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A COMBINATORIAL CONSTRUCTION OF SYMPLECTIC EXPANSIONS

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ABSTRACT. The notion of a symplectic expansion directly relates the topology of a surface to formal symplectic geometry. We give a method to construct a symplectic expansion by solving a recurrence formula given in terms of the Baker-Campbell-Hausdorff series.

1. INTRODUCTION

Let Σ be a compact connected oriented surface of genus $g > 0$ with one boundary component. Choose a basepoint $*$ on the boundary $\partial\Sigma$ and let $\pi = \pi_1(\Sigma, *)$ be the fundamental group of Σ.

The notion of (*generalized*) *Magnus expansions* was introduced by Kawazumi [5] in his study of the mapping class group of a surface. By definition, the mapping class group $\mathcal{M}_{g,1}$ is the group of homomorphisms of Σ fixing $\partial\Sigma$ pointwise, modulo isotopies fixing $\partial\Sigma$ pointwise. The group $\mathcal{M}_{g,1}$ faithfully acts on π, a free group of rank $2g$, and it is known as *the theorem of Dehn-Nielsen* that $\mathcal{M}_{g,1}$ is identified with a subgroup of the automorphism group of a free group:

$$\mathcal{M}_{g,1} = \{\varphi \in \mathrm{Aut}(\pi); \varphi(\zeta) = \zeta\}.$$

Here, $\zeta \in \pi$ is the element corresponding to the boundary. See §2. By choosing a Magnus expansion, the completed group ring of π (with respect to the augmentation ideal) is identified with the completed tensor algebra generated by the first homology of the surface. In this way we obtain a tensor expression of the action of $\mathcal{M}_{g,1}$ on π. From this point of view, Kawazumi obtained extensions of the Johnson homomorphisms τ_k introduced by Johnson [3], [4]. For details, see [5].

Actually the treatment in [5] is on the automorphism group of a free group, rather than the mapping class group. There are infinitely many Magnus expansions, and the arguments in [5] hold for any Magnus expansions. Recently, Massuyeau [11] introduced the notion of *symplectic expansions*, which are Magnus expansions satisfying a certain kind of boundary condition, which comes from the fact that π has a particular element corresponding to the boundary $\partial\Sigma$. Some nice properties of symplectic expansions are clarified by [7]. In particular, it is shown that there is a Lie algebra homomorphism from the Goldman Lie algebra of Σ

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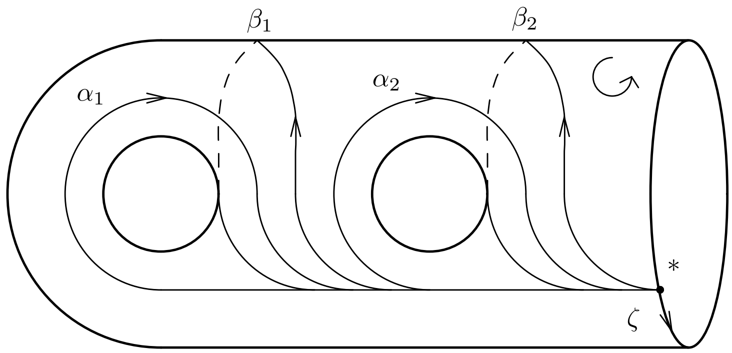

FIGURE 1. Symplectic generators for $g = 2$

(see Goldman [2]) to "associative", one of the three Lie algebras in formal symplectic geometry by Kontsevich [8], via a symplectic expansion (see [7], Theorem 1.2.1).

Although there are infinitely many symplectic expansions (see [7], Proposition 2.8.1), there are not so many known examples. The boundary condition is too strong to be satisfied. For instance, *the fatgraph Magnus expansion* given by Bene-Kawazumi-Penner [1] is, unfortunately, not symplectic. Kawazumi [6], §6, first constructed an $\mathbb{R}$-valued symplectic expansion, called *the harmonic Magnus expansion*, by a transcendental method. Massuyeau [11], Proposition 5.6, also gave a $\mathbb{Q}$-valued symplectic expansion using *the LMO functor*.

The purpose of this paper is to present another construction of symplectic expansions. Our construction is elementary and suitable for computer-aided calculation.

Theorem 1.1. *There is an algorithm to construct a symplectic expansion $\theta^{\mathcal{S}}$ associated to any free generating set $\mathcal{S}$ for π.*

It should be remarked here that in the proof of the existence of symplectic expansions ([11], Lemma 2.16), Massuyeau already showed how to construct a symplectic expansion degree after degree. Our construction is also inductive, but by using *the Dynkin idempotents* it fixes the choices that had to be done in the inductive step of [11], Lemma 2.16, hence is canonical. Moreover, our construction works for any free generating set for π whereas [11], Lemma 2.16, only deals with symplectic generators.

In §2, we recall Magnus expansions and symplectic expansions. Theorem 1.1 will be proved in §3. In §4, we show a naturality of our construction under the action of a subgroup of $\mathrm{Aut}(\pi)$ including the mapping class group $\mathcal{M}_{g,1}$. In §5, we discuss the symplectic expansion associated to symplectic generators.

