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A COMBINATORIAL VERSION OF THE GROTHENDIECK CONJECTURE

SHINICHI MOCHIZUKI

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Abstract. We study the "*combinatorial anabelian geometry*" that governs the relationship between the dual semi-graph of a pointed stable curve and various associated profinite fundamental groups of the pointed stable curve. Although many results of this type have been obtained previously in various particular situations of interest under unnecessarily strong hypotheses, the goal of the present paper is to step back from such "typical situations of interest" and instead to consider this topic in the *abstract*—a point of view which allows one to prove results of this type in *much greater generality* under *very weak hypotheses*.

Introduction. In this paper, we apply the language of *anabelioids* [cf. [Mzk5], [Mzk7]] to study the "*profinite combinatorial group theory*" arising from the relationship between the *semi-graph of anabelioids associated to a pointed stable curve* [i.e., a "semi-graph of anabelioids of PSC-type"—cf. Definition 1.1, (i), below for more details] and a certain associated profinite fundamental group [cf. Definition 1.1, (ii)]. In particular, we show that:

(i) The *cuspidal* portion of the semi-graph may be recovered group-theoretically from the associated profinite fundamental group, together with certain *numerical information* [roughly speaking, the number of cusps of the various finite étale coverings of the given semi-graph of anabelioids]—cf. Theorem 1.6, (i).

(ii) The entire "*semi-graph of anabelioids structure*" may be recovered group-theoretically from the associated profinite fundamental group, together with a certain *filtration* [arising from this "semi-graph of anabelioids structure"] of the abelianizations of the various finite étale coverings of the given semi-graph of anabelioids—cf. Theorem 1.6, (ii).

Moreover, we show that from the point of view of "*weights*" [i.e., logarithms of absolute values of eigenvalues of the action of the Frobenius element of the Galois group of a finite field], the data necessary for (i) (respectively, (ii)) above may be recovered from *very weak* assumptions concerning the "weights"—cf. Corollary 2.7, (i), (ii). In particular, [unlike the techniques of [Mzk4], Lemmas 1.3.9, 2.3, for example] these very weak assumptions do not even require the existence of a *particular Frobenius element*. Alternatively, when there are no cusps, the data necessary for (ii) may be recovered from *very weak* assumptions concerning the *l-adic inertia* action [cf. Corollary 2.7, (iii)]—i.e., one does not even need to consider weights. This sort of result may be regarded as a *strengthening* of various results to the effect

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that a curve has good reduction if and only if the l-adic inertia action is *trivial* [cf., e.g., [Tama1], Theorem 0.8]

One consequence of this theory is the result [cf. Corollary 2.7, (iv)] that the subgroup of the group of outer automorphisms of the associated fundamental group consisting of the *graphic* outer automorphisms [i.e., the automorphisms that are compatible with the "semi-graph of anabelioids structure"] is *equal to its own commensurator* within the entire group of outer automorphisms. This result may be regarded as a sort of "anabelian analogue" of a well-known "linear algebra fact" concerning the general linear group [cf. Remark 2.7.1].

The original motivation for the development of the theory of the present paper is as follows: Frequently, in discussions of the *anabelian geometry* of hyperbolic curves, one finds it necessary to reconstruct the *cusps* [cf., e.g., [Naka1], Theorem 3.4; [Mzk4], Lemma 1.3.9; [Tama2], Lemma 2.3, Proposition 2.4] or the entire *dual semi-graph* associated to a pointed stable curve [cf., e.g., [Mzk2], §1–5; [Mzk4], Lemma 2.3] *group-theoretically* from some associated profinite fundamental group. Moreover, although the techniques for doing this in various diverse situations are *quite similar* and only require *much weaker assumptions* than the assumptions that often hold in particular situations of interest, up till now, there was no *unified presentation* or *general results* concerning this topic—only a collection of papers covering various "particular situations of interest". Thus, the goal of the present paper is to prove results concerning this topic in *maximum possible generality*, in the hope that this may prove *useful in applications* to situations not covered in previous papers [cf., e.g., Corollaries 2.8, 2.9, 2.10; Remarks 2.8.1, 2.8.2].

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0. Notation and conventions.

Numbers: The notation $\boldsymbol{Q}$ will be used to denote the field of *rational numbers*. The notation $\boldsymbol{Z} \subseteq \boldsymbol{Q}$ will be used to denote the set, group, or ring of *rational integers*. The notation $\boldsymbol{N} \subseteq \boldsymbol{Z}$ will be used to denote the submonoid of integers ≥ 0. If l is a prime number, then the notation $\boldsymbol{Q}_l$ (respectively, $\boldsymbol{Z}_l$) will be used to denote the *l-adic completion* of $\boldsymbol{Q}$ (respectively, $\boldsymbol{Z}$).

Topological Groups: Let G be a *Hausdorff topological group*, and $H \subseteq G$ a *closed subgroup*. Let us write

$$Z_G(H) \stackrel{\text{def}}{=} \{g \in G \mid g \cdot h = h \cdot g \quad \text{for any } h \in H\}$$

for the *centralizer* of H in G;

$$N_G(H) \stackrel{\text{def}}{=} \{g \in G \mid g \cdot H \cdot g^{-1} = H\}$$

for the *normalizer* of H in G; and

$$C_G(H) \stackrel{\text{def}}{=} \{g \in G \mid (g \cdot H \cdot g^{-1}) \cap H \text{ has finite index in } H,\, g \cdot H \cdot g^{-1}\}$$

for the *commensurator* of H in G. Note that: (i) $Z_G(H)$, $N_G(H)$ and $C_G(H)$ are *subgroups of* G; (ii) we have *inclusions*

$$H,\ Z_G(H) \subseteq N_G(H) \subseteq C_G(H)$$

and (iii) H is *normal* in $N_G(H)$. We shall say that H is *centrally* (respectively, *normally*; *commensurably*) *terminal* in G if $Z_G(H) = H$ (respectively, $N_G(H) = H$; $C_G(H) = H$).

We shall denote the group of automorphisms of G by Aut(G). Conjugation by elements of G determines a homomorphism $G \to \text{Aut}(G)$ whose image consists of the *inner automorphisms* of G. We shall denote by Out(G) the quotient of Aut(G) by the [normal] subgroup consisting of the inner automorphisms.

Curves: Suppose that $g \geq 0$ is an *integer*. Then if S is a scheme, a *family of curves of genus* g

$$X \to S$$

is defined to be a smooth, proper, geometrically connected morphism of schemes $X \to S$ whose geometric fibers are curves of genus g.

Suppose that $g, r \geq 0$ are *integers* such that $2g - 2 + r > 0$. We shall denote the *moduli stack of* r*-pointed stable curves of genus* g (where we assume the points to be *unordered*) by $\bar{\mathcal{M}}_{g,r}$ [cf. [DM], [Knud] for an exposition of the theory of such curves; strictly speaking, [Knud] treats the finite étale covering of $\bar{\mathcal{M}}_{g,r}$ determined by *ordering* the marked points]. The open substack $\mathcal{M}_{g,r} \subseteq \bar{\mathcal{M}}_{g,r}$ of smooth curves will be referred to as the *moduli stack of smooth* r*-pointed stable curves of genus* g or, alternatively, as the *moduli stack of hyperbolic curves of type* (g, r). The *divisor at infinity* $\bar{\mathcal{M}}_{g,r} \backslash \mathcal{M}_{g,r}$ of $\bar{\mathcal{M}}_{g,r}$ determines a *log structure* on $\bar{\mathcal{M}}_{g,r}$; denote the resulting log stack by $\bar{\mathcal{M}}_{g,r}^{\log}$.

A *family of hyperbolic curves of type* (g, r)

$$X \to S$$

is defined to be a morphism which factors $X \hookrightarrow Y \to S$ as the composite of an open immersion $X \hookrightarrow Y$ onto the complement $Y \backslash D$ of a relative divisor $D \subseteq Y$ which is finite étale over S of relative degree r, and a family $Y \to S$ of curves of genus g. One checks easily that, if S is *normal*, then the pair (Y, D) is *unique up to canonical isomorphism*. (Indeed, when S is the spectrum of a field, this fact is well-known from the elementary theory of algebraic curves. Next, we consider an arbitrary *connected normal* S on which a prime l is *invertible* (which, by Zariski localization, we may assume without loss of generality). Denote by $S' \to S$ the finite étale covering parametrizing *orderings of the marked points* and *trivializations of the* l*-torsion points of the Jacobian of* Y. Note that $S' \to S$ is *independent* of the choice of (Y, D), since (by the normality of S), S' may be constructed as the *normalization* of S in the function field of S' (which is independent of the choice of (Y, D), since the restriction of (Y, D) to the generic point of S has already been shown to be unique). Thus, the uniqueness of (Y, D) follows by considering the classifying morphism (associated to (Y, D)) from S' to the finite étale covering of $(\mathcal{M}_{g,r})_{\mathbf{Z}[1/l]}$ parametrizing orderings of the marked points and trivializations of the l-torsion points of the Jacobian [since this covering is well-known to be a scheme, for l sufficiently large].) We shall refer to Y (respectively, D; D; D) as the *compactification* (respectively, *divisor at infinity*; *divisor of cusps*; *divisor of marked points*) of X. A *family of hyperbolic curves* $X \to S$ is defined to be a morphism $X \to S$ such that the restriction of this morphism to each connected component of S is a *family of hyperbolic curves of type* (g, r) for some integers (g, r) as above.

Write

$$\bar{\mathcal{C}}_{g,r} \to \bar{\mathcal{M}}_{g,r}$$

for the *tautological curve* over $\bar{\mathcal{M}}_{g,r}$; $\bar{\mathcal{D}}_{g,r} \subseteq \bar{\mathcal{M}}_{g,r}$ for the corresponding *tautological divisor of marked points*. The divisor given by the union of $\bar{\mathcal{D}}_{g,r}$ with the inverse image in $\bar{\mathcal{C}}_{g,r}$ of the divisor at infinity of $\bar{\mathcal{M}}_{g,r}$ determines a *log structure* on $\bar{\mathcal{C}}_{g,r}$; denote the resulting log stack by $\bar{\mathcal{C}}_{g,r}^{\log}$. Thus, we obtain a morphism of log stacks

$$\bar{\mathcal{C}}_{g,r}^{\log} \to \bar{\mathcal{M}}_{g,r}^{\log},$$

which we refer to as the *tautological log curve* over $\bar{\mathcal{M}}_{g,r}^{\log}$. If $S^{\log}$ is *any log scheme*, then we shall refer to a morphism

$$C^{\log} \to S^{\log},$$

which is obtained as the pull-back of the tautological log curve via some [necessarily *uniquely determined*—cf., e.g., [Mzk1], §3] *classifying morphism* $S^{\log} \to \bar{\mathcal{M}}_{g,r}^{\log}$ as a *stable log curve*. If C has *no nodes*, then we shall refer to $C^{\log} \to S^{\log}$ as a *smooth log curve*.

If X_K (respectively, Y_L) is a *hyperbolic curve over a field* K (respectively, L), then we shall say that X_K is *isogenous* to Y_L if there exists a hyperbolic curve Z_M over a field M together with *finite étale morphisms* $Z_M \to X_K$, $Z_M \to Y_L$.

1. Criterion for graphicity. In the present §1, we state and prove a criterion for an isomorphism between the *profinite fundamental groups of pointed stable curves* to arise from an isomorphism of [*semi-*] *graphs of groups*. To do this, we shall find it convenient to use the language of *anabelioids* [cf. [Mzk5]], together with the theory of *semi-graphs of anabelioids* of [Mzk7].

Let Σ be a *nonempty set of prime numbers*. Denote by

$$\hat{\boldsymbol{Z}}^{\Sigma}$$

the *pro-Σ completion* of $\boldsymbol{Z}$. Let $\mathcal{G}$ be a *semi-graph of anabelioids* [cf. [Mzk7], Definition 2.1], whose underlying semi-graph we denote by $\boldsymbol{G}$. Thus, for each vertex v (respectively, edge e) of $\boldsymbol{G}$, we are given a connected anabelioid [i.e., a Galois category] $\mathcal{G}_v$ (respectively, $\mathcal{G}_e$), and for each branch b of an edge e abutting to a vertex v, we are given a morphism of anabelioids $\mathcal{G}_e \to \mathcal{G}_v$.

