

A combinatorial formula for rank 2 cluster variables

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Abstract Let r be any positive integer, and let x_1, x_2 be indeterminates. We consider the sequence $\{x_n\}$ defined by the recursive relation

$$x_{n+1} = (x_n^r + 1)/x_{n-1}$$

for any integer n . Finding a combinatorial expression for x_n as a rational function of x_1 and x_2 has been an open problem since 2001. We give a direct elementary formula for x_n in terms of subpaths of a specific lattice path in the plane. The formula is manifestly positive, providing a new proof of a result by Nakajima and Qin.

Keywords Laurent polynomials · Cluster algebras · Lattice paths

1 Introduction

Let r be any positive integer, and x_1, x_2 be indeterminates. We consider the sequence $\{x_n\}$ defined by the recursive relation

$$x_{n+1} = (x_n^r + 1)/x_{n-1}$$

for any integer n . These seemingly elementary sequences have turned out to be stubbornly hard to analyze. In the case $r = 1$, surprisingly, we obtain a periodic sequence

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of Laurent polynomials of x_1 and x_2 :

$$x_3 = \frac{x_2 + 1}{x_1}, \quad x_4 = \frac{x_1 + x_2 + 1}{x_1 x_2}, \quad x_5 = \frac{x_1 + 1}{x_2}, \quad x_6 = x_1, \quad x_7 = x_2, \quad \dots$$

In all other cases $r > 1$, the sequence is without repetition. It is an important problem in the theory of cluster algebras to understand the sequence for an arbitrary positive integer r , because each sequence forms the set of cluster variables of a rank two cluster algebra.

Cluster algebras have been introduced by Fomin and Zelevinsky in [10] in the context of total positivity and canonical bases in Lie theory. Since then cluster algebras have been shown to be related to various fields in mathematics including representation theory of finite dimensional algebras, Teichmüller theory, Poisson geometry, combinatorics, Lie theory, tropical geometry and mathematical physics.

A cluster algebra is a subalgebra of a field of rational functions in n variables x_1, x_2, \dots, x_n , given by specifying a set of generators, the so-called *cluster variables*. These generators are constructed in a recursive way, starting from the initial variables x_1, x_2, \dots, x_n , by a procedure called *mutation*, which is determined by the choice of a skew-symmetric $n \times n$ integer matrix B . Although each mutation is an elementary operation, it is very difficult to compute cluster variables in general, because of the recursive character of the construction.

Finding explicit computable direct formulas for the cluster variables is one of the main open problems in the theory of cluster algebras and has been studied by many mathematicians. Fomin and Zelevinsky showed in [10] that every cluster variable is a Laurent polynomial in the initial variables x_1, x_2, \dots, x_n , and they conjectured that this Laurent polynomial has positive coefficients. This *positivity conjecture* has been proved in several special cases; the most general ones being cluster algebras from surfaces, see [17], cluster algebras that have a bipartite seed, see [14, 18] and cluster algebras whose initial seed is acyclic [19].

Direct formulas for the Laurent polynomials have been obtained in several special cases. The most general results are the following:

- a formula involving the Euler–Poincaré characteristic of quiver Grassmannians obtained in [12] using categorification and generalizing results in [5, 6]. While this formula shows a very interesting connection between cluster algebras and geometry, it is of limited computational use, since the Euler–Poincaré characteristics of quiver Grassmannians are hard to compute.
- an elementary combinatorial formula for cluster algebras from surfaces given in [17] building on [20–22].
- a formula for cluster variables corresponding to string modules as a product of 2×2 matrices obtained in [2], generalizing a result in [3].

In this paper, we consider cluster algebras of rank $n = 2$, that is, the integer matrix B is of the form

$$\begin{bmatrix} 0 & r \\ -r & 0 \end{bmatrix},$$

and the cluster variables form the sequence $\{x_n\}$ given by the recursion above. The rank 2 case is considerably simpler than the general case, but even so, the problem of

finding an elementary formula for the cluster variables which also shows positivity had not been solved yet. In [7, 8, 16, 23] such formulas are given in the case $r = 2$, which is also a special case of the ones considered in [2, 17, 22]. In [18, 19] the positivity conjecture was proved for arbitrary r using Euler–Poincaré characteristics, and in [15] an elementary formula is proved but it does not show positivity. We also want to point out that the positivity conjecture has been shown in the case where the 2×2 matrix B is skew-symmetrizable in [9], again using Euler–Poincaré characteristics.

The main result of this paper is a direct combinatorial formula which computes the cluster variables as a sum of monomials each with coefficient 1. Clearly, this formula shows positivity. Moreover, each monomial is explicitly given, and the sum is parametrized by subpaths of a specific lattice path in the plane.

2 Main result

Fix a positive integer $r \geq 2$.

Definition 1 Let $\{c_n\}$ be the sequence defined by the recurrence relation

$$c_n = rc_{n-1} - c_{n-2},$$

with the initial condition $c_1 = 0, c_2 = 1$. When $r = 2, c_n = n - 1$. When $r > 2$, it is easy to see that

$$\begin{aligned} c_n &= \frac{1}{\sqrt{r^2 - 4}} \left(\frac{r + \sqrt{r^2 - 4}}{2} \right)^{n-1} - \frac{1}{\sqrt{r^2 - 4}} \left(\frac{r - \sqrt{r^2 - 4}}{2} \right)^{n-1} \\ &= \sum_{i \geq 0} (-1)^i \binom{n-2-i}{i} r^{n-2-2i}. \end{aligned}$$

For example, for $r = 3$, the sequence c_n takes the following values:

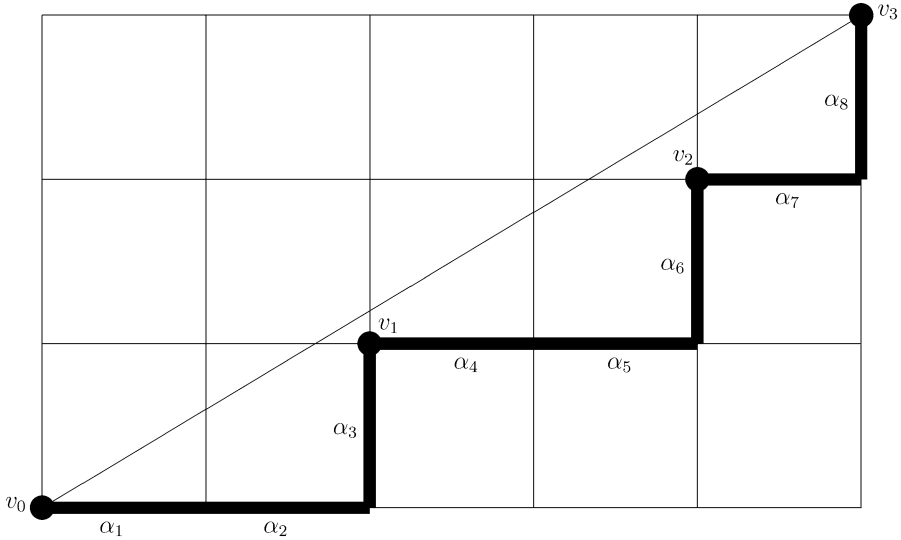
$$0, 1, 3, 8, 21, 55, 144, \dots$$

In order to state our theorem, we fix an integer $n \geq 4$. Consider a rectangle with vertices $(0, 0), (0, c_{n-2}), (c_{n-1} - c_{n-2}, c_{n-2})$ and $(c_{n-1} - c_{n-2}, 0)$. In what follows, by the diagonal we mean the line segment from $(0, 0)$ to $(c_{n-1} - c_{n-2}, c_{n-2})$. A Dyck path is a lattice path from $(0, 0)$ to $(c_{n-1} - c_{n-2}, c_{n-2})$ that proceeds by NORTH or EAST steps and never goes above the diagonal.