2. BASIC NOTIONS

We denote by ζ the loop parallel to $\partial\Sigma$ and going in a counterclockwise manner. Explicitly, if we take symplectic generators $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g \in \pi$ as shown in Figure 1, $\zeta = \prod_{i=1}^{g} [\alpha_i, \beta_i]$. Here our notation for commutators is $[x, y] := xyx^{-1}y^{-1}$.

Let $H_{\mathbb{Z}} := H_1(\Sigma; \mathbb{Z})$ be the first integral homology group of Σ. We denote $H := H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$. $H_{\mathbb{Z}}$ is naturally isomorphic to $\pi/[\pi, \pi]$, the abelianization of π.

With this identification in mind, we denote $[x] := x \bmod [\pi,\pi] \in H_{\mathbb{Z}}$, or $[x] := (x \bmod [\pi,\pi]) \otimes_{\mathbb{Z}} 1 \in H$, for $x \in \pi$.

Let $\widehat{T}$ be the completed tensor algebra generated by H. Namely $\widehat{T} = \prod_{m=0}^{\infty} H^{\otimes m}$, where $H^{\otimes m}$ is the tensor space of degree m. For each $p \geq 1$, denote $\widehat{T}_p := \prod_{m \geq p}^{\infty} H^{\otimes m}$. Note that the subset $1 + \widehat{T}_1$ constitutes a subgroup of the multiplicative group of the algebra $\widehat{T}$.

Definition 2.1 (Kawazumi [5])**.** A map $\theta\colon \pi \to 1 + \widehat{T}_1$ is called a ($\mathbb{Q}$-valued) Magnus expansion if

(1) $\theta\colon \pi \to 1 + \widehat{T}_1$ is a group homomorphism, and
(2) $\theta(x) \equiv 1 + [x] \bmod \widehat{T}_2$, for any $x \in \pi$.

The *standard* Magnus expansion defined by $\theta(s_i) = 1 + [s_i]$, for some free generating set $\{s_i\}_i$ for π, is the simplest example of a Magnus expansion. This is introduced by Magnus [9] and is often used in combinatorial group theory.

Let $\widehat{\mathcal{L}} \subset \widehat{T}$ be the completed free Lie algebra generated by H. The bracket is given by $[u,v] := u \otimes v - v \otimes u$, and its degree p-part $\mathcal{L}_p = \widehat{\mathcal{L}} \cap H^{\otimes p}$ is successively given by $\mathcal{L}_1 = H$ and $\mathcal{L}_p = [H, \mathcal{L}_{p-1}]$, $p \geq 2$. Via the intersection form $(\ \cdot\ )\colon H \times H \to \mathbb{Q}$ on Σ, H and its dual $H^* = \mathrm{Hom}_{\mathbb{Q}}(H,\mathbb{Q})$ are canonically identified by the map $H \cong H^*$, $X \mapsto (Y \mapsto (Y \cdot X))$. Let $\omega \in \mathcal{L}_2 \subset H^{\otimes 2}$ be the symplectic form, namely the tensor corresponding to $-1_H \in \mathrm{Hom}_{\mathbb{Q}}(H,H) = H^* \otimes H = H \otimes H$. Explicitly, if we take symplectic generators as in Figure 1, then $A_i = [\alpha_i]$ and $B_i = [\beta_i]$ satisfy $(A_i \cdot B_j) = -(B_j \cdot A_i) = \delta_{ij}$ and $(A_i \cdot A_j) = (B_i \cdot B_j) = 0$; hence we have

$$\omega = \sum_{i=1}^{g} A_i \otimes B_i - B_i \otimes A_i = \sum_{i=1}^{g} [A_i, B_i]. \tag{2.1}$$

For a Magnus expansion θ, let $\ell^\theta := \log \theta$. Here, log is the formal power series

$$\log(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (x-1)^n$$

defined on the set $1 + \widehat{T}_1$. The inverse of log is given by the exponential $\exp(x) = \sum_{n=0}^{\infty} (1/n!) x^n$. Note that the Baker-Campbell-Hausdorff formula

$$\begin{aligned} u \star v := \log(\exp(u)\exp(v)) &= u + v + \frac{1}{2}[u,v] + \frac{1}{12}[u-v,[u,v]] \\ &\quad - \frac{1}{24}[u,[v,[u,v]]] + \cdots \end{aligned} \tag{2.2}$$

endows the underlying set of $\widehat{\mathcal{L}}$ with a group structure. A priori, ℓ^θ is a map from π to $\widehat{T}_1$.

Definition 2.2 (Massuyeau [11])**.** A Magnus expansion θ is called symplectic if

(1) θ is group-like, i.e., $\ell^\theta(\pi) \subset \widehat{\mathcal{L}}$, and
(2) $\theta(\zeta) = \exp(\omega)$, or equivalently, $\ell^\theta(\zeta) = \omega$.

