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A COMBINATORIAL PROOF OF THE ALL MINORS MATRIX TREE THEOREM*

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Abstract. Let (A_{ij}), $i, j \in V$ be the matrix with entries $-a_{ij}$ if $i \neq j$ and diagonal entries such that all the column sums are zero. Let a_{ij} be a variable associated with arc ij in the complete digraph G on vertices V. Let $A(\bar{W}|\bar{U})$ be the matrix that results from deleting sets of k rows W and columns U from A. The all minors matrix tree theorem states that $|A(\bar{W}|\bar{U})|$ enumerates the forests in G that have (a) k trees, (b) each tree contains exactly one vertex in U and exactly one vertex in W, and (c) each arc is directed away from the vertex in U of the tree containing the arc. We give an elementary combinatorial proof in which we show that each of the terms in $|A(\bar{W}|\bar{U})|$ that corresponds to an enumerated forest occurs just once and the other terms cancel. The sign of each term is determined by the parity of the linking from U to W contained in the forest, and is easy to calculate explicitly in the proof.

The results are extended to signed graphs. The theorem provides a coordinatization (linear representation) of gammoids that is in a certain sense natural.

1. Introduction. This paper describes an elementary, combinatorial proof of the matrix tree theorem, an extension of it to signed and voltage graphs, and its applicability to the coordinatization of gammoids. We begin with a statement of the theorem.

Let the variables a_{ij}, for $i, j \in S$ and $i \neq j$ be weights on the arcs ij of the complete, loopless directed graph on a finite set of vertices S. Define matrix A by

$$A_{ij} = \begin{cases} -a_{ij} & \text{if } i \neq j, \\ \sum_k a_{kj} & \text{if } i = j. \end{cases} \tag{1}$$

A can be regarded as a "special" weighted adjacency matrix in which the jth diagonal entry is the sum of the weights of arcs directed into vertex j. Let $A(\bar{W}|\bar{U})$ be the submatrix of A obtained by deleting the rows indexed by the elements of $W \subset S$ and the columns indexed by $U \subset S$. Assume S is linearly ordered; for example, it may be $\{1, 2, \cdots, N\}$. Assume $|W| = |U|$. When F is a set of arcs, a_F denotes the product of their weights.

(ALL MINORS) MATRIX TREE THEOREM.

$$\det A(\bar{W}|\bar{U}) = \varepsilon(W, S)\varepsilon(U, S) \sum_F \varepsilon(\pi^*)a_F, \tag{2}$$

where the $\varepsilon(\cdot)$ denote signs which are defined in § 2. The sum is over all forests F such that

(i) *F contains exactly $|W| = |U|$ trees.*
(ii) *Each tree in F contains exactly one vertex in U and exactly one vertex in W.*
(iii) *Each arc in F is directed away from the vertex in U of the tree containing that arc.*

F defines a bijection or matching $\pi^*: W \to U$ so $\pi^*(j) = i$ if and only if i and j are in the same tree of F.

The all minors matrix tree theorem was given in a form similar to that here by W. K. Chen [4]. The rooted, directed forests enumerated in this theorem are sometimes called branchings, the components of which are called arborescences.

One should observe that every forest enumerated by (2) contains a collection of $|U|$ disjoint, simple, directed paths each of which starts at a vertex $i \in U$ and ends at

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a vertex $\pi^{*-1}(i) \in W$. Each element of $U \cap W$ comprises a trivial path of one vertex. π^*, and therefore the relative signs of the terms in (2) are completely determined by the pairs defined by the start and end vertices of these paths.

When $U = W$ every path above degenerates to a single vertex. Every sign in (2) becomes $+1$. If we replace the a_{ij} by 0s or 1s, the theorem gives us a way to count the forests rooted and directed away from the vertices U in an arbitrary directed graph. The resulting theorem is an easy generalization of the classical directed graph version of the matrix tree theorem, for which $|U| = 1$. The latter was probably first described by Sylvester [23], [17], and was proved by Borchardt [2] and Tutte [24]. The undirected graph version is a special case for which $a_{ij} = a_{ji}$. When a_{ij} is given the value of the electrical conductance of the resistor joining nodes i and j in an electrical network, (2) for $|U| = 1$ and $|U| = 2$ can be used to solve the electrical network equations. The use of the duals of these "tree sums" for this purpose was given by Kirchhoff [9]. Maxwell [14, Ch. 6 and appendix] described this application of (2) which is called Maxwell's rule. See [16] for an historical survey and applications. The application of the matrix tree theorem and similar theorems to electrical network theory is detailed by Chen [4]. The interested reader should also see [13] and [22].