DEFINITION 1.1. (i) We shall refer to $\mathcal{G}$ as being *of pro-Σ PSC-type* [i.e., "pointed stable curve type"] if it arises as the pro-Σ completion [cf. [Mzk7], Definition 2.9, (ii)] of the semi-graph of anabelioids determined by the "dual semi-graph of profinite groups with compact structure" [i.e., the object denoted "$\mathcal{G}_X^{\mathrm{c}}$" in the discussion of *pointed stable curves* in

[Mzk4], Appendix] of a pointed stable curve over an algebraically closed field whose characteristic $\notin \Sigma$. [Thus, the vertices (respectively, closed edges; open edges) of $\boldsymbol{G}$ correspond to the *irreducible components* (respectively, *nodes*; *cusps* [i.e., marked points]) of the pointed stable curve.] We shall refer to $\mathcal{G}$ as being *of PSC-type* if it is of pro-Σ PSC-type for some nonempty set of prime numbers Σ. If $\mathcal{G}$ is a semi-graph of anabelioids of PSC-type, then we shall refer to the open (respectively, closed) edges of the underlying semi-graph $\boldsymbol{G}$ of $\mathcal{G}$ as the *cusps* (respectively, *nodes*) of $\mathcal{G}$ [or $\boldsymbol{G}$] and write $\underline{r}(\mathcal{G})$ (respectively, $\underline{n}(\mathcal{G})$) for the cardinality of the set of cusps (respectively, nodes) of $\mathcal{G}$; if $\underline{r}(\mathcal{G}) = 0$ (respectively, $\underline{n}(\mathcal{G}) = 0$), then we shall say that $\mathcal{G}$ is *noncuspidal* (respectively, *nonnodal*). Also, we shall write $\underline{i}(\mathcal{G})$ for the cardinality of the set of vertices of $\boldsymbol{G}$.

(ii) Suppose that $\mathcal{G}$ is of pro-Σ PSC-type. Then we shall denote by

$$\Pi_{\mathcal{G}}$$

and refer to as the *PSC-fundamental group* of $\mathcal{G}$ the maximal pro-Σ quotient of the profinite fundamental group of $\mathcal{G}$ [cf. [Mzk7], the discussion following Definition 2.2]; we shall refer to a finite étale covering of $\mathcal{G}$ that arises from an open subgroup of $\Pi_{\mathcal{G}}$ as a [finite étale] *$\Pi_{\mathcal{G}}$-covering* of $\mathcal{G}$. A vertex (respectively, edge) of $\boldsymbol{G}$ determines, up to conjugation, a closed subgroup of $\Pi_{\mathcal{G}}$; we shall refer to such subgroups as *verticial* (respectively, *edge-like*). An edge-like subgroup that arises from a closed edge will be referred to as *nodal*; an edge-like subgroup that arises from an open edge will be referred to as *cuspidal*. Write $M_{\mathcal{G}}$ for the *abelianization* of $\Pi_{\mathcal{G}}$. Then the cuspidal, edge-like, and verticial subgroups of $\Pi_{\mathcal{G}}$ determine submodules

$$M_{\mathcal{G}}^{\mathrm{cusp}} \subseteq M_{\mathcal{G}}^{\mathrm{edge}} \subseteq M_{\mathcal{G}}^{\mathrm{vert}} \subseteq M_{\mathcal{G}}$$

of $M_{\mathcal{G}}$, which we shall refer to as cuspidal, edge-like, and verticial, respectively. We shall refer to any *cyclic* finite étale covering of $\mathcal{G}$ which arises from a finite quotient $M_{\mathcal{G}} \twoheadrightarrow Q$ that factors through $M_{\mathcal{G}}/M_{\mathcal{G}}^{\mathrm{cusp}}$ and induces a surjection $M_{\mathcal{G}}^{\mathrm{edge}}/M_{\mathcal{G}}^{\mathrm{cusp}} \twoheadrightarrow Q$ as *module-wise nodal*. If one forms the quotient of $\Pi_{\mathcal{G}}$ by the closed normal subgroup generated by the *cuspidal* [cf. the first "$\twoheadrightarrow$" in the following display], *edge-like* [cf. the composite of the first two "$\twoheadrightarrow$'s" in the following display], or *verticial* [cf. the composite of the three "$\twoheadrightarrow$'s" in the following display] subgroups, then one obtains arrows as follows:

$$\Pi_{\mathcal{G}} \twoheadrightarrow \Pi_{\mathcal{G}}^{\mathrm{cpt}} \twoheadrightarrow \Pi_{\mathcal{G}}^{\mathrm{unr}} \twoheadrightarrow \Pi_{\mathcal{G}}^{\mathrm{grph}}.$$

We shall refer to $\Pi_{\mathcal{G}}^{\mathrm{cpt}}$ (respectively, $\Pi_{\mathcal{G}}^{\mathrm{unr}}$; $\Pi_{\mathcal{G}}^{\mathrm{grph}}$) as the *compactified* (respectively, *unramified*; *graph-theoretic*) quotient of $\Pi_{\mathcal{G}}$. We shall refer to a $\Pi_{\mathcal{G}}$-covering of $\mathcal{G}$ that arises from an open subgroup of $\Pi_{\mathcal{G}}^{\mathrm{cpt}}$ (respectively, $\Pi_{\mathcal{G}}^{\mathrm{unr}}$; $\Pi_{\mathcal{G}}^{\mathrm{grph}}$) as a $\Pi_{\mathcal{G}}^{\mathrm{cpt}}$- (respectively, $\Pi_{\mathcal{G}}^{\mathrm{unr}}$-; $\Pi_{\mathcal{G}}^{\mathrm{grph}}$-) *covering* of $\mathcal{G}$. We shall refer to the images of the verticial (respectively, verticial; edge-like) subgroups of $\Pi_{\mathcal{G}}$ in $\Pi_{\mathcal{G}}^{\mathrm{cpt}}$ (respectively, $\Pi_{\mathcal{G}}^{\mathrm{unr}}$; $\Pi_{\mathcal{G}}^{\mathrm{cpt}}$) as the *compactified verticial* (respectively, *unramified verticial*; *compactified edge-like*) subgroups. If the abelianization of every unramified verticial subgroup of $\Pi_{\mathcal{G}}^{\mathrm{unr}}$ is free of rank ≥ 2 over $\hat{\boldsymbol{Z}}^{\Sigma}$, then we shall say that $\mathcal{G}$ is *sturdy*.

REMARK 1.1.1. It is immediate from the definitions that any *connected finite étale covering* of a semi-graph of anabelioids of PSC-type is again a semi-graph of anabelioids of PSC-type.

REMARK 1.1.2. Note that if $\mathcal{G}$ is a semi-graph of anabelioids of pro-Σ PSC-type, with associated PSC-fundamental group $\Pi_{\mathcal{G}}$, then Σ may be *recovered* either from $\Pi_{\mathcal{G}}$ or from any verticial or edge-like subgroup of $\Pi_{\mathcal{G}}$ as the set of prime numbers that occur as factors of orders of finite quotients of $\Pi_{\mathcal{G}}$ or a verticial or edge-like subgroup of $\Pi_{\mathcal{G}}$.

REMARK 1.1.3. It is immediate [cf. the discussion in [Mzk4], Appendix] that $\Pi_{\mathcal{G}}$ is the pro-Σ fundamental group of some *hyperbolic curve* over an algebraically closed field of characteristic $\notin \Sigma$ [or, alternatively, of some hyperbolic Riemann surface of finite type], and that every open subgroup of an *edge-like* (respectively, *verticial*) subgroup of $\Pi_{\mathcal{G}}$ is isomorphic to $\hat{\boldsymbol{Z}}^{\Sigma}$ (respectively, nonabelian). In particular, [by [Naka2], Corollary 1.3.4] $\Pi_{\mathcal{G}}$ is *center-free* [cf. also [Mzk4], Lemma 1.3.1, for the case where Σ is the set of all prime numbers; the case of arbitrary Σ may be proven similarly]. Moreover, $\mathcal{G}$ has *cusps* if and only if $\Pi_{\mathcal{G}}$ is a *finitely generated*, *free pro-*Σ *group*. On the other hand, $\Pi_{\mathcal{G}}^{\mathrm{grph}}$ is naturally isomorphic to the pro-Σ fundamental group of the *underlying semi-graph* $\boldsymbol{G}$. In particular, $\Pi_{\mathcal{G}}^{\mathrm{grph}}$ is a *finitely generated*, *free pro-*Σ *group* of rank $\underline{n}(\mathcal{G}) - \underline{i}(\mathcal{G}) + 1$.

REMARK 1.1.4. It is immediate from the well-known structure of fundamental groups of Riemann surfaces that, in the notation of Definition 1.1, (ii), the $\hat{\boldsymbol{Z}}^{\Sigma}$-modules $M_{\mathcal{G}}$, $M_{\mathcal{G}}/M_{\mathcal{G}}^{\mathrm{cusp}}$ [i.e., the abelianization of $\Pi_{\mathcal{G}}^{\mathrm{cpt}}$], $M_{\mathcal{G}}/M_{\mathcal{G}}^{\mathrm{vert}}$ [i.e., the abelianization of $\Pi_{\mathcal{G}}^{\mathrm{grph}}$], $M_{\mathcal{G}}^{\mathrm{vert}}/M_{\mathcal{G}}^{\mathrm{edge}}$ [i.e., the direct sum, over the set of vertices of $\boldsymbol{G}$, of the abelianizations of the corresponding unramified verticial subgroups of $\Pi_{\mathcal{G}}^{\mathrm{unr}}$] are all *free* and *finitely generated* over $\hat{\boldsymbol{Z}}^{\Sigma}$. That is to say, all of the subquotients of the following filtration are *free* and *finitely generated* over $\hat{\boldsymbol{Z}}^{\Sigma}$:

$$M_{\mathcal{G}}^{\mathrm{cusp}} \subseteq M_{\mathcal{G}}^{\mathrm{edge}} \subseteq M_{\mathcal{G}}^{\mathrm{vert}} \subseteq M_{\mathcal{G}} .$$

REMARK 1.1.5. From the point of view of Definition 1.1, (i), the condition that a semi-graph of anabelioids $\mathcal{G}$ of PSC-type be *sturdy* corresponds to the condition that every irreducible component of the pointed stable curve that gives rise to $\mathcal{G}$ be of *genus* ≥ 2. [Indeed, this follows immediately from the well-known structure of fundamental groups of Riemann surfaces.] In particular, one verifies immediately that, even if $\mathcal{G}$ is *not* sturdy, there always exists a *characteristic open subgroup* $H \subseteq \Pi_{\mathcal{G}}$ which satisfies the following property: *Every* $\mathcal{G}'$ which arises as a $\Pi_{\mathcal{G}}$-covering $\mathcal{G}' \to \mathcal{G}$ such that $\Pi_{\mathcal{G}'} \subseteq H \subseteq \Pi_{\mathcal{G}}$ is *sturdy*. In fact, [it is a routine exercise to show that] one may even *bound* the index $[\Pi_{\mathcal{G}} : H]$ *explicitly* in terms of say, the rank [over $\hat{\boldsymbol{Z}}^{\Sigma}$] of $M_{\mathcal{G}}$.

REMARK 1.1.6. Suppose that $\mathcal{G}$ is *sturdy*. Then observe that the quotient $\Pi_{\mathcal{G}} \twoheadrightarrow \Pi_{\mathcal{G}}^{\mathrm{cpt}}$ determines a new semi-graph of anabelioids $\mathcal{G}'$ of PSC-type, which we shall refer to as the *compactification* of $\mathcal{G}$. That is to say, the underlying semi-graph $\boldsymbol{G}'$ of $\mathcal{G}'$ is obtained from the underlying semi-graph $\boldsymbol{G}$ of $\mathcal{G}$ by *omitting the cusps*. The anabelioids at the vertices and edges of $\mathcal{G}'$ are then obtained from $\mathcal{G}$ as the subcategories of the corresponding anabelioids of $\mathcal{G}$ determined by the quotients of the corresponding verticial and edge-like subgroups of $\Pi_{\mathcal{G}}$ induced by the quotient $\Pi_{\mathcal{G}} \twoheadrightarrow \Pi_{\mathcal{G}}^{\mathrm{cpt}}$. Thus, it follows immediately that we obtain a natural isomorphism $\Pi_{\mathcal{G}}^{\mathrm{cpt}} \xrightarrow{\sim} \Pi_{\mathcal{G}'}$.