Definition 2 A Dyck path below the diagonal is said to be maximal if no subpath of any other Dyck path lies above it. The maximal Dyck path, denoted by \mathcal{D}_n , consists of $(w_0, \alpha_1, w_1, \dots, \alpha_{c_{n-1}}, w_{c_{n-1}})$, where $w_0, \dots, w_{c_{n-1}}$ are vertices and $\alpha_1, \dots, \alpha_{c_{n-1}}$ are edges, such that $w_0 = (0, 0)$ is the south-west corner of the rectangle, α_i connects w_{i-1} and w_i , and $w_{c_{n-1}} = (c_{n-1} - c_{n-2}, c_{n-2})$ is the north-east corner of the rectangle.

Remark 3 The word obtained from \mathcal{D}_n by forgetting the vertices w_i and replacing each horizontal edge by the letter x and each vertical edge by the letter y is the Christoffel word of slope $c_{n-2}/(c_{n-1} - c_{n-2})$, see for example [4].

Example 4 Let $r = 3$ and $n = 5$. Then \mathcal{D}_5 is illustrated as follows:



Definition 5 Let $i_1 < \dots < i_{c_{n-2}}$ be the sequence of integers such that α_{i_j} is vertical for any $1 \leq j \leq c_{n-2}$. Define a sequence $v_0, v_1, \dots, v_{c_{n-2}}$ of vertices by $v_0 = (0, 0)$ and $v_j = w_{i_j}$.

Definition 6 For any $i < j$, let $s_{i,j}$ be the slope of the line through v_i and v_j . Let s be the slope of the diagonal, that is, $s = s_{0,c_{n-2}}$.

Definition 7 For any $0 \leq i < k \leq c_{n-2}$, we define a *colored subpath* $\alpha(i, k)$ to be the subpath of \mathcal{D}_n defined as follows (for illustrations see Example 11).

- (1) If $s_{i,t} \leq s$ for all t such that $i < t \leq k$, then let $\alpha(i, k)$ be the subpath from v_i to v_k . Each of these subpaths will be called a BLUE subpath. See Example 11.
- (2) If $s_{i,t} > s$ for some $i < t \leq k$, then
 - (2-a) If the smallest such t is of the form $i + c_m - wc_{m-1}$ for some integers $3 \leq m \leq n - 2$ and $1 \leq w < r - 1$, then let $\alpha(i, k)$ be the subpath from v_i to v_k . Each of these subpaths will be called a GREEN subpath. When m and w are specified, it will be said to be (m, w) -green;
 - (2-b) Otherwise, let $\alpha(i, k)$ be the subpath from the immediate predecessor of v_i to v_k . Each of these subpaths will be called a RED subpath.

Note that every pair (i, k) defines exactly one subpath $\alpha(i, k)$. We denote the set of all these subpaths together with the single edges α_i by $\mathcal{P}(\mathcal{D}_n)$, that is,

$$\mathcal{P}(\mathcal{D}_n) = \{ \alpha(i, k) \mid 0 \leq i < k \leq c_{n-2} \} \cup \{ \alpha_1, \dots, \alpha_{c_{n-1}} \}.$$

Now we define a set $\mathcal{F}(\mathcal{D}_n)$ of certain sequences of non-overlapping subpaths of \mathcal{D}_n . This set will parametrize the monomials in our expansion formula.

Definition 8 Let

$$\begin{aligned} \mathcal{F}(\mathcal{D}_n) = \{ & \{\beta_1, \dots, \beta_t\} \mid t \geq 0, \beta_j \in \mathcal{P}(\mathcal{D}_n) \text{ for all } 1 \leq j \leq t, \\ & \text{if } j \neq j' \text{ then } \beta_j \text{ and } \beta_{j'} \text{ have no common edge,} \\ & \text{if } \beta_j = \alpha(i, k) \text{ and } \beta_{j'} = \alpha(i', k') \text{ then } i \neq k' \text{ and } i' \neq k, \\ & \text{and if } \beta_j \text{ is } (m, w)\text{-green then at least one of the } (c_{m-1} - wc_{m-2}) \\ & \text{preceding edges of } v_i \text{ is contained in some } \beta_{j'} \}. \end{aligned}$$

For any $\beta = \{\beta_1, \dots, \beta_t\}$, let $|\beta|_2$ be the total number of edges in β_1, \dots, β_t , and $|\beta|_1 = \sum_{j=1}^t |\beta_j|_1$, where

$$|\beta_j|_1 = \begin{cases} 0, & \text{if } \beta_j = \alpha_i \text{ for some } 1 \leq i \leq c_{n-1} \\ k - i, & \text{if } \beta_j = \alpha(i, k) \text{ for some } 0 \leq i < k \leq c_{n-2}. \end{cases}$$

Theorem 9 Let $r \geq 2$ be a positive integer. Let x_1, x_2 be indeterminates. Define the sequence $\{x_n\}$ by

$$x_{n+1} = (x_n^r + 1)/x_{n-1} \quad \text{for any integer } n.$$

Then for $n \geq 4$,

$$x_n = x_1^{-c_{n-1}} x_2^{-c_{n-2}} \sum_{\beta \in \mathcal{F}(\mathcal{D}_n)} x_1^{r|\beta|_1} x_2^{r(c_{n-1}-|\beta|_2)} \tag{2.1}$$

and

$$x_{3-n} = x_2^{-c_{n-1}} x_1^{-c_{n-2}} \sum_{\beta \in \mathcal{F}(\mathcal{D}_n)} x_2^{r|\beta|_1} x_1^{r(c_{n-1}-|\beta|_2)}. \tag{2.2}$$

Remark 10 We can rewrite (2.1) as

$$x_n = x_1^{-c_{n-1}} x_2^{-c_{n-2}} \sum \sum x_1^{r|\beta|_1} x_2^{r(c_{n-1}-|\beta|_2)}$$

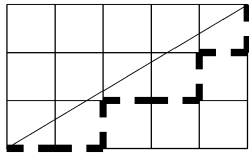
where the first sum is over all sequences

$$0 \leq i_1 < k_1 < i_2 < k_2 < \dots < i_e < k_e \leq c_{n-2}$$

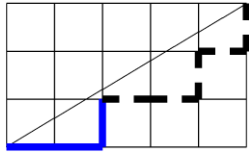
and the second sum is over all $\beta \in \mathcal{F}(\mathcal{D}_n)$ whose colored subpaths are precisely the $\alpha(i_j, k_j)$ for $1 \leq j \leq e$.

Here x_n ($n \in \mathbb{Z}$) are called cluster variables, and the cluster algebra of rank 2 is the \mathbb{Q} -subalgebra generated by all cluster variables x_n in the field of rational functions in the commutative variables x_1 and x_2 . Note that x_3 and x_0 are directly given by the recurrence.

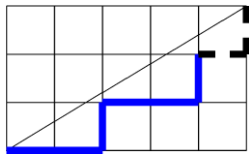
Example 11 Let $r = 3$ and $n = 5$. The illustrations below show the possible configurations for $\beta \in \mathcal{F}(\mathcal{D}_n)$. If the edge α_i is marked \blacksquare , then α_i can occur in β .



