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A COMBINATORIAL FORMULA FOR THE PONTRJAGIN CLASSES

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ABSTRACT. A combinatorial formula for the Pontrjagin classes of a triangulated manifold is given. The main ingredients are oriented matroid theory and a modified formulation of Chern-Weil theory.

1. INTRODUCTION

The problem of finding a combinatorial formula for the Pontrjagin classes of a polyhedral manifold X is forty-five years old and has stimulated much research (see [M] and [L] for references). Chern-Weil theory provides a formula for the Pontrjagin classes of a Riemannian manifold: They are represented by differential forms that measure certain types of curvature of the manifold. The problem is to find an analogous theory for polyhedra, which have "infinite curvature" at the corners.

In this note, we announce a formula that holds in all dimensions, is completely explicit, and can be calculated using combinatorial constructions and the operations of finite-dimensional linear algebra over $\mathbb{Q}$. For each i, the formula gives a rational simplicial cycle ζ_i in the barycentric subdivision of X, whose Poincaré dual represents the ith inverse Pontrjagin class $\tilde{p}_i(X)$. (The inverse Pontrjagin classes of X are defined from the usual Pontrjagin classes $p_i(X)$ by $(1 + p_1(X) + p_2(X) + \cdots) \smile (1 + \tilde{p}_1(X) + \tilde{p}_2(X) + \cdots) = 1$. Like the p_i, the $\tilde{p}_i$ generate the Pontrjagin ring.) The cycle ζ_i depends on the choice of certain additional combinatorial data called a *fixing cycle*. We think of a fixing cycle as a combinatorial analogue of a smooth structure on X. In fact, a smooth structure on X induces a canonical fixing cycle.

Two ideas make this formula possible: the systematic exploitation of oriented matroids and a reformulation of Chern-Weil curvature theory.

The only other general and explicit combinatorial formula for the Pontrjagin classes is Cheeger's [C]. It uses the asymptotics of the spectrum of a differential operator, so it is difficult to compute and its rationality properties are not clear. However, Cheeger's formula is clearly the best one for the context of the Hodge operator constructed from the metric.

2. REVIEW OF ORIENTED MATROIDS

Oriented matroid theory is a well-developed branch of combinatorics with many applications. One of our goals is to establish a link between this theory and differential topology. We include here only the definitions from oriented matroid theory that we need. The best general reference is [BLSWZ].

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Definition. An *oriented matroid* M is a finite set V called the *elements* of M together with a finite collection of maps $c_i: V \to \{-, 0, +\}$ called the *covectors* of M subject to the following axioms:

1. The constant function with value 0 is a covector.
2. If c is a covector, then $-c$ is a covector where $-(-) = +$, $-(0) = 0$, and $-(+) = -$.
3. If c and d are covectors, then $c \circ d$ is a covector where $c \circ d$ is defined by
$$c \circ d(v) = \begin{cases} c(v) & \text{if } c(v) \neq 0, \\ d(v) & \text{otherwise.} \end{cases}$$
4. For all covectors c and d, if v is an element such that $c(v) = +$ and $d(v) = -$, then there exists a covector e such that
   - $e(v) = 0$;
   - If $c(w) = d(w) = 0$, then $e(w) = 0$;
   - If $c(w) \neq -$, $d(w) \neq -$ but $c(w)$ and $d(w)$ are not both 0, then $e(w) = +$;
   - If $c(w) \neq +$, $d(w) \neq +$ but $c(w)$ and $d(w)$ are not both 0, then $e(w) = -$.

The idea behind this definition is that an oriented matroid is a combinatorial abstraction of a finite set V of vectors (which are not assumed to be distinct or to be nonzero) in a finite-dimensional real vector space W. The covectors c correspond to linear functionals $\bar{c}: W \to \mathbb{R}$, but $c(v)$ remembers only whether $\bar{c}(v)$ is negative, zero, or positive.

An element v of V is *nonzero* if $c(v) \neq 0$ for some covector c. A subset $\{v_1, v_2, \ldots, v_j\}$ of V is said to be *independent* if there exists a set of covectors $\{c_1, c_2, \ldots, c_j\}$ such that $c_i(v_k) = 0$ if and only if $i \neq k$. The *rank* of x is the cardinality of any (and hence every) maximal independent subset of V. The *convex hull* of a set S of elements is $\{v \in V | - \in c(S) \text{ if } c(v) = -\}$. Suppose N and M are two matroids with the same set of elements V. We say that N is a *strong quotient* of M, symbolized $M \Rightarrow N$, if every covector of N is a covector of M. If M and N have the same rank, we say that the matroid N is a *weak specialization* of M, symbolized $M \rightsquigarrow N$ if every covector of N is obtained from some covector of M by setting nonzero values equal to zero.