Remark 2.3. Let $I\pi$ be the augmentation ideal of the group ring $\mathbb{Q}\pi$, and $\widehat{\mathbb{Q}\pi} := \varprojlim_m \mathbb{Q}\pi / I\pi^m$ the completed group ring of π. Any Magnus expansion θ induces an isomorphism $\theta\colon \widehat{\mathbb{Q}\pi} \xrightarrow{\cong} \widehat{T}$ of complete augmented algebras. See [5], Theorem 1.3. Moreover, let $\langle \zeta \rangle$ be the cyclic subgroup of π generated by ζ, and $\mathbb{Q}[[\omega]]$ the ring of formal power series in the symplectic form ω, which is regarded as a subalgebra of $\widehat{T}$ in an obvious way. Then any symplectic expansion θ induces an isomorphism $\theta\colon (\widehat{\mathbb{Q}\pi}, \widehat{\mathbb{Q}\langle\zeta\rangle}) \to (\widehat{T}, \mathbb{Q}[[\omega]])$ of complete Hopf algebras. See [7], §6.2.

3. Main construction

We fix a free generating set $\mathcal{S} = \{s_1, \dots, s_{2g}\}$ for π. We denote $S_i := [s_i] \in H$, $1 \leq i \leq 2g$. Let $x_1 x_2 \cdots x_p$ be the unique reduced word in $\mathcal{S}$ representing ζ.

Definition 3.1. Fix an integer $n \geq 1$. A set $\{\ell_j(s_i) : 1 \leq i \leq 2g, 1 \leq j \leq n\} \subset \widehat{\mathcal{L}}$ is called a partial symplectic expansion up to degree n if

(1) $\ell_1(s_i) = S_i$, for $1 \leq i \leq 2g$,
(2) $\ell_j(s_i) \in \mathcal{L}_j$, for $1 \leq i \leq 2g$, $1 \leq j \leq n$, and
(3) if we set $\bar{\ell}_n(s_i) = \sum_{j=1}^n \ell_j(s_i)$ for $1 \leq i \leq 2g$, then

$$\bar{\ell}_n(x_1) \star \bar{\ell}_n(x_2) \star \cdots \star \bar{\ell}_n(x_p) \equiv \omega \bmod \widehat{T}_{n+2}. \tag{3.1}$$

Here, we understand $\bar{\ell}_n(s_i^{-1}) = -\bar{\ell}_n(s_i)$.

This notion could be thought of as an approximation to a symplectic expansion. In this section we give a method to refine an approximation up to degree $n-1$, to the one up to degree n. Repeating this process, we will obtain a symplectic expansion.

We need two lemmas.

Lemma 3.2. *Suppose $4g$ elements $Y_1, \dots, Y_{2g}, Z_1, \dots, Z_{2g} \in H$ satisfy $\sum_{i=1}^{2g} Y_i \otimes Z_i = \omega \in H^{\otimes 2}$. Then $Z_1, \dots, Z_{2g}$ constitute a basis for H.*

Proof. Since ω corresponds to $-1_H \in \mathrm{Hom}_{\mathbb{Q}}(H, H)$ (see §2), for any $X \in H$, we have

$$X = \omega(-X) = \sum_{i=1}^{2g} (-X \cdot Y_i) Z_i.$$

This shows that the $2g$ elements $Z_1, \dots, Z_{2g}$ generate H. This proves the lemma. $\square$

Since π is free, the quotient $[\pi,\pi]/[\pi,[\pi,\pi]]$ is naturally isomorphic to $\Lambda^2 H_{\mathbb{Z}}$, the second exterior product of $H_{\mathbb{Z}}$. The isomorphism is induced by the homomorphism $f\colon [\pi,\pi] \to \Lambda^2 H_{\mathbb{Z}}$ which maps the commutator $[x,y]$ to $[x] \wedge [y]$. Note that $\Lambda^2 H_{\mathbb{Z}}$ is naturally identified with a subgroup of $H^{\otimes 2}$ by

$$\Lambda^2 H_{\mathbb{Z}} \to H^{\otimes 2},\ X \wedge Y \mapsto X \otimes Y - Y \otimes X,$$

and under this identification we have $f(\zeta) = \omega$.

Lemma 3.3. *Let $y_1 \cdots y_q$ be a word in $\mathcal{S}$ and suppose $y_1 \cdots y_q$ lies in the commutator subgroup $[\pi,\pi]$. Then*

$$f(y_1 \cdots y_q) = \frac{1}{2} \sum_{i<j} [y_i] \wedge [y_j].$$

Proof. We may assume $q \geq 2$. We prove the lemma by induction on q. The case $q = 2$ is trivial. Suppose $q > 2$. Then there must exist $i \geq 1$ such that $y_{i+1} = y_1^{-1}$, and

$$y_1 \cdots y_q = y_1 y_2 \cdots y_i y_1^{-1} y_{i+2} \cdots y_q = [y_1, y_2 \cdots y_i] y_2 \cdots y_i y_{i+2} \cdots y_q.$$