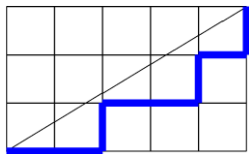
$$\sum_{\beta \subset \{\alpha_1, \dots, \alpha_8\}} x_1^{r|\beta|_1} x_2^{r(c_{n-1}-|\beta|_2)} = (1 + x_2^3)^8$$



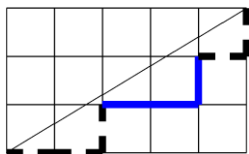
$$\sum_{\{\alpha(0,1)\} \subset \beta \subset \{\alpha(0,1)\} \cup \{\alpha_4, \dots, \alpha_8\}} x_1^{r|\beta|_1} x_2^{r(c_{n-1}-|\beta|_2)} = x_1^3 (1 + x_2^3)^5$$



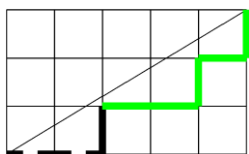
$$\sum_{\{\alpha(0,2)\} \subset \beta \subset \{\alpha(0,2)\} \cup \{\alpha_7, \alpha_8\}} x_1^{r|\beta|_1} x_2^{r(c_{n-1}-|\beta|_2)} = x_1^6 (1 + x_2^3)^2$$



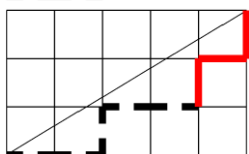
$$\sum_{\beta = \{\alpha(0,3)\}} x_1^{r|\beta|_1} x_2^{r(c_{n-1}-|\beta|_2)} = x_1^9$$



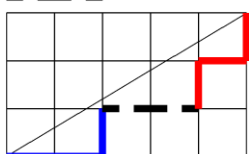
$$\sum_{\{\alpha(1,2)\} \subset \beta \subset \{\alpha(1,2)\} \cup \{\alpha_1, \alpha_2, \alpha_3, \alpha_7, \alpha_8\}} x_1^{r|\beta|_1} x_2^{r(c_{n-1}-|\beta|_2)} = x_1^3 (1 + x_2^3)^5$$



$$\sum_{\{\alpha(1,3)\} \cup \{\alpha_3\} \subset \beta \subset \{\alpha(1,3)\} \cup \{\alpha_1, \alpha_2, \alpha_3\}} x_1^{r|\beta|_1} x_2^{r(c_{n-1}-|\beta|_2)} = x_1^6 (1 + x_2^3)^2$$



$$\sum_{\{\alpha(2,3)\} \subset \beta \subset \{\alpha(2,3)\} \cup \{\alpha_1, \dots, \alpha_5\}} x_1^{r|\beta|_1} x_2^{r(c_{n-1}-|\beta|_2)} = x_1^3 (1 + x_2^3)^5$$



$$\sum_{\{\alpha(0,1), \alpha(2,3)\} \subset \beta \subset \{\alpha(0,1), \alpha(2,3)\} \cup \{\alpha_4, \alpha_5\}} x_1^{r|\beta|_1} x_2^{r(c_{n-1}-|\beta|_2)} = x_1^6 (1 + x_2^3)^2$$

These configurations are grouped according to Remark 10: in the first picture there are no colored subpath, each of the next four pictures has precisely one blue subpath,

the sixth picture has a (3, 1)-green subpath forcing the preceding edge to be included in β , the seventh picture has a red subpath and the last picture has a blue subpath and a red subpath.

Adding the above eight polynomials together gives

$$\begin{aligned} &x_2^{24} + 8x_2^{21} + 3x_1^3x_2^{15} + 28x_2^{18} + 15x_1^3x_2^{12} + 56x_2^{15} + 3x_1^6x_2^6 \\ &+ 30x_1^3x_2^9 + 70x_2^{12} + x_1^9 + 6x_1^6x_2^3 + 30x_1^3x_2^6 + 56x_2^9 + 3x_1^6 \\ &+ 15x_1^3x_2^3 + 28x_2^6 + 3x_1^3 + 8x_2^3 + 1. \end{aligned} \tag{2.3}$$

Then x_5 is obtained by dividing (2.3) by $x_1^8x_2^3$.

We also obtain the following formula for the F -polynomials. Let g_ℓ be the g -vector and let F_ℓ be the F -polynomial of x_ℓ , for all integers ℓ . Then $g_3 = (-1, r)$, $g_0 = (0, -1)$, $F_3 = y_1 + 1$ and $F_0 = y_2 + 1$, and all other cases are described in the following result.

Corollary 12 *Let $n \geq 4$. Then*

$$\begin{aligned} g_n &= (-c_{n-1}, c_n), \quad g_{3-n} = (-c_{n-2}, c_{n-3}), \quad \text{and} \\ F_n &= \sum_{\beta \in \mathcal{F}(\mathcal{D}_n)} y_1^{|\beta|_2} y_2^{|\beta|_1}, \quad F_{3-n} = \sum_{\beta \in \mathcal{F}(\mathcal{D}_n)} y_1^{c_{n-2}-|\beta|_1} y_2^{c_{n-1}-|\beta|_2}. \end{aligned}$$

Proof The formulas for the g -vectors follow easily from the recursive relations for g -vectors given in [11, Proposition 6.6]. To prove the formulas for the F -polynomials, we work in the cluster algebra with principal coefficients at the seed

$$\Sigma = \left((x_1, x_2), (y_1, y_2), \begin{bmatrix} 0 & r \\ -r & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \right).$$

In this cluster algebra, the Laurent expansion in Σ of any cluster variable x_ℓ is homogeneous with respect to the \mathbb{Z}^2 -grading

$$\deg x_1 = (1, 0), \quad \deg x_2 = (0, 1), \quad \deg y_1 = (0, r), \quad \deg y_2 = (-r, 0),$$

and, moreover, $\deg x_\ell = g_\ell$, see [11, Proposition 6.1]. It follows that the expansion formulas with principal coefficients are of the form

$$x_n = x_1^{-c_{n-1}} x_2^{-c_{n-2}} \sum_{\beta \in \mathcal{F}(\mathcal{D}_n)} x_1^{r|\beta|_1} x_2^{r(c_{n-1}-|\beta|_2)} y_1^{a_1} y_2^{a_2} \tag{2.4}$$

and

$$x_{3-n} = x_2^{-c_{n-1}} x_1^{-c_{n-2}} \sum_{\beta \in \mathcal{F}(\mathcal{D}_n)} x_2^{r|\beta|_1} x_1^{r(c_{n-1}-|\beta|_2)} y_1^{b_1} y_2^{b_2}, \tag{2.5}$$

where a_1, a_2, b_1 and b_2 are integers such that

$$g_n = \deg x_n = (-c_{n-1} + r|\beta|_1 - r a_2, -c_{n-2} + r(c_{n-1} - |\beta|_2) + r a_1)$$

and

$$g_{3-n} = \deg x_{3-n} = (-c_{n-2} + r(c_{n-1} - |\beta|_2) - r b_2, -c_{n-1} + r|\beta|_1 + r b_1).$$

Now, using the formulas for the g -vectors in the corollary and the recurrence $c_n = r c_{n-1} - c_{n-2}$, we get $a_1 = |\beta|_2, a_2 = |\beta|_1, b_1 = c_{n-2} - |\beta|_1$ and $b_2 = c_{n-1} - |\beta|_2$. The formulas for the F -polynomials now follow by setting $x_1 = x_2 = 1$ in (2.4) and (2.5). □

Theorem 9 also enables us to compute the Euler–Poincaré characteristics of certain quiver Grassmannians. Let Q_r be the generalized Kronecker quiver with two vertices 1 and 2, and r arrows from 1 to 2. For $n \geq 3$, let $M(n)$ (resp. $M(3 - n)$) be the unique (up to an isomorphism) indecomposable representation of dimension vector (c_{n-1}, c_{n-2}) (resp. (c_{n-2}, c_{n-1})). Then the indecomposable projective representations are $P(2) = M(0)$ and $P(1) = M(-1)$, and any indecomposable preprojective representation is of the form $M(3 - n)$. Similarly, the indecomposable injective representations are $I(1) = M(3)$ and $I(2) = M(4)$, and any indecomposable preinjective representation is of the form $M(n)$. Let $\text{Gr}_{(e_1, e_2)}(M(n))$ (resp. $\text{Gr}_{(e_1, e_2)}(M(3 - n))$) be the variety parametrizing all subrepresentations of $M(n)$ (resp. $M(3 - n)$) of dimension vector (e_1, e_2) . We use a result of Caldero and Zelevinsky [7, Theorem 3.2 and (3.5)].