3. The formula

Let X be a simplicial manifold of dimension n. For simplicity, we assume that X is oriented (with orientation class $[X]$) and that n is odd. The modifications necessary for the general case are noted at the end.

Definition. The *associated complex* Z of X is the simplicial complex constructed as follows: The vertices of Z are quadruples (Δ, t, y, z) where

- $\Delta \subset V$ is a simplex of X, where V is the set of vertices of X.
- t, y, and z are oriented matroids of rank $n+1$, 2, and 1 whose set of elements is V.
- The matroid t has a covector that does not take the value $-$ on any element of V.
- The simplex Δ is related to the matroid t by the following two conditions:

1. The nonzero elements of t are exactly the vertices of the star $\operatorname{St}\Delta$ of Δ.
2. For each simplex Δ' in $\operatorname{St}\Delta$, let $V(\Delta')$ be the set of vertices of Δ'. Then $V(\Delta')$ is linearly independent in t and the set of nonzero elements of t in the convex hull of $V(\Delta')$ is just $V(\Delta')$ itself.

- We have strong quotients $t \Rightarrow y \Rightarrow z$.

The k-simplices of Z are diagrams of weak specializations and inclusions

$$\begin{array}{ccccccc} t_0 & \rightsquigarrow & t_1 & \rightsquigarrow & \cdots & \rightsquigarrow & t_k \\ \Downarrow & & \Downarrow & & & & \Downarrow \\ y_0 & \rightsquigarrow & y_1 & \rightsquigarrow & \cdots & \rightsquigarrow & y_k \\ \Downarrow & & \Downarrow & & & & \Downarrow \\ z_0 & \rightsquigarrow & z_1 & \rightsquigarrow & \cdots & \rightsquigarrow & z_k \\ & & & & & & \\ \Delta_0 & \subseteq & \Delta_1 & \subseteq & \cdots & \subseteq & \Delta_k \end{array}$$

If we delete the matroids z *resp.* all matroids (t, y, and z) in this definition, we get additional associated simplicial complexes, which we denote by Y *resp.* $\widetilde{X}$, equipped with simplicial maps $Z \xrightarrow{\rho} Y \xrightarrow{\pi} \widetilde{X}$. Note that $\widetilde{X}$ is just the barycentric subdivision of X.

Remarks. The matroids t are combinatorial abstractions of $T_pX \oplus \mathbb{R}$ where T_pX is the tangent space to X at a point p in the simplex Δ. The map $Y \xrightarrow{\pi} \widetilde{X}$ is an analogue of the Grassmannian bundle of two-dimensional quotients of $TX \oplus \mathbb{R}$, and $Z \xrightarrow{\rho} Y$ is the circle bundle of one dimensional quotients of the two plane.

Proposition 1. *The map $\rho: Z \to Y$ is topologically a fibration with a circle as fiber.*

Idea of proof. That the fibers over the vertices of Y are circles is a special case of the Folkman-Lawrence representation theorem for the oriented matroid y.

We now give a combinatorial formula for the first Chern class of a triangulated circle bundle. Let $\mathscr{O}$ be the local system on Y with fiber $\mathbb{Q}$ and twisting given by the fiber orientation of Z. Define a 1-cocycle Θ on Z with coefficients in $\rho^*\mathscr{O}$ as follows: For each vertex v of Y, $\Theta|\rho^{-1}v$ is the class that integrates to 1 around the circle and has the same value on each 1-simplex. Having fixed this, for each edge e of Y, $\Theta|\rho^{-1}e$ is the cocycle such that the sum of the squares of the coefficients is minimum. It is rational, since the problem of minimizing a quadratic expression subject to linear constraints (the cocycle condition) can be solved by linear equations. Now, define the 2-cocycle Ω on Y with coefficients in $\mathscr{O}$ by $\rho^*\Omega = \delta\Theta$.

Proposition 2. *The cohomology class $\{\Omega\}$ is the first Chern class of the circle bundle Z.*

Proof. By construction, $\{\Omega\}$ is the δ_2 differential (transgression) of the fiber orientation in the spectral sequence for Z.