Let G be a directed graph with vertices S. A *linking* in G from $U \subset S$ onto $W \subset S$ is a subgraph of G consisting of $|U|$ disjoint, directed paths each of which starts at a vertex in U and ends at a vertex in W. If the a_{ij} are set to appropriate values derived from a simple modification of G, a matrix $M(S \mid S)$ is obtained for which $M(\bar{W} \mid \bar{U})$ is nonsingular if and only if there is a linking from U onto W in G. Thus submatrices of M^{-1} are coordinatizations (linear representations) of gammoids defined by G. The coordinatizations so obtained are such that (up to a $(\det M)^k$ factor, which is a polynomial with all positive terms) determinants of their minors are generating functions for directed forests that contain linkings. These generating functions have the property that the sign of each term is determined by the parity of the "permutation" defined by the linking. In § 5 this coordinatization is contrasted with two other known coordinatizations. See [25] and [21] as general references for matroid theory and linking systems.

The notion of parity as used above is made precise in § 2. In fact, our proof of the matrix tree theorem is the result of a modification and strengthening of the linkage lemma of Ingleton and Piff [8] to take parity into account, along with an application of the principle of inclusion and exclusion as used by Orlin [19] in a proof of the theorem for $U = W = \{N\}$.

It is straightforward to extend the matrix tree theorem to graphs with multiple arcs. We omit these details except in § 4 where the results are extended to signed graphs. There the results apply nontrivially even to the loops and half-arcs that may belong to such graphs.

Our proofs are purely combinatorial in that we show every expression we deal with is a generating function for a set of combinatorial objects. We classify and count, with sign, the objects that correspond to a given monomial in order to compute its coefficient. This way we can see why the subgraphs enumerated by (2) contain linkings and have no cycles. We also see that the weights of the arcs in the linking only come from off-diagonal matrix entries and all the other weights come from diagonal entries. These insights lead us to proofs of extensions of the matrix tree theorem to signed and voltage graphs ([27], [6] and [7]) which are discussed in § 4.

The author's study of the matrix tree theorem and the work in § 2 and § 5 is mostly from [3], but §§ 3 and 4 are new. [1] is a general reference for the elementary graph theory notions which we do not define explicitly.

2. Matchings, paths, cycles and signs. Let A and B be equicardinal and not necessarily disjoint subsets of a set S. All sets in this paper are finite. A bijection $\pi: A \to B$ is called a *matching*. A *k-path* in π is a sequence $(x_0, x_1, \cdots, x_k)$ for which $x_0 \in A \backslash B$, $x_k \in B \backslash A$, and $\pi(x_i) = x_{i+1}$ for $0 \leqq i < k$. A *0-path* or trivial path (x_0) in π must satisfy $x_0 \notin A \cup B$. For nontrivial k-paths, $k > 0$, the elements $x_0, x_1, \cdots, x_k$ are distinct, and $x_i \in A \cap B$ for $0 < i < k$. For $n > 0$, an *n-cycle* in π is a set of distinct x_i, $\{x_1, x_2, \cdots, x_n\}$, for which $\pi(x_i) = x_{i+1}$ for $1 \leqq i < n$ and $\pi(x_n) = x_1$. Every element of an n-cycle in π belongs to $A \cap B$. A 1-cycle is called a trivial cycle.

We can view the matching π as a directed graph on S in which ij is an arc if and only if $i \in A$ and $\pi(i) = j$. Given a directed graph G, we say π is a matching in G when $\pi(i) = j$ only if $ij \in G$. Unless otherwise specified, a cycle or path will always mean a directed cycle or path. When we use the terms circuit or (connected) component, we ignore the arc directions.

It is clear that every matching decomposes into disjoint paths and cycles. (To be technical, we should note that the trivial paths depend upon the underlying set S.) The outdegree (resp. indegree) of i in π is 1 if $i \in A$ (resp. $i \in B$) and is 0 otherwise. When $A = B$ there are no nontrivial paths in π and we get the familiar decomposition of a permutation of A into cycles.

For completeness, we state the linkage lemma [8]. A *linking* of U onto W is a collection of $|U|$ disjoint directed paths each of which starts at an element of U and ends at an element of W.

LEMMA. *Suppose G is a directed graph of S. Let $G' = G \cup \{ii \mid i \in S\}$. Suppose U, $W \subset S$. Then, there is a linking in G from U onto W if and only if there is a matching $\pi: S \backslash W \to S \backslash U$ in G'.*

For a proof, see [25].