PROPOSITION 1.2 (Commensurable terminality). *Suppose that $\mathcal{G}$ is of PSC-type, with associated PSC-fundamental group $\Pi_{\mathcal{G}}$. For $i = 1, 2$, let $A_i \subseteq \Pi_{\mathcal{G}}$ be a* **verticial** *(respectively,* **edge-like***) subgroup of $\Pi_{\mathcal{G}}$ arising from a vertex v_i (respectively, an edge e_i) of $\Pi_{\mathcal{G}}$; write B_i for the image of A_i in $\Pi_{\mathcal{G}}^{\mathrm{unr}}$. Then the following hold.*

(i) *If $A_1 \cap A_2$ is* **open** *in A_1, then $v_1 = v_2$ (respectively, $e_1 = e_2$). In the non-resp'd case, under the further assumption that $\mathcal{G}$ is* **sturdy***, if $B_1 \cap B_2$ is* **open** *in B_1, then $v_1 = v_2$.*

(ii) *The A_i are* **commensurably terminal** *[cf. §0] in $\Pi_{\mathcal{G}}$. In the non-resp'd case, under the further assumption that $\mathcal{G}$ is* **sturdy***, the B_i are* **commensurably terminal** *in $\Pi_{\mathcal{G}}^{\mathrm{unr}}$.*

PROOF. First, we observe that assertion (ii) follows formally from assertion (i) [cf. the derivation of [Mzk7], Corollary 2.7, (i), from [Mzk7], Proposition 2.6]. Now the proof of assertion (i) is entirely similar to the proof of [Mzk7], Proposition 2.6: That is to say, upon *translating* the group-theory of $\Pi_{\mathcal{G}}$ into the language of *finite étale coverings* of $\mathcal{G}$ and possibly *replacing* $\mathcal{G}$ by some finite étale covering of $\mathcal{G}$ [which allows us, in particular, to replace the words "open in" in assertion (i) by the words "equal to"], one sees that to prove assertions (i), (ii), it suffices to prove, under the further assumption that $\mathcal{G}$ is *sturdy* [cf. Remark 1.1.5], that if $v_1 \neq v_2$ (respectively, $e_1 \neq e_2$), then there exists a finite étale $\Pi_{\mathcal{G}}^{\mathrm{unr}}$- (respectively, $\Pi_{\mathcal{G}}$-) covering $\mathcal{G}' \to \mathcal{G}$ whose restriction to the anabelioid $\mathcal{G}_{v_2}$ (respectively, $\mathcal{G}_{e_2}$) is *trivial* [i.e., isomorphic to a disjoint union of copies of $\mathcal{G}_{v_2}$ (respectively, $\mathcal{G}_{e_2}$)], but whose restriction to the anabelioid $\mathcal{G}_{v_1}$ (respectively, $\mathcal{G}_{e_1}$) is *nontrivial*. But, in light of our assumption that $\mathcal{G}$ is *sturdy*, one verifies immediately that by gluing together appropriate finite étale coverings of the anabelioids $\mathcal{G}_v$, $\mathcal{G}_e$, one may construct a finite étale covering $\mathcal{G}' \to \mathcal{G}$ with the desired properties. $\square$

PROPOSITION 1.3 (Duality). *Let $\mathcal{G}$ be a* **noncuspidal** *semi-graph of anabelioids of PSC-type. Then the cup product in group cohomology*

$$H^1(\Pi_{\mathcal{G}}, \hat{\mathbf{Z}}^{\Sigma}) \times H^1(\Pi_{\mathcal{G}}, \hat{\mathbf{Z}}^{\Sigma}) \to H^2(\Pi_{\mathcal{G}}, \hat{\mathbf{Z}}^{\Sigma}) \cong \hat{\mathbf{Z}}^{\Sigma}$$

determines a perfect pairing on $M_{\mathcal{G}} \cong \mathrm{Hom}(H^1(\Pi_{\mathcal{G}}, \hat{\mathbf{Z}}^{\Sigma}), \hat{\mathbf{Z}}^{\Sigma})$, well-defined up to multiplication by a unit of $\hat{\mathbf{Z}}^{\Sigma}$. Moreover, relative to this perfect pairing, the submodules $M_{\mathcal{G}}^{\mathrm{edge}}$, $M_{\mathcal{G}}^{\mathrm{vert}}$ of $M_{\mathcal{G}}$ are **mutual annihilators**.

PROOF. Since $\mathcal{G}$ is *noncuspidal*, it follows [cf. Remark 1.1.3] that $\Pi_{\mathcal{G}}$ is the pro-Σ fundamental group of some *compact Riemann surface*, so the existence of a perfect pairing as asserted follows from the well-known *Poincaré duality* of such a compact Riemann surface. To see that the submodules $M_{\mathcal{G}}^{\mathrm{edge}}$, $M_{\mathcal{G}}^{\mathrm{vert}}$ of $M_{\mathcal{G}}$ are *mutual annihilators*, we reason as follows: Since the isomorphism class of $\mathcal{G}$ is manifestly determined by *purely combinatorial data*, we may assume without loss of generality [by possibly replacing $\mathcal{G}$ by the "pro-Σ' completion" of $\mathcal{G}$, for some subset $\Sigma' \subseteq \Sigma$] that $\mathcal{G}$ arises from a stable curve over a *finite field* k whose characteristic $\notin \Sigma$. Write q for the cardinality of k; G_k for the *absolute Galois group* of k. We shall say that an action of G_k on a finitely generated, free $\hat{\mathbf{Z}}^{\Sigma}$-module is of *weight* w if the eigenvalues of the Frobenius element $\in G_k$ are algebraic integers all of whose complex absolute values are equal to $q^{w/2}$. Now one has a natural action of G_k on $\mathcal{G}$ [cf. Remark 2.5.1 below for a more detailed description of this action], and hence a natural action on $M_{\mathcal{G}}$ which preserves $M_{\mathcal{G}}^{\text{edge}}$, $M_{\mathcal{G}}^{\text{vert}}$. By replacing k by a finite extension of k, we may assume that G_k acts trivially on the underlying semi-graph $\mathbf{G}$. Thus, the action of G_k on $M_{\mathcal{G}}/M_{\mathcal{G}}^{\text{vert}}$ (respectively, $M_{\mathcal{G}}^{\text{edge}}$) is *trivial* [cf. Remark 1.1.3] (respectively, of *weight* 2). On the other hand, by the "Riemann hypothesis" for abelian varieties over finite fields [cf., e.g., [Mumf], p. 206], it follows [cf. Remark 1.1.4] that the action of G_k on $M_{\mathcal{G}}^{\text{vert}}/M_{\mathcal{G}}^{\text{edge}}$ is of *weight* 1. Note, moreover, that the action of G_k on $H^2(\Pi_{\mathcal{G}}, \hat{\mathbf{Z}}^{\Sigma})$ is of *weight* -2. [Indeed, this follows by considering the *first Chern class* [cf., e.g., [FK], Chapter II, Definition 1.2] of a line bundle of degree one on some irreducible component of the given stable curve over k—cf., e.g., [Mzk4], the proof of Lemma 2.6.] Thus, since the subquotients of the filtration $M_{\mathcal{G}}^{\text{edge}} \subseteq M_{\mathcal{G}}^{\text{vert}} \subseteq M_{\mathcal{G}}$ are all *free* over $\hat{\mathbf{Z}}^{\Sigma}$, the fact that $M_{\mathcal{G}}^{\text{edge}}$ and $M_{\mathcal{G}}^{\text{vert}}$ are mutual annihilators follows immediately by consideration of the *weights* of the modules involved. □

REMARK 1.3.1. By Proposition 1.3 [applied to the semi-graph of anabelioids of PSC-type $\mathcal{G}'$ obtained by "*compactifying*" $\mathcal{G}$—cf. Remark 1.1.6], it follows that if $\mathcal{G}$ is a [not necessarily noncuspidal!] *sturdy* semi-graph of anabelioids of PSC-type, then the ranks [over $\hat{\mathbf{Z}}^{\Sigma}$] of $M_{\mathcal{G}}^{\text{edge}}/M_{\mathcal{G}}^{\text{cusp}}$, $M_{\mathcal{G}}/M_{\mathcal{G}}^{\text{vert}}$ *coincide*. This implies that the rank [over $\hat{\mathbf{Z}}^{\Sigma}$] of $M_{\mathcal{G}}^{\text{cusp}}$ may be computed as the *difference* between the ranks [over $\hat{\mathbf{Z}}^{\Sigma}$] of $M_{\mathcal{G}}^{\text{edge}}$, $M_{\mathcal{G}}/M_{\mathcal{G}}^{\text{vert}}$. Moreover, it follows immediately from the definitions that if $\mathcal{G}$ has *cusps*, then the rank of $M_{\mathcal{G}}^{\text{cusp}}$ is equal to $\underline{r}(\mathcal{G}) - 1$. Also, [again it follows immediately from the definitions that] $\mathcal{G}$ is *noncuspidal* if and only if $M_{\mathcal{G}'}^{\text{cusp}} = 0$ for all finite étale $\Pi_{\mathcal{G}}$-coverings $\mathcal{G}' \to \mathcal{G}$. Thus, in summary, it follows that one may *compute* $\underline{r}(\mathcal{G})$ as soon as one knows the *difference* between the ranks [over $\hat{\mathbf{Z}}^{\Sigma}$] of $M_{\mathcal{G}'}^{\text{edge}}$, $M_{\mathcal{G}'}/M_{\mathcal{G}'}^{\text{vert}}$ for all finite étale $\Pi_{\mathcal{G}}$-coverings $\mathcal{G}' \to \mathcal{G}$.

DEFINITION 1.4. Suppose that $\mathcal{G}$, $\mathcal{H}$ are of PSC-type; denote the respective associated PSC-fundamental groups by $\Pi_{\mathcal{G}}$, $\Pi_{\mathcal{H}}$ and the respective underlying semi-graphs by $\mathbf{G}, \mathbf{H}$. Let

$$\alpha : \Pi_{\mathcal{G}} \xrightarrow{\sim} \Pi_{\mathcal{H}}; \quad \beta : \Pi_{\mathcal{G}}^{\text{unr}} \xrightarrow{\sim} \Pi_{\mathcal{H}}^{\text{unr}}$$

be isomorphisms of profinite groups.

(i) We shall say that α is *graphic* if it arises from an isomorphism of semi-graphs of anabelioids $\mathcal{G} \xrightarrow{\sim} \mathcal{H}$.

(ii) We shall say that α is *numerically cuspidal* if, for any pair of finite étale coverings $\mathcal{G}' \to \mathcal{G}$, $\mathcal{H}' \to \mathcal{H}$ which correspond via α, we have $\underline{r}(\mathcal{G}') = \underline{r}(\mathcal{H}')$.

(iii) We shall say that α is *graphically filtration-preserving* (respectively, *verticially filtration-preserving*; *edge-wise filtration-preserving*) if, for any pair of finite étale coverings

$\mathcal{G}' \to \mathcal{G}$, $\mathcal{H}' \to \mathcal{H}$ which correspond via α, the isomorphism

$$M_{\mathcal{G}'} \xrightarrow{\sim} M_{\mathcal{H}'}$$

induced by α induces an isomorphism between the respective verticial and edge-like (respectively, verticial; edge-like) submodules. We shall say that β is *verticially filtration-preserving* if, for any pair of finite étale coverings $\mathcal{G}' \to \mathcal{G}$, $\mathcal{H}' \to \mathcal{H}$ which correspond via β, the isomorphism

$$M_{\mathcal{G}'}/M^{\text{edge}}_{\mathcal{G}'} \xrightarrow{\sim} M_{\mathcal{H}'}/M^{\text{edge}}_{\mathcal{H}'}$$

induced by β induces an isomorphism $M^{\text{vert}}_{\mathcal{G}'}/M^{\text{edge}}_{\mathcal{G}'} \xrightarrow{\sim} M^{\text{vert}}_{\mathcal{H}'}/M^{\text{edge}}_{\mathcal{H}'}$.