Hence $f(y_1 \cdots y_q) = f([y_1, y_2 \cdots y_i]) + f(y_2 \cdots y_i y_{i+2} \cdots y_q)$. The first term equals

$$[y_1] \wedge ([y_2] + \cdots + [y_i]) = \frac{1}{2}\left([y_1] \wedge ([y_2] + \cdots + [y_i]) + ([y_2] + \cdots + [y_i]) \wedge [y_{i+1}]\right)$$

since $[y_1] = -[y_{i+1}]$, and the second term equals

$$\frac{1}{2} \sum_{\substack{k<\ell; \\ k,\ell \neq 1, i+1}} [y_k] \wedge [y_\ell],$$

by the inductive assumption. This proves the lemma. □

Let $\Phi\colon \widehat{T}_1 \to \widehat{\mathcal{L}}$ be the linear map defined by $\Phi(Y_1 \otimes \cdots \otimes Y_m) = [Y_1, [\cdots [Y_{m-1}, Y_m] \cdots]]$, $Y_i \in H$, $m \geq 1$. We have $\Phi(u) = mu$ and $\Phi(uv) = [u, \Phi(v)]$ for any $u \in \mathcal{L}_m$, $v \in \widehat{T}_1$. See Serre [12], Part I, Theorem 8.1, p. 28. The maps $(1/m)\Phi|_{H^{\otimes m}}$ are called *the Dynkin idempotents*. From these two properties we see that the restriction of the map

$$\frac{1}{m+1}(\mathrm{id} \otimes \Phi)\colon H^{\otimes m+1} \to H \otimes \mathcal{L}_m \tag{3.2}$$

to $\mathcal{L}_{m+1}$ gives a right inverse of the bracket $[\ ,\ ]\colon H \otimes \mathcal{L}_m \twoheadrightarrow \mathcal{L}_{m+1}$.

Let $n \geq 2$ and let $\{\ell_j(s_i) : 1 \leq j \leq n-1,\ 1 \leq i \leq 2g\}$ be a partial symplectic expansion up to degree $n-1$. We have

$$\bar{\ell}_{n-1}(x_1) \star \bar{\ell}_{n-1}(x_2) \star \cdots \star \bar{\ell}_{n-1}(x_p) \equiv \omega \bmod \widehat{T}_{n+1}. \tag{3.3}$$

Let $V_{n+1} \in \mathcal{L}_{n+1}$ be the degree $(n+1)$-part of $\bar{\ell}_{n-1}(x_1) \star \bar{\ell}_{n-1}(x_2) \star \cdots \star \bar{\ell}_{n-1}(x_p)$. By Lemma 3.3 we have $\omega = f(\zeta) = f(x_1 \cdots x_p) = \frac{1}{2}\sum_{i<j} X_i \wedge X_j = \frac{1}{2}\sum_{i<j}(X_i \otimes X_j - X_j \otimes X_i)$, where $X_i = [x_i]$. Since $S_1, \ldots, S_{2g}$ constitute a basis for H, we can uniquely write

(3.4)

$$\omega = \frac{1}{2}\sum_{i<j}(X_i \otimes X_j - X_j \otimes X_i) = \sum_{i=1}^{2g} S_i \otimes Z_i, \quad \text{where } Z_i = \sum_k c_{ik} S_k, \quad c_{ik} \in \mathbb{Z}.$$

Also, in view of applying (3.2) we write $V_{n+1} \in \mathcal{L}_{n+1} \subset H^{\otimes n+1}$ as

$$V_{n+1} = \sum_{i=1}^{2g} S_i \otimes V_n^{S_i}, \quad V_n^{S_i} \in H^{\otimes n}.$$

Now by Lemma 3.2, $Z_1, \ldots, Z_{2g}$ constitute a basis for H; hence the matrix $\{c_{ik}\}_{i,k}$ is of full rank. Let $\{d_{ik}\}_{i,k}$ be the inverse matrix of $\{c_{ik}\}_{i,k}$.

Proposition 3.4. *Keep the same notation as above. Set* $W_i := (-1/(n+1))\Phi(V_n^{S_i}) \in \mathcal{L}_n$ *for* $1 \leq i \leq 2g$, *and* $\ell_n(s_i) := \sum_k d_{ik} W_k$ *for* $1 \leq i \leq 2g$. *Then* $\{\ell_j(s_i) : 1 \leq j \leq n-1,\ 1 \leq i \leq 2g\} \cup \{\ell_n(s_i) : 1 \leq i \leq 2g\}$ *is a partial symplectic expansion up to degree* n.