Now suppose A and B are linearly ordered; for example, suppose A and B are sets of integers. The pair $\{i, j\} \subset A$ is an *inversion* in π if $i < j$ and $\pi(i) > \pi(j)$. Let $n(\pi)$ denote the number of inversions in π. We define the *sign* $\varepsilon(\pi)$ of the matching by

$$\varepsilon(\pi) = (-1)^{n(\pi)}.$$

When π is a permutation, it is well known that $\varepsilon(\pi)$ is its sign, that $\varepsilon(\pi)$ does not depend on the ordering of $A = B$, and that when π is decomposed into cycles,

$$\varepsilon(\pi) = \prod_{n\text{-cycles}} (-1)^{n-1}. \tag{3}$$

Let Y be a linearly ordered set and $X \subset Y$. We define

$$n(X, Y) = |\{\{i, j\} \mid i < j, i \in Y \backslash X, j \in X\}|$$

and

$$\varepsilon(X, Y) = (-1)^{n(X, Y)}.$$

When $Y = \{1, 2, \cdots, N\}$, $n(X, Y)$ equals $\sum X - |X| - \binom{|X|}{2}$. Hence $\varepsilon(X, Y)\varepsilon(X', Y)$ commonly appears as $(-1)^{\sum X + \sum X'}$, when $|X| = |X'|$.

Suppose S, T are linearly ordered sets and $S \cap T = \phi$. Suppose $\pi: A \to B$ and $\pi': \bar{A} \to \bar{B}$ are matchings where $A \subset S$, $\bar{A} = S \backslash A$, $B \subset T$, and $\bar{B} = S \backslash B$. We can combine π and π' to form a matching $\pi \oplus \pi': S \to T$ for which

$$\pi \oplus \pi'(i) = \begin{cases} \pi(i) & \text{if } i \in A, \\ \pi'(i) & \text{if } i \in \bar{A}. \end{cases}$$

It is easy to prove by induction on $n(A, S)+n(B, T)$ that

$$\varepsilon(\pi \oplus \pi')=\varepsilon(A, S)\varepsilon(B, T)\varepsilon(\pi)\varepsilon(\pi').$$

COROLLARY. *Suppose S is linearly ordered, $A \subset S$, $\bar{A}=S\backslash A$, $B \subset S$, and $\bar{B}=S\backslash B$. Let $\pi : A \to B$ and $\pi' : \bar{A} \to \bar{B}$ be matchings. Then*

$$\varepsilon(\pi \oplus \pi')=\varepsilon(A, S)\varepsilon(B, S)\varepsilon(\pi)\varepsilon(\pi'). \tag{4}$$

Proof. Let T be a disjoint copy of S. Redefine B, $\bar{B}$ appropriately and apply the above remark. □

Let A and B be subsets of a linearly ordered set S and $\pi : A \to B$ be a matching. The paths in π determine a matching $\pi^* : \bar{A} \to \bar{B}$ as follows: For each (possibly trivial) path $(x_0, x_1, \cdots, x_k)$, we have $\pi^*(x_k)=x_0$. The linkage lemma asserts that there is a matching $\pi : A \to B$ in a certain digraph G' if and only if there is a linking in G from $\bar{B}$ onto $\bar{A}$ which defines π^* as shown. Our strengthening of this lemma shows how the signs of any such pair π, π^* must be related.

THEOREM. *Suppose $\pi : A \to B$ and $\pi^* : \bar{A} \to \bar{B}$ are given as above. Then*

$$\varepsilon(\pi)=\varepsilon(\pi^*)\varepsilon(A, S)\varepsilon(B, S) \prod_{\substack{k\text{-paths}\\ \text{in } \pi}} (-1)^k \prod_{\substack{n\text{-cycles}\\ \text{in } \pi}} (-1)^{n-1}. \tag{5}$$

Proof. $\pi \oplus \pi^* : S \to S$ is a permutation. Its cycles consist of one $(k+1)$-cycle for each k-path in π, along with all the n-cycles of π. Hence, when we apply (3) we obtain

$$\varepsilon(\pi \oplus \pi^*)=\prod_{\substack{k\text{-paths}\\ \text{in } \pi}} (-1)^k \prod_{n\text{-cycles in } \pi} (-1)^{n-1}.$$

The identity follows immediately from (4). □

The matrix tree theorem will be an easy consequence of the decomposition of π into paths and cycles, formula (5), and the definition of the determinant

$$\det M(A \mid B)=\sum_{\pi : A \to B} \varepsilon(\pi) \prod_{i \in A} M_{i,\pi(i)}.$$

The sum is taken over all matchings $\pi : A \to B$.

3. Proof of the matrix tree theorem. For convenience, we here restate the matrix tree theorem.

ALL MINORS MATRIX TREE THEOREM.