(iv) We shall say that α is *group-theoretically cuspidal* (respectively, *group-theoretically edge-like*; *group-theoretically verticial*) if and only if it maps each cuspidal (respectively, edge-like; verticial) subgroup of $\Pi_{\mathcal{G}}$ isomorphically onto a cuspidal (respectively, edge-like; verticial) subgroup of $\Pi_{\mathcal{H}}$, and, moreover, every cuspidal (respectively, edge-like; verticial) subgroup of $\Pi_{\mathcal{H}}$ arises in this fashion. We shall say that β is *group-theoretically verticial* if and only if it maps each unramified verticial subgroup of $\Pi^{\text{unr}}_{\mathcal{G}}$ isomorphically onto an unramified verticial subgroup of $\Pi^{\text{unr}}_{\mathcal{H}}$, and, moreover, every verticial subgroup of $\Pi^{\text{unr}}_{\mathcal{H}}$ arises in this fashion.

(v) Let $\mathcal{G}' \to \mathcal{G}$ be a Galois finite étale covering. Then we shall say that $\mathcal{G}' \to \mathcal{G}$ is *cuspidally* (respectively, *nodally*; *verticially*) *purely totally ramified* if there exists a cusp e (respectively, node e; vertex v) of $\boldsymbol{G}$ such that $\mathcal{G}' \to \mathcal{G}$ restricts to a *trivial* covering over $\mathcal{G}_{e'}$ (respectively, $\mathcal{G}_{e'}$; $\mathcal{G}_{v'}$) for all cusps $e' \neq e$ (respectively, nodes $e' \neq e$; vertices $v' \neq v$) of $\boldsymbol{G}$ and to a *connected* covering over $\mathcal{G}_e$ (respectively, $\mathcal{G}_e$; $\mathcal{G}_v$). We shall say that $\mathcal{G}' \to \mathcal{G}$ is *cuspidally* (respectively, *nodally*; *verticially*) *totally ramified* if there exists a cusp e (respectively, node e; vertex v) of $\boldsymbol{G}$ such that $\mathcal{G}' \to \mathcal{G}$ restricts to a *connected* covering over $\mathcal{G}_e$ (respectively, $\mathcal{G}_e$; $\mathcal{G}_v$).

(vi) If $A \subseteq \Pi_{\mathcal{G}}$ is a closed subgroup, and $A' \subseteq A$ is an open subgroup of A, then we shall say that the inclusion $A' \subseteq A$ *descends to a finite étale covering* $\mathcal{G}' \to \mathcal{G}''$ if the arrow $\mathcal{G}' \to \mathcal{G}''$ is a morphism of finite étale $\Pi_{\mathcal{G}}$-coverings of $\mathcal{G}$ such that the corresponding open subgroups $\Pi_{\mathcal{G}'} \subseteq \Pi_{\mathcal{G}''} \subseteq \Pi_{\mathcal{G}}$ satisfy $A \subseteq \Pi_{\mathcal{G}''}$, $A \cap \Pi_{\mathcal{G}'} = A'$, $[A : A'] = [\Pi_{\mathcal{G}''} : \Pi_{\mathcal{G}'}]$. We shall use similar terminology when, in the preceding sentence, "Π" is replaced by "Π^{unr}".

REMARK 1.4.1. Thus, by Proposition 1.3, it follows that, if, in the notation of Definition 1.4, $\mathcal{G}$, $\mathcal{H}$ are *noncuspidal*, then the following three conditions on α are *equivalent*: (a) α is *graphically filtration-preserving*; (b) α is *verticially filtration-preserving*; (c) α is *edge-wise filtration-preserving*.

REMARK 1.4.2. Let $\mathcal{G}' \to \mathcal{G}$ be a Galois finite étale covering of degree a *power of* l, where $\mathcal{G}$ is of pro-Σ PSC-type, $\Sigma = \{l\}$. Then one verifies immediately that $\mathcal{G}' \to \mathcal{G}$ is *cuspidally purely totally ramified* if and only if the equality

$$\underline{r}(\mathcal{G}') = \deg(\mathcal{G}'/\mathcal{G}) \cdot (\underline{r}(\mathcal{G}) - 1) + 1$$

is satisfied. Similarly, if $\mathcal{G}' \to \mathcal{G}$ is a finite étale $\Pi_{\mathcal{G}}^{\mathrm{unr}}$-covering [so $\underline{n}(\mathcal{G}') = \underline{n}(\mathcal{G}) \cdot \deg(\mathcal{G}'/\mathcal{G})$], then one verifies immediately that $\mathcal{G}' \to \mathcal{G}$ is *verticially purely totally ramified* if and only if the equality

$$\underline{i}(\mathcal{G}') = \deg(\mathcal{G}'/\mathcal{G}) \cdot (\underline{i}(\mathcal{G}) - 1) + 1$$

is satisfied. Also, we observe that this last equality is equivalent to the following equality involving the expression "$\underline{i}(\dots) - \underline{n}(\dots)$" [cf. Remark 1.1.3]:

$$\underline{i}(\mathcal{G}') - \underline{n}(\mathcal{G}') = \deg(\mathcal{G}'/\mathcal{G}) \cdot (\underline{i}(\mathcal{G}) - \underline{n}(\mathcal{G}) - 1) + 1\,.$$

REMARK 1.4.3. Let $\mathcal{G}' \to \mathcal{G}$ be as in Remark 1.4.2; assume further that this covering is a *cuspidally* (respectively, *nodally*; *verticially*) *totally ramified* $\Pi_{\mathcal{G}}$- (respectively, $\Pi_{\mathcal{G}}$-; $\Pi_{\mathcal{G}}^{\mathrm{unr}}$-) *covering*, and that $\mathcal{G}$ is arbitrary (respectively, arbitrary; *sturdy*). Let e (respectively, e; v) be a cusp (respectively, node; vertex) of $\boldsymbol{G}$ such that $\mathcal{G}' \to \mathcal{G}$ restricts to a *connected* covering of $\mathcal{G}_e$ (respectively, $\mathcal{G}_e$; $\mathcal{G}_v$). Then observe that:

There exists a finite étale $\Pi_{\mathcal{G}}$- (respectively, $\Pi_{\mathcal{G}}$-; $\Pi_{\mathcal{G}}^{\mathrm{unr}}$-) covering $\mathcal{G}'' \to \mathcal{G}$ such that: (a) $\mathcal{G}'' \to \mathcal{G}$ is *trivial* over $\mathcal{G}_e$ (respectively, $\mathcal{G}_e$; $\mathcal{G}_v$); (b) the subcovering $\mathcal{G}''' \to \mathcal{G}''$ of the composite covering $\mathcal{G}''' \to \mathcal{G}$ of the coverings $\mathcal{G}'' \to \mathcal{G}$ and $\mathcal{G}' \to \mathcal{G}$ is *cuspidally* (respectively, *nodally*; *verticially*) *purely totally ramified.*

Indeed, the construction of such a covering is immediate [cf. the proof of Proposition 1.2].

REMARK 1.4.4. Let $\mathcal{G}' \to \mathcal{G}$ be as in Remark 1.4.2; assume further that this covering is *cyclic*, and that $\mathcal{G}$ is *noncuspidal*. Then it is immediate that $\mathcal{G}' \to \mathcal{G}$ is *module-wise nodal* if and only if it is *nodally totally ramified.* In particular, it follows that:

(i) Any closed subgroup $B \subseteq \Pi_{\mathcal{G}}$ is *contained* in some nodal edge-like subgroup if and only if, for every open normal subgroup $B' \subseteq B$, the inclusion $B' \subseteq B$ descends to a module-wise nodal finite étale covering.

(ii) A closed subgroup $A \subseteq \Pi_{\mathcal{G}}$ is a *nodal edge-like subgroup* of $\Pi_{\mathcal{G}}$ if and only if it satisfies the condition of (i) above [i.e., where one takes "B" to be A], and, moreover, is *maximal* among closed subgroups $B \subseteq \Pi_{\mathcal{G}}$ satisfying the condition of (i).

Indeed, the *necessity* of (i) is immediate. The *sufficiency* of (i) follows by observing that since the set of nodes of a finite étale covering of $\mathcal{G}$ is always *finite*, an exhaustive collection of open normal subgroups of B thus determines—by considering the nodes at which the "total ramification" occurs—[at least one] compatible system of nodes of the finite étale $\Pi_{\mathcal{G}}$-coverings of $\mathcal{G}$; but this implies that B is contained in *some* nodal edge-like subgroup. In light of (i), the *necessity* of (ii) is immediate from the definitions and Proposition 1.2, (i) [which implies *maximality*], while the *sufficiency* of (ii) follows immediately from the assumption of *maximality*.

PROPOSITION 1.5 (Incidence relations). *We maintain the notation of Definition* 1.4. *Then the following hold.*

(i) *An edge-like subgroup of* $\Pi_{\mathcal{G}}$ *is* **cuspidal** (*respectively*, **not cuspidal**) *if and only if it is contained in* **precisely one** (*respectively*, **precisely two**) **verticial subgroup(s)**.

(ii) *α is* **graphic** *if and only if it is* **group-theoretically edge-like** *and* **group-theoretically verticial**. *Moreover, in this case, α arises from a* **unique** *isomorphism of semi-graphs of anabelioids $\mathcal{G} \xrightarrow{\sim} \mathcal{H}$.*

PROOF. First, we consider assertion (i). Observe that it is immediate from the definitions that a cuspidal (respectively, noncuspidal) edge-like subgroup of $\Pi_{\mathcal{G}}$ is contained in *at least one* (respectively, *at least two*) *verticial subgroup(s).* To prove that these lower bounds also serve as upper bounds, it suffices [by possibly replacing $\mathcal{G}$ by a finite étale covering of $\mathcal{G}$] to show that if e is a cuspidal (respectively, nodal) edge of $\boldsymbol{G}$ that does *not* abut to a vertex v, then there exists a finite étale $\Pi_{\mathcal{G}}$-covering $\mathcal{G}' \to \mathcal{G}$ which is *trivial* over $\mathcal{G}_v$, but *nontrivial* over $\mathcal{G}_e$. But this is immediate [cf. the proof of Proposition 1.2, (i)].

Next, we consider assertion (ii). *Necessity* is immediate. To prove *sufficiency*, we reason as follows: The assumption that α is *group-theoretically edge-like* and *group-theoretically verticial* implies, by considering conjugacy classes of verticial and edge-like subgroups [and applying Proposition 1.2, (i)], that α induces a bijection between the vertices (respectively, edges) of the underlying semi-graphs $\boldsymbol{G}$, $\boldsymbol{H}$. By assertion (i), this bijection maps cuspidal (respectively, nodal) edges to cuspidal (respectively, nodal) edges and is compatible with the various *"incidence relations"* that define the semi-graph structure [i.e., the data of which vertices an edge abuts to]. Thus, α induces an isomorphism of semi-graphs $\boldsymbol{G} \xrightarrow{\sim} \boldsymbol{H}$. Finally, by Proposition 1.2, (ii), one concludes that α arises from a *unique* isomorphism $\mathcal{G} \xrightarrow{\sim} \mathcal{H}$, as desired. □

THEOREM 1.6 (Criterion for graphicity). *We maintain the notation of Definition* 1.4. *Then the following hold.*

(i) *α is* **numerically cuspidal** *if and only if it is* **group-theoretically cuspidal**.

(ii) *α is* **graphic** *if and only if it is* **graphically filtration-preserving**.

(iii) *Assume that $\mathcal{G}$, $\mathcal{H}$ are* **sturdy**. *Then β is* **verticially filtration-preserving** *if and only if it is* **group-theoretically verticial**.

PROOF. First, we consider assertion (i). *Sufficiency* is immediate [cf. Proposition 1.2, (i)]. The proof of *necessity* is entirely similar to the latter half of the proof of [Mzk4], Lemma 1.3.9: Let $l \in \Sigma$ [where $\mathcal{G}$, $\mathcal{H}$ are of pro-Σ PSC-type]. Since the cuspidal edge-like subgroups may be recovered as the stabilizers of cusps of finite étale coverings of $\mathcal{G}$, $\mathcal{H}$, it suffices to show that α induces a *functorial bijection* between the sets of cusps of $\mathcal{G}$, $\mathcal{H}$. In particular, we may assume, without loss of generality, that $\Sigma = \{l\}$.