Proof. Set $\bar{\ell}_n(s_i) = \bar{\ell}_{n-1}(s_i) + \ell_n(s_i)$. Understanding $\ell_n(s_i^{-1}) = -\ell_n(s_i)$, we have $\sum_{i=1}^p \ell_n(x_i) = 0$ since $\zeta \in [\pi, \pi]$. Hence we have $\bar{\ell}_n(x_1) \star \bar{\ell}_n(x_2) \star \cdots \star \bar{\ell}_n(x_p) \equiv \omega \bmod \widehat{T}_{n+1}$ from (3.3). By (2.2) we see that the degree $(n+1)$-part of $\bar{\ell}_n(x_1) \star \bar{\ell}_n(x_2) \star \cdots \star \bar{\ell}_n(x_p)$ equals

$$V_{n+1} + \frac{1}{2}\sum_{i<j}([X_i, \ell_n(x_j)] + [\ell_n(x_i), X_j]). \tag{3.5}$$

Let $\lambda\colon H \to \mathcal{L}_n$ be the linear map defined by $\lambda(S_i) = \ell_n(s_i)$ and apply the linear map $[\mathrm{id}, \lambda]\colon H^{\otimes 2} \to H^{\otimes n+1}$ to (3.4). Then we obtain

$$\frac{1}{2}\sum_{i<j}([X_i, \ell_n(x_j)] - [X_j, \ell_n(x_i)]) = \sum_{i=1}^{2g}[S_i, W_i'], \quad W_i' = \sum_k c_{ik}\ell_n(s_k).$$

But $W_i' = \sum_k \sum_j c_{ik} d_{kj} W_j = W_i$. Hence (3.5) is equal to

$$V_{n+1} + \sum_{i=1}^{2g}[S_i, W_i] = V_{n+1} - \frac{1}{n+1}\sum_{i=1}^{2g}[S_i, \Phi(V_n^{S_i})] = V_{n+1} - \frac{1}{n+1}\Phi(V_{n+1}) = 0,$$

since $V_{n+1} \in \mathcal{L}_{n+1}$. Therefore, we have $\bar{\ell}_n(x_1)\star\bar{\ell}_n(x_2)\star\cdots\star\bar{\ell}_n(x_p) \equiv \omega \bmod \widehat{T}_{n+2}$. This completes the proof. □

We can now conclude the proof of Theorem 1.1. Denote $\mathcal{S} = \{s_1, \ldots, s_{2g}\}$ and set $\ell_1(s_i) := S_i$, $1 \le i \le 2g$. By the Baker-Campbell-Hausdorff formula (2.2) and Lemma 3.3, $\{\ell_1(s_i)\}_{1\le i\le 2g}$ is a partial symplectic expansion up to degree 1. Applying Proposition 3.4, we obtain $\{\ell_j(s_i); 1 \le i \le 2g, j \ge 1\}$ satisfying (3.1) for any $n \ge 1$. Setting $\ell^{\mathcal{S}}(s_i) := \sum_{j=1}^{\infty} \ell_j(s_i) \in \widehat{\mathcal{L}}$ and $\theta^{\mathcal{S}}(s_i) := \exp(\ell^{\mathcal{S}}(s_i))$, we extend $\theta^{\mathcal{S}}$ to a homomorphism from π using the universality of the free group π. Then $\theta^{\mathcal{S}}$ is the desired symplectic expansion. Note that the result $\theta^{\mathcal{S}}$ does not depend on the total ordering on the set $\mathcal{S}$. This completes the proof of Theorem 1.1.

Remark 3.5. For a group-like expansion θ, we denote $\ell^\theta(x) = \sum_{j=1}^{\infty} \ell_j^\theta(x)$, $\ell_j^\theta(x) \in \mathcal{L}_j$, for $x \in \pi$. Proposition 3.4 can be phrased shortly as: a choice of a free generating set for π gives a canonical way of modifying any group-like expansion θ satisfying $\ell^\theta(\zeta) \equiv \omega \bmod \widehat{T}_{n+1}$ for some $n \ge 2$ into a group-like expansion satisfying the same congruence with $n+1$ replaced by $n+2$, without changing $\ell_j^\theta(x)$, for $1 \le j \le n-1$.

4. Naturality

Let $\mathrm{Aut}(\pi)$ be the automorphism group of π. For $\varphi \in \mathrm{Aut}(\pi)$, let $|\varphi|$ be the filter-preserving algebra automorphism of $\widehat{T}$ induced by the action of φ on the first homology H. If θ is a Magnus expansion, then the composite $|\varphi| \circ \theta \circ \varphi^{-1}$ is again a Magnus expansion.

We show a naturality of the symplectic expansion $\theta^{\mathcal{S}}$ given in Theorem 1.1. Note that fatgraph Magnus expansions have a similar property (see [1], Theorem 4.2).

Proposition 4.1. *Suppose $\varphi \in \mathrm{Aut}(\pi)$ satisfies $\varphi(\zeta) = \zeta$, or $\varphi(\zeta) = \zeta^{-1}$. Then*

$$\theta^{\varphi(\mathcal{S})} = |\varphi| \circ \theta^{\mathcal{S}} \circ \varphi^{-1}.$$

Proof. Let $\mathcal{S} = \{s_1, \ldots, s_{2g}\}$. We shall put $\mathcal{S}$ on the upper right of the objects V_{n+1}, ℓ_j, c_{ik}, etc., in the proof of Proposition 3.4 to indicate their dependence on $\mathcal{S}$.