THEOREM. *Suppose $A(S \mid S)$ is given by* (1), *the $\varepsilon(\ )$ are defined in* § 2, *and U, $W \subset S$ with $|U|=|W|$. Then*

$$\det A(\bar{W} \mid \bar{U})=\varepsilon(W, S)\varepsilon(U, S) \sum_F \varepsilon(\pi^*)a_F \tag{2}$$

where the sum is over all forests F on S such that

(i) *F contains exactly $|U|=|W|$ trees.*

(ii) *Each tree in F contains exactly one vertex in U and exactly one vertex in W.*

(iii) *Each arc in F is directed away from the vertex in U of the tree containing that arc.*

F defines a matching $\pi^* : W \to U$ so $\pi^*(j)=i$ if and only if i and j are in the same tree of F.

Proof. By definition of $\det A(\bar{W} \mid \bar{U})$,

$$\det A(\bar{W} \mid \bar{U})=\sum_{\pi : \bar{W} \to \bar{U}} \varepsilon(\pi) \prod_{i \in \bar{W}} A_{i,\pi(i)}. \tag{6}$$

Suppose in (6), for each matching π, we distinguish the diagonal entries, which have the form A_{jj}, from the off-diagonal entries of A. If we apply the definition of A, we obtain

$$\det A(\bar{W}\,|\,\bar{U}) = \sum_{(\pi,\sigma)} \varepsilon(\pi)\left[\prod_{ij\in\sigma} a_{ij}\right] \prod_{\substack{\pi(i)=j\\ i\neq j}} (-a_{ij}) \ . \tag{7}$$

Here, the determinant is expressed as a sum of terms $\pm a_H$, one for each pair (π, σ) such that π is a matching $\pi: \bar{W} \to \bar{U}$ and σ is a set of arcs consisting of one and only one arc ij for each j such that $\pi(j) = j$.

Let H be any subgraph defined by a pair (π, σ) as above. In H, for all $j \in S$,

$$\operatorname{indeg}(j) = \begin{cases} 1 & \text{if } j \in \bar{U}, \\ 0 & \text{if } j \in U. \end{cases} \tag{8}$$

The indegrees in H are all at most one. Hence, any circuit in H must be a (directed) cycle. Furthermore, the cycles in H are disjoint. Now consider any path P in π as a subgraph of H. No arc in P can belong to a cycle in H. This is because the indegree in P of each vertex in P is equal to its indegree in H. Therefore, only arcs in P may be directed into vertices in P. We conclude that each cycle in H either belongs to σ or is a nontrivial cycle in π.

We can now conclude that if H has no cycles, then H is a forest F that satisfies (i), (ii), and (iii). Let us therefore write $\det A(\bar{W}\,|\,\bar{U})$ as $\sum_H c_H a_H$. The theorem will be proved when we show that $c_H = 0$ when H contains a cycle, that c_H is given by (2) otherwise, and that there is a pair (π, σ) that defines $H = F$ for every forest that satisfies (i), (ii), and (iii).

Let π^* be the matching $\pi^*: W \to U$ defined in (2) by the paths in π. When we apply (5) to (7) we obtain

$$\det A(\bar{W}\,|\,\bar{U}) = \varepsilon(\bar{W}, S)\varepsilon(\bar{U}, S) \sum_{(\pi,\sigma)} \varepsilon(\pi^*)(-1)^{cy(\pi)}\left[\prod_{ij\in\sigma} a_{ij}\right] \prod_{\substack{\pi(i)=j\\ i\neq j}} a_{ij} \tag{9}$$

where $cy(\pi)$ is the number of nontrivial cycles in π.

Let H be a subgraph with K cycles that is defined by some (π_1, σ_1). Let us consider all pairs (π, σ) that define H. In each pair, π has the same paths as π_1. All the arcs that are neither in a cycle nor in a path in π_1 belong to σ. Each cycle in H can be either a nontrivial cycle in π or a cycle in σ. Hence, there are 2^K pairs (π, σ) that define H and

$$c_H = \varepsilon(\bar{W}, S)\varepsilon(\bar{U}, S)\varepsilon(\pi^*) \sum_{c=0}^{K} (-1)^c \binom{K}{c} = \pm(1-1)^K = \begin{cases} \pm 1 & \text{if } K = 0, \\ 0 & \text{if } K \neq 0. \end{cases}$$

It is easy to see $\varepsilon(\bar{W}, S)\varepsilon(\bar{U}, S) = \varepsilon(W, S)\varepsilon(U, S)$ when $|W| = |U|$. Hence, c_H is given by (2).

Finally, suppose F is a forest that satisfies (i), (ii), and (iii). F is defined by the pair (π, σ) for which π has the paths linking U to W in F, π has no nontrivial cycles, and σ consists of all the arcs in F not in these paths. $\square$

The last step in the proof tells us each F counted by (2) is due to just one matching π in (6). The weights of the arcs in the linking only come from the off-diagonal entries of A. All the other arc weights come from diagonal entries which correspond to trivial cycles in π.

