_{i,j}A_{i,j} = 1$$

for all i and j, from which we have

$$B_{i,j} = A_{i,j}^{-1}.$$

To these relations we must add the relations that come from rewriting the relations of $\pi_1(\mathbb{C}^2 \setminus B)$. Observe that a set of defining relations for $G_{m,n}$ (notice that we have exchanged n and m) are the following

$$T_{1,n} = \gamma_1, \dots, \gamma_n = T_{h,n} = \gamma_h \dots \gamma_{h+n-1}$$

for each $h = 2, \dots, n$ (see [MP]). Thus we have to rewrite the relators

$$L_i T_{1,\beta+1} T_{h,\beta+1}^{-1} L_i^{-1}$$

for $i = 0, \dots, \beta$ $h = 1, \dots, b\beta$, in terms of the generators $A_{i,j}$.

THEOREM 6.6. *With the above notation, if $\alpha = 1$ $\pi_1(S \setminus \{P\}) = \mathbb{Z}/b\mathbb{Z}$.*

PROOF. Let's apply the rewriting process to the relation $T_{1,\beta+1} = T_{h+s\beta,\beta+1}$ where $1 \le h \le \beta$ and $0 \le s \le b - 1$. Rewriting the first half we get $T_{1,\beta+1} = C_{1,2}C_{1,3} \dots C_{1,\beta}A_{1,1}$, while rewriting the second half we get $T_{h+s\beta,\beta+1} = DA_{h,s}^{-1}A_{h,s+1}D'$ where D and D' are words in the $C_{i,j}$ only, and where we set $A_{i,b} = 1$ $\forall i$. Deleting all $C_{i,j}$'s we obtain

$$A_{i,j} = A_{1,1}^{j} \tag{6.7}$$

for each $j = 1, \dots, b-1$ and

(6.8) $$A_{1,1}^b = 1\,.$$

Now we apply the rewriting process to the relations

$$L_i T_{1,\beta+1} L_i^{-1} = L_i T_{h+i+s\beta,\beta+1} L_i^{-1}$$

for $i = 1, \dots, \beta - 1$, where $1 \le h \le \beta$ and $0 \le s \le b-1$ to get (after deleting all C_{ij}'s)

$$1 = A_{i,s+1} A_{i+1,s+1}^{-1}$$

which are a consequence of 6.7.

The remaining relations are obtained by rewriting the relations

$$L_n T_{1,\beta+1} L_n^{-1} = L_n T_{h+s\beta,\beta+1} L_n^{-1}$$

for $1 \le h \le \beta$ and $0 \le s \le b-1$ which yield (after deleting the $C_{i,j}$'s)

$$A_{1,1}^{-1} = A_{n,s} A_{1,s+1}^{-1}$$

which also are a consequence of 6.7 if $s \ne b-1$ and of 6.8 if $s = b-1$.

Using the relations 6.7 to delete all generators $A_{i,j}$ if i or $j > 1$, we are left with only one generator, $A_{1,1}$, of order b as we wanted. □

Let $\pi : S \longrightarrow \mathbb{C}^2$ be a (not necessarily minimal) ngc branched over $B_{n,m}$ whose monodromy graph is a s-coherently labeled α-ring $\Gamma \in Gr_{n,n}$ (cfr. Section 5). Let $n = \beta\alpha$, h and k be the number of elements of the positive and negative orbit, respectively, and let $b = \frac{m}{hk\beta}(h,k)$. Recall that $(\alpha, s) = 1$, b is an integer and that we may assume that 1 is an edge of the loop c of Γ. Let $\lambda_1, \dots, \lambda_{r\alpha}$ be the edges of c with $\lambda_1 = 1$ and such that λ_{i+1} follows λ_i in the orientation of c and let p be the vertex of 1 in the positive orbit.

Suppressing the edge $\lambda_{r\alpha}$, we obtain a maximal tree Γ' and to calculate $\pi_1(S \setminus P)$ we apply the Reidemeister-Shreier method to the Shreier set of left cosets γ_c for c a p-chain in Γ'. A set of generators for $\pi_1(S \setminus D)$ is given by the following elements (cfr. 6.1)

$$A_i = \gamma_c \gamma_i \gamma_{c'}^{-1}$$

for $1 \ge i \ne \lambda_{r\alpha}$, where $c = (h_k)_{k=1,\dots,t}$ is the p-chain in Γ' such that $h_t = i$ and $c' = c - \{i\}$;

$$B_{j,i} = \gamma_c \gamma_i \gamma_c^{-1}$$

for $0 \ge j \ne \lambda_{r\alpha}$ and, if $j \ge 1$, $c = (h_k)_{k=1,\dots,t}$ the p-chain in Γ' such that $h_t = j$ and i is such that $c \cup \{i\}$ is not a chain, while, if $j = 0$, c is the trivial chain and $i \ne \lambda_{r\alpha}$ is an edge which has not p as vertex;

$$C = \gamma_{\lambda_1} \dots \gamma_{\lambda_{r\alpha}}$$
$$\overline{C} = \gamma_{\lambda_{r\alpha}} (\gamma_{\lambda_1} \dots \gamma_{\lambda_{r\alpha - 1}})^{-1}$$

while a set of defining relators is given by rewriting in terms of the above generators the following

$$\gamma_c T_{i,m} T_{i+1,m}^{-1} \gamma_c^{-1}$$

for all choices of i and c.