Definition. A *fixing cycle* for X is a $(3n-2)$-cycle $\phi \in Z_{3n-2}(Y, \mathbb{Z})$ such that $\pi_*(\Omega^{n-1} \frown \phi) = [X]$.

Remark. A fixing cycle is the combinatorial analogue of an orientation class of the Grassmannian bundle. Unfortunately, it is not unique, as its differential analogue is. The idea evolved from the configuration data of [GGL] and [M].

Theorem 1. *Let* ϕ *be a fixing cycle for* X. *Then*

$$\tilde{p}_i(X) \frown [X] = (-1)^i \pi_\star((\tfrac{1}{2}\Omega)^{n+2i-1} \frown \phi).$$

Note that since n is odd, Ω^{n+2i-1} is a cocycle with coefficients in $\mathbb{Q}$ since $O^{n+2i-1} = \mathbb{Q}$.

Remark. This is a cycle level formula, since the operations in simplicial (co)homology involved in the right-hand side ($\pi_\star$, $\smile$, $\frown$), and in the definitions of Ω, $\rho^\star$ and δ are all chain level operations. The complexity of the formula is contained in the formulas for these operations and the construction of the simplicial complexes Z and Y. Given the fixing cycle ϕ, it is a purely local formula: the value in an open set U of $\widetilde{X}$ depends only on the combinatorial structure of X inside U and on $\phi|\pi^{-1}U$.

4. Construction of the fixing cycle

First, we study the structure of the auxiliary complex Y. For any simplex Δ in X, let U_Δ be the set of diagrams of oriented matroids $y \Rightarrow t$ with t satisfying conditions 1 and 2 of Definition 1 with respect to Δ. This is a poset by the specialization ($\rightsquigarrow$) ordering on the oriented matroids in U_Δ. We denote the order complex of a poset P by $\mathrm{Cx}\, P$. The open dual cell of Δ in $\widetilde{X}$ is denoted $D\Delta$ (so $\widetilde{X}$ is the disjoint union of the $D\Delta$).

Proposition 3. *The subcomplex of* Y *lying over* $D\Delta$ *is canonically homeomorphic to* $\mathrm{Cx}\, U_\Delta \times D\Delta$.

We denote the homeomorphism of the proposition by $c^\Delta \colon \mathrm{Cx}\, U_\Delta \times D\Delta \to Y$.

Remark. Suppose $\Delta \subset \Delta'$. Then the edge of $c(\mathrm{C} \times U_\Delta \times D\Delta)$ is glued to $c(\mathrm{C} \times U_{\Delta'} \times D\Delta')$ by the map $\mathrm{C} \times U_\Delta \to \mathrm{C} \times U_{\Delta'}$ induced by the map of posets $U_\Delta \to U_{\Delta'}$ defined on the matroid level by setting all elements of V in $\mathrm{St}\,\Delta$ but not in $\mathrm{St}\,\Delta'$ equal to zero.

A *smooth structure* on X is a homeomorphism $\sigma \colon X \to M$ to a smooth manifold that is differentiable on each closed simplex in X. Define $\mathscr{Y}$ to be the Grassmannian bundle whose fiber over $x \in M$ is the space of $n-1$-dimensional subspaces $F^{n-1} \subset T_xM \oplus \mathbb{R}$. The map $Y \xrightarrow{\pi} \widetilde{X}$ is a sort of a "combinatorial model" for the map $\mathscr{Y} \to M$.

Let $\mathscr{Y}_\Delta$ be the part of $\mathscr{Y}$ lying over $\sigma(\Delta)$ for a simplex $\Delta \subset X$. Any point y in $\mathscr{Y}_\Delta$ determines an element of U_Δ as follows: Let $\sigma(x)$ be the image of y in M. There is a unique embedding $e \colon \mathrm{St}\,\Delta \hookrightarrow T_xM$ that is linear on each simplex, takes x to 0, and satisfies $d(e|\Delta') = d(\sigma|\Delta')$ for each simplex Δ' in $\mathrm{St}\,\Delta$. Now map V into $T_xM \oplus \mathbb{R}$ by using the embedding $\mathrm{St}\,\Delta \xrightarrow{e} T_xM \xrightarrow{\times 1} T_xM \oplus \mathbb{R}$ for vertices in $\mathrm{St}\,\Delta$ and mapping all other vertices to zero. This gives a representation of the oriented matroid t. The oriented matroid y is represented by projecting the images of these vertices into $(T_xM \oplus \mathbb{R})/F^{n-1}$. By this construction, $\mathscr{Y}_\Delta$ is decomposed into pieces indexed by elements of U_Δ. One can see from stratified transversality theory that if σ is generic, this decomposition can be refined to a Whitney stratification $\mathscr{Y}_\Delta = \bigcup_\alpha S_\alpha$ that is transverse to the boundary. By construction, each stratum S_α determines an element $u(S_\alpha) \in U_\Delta$ by which piece of $\mathscr{Y}_\Delta$ it lies in.

