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A COMBINATORIAL THEOREM ON SYSTEMS OF INEQUALITIES AND ITS APPLICATION TO ANALYSIS

(*Preliminary Communication*)

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Let S be a set. Let us denote by C the set of all functions λ defined on S and fulfilling the following postulates:
1° $\lambda(s) \geqq 0$ for every $s \in S$; 2° the set $N(\lambda)$ of those $s \in S$, where $\lambda(s) > 0$, is finite; 3° $\sum_{s \in S} \lambda(s) = 1$

If $\varepsilon > 0$, $H \subset S$ and W is a system of finite subsets of S, we shall denote by $C(\varepsilon, H, W)$ the set of those $\lambda \in C$ such that $N(\lambda) \subset H$ and $\sum_{s \in w} \lambda(s) < \varepsilon$ for each $w \in W$.

We intend to prove the following

Theorem. *Let S be an infinite set and let T be a system of finite subsets of S. The following two conditions are equivalent to each other:*
(1) *there exists an infinite $H \subset S$ and an $\varepsilon > 0$ such that $C(\varepsilon, H, T) = 0$.*
(2) *there exists a strictly increasing sequence of finite sets $B_n \subset S$ and a sequence $t_n \in T$ such that $B_n \subset t_n$.*

Proof: Suppose first that (2) is fulfilled. Put $H = \cup B_n$ and suppose that $\lambda \in C(\varepsilon, H, T)$ for some $\varepsilon < 1$. There exists an n such that $N(\lambda) \subset B_n$.

It follows that $N(\lambda) \subset t_n$ whence

$$1 = \sum_{s \in N(\lambda)} \lambda(s) \leqq \sum_{s \in t_n} \lambda(s) < \varepsilon < 1,$$

which is a contradiction.

For the proof of the other part of the theorem, it is convenient to introduce the following abbreviations. If $K \subset S$, denote by $T(K)$ the set of those $t \in T$ for which $K \cap t \neq 0$. If $j \in S$, denote by $T(j)$ the set of those $t \in T$ for which $j \in t$. The proof is based on the following simple proposition:

Let $C(\varepsilon, H, T) = 0$. Let F be an arbitrary finite subset of S and let $\varepsilon_1 < \varepsilon$. Then there exists a nonvoid finite $K_1 \subset S$ disjoint from F such that $C(\varepsilon_1, H - F, T(K_1)) = 0$.

To prove this proposition, take a fixed finite $F \subset S$ and a fixed ε_1 with $0 < \varepsilon_1 < \varepsilon$. Suppose that $C(\varepsilon_1, H-F, T(K)) \neq 0$ for each finite $K \subset S$ disjoint from F. Choose a finite nonvoid $A_1 \subset H-F$. Take $\lambda_1 \in C\ (\varepsilon_1, H-F, T(A_1))$ and put $A_2 = A_1 \cup N(\lambda_1)$ so that $A_2 \subset H-F$. Take $\lambda_2 \in C(\varepsilon_1, H-F, T(A_2))$ and put $A_3 = A_2 \cup N(\lambda_2) \subset H-F$. Putting $A_{n+1} = A_n \cup N(\lambda_n)$ and choosing $\lambda_{n+1} \in C(\varepsilon_1, H-F, T(A_{n+1}))$a sequence λ_n and $A_n \subset H-F$ may be constructed.

Let $t \in T$. Let us show now that the sequence $\alpha_n = \sum_{s \in t} \lambda_n(s)$ contains at most one term $\geqq \varepsilon_1$. Indeed, let $\alpha_p \geqq \varepsilon_1$ for some p. Since $N(\lambda_p) \subset A_{p+1}$, we have $A_{p+1} \cap t \neq 0$ whence $t \in T(A_{p+1})$. If $q > p$, we have $t \in T(A_{p+1}) \subset T(A_q)$. Since $\lambda_q \in C(\varepsilon_1, H-F, T(A_q))$, we have $\alpha_q < \varepsilon_1$. It follows immediately that, for n large enough,

$$n^{-1}(\lambda_1 + \ldots + \lambda_n) \in C(\varepsilon, H-F, T)\ ,$$

which is a contradiction. The proof of our proposition is complete.

Apply now the same proposition with $\varepsilon_1, K_1, T(K_1)$ instead of ε, F, T. It follows that there exists a nonvoid finite K_2 disjoint from K_1 and an $\varepsilon_2 > 0$ such that $C\ (\varepsilon_2, H-K_1, T(K_1) \cap T(K_2)) = 0$. Continuing this process, we obtain a sequence $K_1, K_2, \ldots$ of mutually disjoint finite nonvoid sets such that $T(K_1) \cap \ldots \cap T(K_r) \neq 0$ for each r.

Since $T(K) = \bigcup_{k \in K} T(k)$, it is easy to see that there exist points $i_1 \in K_1$, $i_2 \in K_2, \ldots$ such that $T(i_1) \cap \ldots \cap T(i_r) \neq 0$ for each r. Let B_n be the set consisting of $i_1, i_2, \ldots, i_n$. The sets K_r being mutually disjoint, the sequence B_n is strictly increasing; if t_n is chosen in $T(i_1) \cap \ldots \cap T(i_n)$, we have clearly $B_n \subset t_n$. The proof is complete.

The preceding theorem yields several theorems connecting convergence and uniform convergence in double sequences and may be applied to obtain results on weak compactness in Banach spaces.

Резюме

КОМБИНАТОРНАЯ ТЕОРЕМА О СИСТЕМАХ НЕРАВЕНСТВ И ЕЕ ПРИМЕНЕНИЯ В АНАЛИЗЕ

(Предварительное сообщение)

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Приводится комбинаторная теорема о разрешимости некоторых систем линейных неравенств и указываются возможности ее применения в анализе.