Observe that $\pi_1(S \setminus \{P\})$ is obtained from $\pi_1(S \setminus D)$ by adding the relations

$$A_i = B_{i,j} = C\overline{C} = e$$

for all choices of i and j (they represent loops around all the components of D). Thus $\pi_1(S \setminus \{P\})$ is generated by C and to compute its order we have to go through the rewriting process for the relations.

THEOREM 6.9. *With the above notation*, $\pi_1(S \setminus \{P\}) = \mathbb{Z}/\frac{m}{hk}\mathbb{Z}$.

PROOF. First observe that if $\frac{hk}{(h,k)}\beta < n$ then $\frac{h\beta}{(h,k)}\frac{k\beta}{(h,k)} < \frac{h\beta}{(h,k)} + \frac{k\beta}{(h,k)}$ so that $\frac{h\beta}{(h,k)} = 1$ or $\frac{k\beta}{(h,k)} = 1$, thus $\beta = 1$. Since if $\beta = 1$ then Γ is a polygon, the result follows from Theorems 6.4, 6.6 and we may suppose $m \geq n$.

As before, we have to rewrite the relations

$$\gamma_c T_{1,m}(\gamma_c)^{-1} = \gamma_c T_{i+1,m}(\gamma_c)^{-1}$$

for $i = 2, \dots, m$ and each chain c as above. Observe that, if c is a p-chain, then $q = \gamma_c(p)$ is its ending vertex, so

- $\gamma_c\gamma_i = A_i\gamma_{c-\{i\}}$ if i is the last edge of c,
- $\gamma_c\gamma_i = B_{j,i}\gamma_c$ if j is the last edge of c and i has not q as vertex,
- $\gamma_c\gamma_i = C$ if $c = (\lambda_1, \dots, \lambda_{r\alpha-1})$ and $i = \lambda_{r\alpha}$,
- $\gamma_c\gamma_i = \overline{C}$ if c is the trivial chain and $i = \lambda_{r\alpha}$.

Form this we immediately get that, in order to calculate the power of C or of $\overline{C}$ that appears rewriting $\gamma(c)T_{i,m}$ after killing all the A_i and $B_{j,i}$, we can count how many times the motion of q under the action of $T_{i,m}$ contains the edge $\lambda_{r\alpha}$.

Since for each $k \geq 0$ we have that $T_{i,k}(q) = T_{1,k+i-1}(q_i')$, where $q_i' = (T_{1,i-1})^{-1}(q)$, then q will move in the positive (resp. negative) direction if and only if q_i' belongs to the positive (resp. negative) orbit, thus, rewriting $\gamma(c)T_{i,m}$ we will get C^u (resp. $\overline{C}^v$) for a certain u (resp. v). We shall prove that u (resp. v) does not depend on i and that the sum $u + v$ does not depend on q either.

It may be that q_i' belongs to the positive (resp. negative) orbit for every i and we obtain the trivial relation $C^u = C^u$ (resp. $\overline{C}^v = \overline{C}^v$). On the other hand, if q_i' is not contained in the same orbit for every i (as is the case when $q = p$, $m \geq n$), then we will have the relation $C^u = \overline{C}^v$, i.e. $C^{u+v} = 1$.

Define $Q = T_{i,m}(q)$ and notice that Q does not depend on i. Also, Q is the terminal point of the motion of q_i' under the action of $T_{1,m+i-1}$, i.e. $T_{1,m+i-1}(q_i') = Q$. If $Q = q$ then

$$T_{i,m}(q_i') = T_{1,i-1}^{-1} T_{1,m+i-1}(q_i') = T_{1,i-1}^{-1}(Q) = T_{1,i-1}^{-1}(q) = q'$$

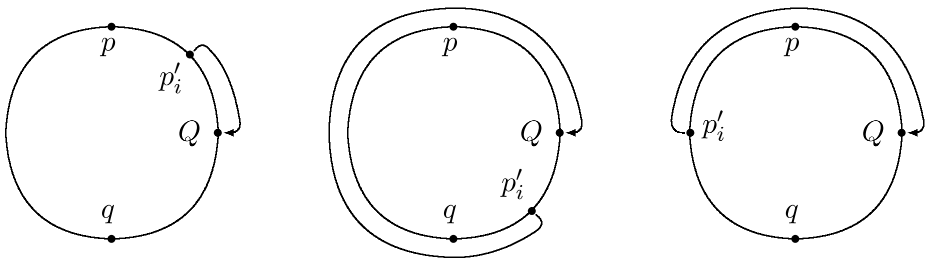
i.e. $T_{1,m}$ acts as the identity on p_i' for every i, which implies that it acts as the identity on all vertices of Γ. Hence, the covering is a pull back of the standard one and $n|m$. Notice that this is true if and only if $\alpha|b$. Since each point in the positive (resp. negative) orbit undergoes a complete loop under $T_{1,hn}$ (resp. $T_{1,kn}$) it is easy to see that, in this case, $u = \frac{m}{hn}$ and $v = \frac{m}{kn}$, so that $u + v = \frac{m}{hk}$.