Then given pairs of finite étale $\Pi_{\mathcal{G}}$- or $\Pi_{\mathcal{H}}$-coverings that correspond via α

$$\mathcal{G}'' \to \mathcal{G}' \to \mathcal{G}\,; \quad \mathcal{H}'' \to \mathcal{H}' \to \mathcal{H}$$

such that $\mathcal{G}''$ is Galois over $\mathcal{G}'$, and $\mathcal{H}''$ is Galois over $\mathcal{H}'$, it follows from the assumption that α is *numerically cuspidal* that $\mathcal{G}'' \to \mathcal{G}'$ is *cuspidally purely totally ramified* if and only if $\mathcal{H}'' \to \mathcal{H}'$ is [cf. Remark 1.4.2]. Now observe that the cuspidal edge-like subgroups of $\Pi_{\mathcal{G}}$ (respectively, $\Pi_{\mathcal{H}}$) are precisely the maximal closed subgroups A such that, for every open normal subgroup $A' \subseteq A$, the inclusion $A' \subseteq A$ descends to a cuspidally purely totally ramified Galois finite étale covering. Indeed, in light of Remark 1.4.3 [which implies that, in the preceding sentence, one may remove the word "purely" without affecting the validity of the assertion contained in this sentence], this follows by a similar argument to the argument applied in the case of nodes in Remark 1.4.4. Thus, we thus conclude that α is *group-theoretically cuspidal*, as desired.

Next, we consider assertion (ii). *Necessity* is immediate. To prove *sufficiency*, let us first observe that by *functoriality*; Proposition 1.2, (ii); Proposition 1.5, (ii), it follows that we may always *replace* $\mathcal{G}$, $\mathcal{H}$ by finite étale $\Pi_{\mathcal{G}}$- or $\Pi_{\mathcal{H}}$-coverings that correspond via α. In particular, by Remark 1.1.5, we may assume without loss of generality that $\mathcal{G}$, $\mathcal{H}$ are *sturdy*. Next, let us observe that by Proposition 1.3 [cf. Remark 1.3.1], the assumption that α is *graphically filtration-preserving* implies that α is *numerically cuspidal*, hence [by assertion (i)] that α is *group-theoretically cuspidal*. Thus, by replacing $\mathcal{G}$, $\mathcal{H}$ by their respective *compactifications* [cf. Remark 1.1.6], and replacing α by the isomorphism induced by α between the respective quotients $\Pi_{\mathcal{G}} \twoheadrightarrow \Pi_{\mathcal{G}}^{\mathrm{cpt}}$, $\Pi_{\mathcal{H}} \twoheadrightarrow \Pi_{\mathcal{H}}^{\mathrm{cpt}}$, we may assume, without loss of generality, that $\mathcal{G}$, $\mathcal{H}$ are *noncuspidal* and *sturdy*. Also, as in the proof of assertion (i), we may assume that $\Sigma = \{l\}$. Now by Proposition 1.5, (ii), it suffices to prove that α is *group-theoretically edge-like* and *group-theoretically verticial*. But by Remark 1.4.4, the assumption that α is *edge-wise filtration-preserving* implies that α is *group-theoretically edge-like*. In particular, α induces a *verticially filtration-preserving* isomorphism $\Pi_{\mathcal{G}}^{\mathrm{unr}} \xrightarrow{\sim} \Pi_{\mathcal{H}}^{\mathrm{unr}}$. Now to prove that α is *group-theoretically verticial*, it suffices to prove [cf. the proof of assertion (i)] that α induces a *functorial bijection* between the sets of vertices of $\mathcal{G}$, $\mathcal{H}$. Thus, to complete the proof of assertion (ii), it suffices to prove that β is *group-theoretically verticial*, that is to say, it suffices to verify assertion (iii).

Finally, we consider assertion (iii). *Sufficiency* is immediate. On the other hand, *necessity* follows from Remark 1.4.2, by observing that the unramified verticial subgroups are precisely the maximal closed subgroups A of $\Pi_{\mathcal{G}}^{\mathrm{unr}}$ or $\Pi_{\mathcal{H}}^{\mathrm{unr}}$ such that, for every open normal subgroup $A' \subseteq A$, the inclusion $A' \subseteq A$ descends to a verticially purely totally ramified Galois finite étale covering. Indeed, in light of Remark 1.4.3 [which implies that, in the preceding sentence, one may remove the word "purely" without affecting the validity of the assertion contained in this sentence], this follows by a similar argument to the argument applied in the case of nodes in Remark 1.4.4. This completes the proof of assertion (ii). $\square$

REMARK 1.6.1. The essential content of Theorem 1.6, (i), is, in many respects, similar to the essential content of [Tama2], Lemma 2.3 [cf. the use of this lemma in [Tama2], Proposition 2.4].

2. The group of graphic outer automorphisms. In this Section, we study the consequences of the theory of §1 for the *group of automorphisms* of a semi-graph of anabelioids of PSC-type.

Let $\mathcal{G}$ be a *semi-graph of anabelioids of pro-Σ PSC-type* [with underlying semi-graph $\mathbb{G}$]. In the following discussion, $\mathcal{G}$, $\mathbb{G}$ will remain *fixed* until further notice to the contrary [in Corollary 2.7].

Denote by $\mathrm{Aut}(\mathcal{G})$ the group of automorphisms of the semi-graph of anabelioids $\mathcal{G}$. Here, we recall that an automorphism of a semi-graph of anabelioids consists of an automorphism of the underlying semi-graph, together with a compatible system of isomorphisms between the various anabelioids at each of the vertices and edges of the underlying semi-graph, which are compatible with the various morphisms of anabelioids associated to the branches of the underlying semi-graph—cf. [Mzk7], Definition 2.1; [Mzk7], Remark 2.4.2. Then, by Proposition 1.5, (ii), we obtain an *injective homomorphism*

$$\mathrm{Aut}(\mathcal{G}) \hookrightarrow \mathrm{Out}(\Pi_{\mathcal{G}}),$$

whose image we shall denote by

$$\mathrm{Out}_{\mathrm{grph}}(\Pi_{\mathcal{G}}) \subseteq \mathrm{Out}(\Pi_{\mathcal{G}})$$

and refer to as the *group of graphic outer automorphisms* of $\Pi_{\mathcal{G}}$. Since $\Pi_{\mathcal{G}}$ is *topologically finitely generated* [cf. Remark 1.1.3], it follows that $\mathrm{Out}(\Pi_{\mathcal{G}})$ is equipped with a natural *profinite topology*, which thus induces a *natural topology* on the subgroup $\mathrm{Out}_{\mathrm{grph}}(\Pi_{\mathcal{G}}) \subseteq \mathrm{Out}(\Pi_{\mathcal{G}})$, which is *manifestly closed*, by Proposition 1.5, (ii). In particular, $\mathrm{Aut}(\mathcal{G}) \cong \mathrm{Out}_{\mathrm{grph}}(\Pi_{\mathcal{G}})$ is equipped with a natural *profinite topology*.

Since $\Pi_{\mathcal{G}}$ is *center-free* [cf. Remark 1.1.3], we have a natural exact sequence $1 \to \Pi_{\mathcal{G}} \to \mathrm{Aut}(\Pi_{\mathcal{G}}) \to \mathrm{Out}(\Pi_{\mathcal{G}}) \to 1$, which we may pull-back via $\mathrm{Aut}(\mathcal{G}) \hookrightarrow \mathrm{Out}(\Pi_{\mathcal{G}})$ to obtain an exact sequence as follows:

$$1 \to \Pi_{\mathcal{G}} \to \Pi_{\mathcal{G}}^{\mathrm{Aut}} \to \mathrm{Aut}(\mathcal{G}) \to 1.$$

If $\mathcal{G}' \to \mathcal{G}$ is a *sturdy* [i.e., $\mathcal{G}'$ is sturdy] finite étale $\Pi_{\mathcal{G}}$-covering which arises from a *characteristic* open subgroup $\Pi_{\mathcal{G}'} \subseteq \Pi_{\mathcal{G}}$, then there is a *natural action* of $\Pi_{\mathcal{G}}^{\mathrm{Aut}}$ on $\mathcal{G}'$. In particular, we obtain, for every $l \in \Sigma$, a natural action of $\Pi_{\mathcal{G}}^{\mathrm{Aut}}$ on the free $\mathbb{Z}_l$-module of rank one [i.e., since $\mathcal{G}'$ is sturdy] $H^2(\Pi_{\mathcal{G}'}^{\mathrm{cpt}}, \mathbb{Z}_l)$.

LEMMA 2.1 (Construction of the cyclotomic character). *This action of* $\Pi_{\mathcal{G}}^{\mathrm{Aut}}$ *on* $H^2(\Pi_{\mathcal{G}'}^{\mathrm{cpt}}, \mathbb{Z}_l)$ *factors through the quotient* $\Pi_{\mathcal{G}}^{\mathrm{Aut}} \twoheadrightarrow \mathrm{Aut}(\mathcal{G})$, *and hence determines a continuous homomorphism* $\mathrm{Aut}(\mathcal{G}) \to \mathbb{Z}_l^{\times}$, *whose* **inverse**

$$\chi_l : \mathrm{Aut}(\mathcal{G}) \to \mathbb{Z}_l^{\times}$$

we shall refer to as the **pro-l cyclotomic character** *of* $\mathrm{Aut}(\mathcal{G})$. *Moreover, this character is* **independent** *of the choice of sturdy* $\Pi_{\mathcal{G}}$*-covering* $\mathcal{G}' \to \mathcal{G}$.

PROOF. To verify the asserted *independence of covering*, it suffices to observe that any two sturdy $\Pi_{\mathcal{G}}$-coverings $\mathcal{G}' \to \mathcal{G}$, $\mathcal{G}'' \to \mathcal{G}$ may be dominated by a third sturdy $\Pi_{\mathcal{G}}$-covering $\mathcal{G}''' \to \mathcal{G}$, which induce *isomorphisms* of free $\mathbb{Q}_l$-modules of rank one

$$H^2(\Pi_{\mathcal{G}'}^{\mathrm{cpt}}, \mathbb{Z}_l) \otimes \mathbb{Q} \to H^2(\Pi_{\mathcal{G}'''}^{\mathrm{cpt}}, \mathbb{Z}_l) \otimes \mathbb{Q}; \quad H^2(\Pi_{\mathcal{G}''}^{\mathrm{cpt}}, \mathbb{Z}_l) \otimes \mathbb{Q} \to H^2(\Pi_{\mathcal{G}'''}^{\mathrm{cpt}}, \mathbb{Z}_l) \otimes \mathbb{Q},$$

which are compatible with the various actions by $\Pi_{\mathcal{G}}^{\mathrm{Aut}}$.

To show that the action of $\Pi_{\mathcal{G}}^{\mathrm{Aut}}$ factors through $\mathrm{Aut}(\mathcal{G})$, we may assume without loss of generality that Σ is the *set of all primes*. On the other hand, by the *independence of covering* already verified, it follows that we may compute the $\Pi_{\mathcal{G}}^{\mathrm{Aut}}$-action in question by using a covering $\mathcal{G}' \to \mathcal{G}$ of degree *prime to* $l(l-1)$. Since the action in question amounts to a continuous homomorphism $\Pi_{\mathcal{G}}^{\mathrm{Aut}} \to \boldsymbol{Z}_l^{\times}$ which clearly factors through $\Pi_{\mathcal{G}}^{\mathrm{Aut}}/\Pi_{\mathcal{G}'}$, the desired factorization follows from the fact that [consideration of *orders* implies that] every homomorphism $\mathrm{Gal}(\mathcal{G}'/\mathcal{G}) \to \boldsymbol{Z}_l^{\times}$ is trivial. □

PROPOSITION 2.2 (The double of a semi-graph of anabelioids of PSC-type). *Suppose that* $\underline{r}(\mathcal{G}) \neq 0$. *Let* $\mathcal{H}$ *be the* **semi-graph of anabelioids** *defined as follows*: *The underlying semi-graph* $\boldsymbol{H}$ *is obtained by taking the disjoint union of two copies of* $\boldsymbol{G}$ *and*, *for each cusp* e *of* $\boldsymbol{G}$ *abutting to a vertex* v *of* $\boldsymbol{G}$, *replacing the corresponding pairs of cusps lying in these two copies of* $\boldsymbol{G}$ *by a node* [*i.e.*, *a closed edge*] *that joins the pairs of vertices corresponding to* v *in these two copies. We shall refer to the newly appended nodes as* **bridges**. *Away from the bridges*, *we take the semi-graph of anabelioids structure of* $\mathcal{H}$ *to be the structure induced by* $\mathcal{G}$, *and*, *at each branch of a bridge of* $\boldsymbol{H}$, *we take the semi-graph of anabelioids structure of* $\mathcal{H}$ *to be the structure induced by* $\mathcal{G}$ *at the corresponding cusp* e *of* $\mathcal{G}$, *by gluing the two copies of* $\mathcal{G}_e$ *in question by means of the* **inversion automorphism** $\mathcal{G}_e \to \mathcal{G}_e$ [*induced by* "*multiplication by* -1" *on the abelian fundamental group of* $\mathcal{G}_e$]. *We shall refer to* $\mathcal{H}$ *as the* **double of** $\mathcal{G}$. *Then the following hold.*

(i) $\mathcal{H}$ *is a* **noncuspidal semi-graph of anabelioids of PSC-type**.