A *full flag of strata* in a manifold is a set $S = S_0, S_1, \ldots, S_d$ where the closure of S_i contains S_{i-1}, the dimension of S_i is i, and d is the dimension of the manifold. If the manifold is oriented, the *sign* εS of S is defined as follows: Map a d-simplex with vertices $v_0, \dots, v_d$ into the manifold so that the vertex v_0 goes to S_0, the edge v_0v_1 goes to S_1, and so on. Then $\varepsilon S = +1$ if the orientation of the simplex agrees with the orientation of the manifold, and $\varepsilon S = -1$ otherwise. If S is a full flag of the strata in $\mathscr{Y}_\Delta$, then denote by $u(S)$ the oriented simplex in $C \times U_\delta$ with vertices $u(S_0), u(S_1), \dots$.

For each simplex Δ of X, choose orientations $[\Delta]$ of Δ and $[D\Delta]$ of $D\Delta$ whose cross product is the orientation of X. Orient $\mathscr{Y}_\Delta$ by the cross product of $[\Delta]$ and the standard orientation of the Grassmannian of $(n-1)$-planes in $(n+1)$ space (remember that n is odd).

Theorem 2. *The generic smoothing σ induces a fixing cycle ϕ by the formula*

$$\phi = \sum_\Delta \sum_S \varepsilon(S) c_\star^\Delta(u(\delta) \times [D\Delta])$$

where the first sum is over all simplices Δ of X and the second over all full flags of strata in $\mathscr{Y}_\Delta$.

The idea of the proof is to construct a continuous map $f: \mathscr{Y} \to Y$ so that $\phi = f_\star[\mathscr{Y}]$.

5. An alternative form of Chern-Weil theory

Let E be a vector bundle with a connection over a differentiable manifold M. Chern-Weil theory gives a formula for the Pontrjagin classes of M as a sum of terms, each of which is a product of curvature 2-forms Ω multiplied by a pattern reflecting the structure of the Lie algebra of $\mathrm{Gl}(n, \mathbb{R})$. Finding a combinatorial analogue of Ω is possible, but it is a singular current. The difficulty in finding a combinatorial analogue for Chern-Weil theory is regularizing the products.

The combinatorial formula of this paper is an analogue of another form of Chern-Weil theory, which we now describe. Let e be the fiber dimension of E and assume that it is even. Let $\pi: \mathscr{Y} \to M$ be the Grassmannian bundle of $(e-2)$-planes in E and $\rho: \mathscr{Z} \to \mathscr{Y}$ be the principle circle bundle of the tautological quotient 2-plane bundle ξ over $\mathscr{Y}$. The connection on E induces a one form Θ on $\mathscr{Z}$ with coefficients twisted by the orientation sheaf $\mathscr{O}$ of $\mathscr{Z}$ and a curvature form Ω on $\mathscr{Y}$ defined by $\rho^\star\Omega = d\Theta$.

Proposition 4. $\tilde{p}_i(E) = (-1)^i \pi_\star \Omega^{(e-2+2i)}$ *where $\pi_\star$ represents integration over the fiber.*

Proposition 4 is proved by an algebraic manipulation using only the Whitney sum formula applied to $\pi^\star E = \xi \oplus \xi^\perp$, the vanishing of high Pontrjagin classes of a low-dimensional bundle, and the projection formula. Theorem 1 is a combinatorial analogue of this formula, where E is $TM \oplus 1$.

Orientations and dimensions. Suppose that X is not orientable and/or not odd dimensional. Let $\mathscr{D}$ be the orientation local system of X, so $[X] \in H_n(X, \mathscr{D})$. The fixing cycle should lie in homology with twisted coefficients: $\phi \in Z_{3n-2}(Y, \pi^\star\mathscr{D} \otimes \mathscr{O}^{\otimes(n-1)})$. The construction of ϕ in §2 still works because $\mathscr{Y}$ has orientation sheaf $\pi^\star\mathscr{D} \otimes \mathscr{O}^{\otimes(n-1)}$ where $\mathscr{O}$ is the orientation sheaf $\mathscr{Z}$.

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