Theorem 13 (Caldero and Zelevinsky) *Let $n \geq 3$. Then the cluster variable x_n is equal to*

$$x_1^{-c_{n-1}} x_2^{-c_{n-2}} \sum_{e_1, e_2} \chi(\text{Gr}_{(e_1, e_2)}(M(n))) x_1^{r(c_{n-2} - e_2)} x_2^{r e_1},$$

and x_{3-n} is equal to

$$x_1^{-c_{n-2}} x_2^{-c_{n-1}} \sum_{e_1, e_2} \chi(\text{Gr}_{(e_1, e_2)}(M(3 - n))) x_1^{r(c_{n-1} - e_2)} x_2^{r e_1}.$$

In the case $n = 3$, the representations $M(3)$ and $M(0)$ are simple representation, meaning that they do not have non-trivial subrepresentations. Therefore the Euler–Poincaré characteristics of their Grassmannians are non-zero if and only if the dimension vector (e_1, e_2) is equal to $(0, 0)$ or equal to the dimension of the simple representation itself, in which cases the Euler–Poincaré characteristic is 1. For $n \geq 4$, we get the following result.

Corollary 14 *Let $n \geq 4$. Then for any integers e_1 and e_2 ,*

$$\chi(\text{Gr}_{(e_1, e_2)}(M(n))) = \#\{\beta \in \mathcal{F}(\mathcal{D}_n) : |\beta|_1 = c_{n-2} - e_2, |\beta|_2 = c_{n-1} - e_1\}$$

and

$$\chi(\text{Gr}_{(e_1, e_2)}(M(3 - n))) = \#\{\beta \in \mathcal{F}(\mathcal{D}_n) : |\beta|_1 = e_1, |\beta|_2 = e_2\}.$$

3 Proofs

Since (2.2) can be easily obtained from (2.1) by interchanging x_1 and x_2 , we will prove (2.1). In addition to our theoretical proof, our formula is checked by Macaulay 2 for any r, n with $r + n \leq 11$. We need more notation.

Definition 15 For integers u, n with $3 \leq u \leq n - 1$, let

$$\begin{aligned}
 \mathcal{T}^{\geq u}(\mathcal{D}_n) := \{ \{ \beta_1, \dots, \beta_t \} \mid t \geq 1, \beta_j \in \mathcal{P}(\mathcal{D}_n) \text{ for all } 1 \leq j \leq t, \\
 \text{if } j \neq j' \text{ then } \beta_j \text{ and } \beta_{j'} \text{ have no common edge,} \\
 \text{if } \beta_j = \alpha(i, k) \text{ and } \beta_{j'} = \alpha(i', k') \text{ then } i \neq k' \text{ and } i' \neq k, \\
 \text{and there exist integers } j, w, m, \text{ with } m \geq u \text{ such that} \\
 \beta_j \text{ is } (m, w)\text{-green and none of the } (c_{m-1} - wc_{m-2}) \\
 \text{preceding edges of } v_i \text{ is contained in any } \beta_{j'} \}.
 \end{aligned}$$

The definition of $\mathcal{T}^{\geq u}(\mathcal{D}_n)$ is similar to the definition of $\mathcal{F}(\mathcal{D}_n)$ with the difference that the elements of $\mathcal{T}^{\geq u}(\mathcal{D}_n)$ violate the condition from green paths which is required for the elements in $\mathcal{F}(\mathcal{D}_n)$.

Definition 16 Let

$$\begin{aligned}
 \tilde{\mathcal{F}}(\mathcal{D}_n) = \{ \{ \beta_1, \dots, \beta_t \} \mid t \geq 0, \beta_j \in \mathcal{P}(\mathcal{D}_n) \text{ for all } 1 \leq j \leq t, \\
 \text{if } j \neq j' \text{ then } \beta_j \text{ and } \beta_{j'} \text{ have no common edge,} \\
 \text{and if } \beta_j = \alpha(i, k) \text{ and } \beta_{j'} = \alpha(i', k') \text{ then } i \neq k' \text{ and } i' \neq k \}.
 \end{aligned}$$

The definition of $\tilde{\mathcal{F}}(\mathcal{D}_n)$ is the same as the definition $\mathcal{F}(\mathcal{D}_n)$ but without the restrictions from green paths. Moreover

$$\mathcal{F}(\mathcal{D}_n) = \tilde{\mathcal{F}}(\mathcal{D}_n) \setminus \mathcal{T}^{\geq 3}(\mathcal{D}_n). \tag{3.1}$$

Lemma 17 *If $m \geq n - 1$, then there do not exist i, w ($1 \leq w < r - 1$) such that $\min\{t \mid i < t \leq c_{n-2}, s_{i,t} > s\}$ is of the form $i + c_m - wc_{m-1}$. In particular, for any $n \geq 4$, the set $\mathcal{T}^{\geq n-1}(\mathcal{D}_n)$ is empty.*

Proof First note that $c_m - wc_{m-1}$ is an increasing sequence; this can be shown by induction using the facts that $w \leq r - 2$ and $c_m = rc_{m-1} - c_{m-2}$. Thus if $m \geq n - 1$ and $\min\{t \mid i < t \leq c_{n-2}, s_{i,t} > s\} = i + c_m - wc_{m-1}$, then $\min\{t \mid i < t \leq c_{n-2}, s_{i,t} > s\} \geq c_{n-1} - wc_{n-2} = c_{n-2}(r - w) - c_{n-3} > c_{n-2}(r - w - 1) \geq c_{n-2}$. But this is a contradiction, because $v_{c_{n-2}}$ is the highest vertex in \mathcal{D}_n . \square

Let $z_3 = x_3$ and

$$z_n = x_1^{-c_{n-1}} x_2^{-c_{n-2}} \sum_{\beta \in \tilde{\mathcal{F}}(\mathcal{D}_n)} x_1^{r|\beta|_1} x_2^{r(c_{n-1}-|\beta|_2)} \tag{3.2}$$

for $n \geq 4$.

Let $K = \mathbb{Q}(x_1, x_2)$ be the field of rational functions in the commutative variables x_1 and x_2 . Let F be the automorphism of K , which is defined by

$$F : \begin{cases} x_1 \mapsto x_2 \\ x_2 \mapsto \frac{1+x_2^r}{x_1} \end{cases}. \quad (3.3)$$

Lemma 18 *Let $n \geq 3$. Then*

$$z_{n+1} = F(z_n) + x_1^{-c_n} x_2^{-c_{n-1}} \sum_{\beta \in T^{\geq 3}(\mathcal{D}_{n+1}) \setminus T^{\geq 4}(\mathcal{D}_{n+1})} x_1^{r|\beta|_1} x_2^{r(c_n - |\beta|_2)}.$$

Lemma 19 *Let $u \geq 3$ and $n \geq u + 2$. Then*

$$\begin{aligned} & F \left(x_1^{-c_{n-1}} x_2^{-c_{n-2}} \sum_{\beta \in T^{\geq u}(\mathcal{D}_n) \setminus T^{\geq u+1}(\mathcal{D}_n)} x_1^{r|\beta|_1} x_2^{r(c_{n-1} - |\beta|_2)} \right) \\ &= x_1^{-c_n} x_2^{-c_{n-1}} \sum_{\beta \in T^{\geq u+1}(\mathcal{D}_{n+1}) \setminus T^{\geq u+2}(\mathcal{D}_{n+1})} x_1^{r|\beta|_1} x_2^{r(c_n - |\beta|_2)}. \end{aligned}$$