(ii) *Restriction of finite étale coverings of* $\mathcal{H}$ *to each of the copies of* $\mathcal{G}$ *used to construct* $\mathcal{H}$ *determines a natural* **injective continuous outer homomorphism** $\Pi_{\mathcal{G}} \hookrightarrow \Pi_{\mathcal{H}}$.

(iii) *The homomorphism of* (ii) *maps* **verticial** (*respectively*, **edge-like**) *subgroups of* $\Pi_{\mathcal{G}}$ *isomorphically onto verticial* (*respectively*, *edge-like*) *subgroups of* $\Pi_{\mathcal{H}}$.

(iv) *The homomorphism of* (ii) *induces an* **injection**

$$M_{\mathcal{G}} \hookrightarrow M_{\mathcal{H}}$$

that maps $M_{\mathcal{G}}^{\mathrm{edge}}$ (*respectively*, $M_{\mathcal{G}}^{\mathrm{vert}}$) *into* $M_{\mathcal{H}}^{\mathrm{edge}}$ (*respectively*, $M_{\mathcal{H}}^{\mathrm{vert}}$).

PROOF. Assertion (i) is immediate from the definitions. Note, relative to Definition 1.1, (i), that there is a corresponding construction of a "double" of a pointed stable curve. This explains the need for "gluing by means of the inversion automorphism" in the definition of $\mathcal{H}$: Over, say, a complete discrete valuation ring A with algebraically closed residue field, the completion of a generically smooth pointed stable curve at a node is isomorphic to the formal spectrum of the complete local ring $A[[x, y]]/(xy - s)$, where x, y are indeterminates and s lies in the maximal ideal of A. Then the action of the local tame Galois group at each of the branches of the node *considered independently* is of the form $x \mapsto \zeta \cdot x$, $y \mapsto \zeta \cdot y$, where ζ is some root of unity. On the other hand, since the Galois action on coverings of the *entire* formal spectrum of $A[[x, y]]/(xy - s)$ [i.e., where one does *not* treat the branches of the node independently] necessarily *fixes* elements of the base ring [i.e., the normalization of

A in some finite extension of its quotient field], it follows that this action must be of the form $x \mapsto \zeta \cdot x$, $y \mapsto \zeta^{-1} \cdot y$.

As for Assertion (ii), it is immediate that we obtain a natural homomorphism $\Pi_{\mathcal{G}} \to \Pi_{\mathcal{H}}$. The asserted *injectivity* may be verified as follows [cf. also the proof of injectivity in [Mzk7], Proposition 2.5, (i)]: Given any finite étale $\Pi_{\mathcal{G}}$-covering $\mathcal{G}' \to \mathcal{G}$, one may construct a finite étale $\Pi_{\mathcal{H}}$-covering $\mathcal{H}' \to \mathcal{H}$, which induces $\mathcal{G}' \to \mathcal{G}$ via the "restriction procedure″ of (ii) by *gluing* together two copies of $\mathcal{G}'$ over the two copies of $\mathcal{G}$ used to construct $\mathcal{H}$. Note that to carry out this gluing, one must *choose* a [noncanonical!] *isomorphism*, at each cusp e of $\boldsymbol{G}$, between the restriction of $\mathcal{G}' \to \mathcal{G}$ to $\mathcal{G}_e$ and the pull-back via the inversion automorphism of this restriction. [Note that it is immediate that such an isomorphism always exists.] Assertion (iii) is immediate from the construction of the double.

Finally, we consider Assertion (iv). To verify that the homomorphism $M_{\mathcal{G}} \to M_{\mathcal{H}}$ induced by the homomorphism of (ii) is an injection, it suffices to observe that the gluing procedure discussed in the proof of the injectivity of (ii) determines a *splitting* of the homomorphism $M_{\mathcal{G}} \to M_{\mathcal{H}}$. Indeed, if the finite étale $\Pi_{\mathcal{G}}$-covering $\mathcal{G}' \to \mathcal{G}$ in question is *abelian*, with Galois group A, then the resulting $\mathcal{H}' \to \mathcal{H}$ admits a natural action by A, by letting A act via the *identity* $A \to A$ on *one* copy of $\mathcal{G}'$ and via "*multiplication by* -1" $A \to A$ on the *other* copy of $\mathcal{G}'$. [Put another way, if we think of the covering $\mathcal{G}' \to \mathcal{G}$ as corresponding to the A-set A, then we glue the set A to the set A at the bridges by means of the automorphism "multiplication by -1".] This completes the proof of injectivity. The fact that this injection maps $M_{\mathcal{G}}^{\text{edge}}$ (respectively, $M_{\mathcal{G}}^{\text{vert}}$) into $M_{\mathcal{H}}^{\text{edge}}$ (respectively, $M_{\mathcal{H}}^{\text{vert}}$) follows immediately from Assertion (iii). □

REMARK 2.2.1. Certain aspects of Proposition 2.2 are related to the results of [Asada].

REMARK 2.2.2. It is quite possible that various aspects of Proposition 2.2 may be generalized from the case of "two copies of $\mathcal{G}$" treated in Proposition 2.2 to the case of gluing together arbitrary finite collections of semi-graphs of anabelioids of PSC-type. This topic, however, lies beyond the scope of the present paper.

DEFINITION 2.3. Let J be a *profinite group* which *acts continuously* on $\mathcal{G}$ [i.e., we are given a continuous homomorphism $J \to \text{Aut}(\mathcal{G})$]. Set

$$\Pi_{\mathcal{G}}^{J} \stackrel{\text{def}}{=} \Pi_{\mathcal{G}}^{\text{Aut}} \times_{\text{Aut}(\mathcal{G})} J .$$

Let M be a continuous $\boldsymbol{Z}_l[J]$-module [i.e., a topological module equipped with continuous actions by $\boldsymbol{Z}_l$, J], where $l \in \Sigma$.

(i) We shall refer to a [continuous] character $\psi : J \to \boldsymbol{Z}_l^{\times}$ as *quasi-cyclotomic* (respectively, $\boldsymbol{Q}$*-cyclotomic*) if ψ (respectively, some *positive power* of ψ) coincides with the restriction to J of the character χ_l (respectively, some *integer* power of the character χ_l) of Lemma 2.1 on some open subgroup $J' \subseteq J$ of J. If $\psi : J \to \boldsymbol{Z}_l^{\times}$ is a [continuous] character, then we shall denote by

$$M(\psi)$$

the *ψ-twist of M*. That is to say, the underlying topological $\boldsymbol{Z}_l$-modules of M, $M(\psi)$ are identical; if the action of $\gamma \in J$ on M maps $m \in M$ to $\gamma \cdot m \in M$, then the action of $\gamma \in J$ on $M(\psi)$ maps $m \mapsto \psi(\gamma) \cdot (\gamma \cdot m) \in M = M(\psi)$. If $n \in \boldsymbol{Z}$, then we shall write $M(n) \stackrel{\text{def}}{=} M((\chi_l|_J)^n)$, where $\chi_l|_J$ denotes the restriction of the cyclotomic character χ_l of Lemma 2.1 to J. We shall say that M is *quasi-trivial* if some open subgroup $J' \subseteq J$ acts trivially on M. We shall say that M is *quasi-toral* if $M(-1)$ is quasi-trivial. If, for some open subgroup $J' \subseteq J$, there exists a *finite filtration* of $\boldsymbol{Z}_l[J']$-submodules

$$M^n \subseteq M^{n-1} \subseteq \cdots \subseteq M^j \subseteq \cdots \subseteq M^1 \subseteq M^0 = M$$

such that each M^j/M^{j+1} is torsion-free and, moreover, either is *quasi-trivial* [over J'] or has *no quasi-trivial J''-subquotients* for any open subgroup $J'' \subseteq J'$, then we shall refer to the [possibly infinite] sum

$$\sum_{M^j/M^{j+1} \text{ quasi-trivial}} \dim_{\boldsymbol{Q}_l}(M^j/M^{j+1} \otimes \boldsymbol{Q}_l)$$

[which is easily verified to be *independent* of the choice of a subgroup $J' \subseteq J$ and a filtration $\{M^j\}$ satisfying the above properties] as the *quasi-trivial rank* of M.

(ii) We shall say that [the action of] J is *l-cyclotomically full* if the image of the homomorphism $\chi_l|_J : J \to \boldsymbol{Z}_l^\times$ is open. Suppose that J is l-cyclotomically full. Then it makes sense to speak of the *weight* w of a $\boldsymbol{Q}$-cyclotomic character $\psi : J \to \boldsymbol{Z}_l^\times$: i.e., w is the *unique* rational number that may be written in the form $2a/b$, where a, b are integers such that $b \neq 0$, $\psi^b = (\chi_l|_J)^a$. If $w > 0$ (respectively, $w = 0$; $w < 0$), then we shall say that ψ is *positive* (respectively, *null*; *negative*). If $w \in \boldsymbol{Q}$, and $\psi : J \to \boldsymbol{Z}_l^\times$ is a $\boldsymbol{Q}$-cyclotomic character of weight w, then we shall refer to the quasi-trivial rank of $M(\psi^{-1})$ as the *l-weight w rank of M*. [One verifies immediately that the l-weight w rank is independent of the choice of ψ.] If the l-weight w rank of M is nonzero, then we shall say that w is an *associated l-weight* of M. Write

$$\underline{w}_l(M) \subseteq \boldsymbol{Q}$$

for the *set of associated l-weights* of M.

(iii) Suppose that J is *l-cyclotomically full*. Observe that if $\Pi_{\mathcal{G}'} \subseteq \Pi_{\mathcal{G}}$ is any *characteristic* open subgroup, then $\Pi_{\mathcal{G}}^J$ acts naturally on $\Pi_{\mathcal{G}'}$, and hence also on $M_{\mathcal{G}'} \otimes \boldsymbol{Z}_l$. Set

$$\underline{w}_l(J) \stackrel{\text{def}}{=} \bigcup_{\mathcal{G}'} \underline{w}_l(M_{\mathcal{G}'} \otimes \boldsymbol{Z}_l)$$

[where the union ranges over characteristic open subgroups $\Pi_{\mathcal{G}'} \subseteq \Pi_{\mathcal{G}}$]. We shall refer to $\underline{w}_l(J)$ as the *set of associated l-weights* of [the action of] J. If every $w \in \underline{w}_l(J)$ satisfies $0 \leq w \leq 2$, then we shall say that [the action of] J is *weakly l-graphically full*. If, for every characteristic open subgroup $\Pi_{\mathcal{G}'} \subseteq \Pi_{\mathcal{G}}$, it holds that

$$\underline{w}_l((M_{\mathcal{G}'}^{\text{vert}}/M_{\mathcal{G}'}^{\text{edge}}) \otimes \boldsymbol{Z}_l) \subseteq (0,2)_{\boldsymbol{Q}} \stackrel{\text{def}}{=} \{w \in \boldsymbol{Q} \mid 0 < w < 2\},$$

then we shall say that [the action of] J is *l-graphically full*. [Thus, "J l-graphically full" implies "J weakly l-graphically full"—cf. Proposition 2.4, (i), (ii), below.]

REMARK 2.3.1. The purpose of the introduction of the notion *"l-cyclotomically full"* is to allow us to describe, in compact form, that situation in which it makes sense to speak of *"weights"* in a fashion similar to the case where the action of J arises from *scheme theory*. Once it makes sense to speak of "weights", one may introduce the notion of *"l-graphically full"* (respectively, *"weakly l-graphically full"*), which asserts, in essence, that the weights behave as one would expect in the case of *precisely one* (respectively, *at least one*, i.e., possibly *two nested*, as in the situations of Corollaries 2.8, 2.10 below) *degeneration*(*s*) of the hypothetical family of hyperbolic curves under consideration.