The equality we are going to prove is equivalent to $\ell^{\varphi(\mathcal{S})}(\varphi(s_i)) = |\varphi|\ell^{\mathcal{S}}(s_i)$, or, $\ell_n^{\varphi(\mathcal{S})}(\varphi(s_i)) = |\varphi|\ell_n^{\mathcal{S}}(s_i)$ for any $n \ge 1$. We prove this by induction on n. Since $\ell_1^{\varphi(\mathcal{S})}(\varphi(s_i)) = [\varphi(s_i)] = |\varphi|[s_i]$, the case $n = 1$ is clear. Suppose $n \ge 2$.

First we assume $\varphi(\zeta) = \zeta$. Then $\varphi(x_1)\cdots\varphi(x_p)$ is a word in $\varphi(\mathcal{S})$ representing ζ, and we have $|\varphi|\omega = \omega$ since $\varphi(\zeta) = \zeta$ and the homomorphism $f\colon [\pi,\pi] \to \Lambda^2 H_{\mathbb{Z}}$

in §3 is $\mathrm{Aut}(\pi)$-equivariant. By the inductive assumption, we have $\bar{\ell}_{n-1}^{\varphi(\mathcal{S})}(\varphi(s_i)) = |\varphi|\bar{\ell}_{n-1}^{\mathcal{S}}(s_i)$; hence applying $|\varphi|$ to the congruence $\bar{\ell}_{n-1}^{\mathcal{S}}(x_1)\star\bar{\ell}_{n-1}^{\mathcal{S}}(x_2)\star\cdots\star\bar{\ell}_{n-1}^{\mathcal{S}}(x_p) \equiv \omega + V_{n+1}^{\mathcal{S}} \bmod \widehat{T}_{n+2}$, we obtain $V_{n+1}^{\varphi(\mathcal{S})} = |\varphi|V_{n+1}^{\mathcal{S}}$. Therefore, writing $V_{n+1}^{\varphi(\mathcal{S})} = \sum_{i=1}^{2g}(|\varphi|S_i) \otimes V_n^{|\varphi|S_i}$, we have $V_n^{|\varphi|S_i} = |\varphi|V_n^{S_i}$.

On the other hand, applying $|\varphi|$ to (3.4), we obtain

$$\omega = \sum_{i=1}^{2g} |\varphi|S_i \otimes Z_i^{\varphi(\mathcal{S})}, \quad Z_i^{\varphi(\mathcal{S})} = \sum_k c_{ik}|\varphi|S_k.$$

This implies that $c_{ik}^{\varphi(\mathcal{S})} = c_{ik}^{\mathcal{S}}$; hence $d_{ik}^{\varphi(\mathcal{S})} = d_{ik}^{\mathcal{S}}$. We conclude that $W_i^{\varphi(\mathcal{S})} = |\varphi|W_i^{\mathcal{S}}$ and $\ell_n^{\varphi(\mathcal{S})}(\varphi(s_i)) = |\varphi|\ell_n^{\mathcal{S}}(s_i)$, as desired.

If $\varphi(\zeta) = \zeta^{-1}$, the same argument shows that $V_{n+1}^{\varphi(\mathcal{S})} = -|\varphi|V_{n+1}^{\mathcal{S}}$ and $c_{ik}^{\varphi(\mathcal{S})} = -c_{ik}^{\mathcal{S}}$ because in this case $\varphi(x_1)\cdots\varphi(x_p)$ is a word in $\varphi(\mathcal{S})$ representing ζ^{-1}, and we have $|\varphi|\omega = -\omega$. Hence we again obtain $\ell_n^{\varphi(\mathcal{S})}(\varphi(s_i)) = |\varphi|\ell_n^{\mathcal{S}}(s_i)$. This completes the induction. □

5. Symplectic generators

Let $\mathcal{S}_0 = \{\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g\}$ be symplectic generators as in §2, and let $\theta^0 = \theta^{\mathcal{S}_0}$ be the symplectic expansion associated to $\mathcal{S}_0$, given by the algorithm of Theorem 1.1. For simplicity we write $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g = \xi_1, \ldots, \xi_{2g}$. Let $T \in \mathrm{Aut}(\pi)$ be the automorphism defined by $T(\xi_i) = \xi_{2g+1-i}$, $1 \leq i \leq 2g$. Then we have $T(\zeta) = \zeta^{-1}$ and $T(\mathcal{S}^0) = \mathcal{S}^0$. By Proposition 4.1, we obtain a certain kind of symmetry for θ^0.