4. Extension to signed graphs. A signed graph is a graph to which each arc has been given a sign. See [27] for a systematic treatment of the definitions, properties, and applications of signed graphs. Broadly speaking, signed graphs differ from ordinary graphs in the matroids they define. For example, a circle (i.e., a circuit in the underlying graph) is a circuit in the signed graphic matroid only if it is *positive*—that is, the product of the signs is + (see [7]), otherwise the circle is an independent set.

A signed directed graph is like an ordinary directed graph, except each arc e is given a sign $s(e) \in \{+, -\}$, and, this time, we allow multiple arcs, *loops* (arcs of the form $e = ii$), and *half-arcs* ($e = i$; the sign of a half-arc is undefined). As in an ordinary directed graph, arc $e = ij$ is said to be directed "out" from i and "into" j (even if $i = j$). If $e = i$, e is said to be directed into i. A directed k-path is a sequence of arcs $(e_1 = x_0x_1, e_2 = x_1x_2, \cdots, e_k = x_{k-1}x_k)$ in which all the x_i are distinct. A directed n-cycle is a set of n arcs $\{e_1 = x_1x_2, e_2 = x_2x_3, \cdots, e_n = x_nx_1\}$ incident on n distinct vertices. Note half-arcs cannot appear in (directed) k-paths or n-cycles, while a loop is a 1-cycle. A signed directed graph differs from a signed graph (as in [27]) in that the fixed order of the endpoints of each arc allows us to define directed paths and cycles in directed graphs. These definitions must not be confused with those involving oriented signed graphs [26].

A path or cycle will be called *positive* if the product of the signs of its arcs is +; it is negative otherwise.

In this chapter we extend the matrix tree theorem and our proof to signed directed graphs. Then, in the same way the undirected graph version of the matrix tree theorem was obtained from the directed graph version, we obtain an extension of the matrix tree theorem to signed graphs by Zaslavsky [27]. We further extend the theorem to voltage graphs [6] over an abelian group.

As for the matrix tree theorem, we assign a weight a_e to each arc in the signed directed graph. One must not confuse the weight of an arc with its sign. Matrix $A(S \mid S)$ is defined as follows.

(10a) $$\text{If } i \neq j, \qquad A_{ij} = -\sum_e s(e) a_e$$

where the sum is over all arcs $e = ij$.

(10b) $$A_{jj} = \sum_e a_e + \sum_l 2a_l + \sum_h a_h$$

where e ranges over arcs ij directed into j for which $i \neq j$, l ranges over negative loops jj, and h ranges over half-arcs into j.

MATRIX TREE THEOREM FOR SIGNED DIRECTED GRAPHS. *Let G be a signed directed graph on S and $A(S \mid S)$ be as above. Suppose $U, W \subset S$, $|U| = |W|$. Then*

(11) $$\det A(\bar{W} \mid \bar{U}) = \varepsilon(U, S)\varepsilon(W, S) \sum_F \varepsilon(\pi^*)(-1)^{np(F)} 2^{nc(F)} a_F$$

where the sum is over all sets of arcs F in G such that

(i) *F contains $|U| = |W|$ components that are trees.*

(ii) *Each tree from* (i) *contains exactly one vertex in U and one vertex in W.*

(iii) *Each arc in each tree from* (i) *is directed away from the vertex in U of the tree containing that arc. Hence these trees together contain a linking from U onto W. This linking defines $\pi^*: W \to U$ as in the matrix tree theorem. $np(F)$ is the number of negative paths in this linking.*

(iv) *Each of the remaining components of F contains exclusively either just one half-arc or just one negative (directed) cycle. There are no other circles and each* *remaining arc is directed away from the half-arc or (directed) cycle of its component. $nc(F)$ is the number of negative cycles.*

Proof. It is easy to verify that

$$\det A(\bar{W}\,|\,\bar{U}) = \sum_H c_H a_H,$$

where the sum is over some subgraphs H in which for all $j \in S$, (8) is satisfied. Since our task is to determine c_H, we can set $a_e = 0$ for $e \notin H$ and write our proof as in § 3. Please note that ij designates a particular arc in $H \subset G$ with a given sign. Equation (7) becomes ($\delta_{ij} = 1$ if $i = j$, $\delta_{ij} = 0$ if $i \neq j$, and $\delta_i = 0$)

$$\det A(W\,|\,U) = \sum_{(\pi,\sigma)} \varepsilon(\pi)\Big[\prod_{ij\in\sigma}(1+\delta_{ij})a_{ij}\Big]\Big[\prod_{\substack{\pi(i)=j\\ i\neq j}}(-s(ij)a_{ij})\Big] \tag{12}$$