Lemma 20 *Let $n \geq 4$. Then*

$$\begin{aligned} x_n &= z_n - \sum_{m=5}^n F^{n-m} \left(x_1^{-c_{m-1}} x_2^{-c_{m-2}} \sum_{\beta \in T^{\geq 3}(\mathcal{D}_m) \setminus T^{\geq 4}(\mathcal{D}_m)} x_1^{r|\beta|_1} x_2^{r(c_{m-1} - |\beta|_2)} \right) \\ &= x_1^{-c_{n-1}} x_2^{-c_{n-2}} \sum_{\beta \in \mathcal{F}(\mathcal{D}_n)} x_1^{r|\beta|_1} x_2^{r(c_{n-1} - |\beta|_2)}. \end{aligned} \quad (3.4)$$

The proof of Lemma 18 will be independent of those of Lemmas 19 and 20. We prove Lemmas 19 and 20 by the following induction:

$$\begin{aligned} & [\text{Lemma 19 holds true for } n \leq d] \\ & \implies [\text{Lemma 20 holds true for } n \leq d + 1] \\ & \implies [\text{Lemma 19 holds true for } n \leq d + 1] \\ & \implies [\text{Lemma 20 holds true for } n \leq d + 2] \cdots \end{aligned} \quad (3.5)$$

Proof of Lemma 20 We use induction on n . It is easy to show that $x_4 = z_4$. Assume that (3.4) holds for n .

Then

$$x_{n+1} = F(x_n)$$

$$\begin{aligned} &\stackrel{\text{F:homomorphism}}{=} F(z_n) \\ &\quad - \sum_{m=5}^n F^{n-m+1} \left(x_1^{-c_{m-1}} x_2^{-c_{m-2}} \sum_{\beta \in \mathcal{T}^{\geq 3}(\mathcal{D}_m) \setminus \mathcal{T}^{\geq 4}(\mathcal{D}_m)} x_1^{r|\beta|_1} x_2^{r(c_{m-1}-|\beta|_2)} \right) \end{aligned}$$

$$\begin{aligned} &\stackrel{\text{Lemma 18}}{=} z_{n+1} \\ &\quad - \sum_{m=5}^{n+1} F^{n-m+1} \left(x_1^{-c_{m-1}} x_2^{-c_{m-2}} \sum_{\beta \in \mathcal{T}^{\geq 3}(\mathcal{D}_m) \setminus \mathcal{T}^{\geq 4}(\mathcal{D}_m)} x_1^{r|\beta|_1} x_2^{r(c_{m-1}-|\beta|_2)} \right) \end{aligned}$$

$$\begin{aligned} &\stackrel{\text{Lemma 19}}{=} z_{n+1} - \sum_{m=5}^{n+1} x_1^{-c_n} x_2^{-c_{n-1}} \sum_{\beta \in \mathcal{T}^{\geq n-m+4}(\mathcal{D}_{n+1}) \setminus \mathcal{T}^{\geq n-m+5}(\mathcal{D}_{n+1})} x_1^{r|\beta|_1} x_2^{r(c_n-|\beta|_2)} \\ &= z_{n+1} - x_1^{-c_n} x_2^{-c_{n-1}} \sum_{\beta \in \mathcal{T}^{\geq 3}(\mathcal{D}_{n+1}) \setminus \mathcal{T}^{\geq n}(\mathcal{D}_{n+1})} x_1^{r|\beta|_1} x_2^{r(c_n-|\beta|_2)} \end{aligned}$$

$$\stackrel{\text{Lemma 17}}{=} z_{n+1} - x_1^{-c_n} x_2^{-c_{n-1}} \sum_{\beta \in \mathcal{T}^{\geq 3}(\mathcal{D}_{n+1})} x_1^{r|\beta|_1} x_2^{r(c_n-|\beta|_2)}$$

$$\stackrel{(3.2)}{=} x_1^{-c_n} x_2^{-c_{n-1}} \sum_{\beta \in \tilde{\mathcal{F}}(\mathcal{D}_{n+1}) \setminus \mathcal{T}^{\geq 3}(\mathcal{D}_{n+1})} x_1^{r|\beta|_1} x_2^{r(c_n-|\beta|_2)}$$

$$\stackrel{(3.1)}{=} x_1^{-c_n} x_2^{-c_{n-1}} \sum_{\beta \in \mathcal{F}(\mathcal{D}_{n+1})} x_1^{r|\beta|_1} x_2^{r(c_n-|\beta|_2)} . \quad \square$$

In order to prove Lemma 18, we need the following notation.

Definition 21 The sequence $\{b_{i,j}\}_{i \in \mathbb{Z}_{\geq 2}, 1 \leq j \leq c_i}$ is defined by

$$b_{i,j} = \begin{cases} r, & \text{if } \alpha_j \text{ is a horizontal edge of } \mathcal{D}_{i+1} \\ r-1, & \text{if } \alpha_j \text{ is a vertical edge of } \mathcal{D}_{i+1}. \end{cases}$$

For integers $i \leq j$, we denote the set $\{i, i+1, i+2, \dots, j\}$ by $[i, j]$. We will always identify $[i, j]$ with the subpath given by $(\alpha_i, \alpha_{i+1}, \dots, \alpha_j)$.

Definition 22 We will need a function f from $\{\text{subsets of } [1, c_{n-1}]\}$ to $\{\text{subsets of } [1, c_n]\}$. For each subset $V \subset [1, c_{n-1}]$, we define $f(V)$ as follows.

If $V = \emptyset$ then $f(\emptyset) = \emptyset$. If $V \neq \emptyset$ then we write V as a disjoint union of maximal connected subsets $V = \bigsqcup_{i=1}^j [e_i, e_i + \ell_i - 1]$ with $\ell_i > 0$ ($1 \leq i \leq j$) and $e_i + \ell_i <$

e_{i+1} ($1 \leq i \leq j - 1$). For each $1 \leq i \leq j$, let

$$W_i = \left[1 + \sum_{k=1}^{e_i-1} b_{n-1,k}, \sum_{k=1}^{e_i+\ell_i-1} b_{n-1,k} \right]$$

and define $f_i(V)$ by

$$f_i(V) := \begin{cases} W_i, & \text{if the subpath given by } W_i \text{ is blue or green;} \\ \{ \sum_{k=1}^{e_i-1} b_{n-1,k} \} \cup W_i, & \text{otherwise.} \end{cases}$$

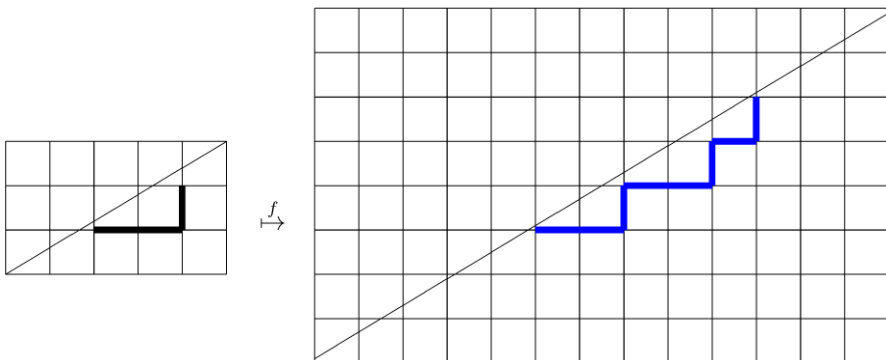
Then $f(V)$ is obtained by taking the union of $f_i(V)$'s:

$$f(V) := \bigcup_{i=1}^j f_i(V).$$

Remark 23

- (1) The subpath given by $f_i(V)$ is always one of blue, green, or red subpaths, every blue, green, or red subpath can be realized as the image of a maximal connected interval under f .
- (2) If V is the union of maximally connected subsets V_i , then the images $f(V_i)$ do not overlap and are disconnected.
- (3) Each horizontal edge in V is mapped under f to a subpath consisting of $r - 1$ horizontal edges and one vertical edge. Each vertical edge in V is mapped to a subpath consisting of $r - 2$ horizontal edges and one vertical edge.