Proposition 5.1. *Let θ^0 be the symplectic expansion as above. Then*

$$\theta^0(\xi_{2g+1-i}) = |T|\theta^0(\xi_i), \quad 1 \leq i \leq 2g.$$

Finally, we give a more explicit formula for $\ell^{\mathcal{S}_0}$ in a form suitable for computer-aided calculation. First we give another description of V_{n+1} which does not involve the Baker-Campbell-Hausdorff series. Let $n \geq 2$ and let $\{\ell_j(s_i) : 1 \leq j \leq n-1,\ 1 \leq i \leq 2g\}$ be a partial symplectic expansion up to degree $n-1$. Set $\bar{\theta}_{n-1}(s_i) := \exp(\bar{\ell}_{n-1}(s_i))$ and $\bar{\theta}_{n-1}(s_i^{-1}) := \exp(-\bar{\ell}_{n-1}(s_i))$. From (3.3), we have $\bar{\ell}_{n-1}(x_1) \star \bar{\ell}_{n-1}(x_2)\star\cdots\star\bar{\ell}_{n-1}(x_p) \equiv \omega + V_{n+1} \bmod \widehat{T}_{n+2}$. Applying the exponential, we obtain $\bar{\theta}_{n-1}(x_1)\bar{\theta}_{n-1}(x_2)\cdots\bar{\theta}_{n-1}(x_p) \equiv \exp(\omega) + V_{n+1} \bmod \widehat{T}_{n+2}$. Hence

$$V_{n+1} = \left(\bar{\theta}_{n-1}(x_1)\bar{\theta}_{n-1}(x_2)\cdots\bar{\theta}_{n-1}(x_p) - \exp(\omega)\right)_{n+1}, \tag{5.1}$$

where the subscript $n+1$ in the right-hand side means taking the degree $(n+1)$-part.

Let us consider the case $\mathcal{S} = \mathcal{S}_0$. Then $\zeta = \prod_{i=1}^{g}[\alpha_i, \beta_i]$. For $X, Y \in \widehat{T}_1$, by a direct computation, we have

$$(1+X)(1+Y)(1+X)^{-1}(1+Y)^{-1} = 1 + \sum_{i,j\geq 0}(-1)^{i+j}[X,Y]X^iY^j. \tag{5.2}$$

See Magnus-Karrass-Solitar [10], §5.5, (7a) for a similar formula. Therefore in case $\mathcal{S} = \mathcal{S}_0$, (5.1) becomes

$$V_{n+1} = \left(\prod_{i=1}^{g} G\left(\bar{\theta}_{n-1}(\alpha_i) - 1, \bar{\theta}_{n-1}(\beta_i) - 1\right) - \exp(\omega)\right)_{n+1},$$

where $G(X,Y)$ is the right-hand side of (5.2). From (2.1) and (3.4), we obtain the following recursive formulas for $\ell^{\mathcal{S}_0}$:

$$\ell_n^{\mathcal{S}_0}(\alpha_i) = \frac{1}{n+1}\Phi(V_n^{B_i}),$$
$$\ell_n^{\mathcal{S}_0}(\beta_i) = \frac{-1}{n+1}\Phi(V_n^{A_i}).$$

In this way we can effectively compute the terms of $\ell^{\mathcal{S}_0}(\xi_i)$. Here we give the first few terms of $\ell^{\mathcal{S}_0}$ for $g=1,2$.

Example 5.2 (the case of genus 1)**.** For simplicity, we write $\alpha_1=\alpha$, $\beta_1=\beta$ and $A_1=A$, $B_1=B$. Modulo $\widehat{T}_6$, we have

$$\begin{aligned}\ell^{\mathcal{S}_0}(\alpha) \equiv A &+ \frac{1}{2}[A,B] + \frac{1}{12}[B,[B,A]] - \frac{1}{8}[A,[A,B]] + \frac{1}{24}[A,[A,[A,B]]]\\
&- \frac{1}{720}[B,[B,[B,[B,A]]]] - \frac{1}{288}[A,[A,[A,[A,B]]]] - \frac{1}{288}[A,[B,[B,[B,A]]]]\\
&- \frac{1}{288}[B,[A,[A,[A,B]]]] + \frac{1}{144}[[A,B],[B,[B,A]]] + \frac{1}{128}[[A,B],[A,[A,B]]];\\
\ell^{\mathcal{S}_0}(\beta) \equiv B &- \frac{1}{2}[A,B] + \frac{1}{12}[A,[A,B]] - \frac{1}{8}[B,[B,A]] + \frac{1}{24}[B,[B,[B,A]]]\\
&- \frac{1}{720}[A,[A,[A,[A,B]]]] - \frac{1}{288}[B,[B,[B,[B,A]]]] - \frac{1}{288}[B,[A,[A,[A,B]]]]\\
&- \frac{1}{288}[A,[B,[B,[B,A]]]] - \frac{1}{144}[[A,B],[A,[A,B]]] - \frac{1}{128}[[A,B],[B,[B,A]]].\end{aligned}$$