Suppose now $Q \neq q$, i.e. $n \nmid m$. Suppose the point q_i' belongs to the positive orbit, and consider its motion under the action of $T_{1,m+i-1}$. Since $i \leq m$, q_i' will do $\bar{u}$ or $\bar{u}+1$ complete loops, where $m = \bar{u}hn + r$ with $r < hn$. If it does $\bar{u}$ complete loops, then it will return to q, if it does $\bar{u}+1$ complete loops then it will not return to q, since

$$\bar{u}hn + r + i - 1 = m + i - 1 \geq (\bar{u}+1)hn + i - 1 \ \Rightarrow\ r \geq hn\,.$$

We will say that Q is before q if, writing $Q = T_{1,j}(p), q = T_{1,l}(p)$ with $j, l \leq hn$, we have $j < l$, i.e. Q is before q in the (oriented) motion of p under the action of $T_{1,hn}$; otherwise we will say that Q is after q (such j and l exist since $p' = T_{1,ns}(p)$ for some s, then $q = T_{1,ns+i-1}(p)$ and $Q = T_{1,ns+i-1+m}(p)$). The same definitions can be given for the position of p_i' with respect to q or Q.

Let's examine the case in which Q is before q. If q_i' is before Q or it is equal to Q, then after $\bar{u}$ complete loops p_i' returns to q and to reach Q it must do another complete loop (and no more), passing through the edge $\lambda_{r\alpha}$ $\bar{u}+1$ times (see picture for a schematic drawing of the motion of p_i' after the complete loops).



p_i' is before Q (after $\bar{u}+1$ loops) | p_i' is before q and after Q (after $\bar{u}$ loops) | p_i' is after q (after $\bar{u}+1$ loops)

Fig. 8. Motion of p_i' in the positive orbit: the case where Q is before q.

If p_i' is after Q but before q or is equal to q, then it must do exactly $\bar{u}$ complete loops, return to q and reach Q passing from the edge $\lambda_{r\alpha}$ $\bar{u}+1$ times. If p_i' is after q, then it must do exactly $\bar{u}+1$ complete loops and reach Q, thus passing from the edge $\lambda_{r\alpha}$ $\bar{u}+2$ times. But since it must cross the edge $\lambda_{r\alpha}$ while acting on it by $T_{1,i-1}$, the number of times the point q crosses the edge $\lambda_{r\alpha}$ is $\bar{u}+1$.

In the same way, examining all the possibilities in the case in which Q is after q, we find that q passes from the edge $\lambda_{r\alpha}$ $\bar{u}$ times (see picture).

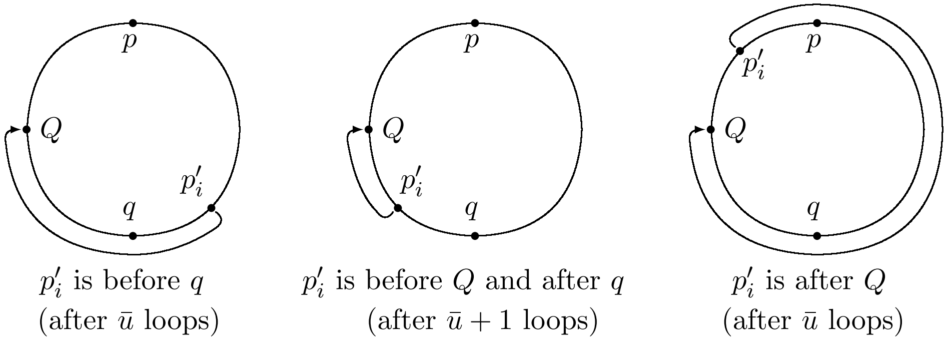


Fig. 9. Motion of p_i' in the positive orbit: the case where Q is after q.

The same conclusions are valid in case p_i' belongs to the negative orbit, provided that we exchange h with k and $\bar{u}$ with $\bar{v}$ such that $m = \bar{v}kn + s$ with $s < kn$ (and we give a slightly different definition of "before" and "after" q). Namely we get that, in case Q is before q, q passes from the edge $\lambda_{r\alpha}$ $\bar{v}$ times, while if Q is after q, $\bar{v}+1$ times.

So the order of C is $\bar{u}+\bar{v}+1 = [\frac{m}{hn}] + [\frac{m}{kn}] + 1$. Since

$$\frac{m}{hk} = \frac{m}{hn} + \frac{m}{kn} = \left[\frac{m}{hn}\right] + \left[\frac{m}{kn}\right] + \left\{\frac{m}{hn}\right\} + \left\{\frac{m}{hn}\right\}$$

and $\frac{m}{hk}$ is an integer, we must have $\{\frac{m}{hn}\} + \{\frac{m}{hn}\} = 0, 1$. Notice, however, that if $\{\frac{m}{hn}\} = \{\frac{m}{hn}\} = 0$, then $n|m$, so $\bar{u}+\bar{v}+1 = \frac{m}{hk}$. □

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