PROPOSITION 2.4 (Quasi-triviality and quasi-torality). *Let J be as in Definition* 2.3; *$l \in \Sigma$. Write $\underline{m}(\mathcal{G})$ for the rank* [*over $\hat{\mathbf{Z}}^{\Sigma}$*] *of the finitely generated, free $\hat{\mathbf{Z}}^{\Sigma}$-module $M_{\mathcal{G}}$. Then the following hold.*

(i) $(M_{\mathcal{G}}/M_{\mathcal{G}}^{\mathrm{vert}}) \otimes \mathbf{Z}_l$ *is* **quasi-trivial**.

(ii) $M_{\mathcal{G}}^{\mathrm{cusp}} \otimes \mathbf{Z}_l$, $M_{\mathcal{G}}^{\mathrm{edge}} \otimes \mathbf{Z}_l$ *are* **quasi-toral**. *In particular, if J is l*-**cyclotomically full**, *and $2 \notin \underline{w}_l(J)$, then the submodule $M_{\mathcal{G}}^{\mathrm{edge}} \otimes \mathbf{Z}_l \subseteq M_{\mathcal{G}} \otimes \mathbf{Z}_l$ is zero.*

(iii) *Assume that $\mathcal{G}$ is* **sturdy**. *Then there exists a positive integer $m \leq 2\underline{m}(\mathcal{G})$ such that* $\det(M_{\mathcal{G}} \otimes \mathbf{Z}_l)^{\otimes 2}(-m)$ *is* **quasi-trivial**.

(iv) *Assume that $\mathcal{G}$ is* **sturdy**. *Then a character $\psi : J \to \mathbf{Z}_l^{\times}$ is* $\mathbf{Q}$-**cyclotomic** *if and only if it admits a positive power that coincides with the a_ψ-th power of the character obtained by the natural action of J on* $\det(M_{\mathcal{G}} \otimes \mathbf{Z}_l)^{\otimes 2}$ *for some $a_\psi \in \mathbf{Z}$. Suppose further that J is l*-**cyclotomically full**. *Then a $\mathbf{Q}$-cyclotomic ψ is* **positive** (*respectively*, **null**; **negative**) *if and only if a_ψ may be taken to be positive* (*respectively, zero; negative*). *Finally, any two $\mathbf{Q}$-cyclotomic characters $J \to \mathbf{Z}_l^{\times}$ of the* **same weight** *necessarily* **coincide** *on some open subgroup $J' \subseteq J$.*

(v) *Assume that the image of J in* $\mathrm{Aut}(\mathcal{G})$ *is* **open**. *Then J is l*-**graphically full**.

(vi) *Assume that J is l*–**cyclotomically full**. *Then $2 \neq w \in \underline{w}_l(J)$ implies $2 - w \in \underline{w}_l(J)$. If, moreover, $\mathcal{G}$ is* **noncuspidal**, *then $w \in \underline{w}_l(J)$ implies $2 - w \in \underline{w}_l(J)$.*

(vii) *Assume that J is l*-**cyclotomically full**, *and that $\mathcal{G}$ has* **cusps** [*i.e., that $\Pi_{\mathcal{G}}$ is* **free**—*cf. Remark* 1.1.3]. *Then for every sufficiently small open subgroup $\Pi_{\mathcal{G}'} \subseteq \Pi_{\mathcal{G}}$, the subset*

$$\{0\} \cup \underline{w}_l(M_{\mathcal{G}'} \otimes \mathbf{Z}_l) \subseteq \mathbf{Q}$$

is **invariant** *with respect to the automorphism $\lambda \mapsto 2 - \lambda$ of $\mathbf{Q}$; in particular, the* **sum** *of the* **maximum** *and* **minimum** *elements of this* [*finite*] *subset is equal to* 2.

(viii) *Assume that J is l*-**graphically full**. *Then $M_{\mathcal{G}}^{\mathrm{edge}} \otimes \mathbf{Z}_l \subseteq M_{\mathcal{G}} \otimes \mathbf{Z}_l$ is the* **maximal quasi-toral $\mathbf{Z}_l[J]$-submodule** *of $M_{\mathcal{G}} \otimes \mathbf{Z}_l$.*

(ix) *Assume that J is l*-**graphically full**. *Then $M_{\mathcal{G}} \otimes \mathbf{Z}_l \twoheadrightarrow (M_{\mathcal{G}}/M_{\mathcal{G}}^{\mathrm{vert}}) \otimes \mathbf{Z}_l$ is the* **maximal torsion-free quasi-trivial $\mathbf{Z}_l[J]$-quotient module** *of $M_{\mathcal{G}} \otimes \mathbf{Z}_l$.*

PROOF. Assertion (i) follows immediately from Remarks 1.1.3, 1.1.4. Now when $\mathcal{G}$ is *noncuspidal*, Assertion (ii) follows from Assertion (i); Proposition 1.3. For arbitrary $\mathcal{G}$, Assertion (ii) follows from Assertion (ii) in the noncuspidal case, together with Proposition 2.2, (iv).

Assertion (iii) follows immediately from Assertion (ii) [applied to $M_{\mathcal{G}}^{\mathrm{cusp}} \otimes \boldsymbol{Z}_l$]; Proposition 1.3 [applied to $(M_{\mathcal{G}}/M_{\mathcal{G}}^{\mathrm{cusp}}) \otimes \boldsymbol{Z}_l$, which is possible in light of the *sturdiness* assumption—cf. Remark 1.1.6]. Assertion (iv) follows formally from Assertion (iii); the definitions; the fact that $\boldsymbol{Z}_l^{\times}$ contains a torsion-free open subgroup.

To verify Assertion (v), it suffices to consider the case where $\mathcal{G}$ arises from a pointed stable curve over a *finite field* k [cf. the proof of Proposition 1.3], and J is equal to an open subgroup of $\mathrm{Aut}(\mathcal{G})$. Then Assertion (v) follows from the fact that [in the notation and terminology of *loc. cit.*] the action of G_k on $M_{\mathcal{G}}^{\mathrm{vert}}/M_{\mathcal{G}}^{\mathrm{edge}}$ is of *weight* 1. Assertion (vi) follows from Assertion (ii); Proposition 1.3, applied to the *compactification* [cf. Remark 1.1.6] of a *sturdy* finite étale $\Pi_{\mathcal{G}}$-covering of $\mathcal{G}$.

Next, we consider Assertion (vii). Suppose that $\Pi_{\mathcal{G}'} \subseteq \Pi_{\mathcal{G}}$ is an open subgroup such that $\underline{r}(\mathcal{G}') \geq 2$. Thus, $M_{\mathcal{G}'}^{\mathrm{cusp}} \neq 0$ [cf. Remark 1.3.1], so [by Assertion (ii)] $0, 2 \in E_{\mathcal{G}'} \stackrel{\mathrm{def}}{=} \{0\} \cup \underline{w}_l(M_{\mathcal{G}'} \otimes \boldsymbol{Z}_l)$. Thus, if we set $E'_{\mathcal{G}'} \stackrel{\mathrm{def}}{=} \underline{w}_l(M_{\mathcal{G}'}^{\mathrm{vert}}/M_{\mathcal{G}'}^{\mathrm{edge}} \otimes \boldsymbol{Z}_l)$, then [by Assertions (i), (ii)], it follows that $E_{\mathcal{G}'} = \{0, 2\} \cup E'_{\mathcal{G}'}$. Moreover, by Proposition 1.3 [applied to the *compactification* [cf. Remark 1.1.6] of $\mathcal{G}$], $E'_{\mathcal{G}'}$ is *invariant* with respect to the automorphism $\lambda \mapsto 2 - \lambda$ of $\boldsymbol{Q}$. But this implies the desired invariance of $E_{\mathcal{G}'}$ with respect to this automorphism of $\boldsymbol{Q}$. This completes the proof of Assertion (vii). Finally, Assertions (viii), (ix) follow immediately from Assertions (i), (ii); the definitions. □

EXAMPLE 2.5. Stable Log Curves over a Logarithmic Point. Let $S^{\log}$ be a *log scheme*, with underlying scheme $S \stackrel{\mathrm{def}}{=} \mathrm{Spec}(k)$, where k is a *field*, and log structure given by a *chart* $\boldsymbol{N} \ni 1 \mapsto 0 \in k$ [cf. the theory of [Kato]]. Let

$$X^{\log} \to S^{\log}$$

be a *stable log curve* over $S^{\log}$ [cf. §0]. Let $T^{\log} \to S^{\log}$ be a "separable closure" of $S^{\log}$, i.e., the underlying scheme T of $T^{\log}$ is of the form $T = \mathrm{Spec}(\bar{k})$, where $\bar{k}$ is a *separable closure* of k; the log structure of $T^{\log}$ is given by a *chart* $\boldsymbol{M} \ni 1 \mapsto 0 \in k$, where $\boldsymbol{M} \subseteq \boldsymbol{Q}$ is the monoid of positive rational numbers with denominators invertible in k; the morphism $T^{\log} \to S^{\log}$ arises from the natural maps $k \hookrightarrow \bar{k}$, $\boldsymbol{N} \hookrightarrow \boldsymbol{M}$. Thus, if we write $G_{k^{\log}} \stackrel{\mathrm{def}}{=} \mathrm{Aut}(T^{\log}/S^{\log})$, then we have a natural exact sequence

$$1 \to I_{k^{\log}} \to G_{k^{\log}} \to G_k \to 1\,,$$

where $G_k \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\bar{k}/k)$; $I_{k^{\log}} \stackrel{\mathrm{def}}{=} \mathrm{Hom}(\boldsymbol{Q}/\boldsymbol{Z}, \bar{k}^{\times})$. Now the *admissible coverings* of $X^{\log}$ [with tame ramification at the cusps] determine an *admissible fundamental group* $\Pi_{X^{\log}}$ which fits into a natural exact sequence:

$$1 \to \Delta_{X^{\log}} \to \Pi_{X^{\log}} \to G_{k^{\log}} \to 1\,.$$

[The *theory of admissible coverings* is discussed in detail in [Mzk1], §3; [Mzk2], §2; [Mzk4], §2; [Mzk4], Appendix. It follows, in particular, from this theory that, if one chooses a *lifting* of $X^{\log} \to S^{\log}$ to some *generically smooth stable log curve*

$$X^{\log}_{\mathrm{lift}} \to S^{\log}_{\mathrm{lift}}$$

—where S_{lift} is the spectrum of a complete discrete valuation ring with residue field k; the log structure on $S^{\log}_{\mathrm{lift}}$ is the log structure determined by the monoid of generically invertible functions—then the coverings arising from $\Pi_{X^{\log}}$ may be realized as coverings of the generically smooth curve $X^{\log}_{\mathrm{lift}}$ that satisfy certain properties.] Moreover, if Σ is a set of primes that does not contain the residue characteristic of k, and we denote by $\mathcal{G}$ the *semi-graph of anabelioids of pro-Σ PSC-type* arising from the pointed stable curve over $\bar{k}$ determined by $X^{\log}$, then the *maximal pro-Σ quotient of $\Delta_{X^{\log}}$* may be naturally identified with the *PSC-fundamental group $\Pi_{\mathcal{G}}$*. In particular, one obtains a natural *outer action* of $G_{k^{\log}}$ on $\Pi_{\mathcal{G}}$, the automorphisms of which are easily seen [by the *functoriality* of the various fundamental groups involved!] to be *graphic*. That is to say, we obtain continuous homomorphisms as follows:

$$G_{k^{\log}} \to \mathrm{Aut}(\mathcal{G}) \cong \mathrm{Out}_{\mathrm{grph}}(\Pi_{\mathcal{G}}) \subseteq \mathrm{Out}(\Pi_{\mathcal{G}})\,.$$

Now suppose that $H \subseteq G_{k^{\log}}$ is a *closed subgroup* such that the restriction to H of the homomorphism $G_{k^{\log}} \to \mathrm{Aut}(\mathcal{G})$ factors through some quotient $H \twoheadrightarrow J$:

$$H \twoheadrightarrow J \to \mathrm{Aut}(\mathcal{G})\,.$$

For $l \in \Sigma$, we shall refer to the image in J of the intersection of H with the pro-l component of $I_{k^{\log}}$ as the *l-inertia subgroup* of J; we shall say that [the action on $\mathcal{G}$ of] J is *l-logarithmically full* if the l-inertia subgroup of J is *infinite* [hence isomorphic to $\mathbf{Z}_l(1)$]. If H is an open subgroup $G_{k^{\log}}$, then we shall say that [the action on $\mathcal{G}$ of] J is *arithmetically full* and refer to k as the *base field*.