where we have abused the notation because σ may contain a half-arc. Still, any nontrivial directed cycle in H is either a nontrivial cycle in π or a nontrivial directed cycle in σ. The arc sign factors $s(\cdot)$ only occur for arcs in π, so the extension of (9) is

$$\det A(\bar{W}\,|\,\bar{U}) = \varepsilon(\bar{W}, S)\varepsilon(\bar{U}, S)$$

$$\cdot \sum_{(\pi,\sigma)} \varepsilon(\pi^*)(-1)^{cy(\pi)}(-1)^{nc'(\pi)}(-1)^{np(\pi)}\Big[\prod_{ij\in\sigma}(1+\delta_{ij})a_{ij}\Big]\Big[\prod_{\substack{\pi(i)=j\\ i\neq j}} a_{ij}\Big]\cdot \tag{13}$$

where $nc'(\pi)$ and $np(\pi)$ are respectively the numbers of negative nontrivial cycles and negative paths in π. If H has K_p positive nontrivial directed cycles and K_n negative nontrivial directed cycles, there are $2^{K_p+K_n}$ pairs (π, σ) that define H. For each trivial cycle jj in H, $jj \in \sigma$ and $\pi(j) = j$ for each (π, σ) that defines H, and so the factor $(1+\delta_{jj}) = 2$ occurs in each term for H in (13). Let K_t be the number of trivial cycles in H.

We conclude

$$c_H = \varepsilon(\bar{W}, S)\varepsilon(\bar{U}, S)\varepsilon(\pi^*)(-1)^{np(H)}2^{K_t}(1+1)^{K_n}(1-1)^{K_p}. \tag{14}$$

Thus, if H has no positive cycles, c_H is given by (11). Finally, suppose F is given which satisfies (i), (ii), (iii), and (iv) with K_n negative nontrivial directed cycles. Again, we set all the as but those in a_F to zero. Then there are 2^{K_n} pairs (π, σ) that define F. In all of them, π contains the linking described in (iii) and σ contains the negative trivial directed cycles and all arcs neither in a cycle nor the linking. Each negative, nontrivial directed cycle belongs to either π or σ exclusively. Thus a_F appears in (11). □

For a signed (undirected) graph G on S, $A(S\,|\,S)$ is a symmetric matrix [27]. To represent G by a signed directed graph G', we represent each undirected arc $e = ij$ by a pair of directed arcs ij and ji with identical weights a_e and signs, even if $i = j$. Half arcs in the undirected graph are represented by only one arc in the directed graph. Hence the analogue of (10) is

$$\text{if } i \neq j, \qquad A_{ij} = -\sum_e s(e)a_e, \qquad A_{jj} = \sum_e a_e + \sum_l 4a_l + \sum_h a_h.$$

The factor of 4 makes more sense when A is written $A = DED^t$ where D is a signed incidence matrix of G and E is the diagonal matrix of arc weights.

MATRIX TREE THEOREM FOR SIGNED (UNDIRECTED) GRAPHS [27].

$$\det A(\bar{A} \mid \bar{U}) = \varepsilon(U, S)\varepsilon(W, S) \sum_F \varepsilon(\pi^*)(-1)^{np(F)}4^{nc(F)}a_F. \tag{15}$$

The sum is over all sets of arcs F that satisfy conditions similar to (i), (ii), (iii) *and* (iv). *The new conditions are obtained by deleting the "directed" qualifier everywhere from the old conditions.*

Proof. Suppose we apply the directed graph version of the theorem to G'. Suppose T is a tree in G that, according to the conditions, contains $u \in U$ or a half arc. Then there is exactly one directed tree T' in G', with $a_T = a_{T'}$, that satisfies the corresponding conditions, and conversely. Now suppose T is a subgraph in G that, according to condition (iv), contains a unique circle (which is negative). Then there are just two subgraphs T' in G', with $a_T = a_{T'}$ that satisfy the corresponding conditions, and conversely. The directed cycles in these two subgraphs are directed oppositely while all the other directed arcs are identical. Thus, each undirected graph F that satisfies (i), (ii), (iii) and (iv) with $nc(F)$ negative circles is counted $2^{nc(F)}$ times by directed graphs F' in G' with $a_F = a_{F'}$. The coefficient for each directed graph F' is $\pm 2^{nc(F)}$ (and is constant), so c in (15) is $\pm 2^{nc(F)}2^{nc(F)} = \pm 4^{nc(F)}$. □