Example 24 Let $r = 3$ and $n = 5$. Then $f(\{4, 5, 6\}) = \{9, 10, 11, 12, 13, 14, 15, 16\}$. As illustrated below, the image of the subpath $(\alpha_4, \alpha_5, \alpha_6)$ under f is the subpath $(\alpha_9, \dots, \alpha_{16})$, which is blue.



Definition 25 Same notation as in Definition 22. For each $1 \leq i \leq j$, we define $\delta_{[e_i, e_i + \ell_i - 1]}$ by

$$\begin{cases} 1, & \text{if the } w\text{-th edge in the subpath corresponding} \\ & \text{to } [e_i, e_i + \ell_i - 1] \text{ is vertical for some } 2 \leq w \leq r - 1; \\ 0, & \text{otherwise.} \end{cases}$$

Let $\delta_V := \sum_{i=1}^j \delta_{[e_i, e_i + \ell_i - 1]}$.

Definition 26 (Pull-back) Let $n > 1$ be any integer. For any subset $W \subset [1, c_n]$, the pull-back $f^*(W)$ by f is defined as follows:

$$f^*(W) = \bigcup_{V: f(V) \subset W} V \subset [1, c_{n-1}].$$

The next lemma will be needed to prove Lemma 18.

Lemma 27 Let V be any subset of $[1, c_{n-1}]$. Then

$$|f(V)| = r|V| - |f^*(V)| - \delta_V.$$

Proof By Remark 23(2), the question is local, so we may assume that V is an interval. First, assume that V is an interval such that the $(r - w)$ -th edge in the corresponding subpath, denoted also by V , is not vertical for any $1 \leq w \leq r - 2$. Observe that $|f(V)|$ is determined by how many horizontal edges and vertical edges are in V and whether the subpath given by $f(V)$ is blue/green or red. Suppose that the subpath given by $f(V)$ is blue or green. Then $|f(V)| = r|V| -$ (the number of vertical edges in V). Moreover, it follows from Remark 23(3), that if the subpath given by $f(V)$ is blue or green, then (the number of vertical edges in V) = $|f^*(V)|$, because the $(w + 1)$ th edge in V is not vertical for any $1 \leq w \leq r - 2$.

Suppose that the subpath given by $f(V)$ is red. Then it follows from Remark 23(3) that $|f(V)| = r|V| -$ (the number of vertical edges in V) + 1. It is not hard to show that (the number of vertical edges in V) = $|f^*(V)| + 1$.

If V is an interval such that the $(r - w)$ -th edge is vertical for some $1 \leq w \leq r - 2$, then it follows, again from Remark 23(3), that $f(V)$ is $(3, w)$ -green. Then $|f(V)| = r|V| -$ (the number of vertical edges in V), and (the number of vertical edges in V) = $|f^*(V)| + 1$. □

The binomial coefficients we will use are generalized binomial coefficients, i.e., for any (possibly negative) integers A, B ,

$$\binom{A}{B} := \begin{cases} \frac{\prod_{i=0}^{B-1} (A-i)}{B!}, & \text{if } B > 0 \\ 1, & \text{if } B = 0 \\ 0, & \text{if } B < 0. \end{cases}$$

Proof of Lemma 18

$$\begin{aligned}
 F(z_n) &= x_2^{-c_{n-1}} \sum_{\beta \in \tilde{\mathcal{F}}(\mathcal{D}_n)} x_2^{r|\beta|_1} \left(\frac{x_2^r + 1}{x_1} \right)^{r(c_{n-1} - |\beta|_2) - c_{n-2}} \\
 &= x_1^{-c_n} x_2^{-c_{n-1}} \sum_{\beta \in \tilde{\mathcal{F}}(\mathcal{D}_n)} x_1^{r|\beta|_2} x_2^{r|\beta|_1} (x_2^r + 1)^{c_n - r|\beta|_2} \\
 &= x_1^{-c_n} x_2^{-c_{n-1}} \sum_{\beta \in \tilde{\mathcal{F}}(\mathcal{D}_n)} \sum_{\theta \in \mathbb{Z}} \binom{c_n - r|\beta|_2}{\theta} x_1^{r|\beta|_2} x_2^{r|\beta|_1} x_2^{r\theta} \\
 &= x_1^{-c_n} x_2^{-c_{n-1}} \sum_{V \subset [1, c_{n-1}]} \sum_{\beta: \cup \beta_i = V} \sum_{\theta \in \mathbb{Z}} \binom{c_n - r|V|}{\theta} x_1^{r|V|} x_2^{r|\beta|_1} x_2^{r\theta}. \quad (\dagger\dagger)
 \end{aligned}$$

For a given $V \subset [1, c_{n-1}]$, consider any subset, say U , of $f^*(V)$. Then the image of each maximal connected subset of U under f is a blue, green, or red subpath. So U uniquely determines β , namely,

$$\beta = \{f(\text{all maximal connected subsets of } U)\} \cup \{\text{all individual edges in } V \setminus f(U)\}.$$

Conversely, β with $\cup \beta_i = V$ uniquely determines a subset U of $f^*(V)$, that is,

$$U = f^{-1}(\text{all blue, green, or red subpaths in } \beta),$$

because f^{-1} (a blue, green, or red subpath) is well-defined and no subpath of $V \setminus f(f^*(V))$ can be a blue, green, or red subpath.

By Remark 23(3), we have $|\beta|_1 = |U|$, and therefore $(\dagger\dagger)$ becomes

$F(z_n)$

$$\begin{aligned}
 &= x_1^{-c_n} x_2^{-c_{n-1}} \sum_{V \subset [1, c_{n-1}]} \sum_{U \subset f^*(V)} \sum_{\theta \in \mathbb{Z}} \binom{c_n - r|V|}{\theta} x_1^{r|V|} x_2^{r|U|} x_2^{r\theta} \\
 &= x_1^{-c_n} x_2^{-c_{n-1}} \sum_{V \subset [1, c_{n-1}]} \sum_{\rho \in \mathbb{Z}} \sum_{\theta \in \mathbb{Z}} \binom{c_n - r|V|}{\theta} \binom{|f^*(V)|}{\rho} x_1^{r|V|} x_2^{r(\theta + \rho)} \\
 &\stackrel{(**)}{=} x_1^{-c_n} x_2^{-c_{n-1}} \sum_{V \subset [1, c_{n-1}]} \sum_{\gamma \in \mathbb{Z}} \binom{c_n - r|V| + |f^*(V)|}{\gamma} x_1^{r|V|} x_2^{r\gamma} \\
 &= x_1^{-c_n} x_2^{-c_{n-1}} \sum_{V \subset [1, c_{n-1}]} \sum_{\gamma \in \mathbb{Z}} \binom{c_n - r|V| + |f^*(V)|}{c_n - r|V| + |f^*(V)| - \gamma} x_1^{r|V|} x_2^{r(c_n - r|V| + |f^*(V)| - \gamma)} \\
 &\stackrel{(\text{Lemma 27})}{=} x_1^{-c_n} x_2^{-c_{n-1}} \sum_{V \subset [1, c_{n-1}]} \sum_{\gamma \in \mathbb{Z}} \binom{c_n - |f(V)| - \delta_V}{c_n - |f(V)| - \delta_V - \gamma} x_1^{r|V|} x_2^{r(c_n - |f(V)| - \delta_V - \gamma)}
 \end{aligned}$$

$$\begin{aligned}
 &= x_1^{-c_n} x_2^{-c_{n-1}} \sum_{V \subset [1, c_{n-1}]} \sum_{h \in \mathbb{Z}} \binom{c_n - |f(V)| - \delta_V}{c_n - |f(V)| - h} x_1^{r|V|} x_2^{r(c_n - |f(V)| - h)} \\
 &= x_1^{-c_n} x_2^{-c_{n-1}} \sum_{V \subset [1, c_{n-1}]} \sum_{h \in \mathbb{Z}} \binom{c_n - |f(V)| - \delta_V}{h - \delta_V} x_1^{r|V|} x_2^{r(c_n - |f(V)| - h)}, \tag{‡}
 \end{aligned}$$

where (**) follows from the Chu–Vandermonde identity (for instance, see [1, pp. 59–60]).