Example 5.3 (the case of genus 2)**.** Modulo $\widehat{T}_5$, we have

$$\begin{aligned}\ell^{\mathcal{S}_0}(\alpha_1) \equiv A_1 &+ \frac{1}{2}[A_1,B_1]\\
&+ \frac{1}{12}[B_1,[B_1,A_1]] - \frac{1}{8}[A_1,[A_1,B_1]] - \frac{1}{4}[A_1,[A_2,B_2]]\\
&+ \frac{1}{24}[A_1,[A_1,[A_1,B_1]]] - \frac{1}{10}[[A_1,B_1],[A_2,B_2]] + \frac{1}{40}[A_1,[B_1,[A_2,B_2]]]\\
&+ \frac{1}{40}[A_1,[B_2,[A_2,B_2]]] + \frac{1}{40}[A_1,[A_1,[A_2,B_2]]] + \frac{1}{40}[A_1,[A_2,[A_2,B_2]]];\\
\ell^{\mathcal{S}_0}(\beta_1) \equiv B_1 &- \frac{1}{2}[A_1,B_1]\\
&+ \frac{1}{12}[A_1,[A_1,B_1]] - \frac{1}{8}[B_1,[B_1,A_1]] - \frac{1}{4}[B_1,[A_2,B_2]]\\
&+ \frac{1}{24}[B_1,[B_1,[B_1,A_1]]] + \frac{1}{10}[[A_1,B_1],[A_2,B_2]] + \frac{1}{40}[B_1,[A_1,[A_2,B_2]]]\\
&+ \frac{1}{40}[B_1,[A_2,[A_2,B_2]]] + \frac{1}{40}[B_1,[B_1,[A_2,B_2]]] + \frac{1}{40}[B_1,[B_2,[A_2,B_2]]];\\
\ell^{\mathcal{S}_0}(\alpha_2) \equiv A_2 &+ \frac{1}{2}[A_2,B_2]\\
&+ \frac{1}{12}[B_2,[B_2,A_2]] - \frac{1}{8}[A_2,[A_2,B_2]] + \frac{1}{4}[A_2,[A_1,B_1]]\\
&+ \frac{1}{24}[A_2,[A_2,[A_2,B_2]]] - \frac{1}{10}[[A_1,B_1],[A_2,B_2]] - \frac{1}{40}[A_2,[B_2,[A_1,B_1]]]\\
&- \frac{1}{40}[A_2,[B_1,[A_1,B_1]]] - \frac{1}{40}[A_2,[A_2,[A_1,B_1]]] - \frac{1}{40}[A_2,[A_1,[A_1,B_1]]];\end{aligned}$$

$$\begin{aligned}\ell^{\mathcal{S}_0}(\beta_2) \equiv B_2 &- \frac{1}{2}[A_2, B_2] \\ &+ \frac{1}{12}[A_2,[A_2,B_2]] - \frac{1}{8}[B_2,[B_2,A_2]] + \frac{1}{4}[B_2,[A_1,B_1]] \\ &+ \frac{1}{24}[B_2,[B_2,[B_2,A_2]]] + \frac{1}{10}[[A_1,B_1],[A_2,B_2]] - \frac{1}{40}[B_2,[A_2,[A_1,B_1]]] \\ &- \frac{1}{40}[B_2,[A_1,[A_1,B_1]]] - \frac{1}{40}[B_2,[B_2,[A_1,B_1]]] - \frac{1}{40}[B_2,[B_1,[A_1,B_1]]].\end{aligned}$$

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References

1. A. J. Bene, N. Kawazumi, and R. C. Penner, Canonical extensions of the Johnson homomorphisms to the Torelli groupoid, Adv. Math. **221**, 627–659 (2009). MR2508933 (2010b:32015)
2. W. M. Goldman, Invariant functions on Lie groups and Hamiltonian flows of surface group representations, Invent. Math. **85**, 263–302 (1986). MR846929 (87j:32069)
3. D. Johnson, An abelian quotient of the mapping class group $\mathcal{I}_g$, Math. Ann. **249**, 225–242 (1980). MR579103 (82a:57008)
4. D. Johnson, A survey of the Torelli group, Contemporary Math. **20**, 165–179, Amer. Math. Soc., Providence, RI (1983). MR718141 (85d:57009)
5. N. Kawazumi, Cohomological aspects of Magnus expansions, preprint, math.GT/0505497 (2005).
6. N. Kawazumi, Harmonic Magnus expansion on the universal family of Riemann surfaces, preprint, math.GT/0603158 (2006).
7. N. Kawazumi and Y. Kuno, The logarithms of Dehn twists, preprint, arXiv:1008.5017 (2010).
8. M. Kontsevich, Formal (non)-commutative symplectic geometry, in: "The Gel'fand Mathematical Seminars, 1990-1992", Birkhäuser, Boston, 173–187 (1993). MR1247289 (94i:58212)
9. W. Magnus, Beziehungen zwischen Gruppen und Idealen in einem speziellen Ring, Math. Ann. **111**, 259–280 (1935). MR1512992
10. W. Magnus, A. Karrass, and D. Solitar, Combinatorial group theory, Dover, New York (1976). MR0422434 (54:10423)
11. G. Massuyeau, Infinitesimal Morita homomorphisms and the tree-level of the LMO invariant, preprint, arXiv:0809.4629 (2008).
12. J. -P. Serre, Lie algebras and Lie groups, Lecture Notes in Mathematics **1500**, Springer-Verlag, Berlin (2006). MR2179691 (2006e:17001)

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