A voltage graph ([27], [6]) is a graph to which each arc has been given an element of a group. Signed graphs are a special case of voltage graphs. Our method can be used to prove a version of the matrix tree theorem for voltage graphs over an abelian group Γ. It is necessary to extend the ring of coefficients for the polynomials in the arc weights to the group ring of Γ. A directed cycle is positive when the product of the voltages on its arcs is 1, the identity of Γ. Suppose we define matrix A for a voltage graph as in (10) except $s(e)$ now stands for the voltage of arc e and the coefficient of a_l in A_{jj} when l is a loop $l = jj$ is $(1 - s(l))$. When E is a set of arcs, let $s(E)$ denote the product of their voltages. The voltage directed graph version goes through as for the signed directed graph theorem except that the notion of positivity is replaced with the notion of positivity for voltage graphs and expression (11) becomes

$$\det A(\bar{W} \mid \bar{U}) = \varepsilon(U, S)\varepsilon(W, S) \sum_F \varepsilon(\pi^*)s(P) \prod_C (1 - s(C))a_F.$$

Here, P is the linking from U onto W in condition (iii). C ranges over the nonpositive directed cycles in F.

5. Gammoids. The matrix tree theorem can be used to give a coordinatization (i.e., representation of a matroid by the column vectors of a matrix over a field) of gammoids that is "natural" with respect to sign in a way that other known coordinatizations are not. We discuss this below. The books by Welsh [25] and Schrijver [21] are our references for matroids and linking systems.

Let G be a directed graph on vertices S and let a_{ij} be an indeterminate when ij is an arc in G and be zero otherwise. The matrix tree theorem implies that $A(\bar{W} \mid \bar{U})$ is nonsingular only if there is a linking in G of U onto W.

Now let $-B$ be the same matrix as A except that its main diagonal entries are all zero. Let I be the identity matrix and $T = I - B$. The linkage lemma of Ingleton and Piff [8] asserts that det $T(\bar{W} \mid \bar{U})$ is nonzero if and only if there is a linking in G of U onto W. The subsets U of S for which there is a linking in from U onto W, where W is a fixed subset of S, comprise the bases of a matroid. Such a matroid is called a *strict gammoid* [20]. The linkage lemma is the key step in the proof that a matroid is a strict gammoid if and only if it is the dual of a transversal matroid.

Linking systems or bimatroids [10] provide an alternative view of matroid theory that is most suitable for the purposes of this section. A linking system (X, Y, Λ) is equivalent to a matroid M on the disjoint union $X \cup Y$ with a distinguished base X. A pair (U, W) belongs to $\Lambda \subset 2^X \times 2^Y$, which is called the set of linked pairs (or nonsingular minors), when $(X \setminus U) \cup W$ is a base in M. The axioms for linking systems given by Schrijver [21] are properties satisfied by the (U, W) such that there is a matching from U onto W in a bipartite graph $G \subset X \times Y$.

(a) If $(U, W) \in \Lambda$ and $x \in U$, then $(U \setminus x, W \setminus y) \in \Lambda$ for some $y \in W$.

(b) If $(U, W) \in \Lambda$ and $y \in W$, then $(U \setminus y, W \setminus y) \in \Lambda$ for some $x \in U$.

(c) If $(U_1, W_1), (U_2, W_2) \in \Lambda$, then there exists $(U', W') \in \Lambda$ with $U_1 \subset U' \subset U_1 \cup U_2$ and $W_2 \subset W' \subset W_1 \cup W_2$.

The third is the Dulmage–Mendelsohn [15] property.

A linking system (X, Y, Λ) is said to be *coordinatized* by a matrix $M(X \mid Y)$ when $(U, W) \in \Lambda$ if and only if $M(U \mid W)$ is nonsingular. Now suppose (X, Y, Λ) is such that $(X, Y) \in \Lambda$. Schrijver shows then that (Y, X, Λ^{-1}) is a linking system, where

$$\Lambda^{-1} = \{(W, U) \mid (X \setminus U, Y \setminus W) \in \Lambda\}.$$

(Y, X, Λ^{-1}) is called the inverse of (X, Y, Λ). It follows from Jacobi's theorem [18] that if $M(X \mid Y)$ coordinatizes (X, Y, Λ), then $M^{-1}(Y \mid X)$ coordinatizes (Y, X, Λ^{-1}). To be more specific in our application of Jacobi's theorem, if $M(S \mid S)$ is any matrix and $\hat{M}(S \mid S)$ is defined by

$$\hat{M}_{ij} = \varepsilon(i, S)\varepsilon(j, S) \det M(\bar{j} \mid \bar{i})$$

(note $\varepsilon(i, S)\varepsilon(j, S) = (-1)^{i+j}$ when $S = \{1, 2, \cdots, N\}$), then

$$\det \hat{M}(U \mid W) = (\det M)^{|U|-1} \varepsilon(U, S)\varepsilon(W, S) \det M(\bar{W} \mid \bar{U}). \tag{16}$$

Let G be a directed graph on S. G defines the *strict gammoid linking system* (S, S, Λ) in which $(U, W) \in \Lambda$ if and only if there is a linking of U onto W in G. Thus, the transposed submatrices of a coordinatization of the strict gammoid linking system (S, S, Λ) comprise coordinatizations of all the gammoid matroids that can be defined by G. We will give three coordinatizations of the strict gammoid linking system defined by G. The coordinatizations will be over any extension field that contains the algebraically independent elements $\{a_e \mid e \text{ is an arc in } G\}$.