Next we analyze z_{n+1} . If $\beta \in \tilde{\mathcal{F}}(\mathcal{D}_{n+1})$, then there exist

$$\left\{ \begin{array}{l} e \in \mathbb{Z}_{\geq 0}, \\ 0 \leq i_1 < k_1 < i_2 < k_2 < \dots < i_e < k_e \leq c_{n-1}, \\ h \in \mathbb{Z}_{\geq 0}, \text{ and} \\ j_1 < \dots < j_h \in [1, c_n] \setminus f\left(\bigsqcup_{\ell=1}^e [1 + i_\ell, k_\ell]\right) \end{array} \right. \tag{3.6}$$

such that

$$\beta = \{\alpha(i_1, k_1), \dots, \alpha(i_e, k_e)\} \cup \{\alpha_{j_1}, \dots, \alpha_{j_h}\}.$$

For a given $\beta \in \tilde{\mathcal{F}}(\mathcal{D}_{n+1})$, this expression is unique. Conversely (3.6) uniquely determines an element β in $\tilde{\mathcal{F}}(\mathcal{D}_{n+1})$. Note that $|\beta|_1 = \sum_{\ell=1}^e (k_\ell - i_\ell)$ and $|\beta|_2 = |f(\bigsqcup_{\ell=1}^e [1 + i_\ell, k_\ell])| + h$.

So we have

$$\begin{aligned}
 z_{n+1} &= x_1^{-c_n} x_2^{-c_{n-1}} \sum_{\beta \in \tilde{\mathcal{F}}(\mathcal{D}_{n+1})} x_1^{r|\beta|_1} x_2^{r(c_n - |\beta|_2)} \\
 &= x_1^{-c_n} x_2^{-c_{n-1}} \sum_{e \in \mathbb{Z}_{\geq 0}} \sum_{1 \leq i_1 < \dots < k_e \leq c_{n-1}} \sum_{h \in \mathbb{Z}_{\geq 0}} \\
 &\quad \sum_{j_1 < \dots < j_h \in [1, c_n] \setminus f(\bigsqcup_{\ell} [1 + i_\ell, k_\ell])} x_1^{r \sum_{\ell} (k_\ell - i_\ell)} x_2^{r(c_n - |f(\bigsqcup_{\ell} [1 + i_\ell, k_\ell])| - h)} \\
 &= x_1^{-c_n} x_2^{-c_{n-1}} \sum_{V \subset [1, c_{n-1}]} \sum_{h \in \mathbb{Z}_{\geq 0}} \sum_{j_1 < \dots < j_h \in [1, c_n] \setminus f(V)} x_1^{r|V|} x_2^{r(c_n - |f(V)| - h)} \\
 &= x_1^{-c_n} x_2^{-c_{n-1}} \sum_{V \subset [1, c_{n-1}]} \sum_{h \in \mathbb{Z}} \binom{c_n - |f(V)|}{h} x_1^{r|V|} x_2^{r(c_n - |f(V)| - h)} \\
 &\stackrel{\text{by } (\ddagger)}{=} F(z_n) + x_1^{-c_n} x_2^{-c_{n-1}} \sum_{V \subset [1, c_{n-1}]} \sum_{h \in \mathbb{Z}} \left(\binom{c_n - |f(V)|}{h} \right. \\
 &\quad \left. - \binom{c_n - |f(V)| - \delta_V}{h - \delta_V} \right) x_1^{r|V|} x_2^{r(c_n - |f(V)| - h)} \\
 &= F(z_n) + x_1^{-c_n} x_2^{-c_{n-1}} \sum_{\beta \in \mathcal{T}^{\geq 3}(\mathcal{D}_{n+1}) \setminus \mathcal{T}^{\geq 4}(\mathcal{D}_{n+1})} x_1^{r|\beta|_1} x_2^{r(c_n - |\beta|_2)},
 \end{aligned}$$

where the last equality is a consequence of the next lemma. □

Lemma 28 *Let V be any subset of $[1, c_{n-1}]$ and h be any integer. Then the number of elements β in $\tilde{\mathcal{F}}(\mathcal{D}_{n+1})$ satisfying*

$$\begin{cases} f(V) \text{ is the union of all blue, green, and red subpaths in } \beta, \\ |\beta|_2 = |f(V)| + h, \text{ and} \\ \beta \in \mathcal{T}^{\geq 3}(\mathcal{D}_{n+1}) \setminus \mathcal{T}^{\geq 4}(\mathcal{D}_{n+1}) \end{cases} \tag{3.7}$$

is equal to

$$\binom{c_n - |f(V)|}{h} - \binom{c_n - |f(V)| - \delta_V}{h - \delta_V}. \tag{3.8}$$

Proof Remembering the definition of (m, w) -green subpaths in Definition 7(2-a), especially for $m = 3$, the following statement is a consequence of Remark 23(3). For an interval $[i, k] \subset [1, c_{n-1}]$, the subpath in \mathcal{D}_n corresponding to $[i + 1, i + r - 1] \cap [i, k]$ contains a vertical edge if and only if the slope in \mathcal{D}_{n+1} between the point v_i and the point corresponding to the upper endpoint of the vertical edge is as large as possible, more precisely,

$$\begin{aligned} &\text{the subpath in } \mathcal{D}_{n+1} \text{ corresponding to } f([i, k]) \text{ is} \\ &(3, w)\text{-green for some } w \in [1, r - 2]. \end{aligned} \tag{3.9}$$

If no maximal connected interval of V satisfies (3.9), then $\delta_V = 0$ and there is no β satisfying (3.7), so the statement holds true. If only one maximal connected interval, say $[i, k]$, of V satisfies (3.9), then $\delta_V = 1$. On the other hand, since

$$\beta \in \mathcal{T}^{\geq 3}(\mathcal{D}_{n+1}) \setminus \mathcal{T}^{\geq 4}(\mathcal{D}_{n+1}),$$

none of the $(c_2 - wc_1)$ preceding edge(s) of v_i is contained in any element $\beta_{j'}$ of β . As $c_2 - wc_1 = 1$, the number of β satisfying (3.7) is obtained by subtracting the number of sequences $j_1 < \dots < j_h \in [1, c_n] \setminus f(V)$ with $(\min f([i, k]) - 1) \in \{j_1, \dots, j_h\}$ from the number of sequences $j_1 < \dots < j_h \in [1, c_n] \setminus f(V)$, which is $\binom{c_n - |f(V)|}{h} - \binom{c_n - |f(V)| - 1}{h - 1}$.

Similarly one can verify the statement in the case that more than one maximal connected interval of V satisfies (3.9). □

It remains to prove Lemma 19.

Sketch of Proof of Lemma 19 Here we will deal only with the case of $n = u + 2$. The case of $n > u + 2$ makes use of the same argument. As we use the induction (3.5), we can assume that

$$x_i = x_1^{-c_{i-1}} x_2^{-c_{i-2}} \sum_{\beta \in \mathcal{F}(\mathcal{D}_i)} x_1^{r|\beta|_1} x_2^{r(c_{i-1} - |\beta|_2)}$$

for $i \leq n$.

