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A Combinatorial Characterization of Hermitian Curves

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Abstract. A unital U with parameter q is a $2-(q^3+1, q+1, 1)$ design. If a point set U in $PG(2, q^2)$ together with its $(q+1)$-secants forms a unital, then U is called a Hermitian arc. Through each point p of a Hermitian arc H there is exactly one line L having with H only the point p in common; this line L is called the tangent of H at p. For any prime power q, the absolute points and nonabsolute lines of a unitary polarity of $PG(2, q^2)$ form a unital that is called the classical unital. The points of a classical unital are the points of a Hermitian curve in $PG(2, q^2)$.

Let H be a Hermitian arc in the projective plane $PG(2, q^2)$. If tangents of H at collinear points of H are concurrent, then H is a Hermitian curve. This result proves a well known conjecture on Hermitian arcs.

Keywords: Hermitian curve, unital, projective plane.

1. Introduction

A unital U with parameter q is a $2-(q^3+1, q+1, 1)$ design. If a point set U in $PG(2, q^2)$ together with its $(q+1)$-secants forms a *unital*, then U is called a *Hermitian arc*. For any prime power q, the absolute points and nonabsolute lines of a unitary polarity of $PG(2, q^2)$ form a unital that is called the *classical unital*. The points of a classical unital are the points of a *Hermitian curve* Γ in $PG(2, q^2)$. This is a curve of degree $q+1$ which is projectively equivalent to the curve $\Delta: X_0^{q+1} + X_1^{q+1} + X_2^{q+1} = 0$.

After Segre's famous theorem that characterizes combinatorially the conics of $PG(2, q)$ with q odd (see Hirschfeld [4], Theorem 8.2.4), some effort was made to characterize the Hermitian curves; in [8] Tallini Scafati did this. Then Buekenhout [2] and Metz [7] constructed Hermitian arcs in $PG(2, q^2)$ that are not Hermitian curves. Since Tallini Scafati's paper, it has been conjectured that the natural condition that tangents at collinear points are concurrent characterizes Hermitian curves. The present paper solves this conjecture in the affirmative. In a recent paper Blokhuis, Brouwer, and Wilbrink [1] proved that a unital that lies in the code of $PG(2, q)$ must be a Hermitian curve; in the course of their proof they showed that the present condition on tangents holds for a unital in the code.

Various good characterizations of a Hermitian curve have already been given. The following characterization is due to Lefèvre-Percsy [6], and independently to Faina and Korchmáros [3], who proved:

Let H be a Hermitian arc of $PG(2,q^2)$. If every $(q+1)$-secant of H meets H in a Baer-subline, then H is a Hermitian curve.

Recently Hirschfeld, Storme, Thas, and Voloch [5] obtained a characterization in terms of algebraic curves:

In $PG(2,q^2), q \neq 2$, any algebraic curve of degree $q+1$, without linear components, and with at least q^3+1 points in $PG(2,q^2)$, must be a Hermitian curve.

As already mentioned we prove here a well known conjecture on Hermitian curves; a main tool is Segre's famous trick, usually called the "Lemma of tangents" (see Hirschfeld [4], Lemma 8.2.2).

2. Lemma

Let H be a Hermitian arc in $PG(2,q^2)$. Let p_0, p_1, p_2 be noncollinear points of H and let L_i be the tangent of H at $p_i, i = 0,1,2$. Coordinates are chosen in such a way that $L_1 \cap L_2 = e_0 = (1,0,0)$, $L_2 \cap L_0 = e_1 = (0,1,0)$, $L_0 \cap L_1 = e_2 = (0,0,1)$. If $p_0 = (0,b,1)$, $p_1 = (1,0,c)$, $p_2 = (a,1,0)$, then

$$(abc)^{q+1} = 1$$

Proof. The points of $p_0p_1 \cap H$ are the points p_0, p_1, and $z_i = (t_0^i, 1, t_2^i)$, $i = 0,1,\cdots,q-1$; the points of $p_1p_2 \cap H$ are the points p_1, p_2, and $x_i = (r_0^i, r_1^i, 1)$, $i = 0,1,\cdots,q-1$; the points of $p_2p_0 \cap H$ are the points p_2, p_0, and $y_i = (1, s_1^i, s_2^i)$, $i = 0,1,\cdots,q-1$. Clearly $t_0^i \neq 0$, $r_1^i \neq 0$, $s_2^i \neq 0$, with $i = 0,1,\cdots,q-1$. By assumption the tangent of H at x_i contains e_0, the tangent of H at y_i contains e_1, and the tangent of H at z_i contains $e_2, i = 0,1,\cdots,q-1$.