The first coordinatization is $\hat{T}$. Essentially, it was described by Schrijver and the proof of its correctness uses the linkage lemma. When we combine (16) with an argument similar to that in § 3, we obtain

$$\det \hat{T}(U \mid W) = (\det T)^{|U|-1} \sum_F \varepsilon(\pi^*)(-1)^{cy(F)} a_F$$

where the sum is over all subgraphs F of G whose connected components consist of a linking from U onto W, isolated vertices, and $cy(F)$ disjoint (directed) cycles. The linking defines a matching $\pi^*: W \to U$ where $\pi^*(j) = i$ when the linking contains a path from i to j.

The second coordinatization is $\hat{H}$, where $H = I + A$ and A is the matrix (1) in the matrix tree theorem.

THEOREM. *$H(\bar{W} \mid \bar{U})$ is nonsingular if and only if there is a disjoint collection of directed paths linking U onto W in G.*

Proof. Let $0 \notin S$. Consider graph G' on $S \cup \{0\}$ which contains all the arcs in G along with all arcs $0j$, $j \in S$. Suppose the latter arcs have weight 1. H is the submatrix of the "special" adjacency matrix (1) of G' obtained by deleting row and column 0.

If there is a linking L in G from U onto W, then there is a term in $\det H(\bar{W} \mid \bar{U})$ corresponding to the forest consisting of the arcs in L along with all arcs $0j$ for j not a vertex in L. Conversely, if $\det H(\bar{W} \mid \bar{U}) \neq 0$ there is a forest in G' that contains a linking in G from U onto W. □

The above proof along with the matrix tree theorem and (16) is used to derive

$$\det \hat{H}(U \mid W) = (\det H)^{|U|-1} \sum_F \varepsilon(\pi^*) a_F.$$

Apart from the $(\det H)^{|U|-1}$ factor, this is the generating function for all directed forests in G that contain linkings from U to W. The sign of each term is the sign of the matching $\pi^*: W \to U$ that the corresponding linking determines. In this sense we remark that the coordinatization $\hat{H}$ is "natural" in a way the first coordinatization fails to be.

The third coordinatization comes from Mason [12]. It is the matrix $P(S \mid S)$ defined by

$$P_{ij} = \sum_P a_P$$

where the sum is over all (simple) directed paths from i to j in G. Suppose $|U| = |W| = l$. Mason's proof uses Menger's theorem to factor $P(U \mid W)$ into a product of an $l \times k$ and a $k \times l$ matrix with $k < l$ when no linking from U to W exists. Lindström [11] attempted to give a proof based upon the claim that $\det P(U \mid W)$ was equal to

$$\sum_L \varepsilon(\pi^*) a_L \tag{17}$$

where the sum is over all linkings from U onto W and $\pi^*: W \to U$ is the matching defined by each. This claim is false when G contains directed cycles. For example, suppose G is itself a directed n-cycle. Then $S = \{1, 2, \cdots, n\}$ and the arcs of G are $\{ij \mid 1 \leqq i \leqq n \text{ and } j = i+1 \bmod n\}$, so

$$P_{ij} = \begin{cases} a_{i,i+1} a_{i+1,i+2} \cdots a_{j-1,j} & \text{if } i \neq j, \\ 1 & \text{if } i = j \end{cases}$$

where the subscripts are taken mod n. We have

$$\det P = (1 - a_G)^{n-1}. \tag{18}$$

We remark that the determinant of a submatrix of P for an acyclic graph has been applied to an enumeration problem for plane partitions by Gessel [5]. There, the relevant $\varepsilon(\pi^*)$ are all equal to 1.

It is tempting to ask whether the coordinatization $\hat{M} = \hat{T}$, $\hat{M} = \hat{H}$ or P can be "fixed up" so that the factor $(\det M)^{|U|-1}$ no longer appears in $\det \hat{M}(U \mid W)$ in the former two or that (17) indeed is the determinant of the $(U \mid W)$ minor in the latter. We remark the answer is no in all cases. The reason is simply that if we require this of the 1×1 minors of the coordinatizations, we obtain the same matrices $\hat{T}$, $\hat{H}$ and P. One can ask, however, for a nice combinatorial description of $\det P(U \mid W)$ for all U, $W \subset S$, $|U| = |W|$, which will provide a combinatorial proof of (18). This question is apparently open.

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