For any $w \in [1, r - 2]$, it is easy to show that the lattice point $(w(c_{n-2} - c_{n-3}), w c_{n-3})$ is below the diagonal from $(0, 0)$ to $(c_{n-1} - c_{n-2}, c_{n-2})$ and that the

points $(w(c_{n-2} - c_{n-3}), 1 + wc_{n-3})$ and $(w(c_{n-2} - c_{n-3}) - 1, wc_{n-3})$ are above the diagonal. So $(w(c_{n-2} - c_{n-3}), wc_{n-3})$ is one of the vertices v_i on \mathcal{D}_n . Actually $v_{wc_{n-3}} = (w(c_{n-2} - c_{n-3}), wc_{n-3})$. Since $u = n - 2$ and $\alpha(wc_{n-3}, c_{n-2})$ is the only $(n - 2, w)$ -green subpath in $\{\alpha(i, k) \mid 0 \leq i < k \leq c_{n-2}\}$, every $\beta \in \mathcal{T}^{\geq u}(\mathcal{D}_n) \setminus \mathcal{T}^{\geq u+1}(\mathcal{D}_n)$ must contain the green subpath from $v_{wc_{n-3}}$ to $v_{c_{n-2}}$. Then none of the $c_{n-3} - wc_{n-4}$ preceding edges of $v_{wc_{n-3}}$ is contained in any element $\beta_{j'}$ of β . The green subpath from $v_{wc_{n-3}}$ to $v_{c_{n-2}}$ corresponds to the interval $[wc_{n-2} + 1, c_{n-1}] \subset [1, c_{n-1}]$. The $c_{n-3} - wc_{n-4}$ preceding edges of $v_{wc_{n-3}}$ are $\alpha_{(rw-1)c_{n-3}+1}, \dots, \alpha_{wc_{n-2}}$.

Thus we have

$$\begin{aligned} & x_1^{-c_{n-1}} x_2^{-c_{n-2}} \sum_{\beta \in \mathcal{T}^{\geq u}(\mathcal{D}_n) \setminus \mathcal{T}^{\geq u+1}(\mathcal{D}_n)} x_1^{r|\beta|_1} x_2^{r(c_{n-1}-|\beta|_2)} \\ &= \sum_{w=1}^{r-2} x_1^{-c_{n-1}} x_2^{-c_{n-2}} \sum_{V \subset [1, (rw-1)c_{n-3}]} \sum_{\beta: \cup \beta_i = V \cup [wc_{n-2}+1, c_{n-1}], \beta \ni \alpha(wc_{n-3}, c_{n-2})} x_1^{r|\beta|_1} x_2^{r(c_{n-1}-|\beta|_2)}. \end{aligned} \tag{*}$$

We observe that the subpath corresponding to $[1, (rw - 1)c_{n-3}]$ consists of $(w - 1)$ copies of \mathcal{D}_{n-1} , $(r - 1)$ copies of \mathcal{D}_{n-2} , and $(w - 1)$ copies of \mathcal{D}_{n-3} . Let $v_{j_0} = (0, 0)$ and v_{j_i} be the end point of each of these copies, i.e.,

$$\begin{aligned} v_{j_i} &= v_{ic_{n-3}} \quad \text{for } 1 \leq i \leq w - 1, \\ v_{j_{w-1+i}} &= v_{(w-1)c_{n-3}+ic_{n-4}} \quad \text{for } 1 \leq i \leq r - 1, \\ v_{j_{w+r-2+i}} &= v_{(w-1)c_{n-3}+(r-1)c_{n-4}+ic_{n-5}} \quad \text{for } 1 \leq i \leq w - 1. \end{aligned}$$

If a (m, w') -green (resp. blue or red) subpath, say $\alpha(i, k)$, in $[1, (rw - 1)c_{n-3}]$ passes through $v_{j_e}, v_{j_{e+1}}, \dots, v_{j_{e+\ell}}$, then $\alpha(i, k)$ can be naturally decomposed into $\alpha(i, j_e), \alpha(j_e, j_{e+1}), \dots, \alpha(j_{e+\ell}, k)$. It is not hard to show that $\alpha(i, j_e)$ is also (m, w') -green (resp. blue or red) and that $\alpha(j_e, j_{e+1}), \dots, \alpha(j_{e+\ell}, k)$ are all blue.

Hence

$$\begin{aligned} (*) &= \sum_{w=1}^{r-2} x_1^{-c_{n-1}} x_2^{-c_{n-2}} \left(\sum_{\beta \in \mathcal{F}(\mathcal{D}_{n-1})} x_1^{r|\beta|_1} x_2^{r(c_{n-2}-|\beta|_2)} \right)^{w-1} \\ &\times \left(\sum_{\beta \in \mathcal{F}(\mathcal{D}_{n-2})} x_1^{r|\beta|_1} x_2^{r(c_{n-3}-|\beta|_2)} \right)^{r-1} \\ &\times \left(\sum_{\beta \in \mathcal{F}(\mathcal{D}_{n-3})} x_1^{r|\beta|_1} x_2^{r(c_{n-4}-|\beta|_2)} \right)^{w-1} x_1^{r(c_{n-2}-wc_{n-3})} x_2^{r(c_{n-3}-wc_{n-4})} \\ &= \sum_{w=1}^{r-2} x_1^{-c_{n-1}} x_2^{-c_{n-2}} (x_{n-1} x_1^{c_{n-2}} x_2^{c_{n-3}})^{w-1} (x_{n-2} x_1^{c_{n-3}} x_2^{c_{n-4}})^{r-1} \\ &\times (x_{n-3} x_1^{c_{n-4}} x_2^{c_{n-5}})^{w-1} x_1^{r(c_{n-2}-wc_{n-3})} x_2^{r(c_{n-3}-wc_{n-4})} \end{aligned}$$

$$= \sum_{w=1}^{r-2} (x_{n-1})^{w-1} (x_{n-2})^{r-1} (x_{n-3})^{w-1}.$$

For the same reason, we get

$$\begin{aligned} & x_1^{-c_n} x_2^{-c_{n-1}} \sum_{\beta \in \mathcal{T}^{\geq u+1}(\mathcal{D}_{n+1}) \setminus \mathcal{T}^{\geq u+2}(\mathcal{D}_{n+1})} x_1^{r|\beta|_1} x_2^{r(c_n-|\beta|_2)} \\ &= \sum_{w=1}^{r-2} (x_n)^{w-1} (x_{n-1})^{r-1} (x_{n-2})^{w-1}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} & F\left(x_1^{-c_n} x_2^{-c_{n-1}} \sum_{\beta \in \mathcal{T}^{\geq u}(\mathcal{D}_n) \setminus \mathcal{T}^{\geq u+1}(\mathcal{D}_n)} x_1^{r|\beta|_1} x_2^{r(c_{n-1}-|\beta|_2)}\right) \\ &= F\left(\sum_{w=1}^{r-2} (x_{n-1})^{w-1} (x_{n-2})^{r-1} (x_{n-3})^{w-1}\right) \\ &= \sum_{w=1}^{r-2} (x_n)^{w-1} (x_{n-1})^{r-1} (x_{n-2})^{w-1} \\ &= x_1^{-c_n} x_2^{-c_{n-1}} \sum_{\beta \in \mathcal{T}^{\geq u+1}(\mathcal{D}_{n+1}) \setminus \mathcal{T}^{\geq u+2}(\mathcal{D}_{n+1})} x_1^{r|\beta|_1} x_2^{r(c_n-|\beta|_2)}. \end{aligned} \quad \square$$

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