Let $p = (y_o, y_1, y_2)$ be a point of $H - \{p_o, p_1, p_2\}$.
The line e_0p has equation $X_1 = \omega_0^p X_2$ with $\omega_0^p = y_1/y_2$; the line e_1p has equation $X_2 = \omega_1^p X_0$ with $\omega_1^p = y_2/y_0$; the line e_2p has equation $X_0 = \omega_2^p X_1$ with $\omega_2^p = y_0/y_1$. Hence

$$\omega_0^p \omega_1^p \omega_2^p = 1$$

Considering all such points p we obtain

$$\prod_p \omega_0^p \omega_1^p \omega_2^p = 1 \tag{1}$$

To see the values taken by $\omega_0^p, \omega_1^p, \omega_2^p$ in (1), let $U = GF(q) - \{0, r_1^1, \cdots, r_1^{q-1}, b\}$, let $V = GF(q) - \{0, s_2^1, \cdots, s_2^{q-1}, c\}$, and let $W = GF(q) - \{0, t_0^1, \cdots, t_0^{q-1}, a\}$. Then the values taken are as follows

	Value	Number of Times
ω_0^p	r_1^i	1 for each $i = 1, \cdots, q-1$
	b	q
	u	$q+1$ for each u in U
ω_1^p	s_2^i	1 for each $i = 1, \cdots, q-1$
	c	q
	v	$q+1$ for each v in V
ω_2^p	t_0^i	1 for each $i = 1, \cdots, q-1$
	a	q
	w	$q+1$ for each w in W.

As the product of all elements of $GF(q) - \{0\}$ is -1, we have

$$b\prod_{i=1}^{q-1} r_1^i \prod_{u\in U} u = c\prod_{i=1}^{q-1} s_2^i \prod_{v\in V} v = a\prod_{i=1}^{q-1} t_0^i \prod_{\omega\in W} \omega = -1 \tag{2}$$

From (1)

$$\left(\prod_{i,j,h} r_1^i s_2^j t_0^h\right)(abc)^q\left(\prod_u u \prod_v v \prod_\omega \omega\right)^{q+1} = 1$$

so

$$\left[\left(\prod_{i,j,h} r_1^i s_2^j t_0^h\right)(abc)\left(\prod_u u \prod_v v \prod_\omega \omega\right)\right]^{q+1} = \left(\prod_{i,j,h} r_1^i s_2^j t_0^h\right)^q (abc) \tag{3}$$

From (2) and (3)

$$(abc)\left(\prod_{i,j,h} r_1^i s_2^j t_0^h\right)^q = 1$$

so

$$(abc)^q \prod_{i,j,h} r_1^i s_2^j t_0^h = 1 \tag{4}$$

Now choose new coordinates in such a way that $p_0 = (1,0,0)$, $p_1 = (0,1,0)$, $p_2 = (0,0,1)$. Transformation matrices are

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & a \\ b & 0 & 1 \\ 1 & c & 0 \end{bmatrix}$$

and

$$\mathbf{A}^A = \begin{bmatrix} -c & ac & 1 \\ 1 & -a & ab \\ bc & 1 & -b \end{bmatrix} \quad \text{with } \mathbf{A}^A = (\det \mathbf{A})\mathbf{A}^{-1}$$

here, **A** multiplied with a column consisting of old coordinates of a point yields a column of new coordinates of the point. Since p_0, p_1, and p_2 are not collinear we have $abc + 1 \neq 0$.

a. Points:

First remark that $x_i \in p_1p_2$, $y_i \in p_2p_0$, $z_i \in p_0p_1$ imply that

$$1 - s_1^i a + s_2^i ab = 1 - t_2^i b + t_0^i bc = 1 - r_0^i c + r_1^i ac = 0 \tag{5}$$

Point	Old Coordinates	New Coordinates
e_0	$(1,0,0)$	$(-c,1,bc)$
e_1	$(0,1,0)$	$(ac,-a,1)$
e_2	$(0,0,1)$	$(1,ab,-b)$
y_i	$(1,s_1^i,s_2^i)$	$(-c+s_1^i ac+s_2^i,0,bc+s_1^i-s_2^i b)=(as_2^i,0,1)$
z_i	$(t_o^i,1,t_2^i)$	$(-t^i c+ac+t_2^i,t_0^i-a+t_2^i ab,0)=(1,bt_0^i,0)$
x_i	$(r_0^i,r_1^i,1)$	$(0,r_0^i-r_1^i a+ab,r_0^i bc+r_1^i-b)=(0,1,cr_1^i)$

b. Lines:

Line	Old Coordinates	New Coordinates
L_0	$[1,0,0]$	$[0,a^{-1},1]$
L_1	$[0,1,0]$	$[1,0,b^{-1}]$
L_2	$[0,0,1]$	$[c^{-1},1,0]$
e_0x_i	$[0,1,-r_1^i]$	$[b-r_1^i,-cr_1^i,1]$
e_1y_i	$[-s_2^i,0,1]$	$[1,c-s_2^i,-as_2^i]$
e_2z_i	$[1,-t_0^i,0]$	$[-bt_0^i,1,a-t_0^i]$

Now applying (4) in dual plane, where the roles of p_0, p_1, and p_2 are taken by L_0, L_1, and L_2, we obtain

$$(a^{-1}b^{-1}c^{-1})^q \prod_i(-cr_1^i)\prod_j(-as_2^j)\prod_h(-bt_0^h) = 1$$

that is,

$$(abc)^{-1} \prod_{i,j,h} r_1^i s_2^j t_0^h = 1 \tag{6}$$

Finally, from (4) and (6)

$$(abc)^{q+1} = 1$$ □

3. Theorem

A Hermitian arc H of $PG(2,q^2)$ is a Hermitian curve if and only if tangents of H at collinear points of H are concurrent.

Proof. We use the notations of the Lemma.

We may assume that in the original reference system the line e_0x_1 has equation $X_1 = X_2$. Then $x_1 = (a + c^{-1}, 1, 1)$, so $r_1^1 = 1$.

Let $e_0x_i \cap e_1e_2 = x_i'$, $i = 1, 2, \cdots, q-1$. Then $x_i' = (0, r_1^i, 1)$, $i = 1, 2, \cdots, q-1$.

Now choose new coordinates in such a way that $e_0 = (1,0,0)$, $e_2 = (0,0,1)$, $x_i' = (0,1,0)$, $i \in \{1, 2, \cdots, q-1\}$. Transformation matrices are

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r_1^i & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

and

$$\mathbf{B}^A = \begin{bmatrix} r_1^i & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & r_1^i \end{bmatrix} \quad \text{with } \mathbf{B}^A = (\det\mathbf{B})\mathbf{B}^{-1}$$

here, **B** multiplied with a column consisting of old coordinates of a point yields a column of new coordinates of the point.

Point	Old Coordinates	New Coordinates
x_i	$(r_0^i, r_1^i, 1)$	$(r_0^i, 1, 0)$
p_0	$(0, b, 1)$	$(0, b/(r_1^i - b), 1)$
p_1	$(1, 0, c)$	$(1, 0, c)$

We remark that $p_0 \notin e_0x_i$ implies $r_1^i - b \neq 0$.

Applying the Lemma we obtain

$$\left(\frac{r_0^i cb}{r_1^i - b}\right)^{q+1} = 1 \tag{7}$$

By (5) the equality (7) becomes

$$\left(\frac{b + abcr_1^i}{r_1^i - b}\right)^{q+1} = 1$$

Hence

$$ab^{q+1}cr_1^i + a^q b^{q+1}c^q(r_1^i)^q + (abc)^{q+1}(r_1^i)^{q+1} = -b^q r_1^i - b(r_1^i)^q + (r_1^i)^{q+1} \qquad (8)$$

Since $(abc)^{q+1} = 1$, we then have

$$b^{q-1} + (r_1^i)^{q-1}(abc + 1)^{q-1} = 0 \qquad (9)$$

Putting $i = 1$, so that $r_1^i = r_1^1 = 1$, (9) becomes

$$b^{q-1} + (abc + 1)^{q-1} = 0 \qquad (10)$$

From (9) and (10)

$$(r_1^i)^{q-1} = 1, i = 1, 2, \cdots, q-1$$

Consequently $\{0, r_1^1 = 1, r_1^2, \cdots, r_1^{q-1}\}$ is the subfield $GF(q)$ of $GF(q^2)$. Hence $p_1p_2 \cap H$ is a Baer-subline of p_1p_2. As p_1p_2 is an arbitrary $(q+1)$-secant of H, so is H a Hermitian curve by the characterization of Lefèvre-Percsy and of Faina and Korchmáros mentioned in the Introduction. □

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