REMARK 2.5.1. Note that from the point of view of Example 2.5, one may think of the action of G_k on $\mathcal{G}$ appearing in the proof of Proposition 1.3 as the restriction of the action of $G_{k^{\log}}$ on $\mathcal{G}$ discussed in Example 2.5 to some section of $G_{k^{\log}} \twoheadrightarrow G_k$.

PROPOSITION 2.6 (The logarithmic inertia action). *In the notation of Example* 2.5, *$I_{k^{\log}}$ acts* **quasi-unipotently** [*i.e.*, *an open subgroup of $I_{k^{\log}}$ acts unipotently*] *on $M_{\mathcal{G}} \otimes \mathbf{Z}_l$*, *and*, *moreover*, *the submodule*

$$M^{\mathrm{vert}}_{\mathcal{G}} \otimes \mathbf{Z}_l \subseteq M_{\mathcal{G}} \otimes \mathbf{Z}_l$$

is the **maximal quasi-trivial $\mathbf{Z}_l[I_{k^{\log}}]$-submodule** *of $M_{\mathcal{G}} \otimes \mathbf{Z}_l$* [*i.e.*, *the maximal submodule on which some open subgroup of $I_{k^{\log}}$ acts trivially*].

PROOF. Let us first observe that if $\mathcal{G}$ is *noncuspidal*, then the asserted quasi-unipotency (respectively, quasi-triviality) of the action of $I_{k^{\log}}$ on $M_{\mathcal{G}} \otimes \mathbf{Z}_l$ (respectively, $M^{\mathrm{vert}}_{\mathcal{G}} \otimes \mathbf{Z}_l$) follows immediately from the well-known theory of Galois actions on torsion points of degenerating abelian varieties [cf., e.g., [FC], Chapter III, Corollary 7.3; here, we note that, in the terminology of *loc. cit.*, the submodule $M^{\mathrm{vert}}_{\mathcal{G}} \otimes \mathbf{Z}_l$ corresponds to the submodule determined by the *"Raynaud extension"*]. Thus, one obtains the asserted quasi-unipotency/quasi-triviality in the case of not necessarily noncuspidal $\mathcal{G}$ by applying the theory of the *"double"* [cf. Proposition 2.2, (iv)]. Now it remains to prove the asserted *maximality*. But this follows again from

[FC], Chapter III, Corollary 7.3 [i.e., the fact that the *period matrix* of a degenerating abelian variety is always *nondegenerate*]. □

Now, by combining Theorem 1.6 with the theory of the present §2 [cf., in particular, Proposition 2.4], we obtain the following result.

COROLLARY 2.7 (Graphicity). *Let* $\mathcal{G}$, $\mathcal{H}$ *be* **semi-graphs of anabelioids of pro-**Σ **PSC-type**; $J_{\mathcal{G}} \to \mathrm{Aut}(\mathcal{G})$, $J_{\mathcal{H}} \to \mathrm{Aut}(\mathcal{H})$ *continuous homomorphisms. Suppose*, *moreover*, *that we have been given* **isomorphisms of profinite groups**

$$\alpha : \Pi_{\mathcal{G}} \xrightarrow{\sim} \Pi_{\mathcal{H}}; \quad \iota : J_{\mathcal{G}} \xrightarrow{\sim} J_{\mathcal{H}}$$

which are **compatible**, *with respect to the respective outer actions of* $J_{\mathcal{G}}$, $J_{\mathcal{H}}$ *on* $\Pi_{\mathcal{G}}$, $\Pi_{\mathcal{H}}$. *Then*, *for* $l \in \Sigma$ *the following hold.*

(i) *Suppose that the respective actions of* $J_{\mathcal{G}}$, $J_{\mathcal{H}}$ *on* $\mathcal{G}$, $\mathcal{H}$ *are* l**-cyclotomically full**. *Then* α *is* **group-theoretically cuspidal**.

(ii) *Suppose that the respective actions of* $J_{\mathcal{G}}$, $J_{\mathcal{H}}$ *on* $\mathcal{G}$, $\mathcal{H}$ *are* l**-graphically full** [*cf.*, *e.g.*, *Proposition* 2.4, (v)]. *Then* α *is* **graphic**.

(iii) *Suppose that the respective actions of* $J_{\mathcal{G}}$, $J_{\mathcal{H}}$ *on* $\mathcal{G}$, $\mathcal{H}$ *arise from data as in Example* 2.5; *that* $\mathcal{G}$, $\mathcal{H}$ *are* **noncuspidal***; and that*, *in the terminology of Example* 2.5, *these actions are* l**-logarithmically full**, *and*, *moreover*, ι *maps the* l**-inertia subgroup** *of* $J_{\mathcal{G}}$ *isomorphically onto that of* $J_{\mathcal{H}}$. *Then* α *is* **graphic**.

(iv) $\mathrm{Out}_{\mathrm{grph}}(\Pi_{\mathcal{G}})$ *is* **commensurably terminal** *in* $\mathrm{Out}(\Pi_{\mathcal{G}})$.

PROOF. First, we consider Assertion (i). By Theorem 1.6, (i), it suffices to prove that α is *numerically cuspidal*, under the further assumption that $\mathcal{G}$, $\mathcal{H}$ have *cusps* [i.e., that $\Pi_{\mathcal{G}}$, $\Pi_{\mathcal{H}}$ are *free*—cf. Remark 1.1.3]. By passing to *sturdy* finite étale coverings of $\mathcal{G}$, $\mathcal{H}$ that correspond via α [cf. Remark 1.1.5], it follows from Proposition 2.4, (iv), that ι preserves *positive* and *null* $\boldsymbol{Q}$-cyclotomic characters to $\mathbf{Z}_l^\times$. Thus, by Proposition 2.4, (vii), it follows that ι preserves the $\boldsymbol{Q}$-cyclotomic characters to $\mathbf{Z}_l^\times$ of *weight* 2. Now, by applying Proposition 1.3 to the *compactifications* [cf. Remark 1.1.6] of sturdy finite étale coverings $\mathcal{G}' \to \mathcal{G}$, $\mathcal{H}' \to \mathcal{H}$ that correspond via α [cf. Remark 1.1.5], we conclude that the rank of $M_{\mathcal{G}'}^{\mathrm{cusp}}$ (respectively, $M_{\mathcal{H}'}^{\mathrm{cusp}}$) may be computed as the *difference* between the l-weight 2 and l-weight 0 ranks of $M_{\mathcal{G}'}$ (respectively, $M_{\mathcal{H}'}$) [cf. Proposition 2.4, (ii); Remark 1.3.1]; moreover, [cf. Remark 1.3.1] this data allows one to compute $\underline{r}(\mathcal{G}')$ (respectively, $\underline{r}(\mathcal{H}')$). This completes the proof of Assertion (i).

Next, we consider Assertion (ii). By Assertion (i), it follows that α is *group-theoretically cuspidal*. Thus, by replacing $\mathcal{G}$, $\mathcal{H}$ by the *compactifications* [cf. Remark 1.1.6] of *sturdy* finite étale coverings of $\mathcal{G}$, $\mathcal{H}$ that correspond via α [cf. Remark 1.1.5], we may assume without loss of generality that $\mathcal{G}$, $\mathcal{H}$ are *noncuspidal*. Thus, by Theorem 1.6, (ii); Remark 1.4.1, it suffices to prove that α is *verticially filtration-preserving*. But this follows from Proposition 2.4, (ix). This completes the proof of Assertion (ii). Assertion (iv) follows formally from Assertion (ii) [by taking $\mathcal{H} \stackrel{\mathrm{def}}{=} \mathcal{G}$; $J_{\mathcal{G}}$, $J_{\mathcal{H}}$ to be open subgroups of $\mathrm{Out}_{\mathrm{grph}}(\Pi_{\mathcal{G}})$—cf. Proposition 2.4, (v)].

Finally, we consider Assertion (iii). By Theorem 1.6, (ii); Remark 1.4.1, it suffices to prove that α is *verticially filtration-preserving*. But this follows from Proposition 2.6 and the assumptions concerning the *l-inertia subgroups*. This completes the proof of Assertion (iii). □

REMARK 2.7.1. Corollary 2.7, (iv), may be regarded as a sort of *anabelian analogue* of the well-known linear algebra fact that, if k is an algebraically closed field, then *parabolic subgroups* of the general linear group $GL_n(k)$, where $n \geq 2$—e.g., the subgroups that preserve *some filtration* of a k-vector space of dimension n—are *normally terminal* in $GL_n(k)$ [cf., e.g., [Hum], p. 179].

REMARK 2.7.2. Note that the *group-theoretic cuspidality* of [Mzk4], Lemma 1.3.9 (respectively, the *graphicity* of [Mzk4], Lemma 2.3) may be regarded as a [rather weak] *special case* of Corollary 2.7, (i) (respectively, Corollary 2.7, (ii))—cf. the proof of Proposition 2.4, (v), above.

COROLLARY 2.8 (Graphicity over an arithmetic logarithmic point). *Let $\mathcal{G}$, $\mathcal{H}$ be* **semi-graphs of anabelioids of pro-Σ PSC-type**; *$J_{\mathcal{G}} \to \mathrm{Aut}(\mathcal{G})$, $J_{\mathcal{H}} \to \mathrm{Aut}(\mathcal{H})$ continuous homomorphisms that arise from data as in Example 2.5 such that [in the terminology of Example 2.5] the resulting actions are* **l-logarithmically full**, *for some $l \in \Sigma$, and* **arithmetically full**, *with* **base field** *isomorphic to a subfield of a finitely generated extension of $\boldsymbol{F}_p$ or $\boldsymbol{Q}_p$, for some prime $p \notin \Sigma$ [where we allow p to differ for $\mathcal{G}$, $\mathcal{H}$]. Suppose, moreover, that we have been given* **isomorphisms of profinite groups**

$$\alpha : \Pi_{\mathcal{G}} \xrightarrow{\sim} \Pi_{\mathcal{H}}; \quad \iota : J_{\mathcal{G}} \xrightarrow{\sim} J_{\mathcal{H}}$$

which are **compatible**, *with respect to the respective outer actions of $J_{\mathcal{G}}$, $J_{\mathcal{H}}$ on $\Pi_{\mathcal{G}}$, $\Pi_{\mathcal{H}}$, and satisfy the property that ι maps the* **l-inertia subgroup** *of $J_{\mathcal{G}}$ isomorphically onto that of $J_{\mathcal{H}}$. Then the respective actions of $J_{\mathcal{G}}$, $J_{\mathcal{H}}$ on $\mathcal{G}$, $\mathcal{H}$ are* **weakly l-graphically full**, *and α is* **graphic**.

PROOF. Indeed, by using the *Frobenius elements* of the Galois group of a finitely generated extension of $\boldsymbol{F}_p$ or $\boldsymbol{Q}_p$ containing the base field in question [cf. the proof of Proposition 2.4, (v)], one obtains that $J_{\mathcal{G}}$, $J_{\mathcal{H}}$ are *weakly l-graphically full*. [Note that, unlike the situation in the proof of Proposition 2.4, (v), the pointed stable curve over a finite field that one uses here to conclude weak l-graphic fullness will, in general, be a *degeneration* of the original pointed stable curve over the base field appearing in Example 2.5. This is the reason why [unlike the situation in the proof of Proposition 2.4, (v)] in the present context, one may only conclude *weak l*-graphic fullness.] By Corollary 2.7, (i), we thus conclude that α is *group-theoretically cuspidal*. Moreover, this allows us [by passing to compactifications of sturdy finite étale coverings] to reduce to the *noncuspidal* case, hence to conclude that α is *graphic* by Corollary 2.7, (iii). □

REMARK 2.8.1. In the situation of Corollary 2.8, suppose further that the base field in question is *sub-p-adic* [i.e., isomorphic to a subfield of a finitely generated extension of

$\boldsymbol{Q}_p$], and that ι lies over an isomorphism between the absolute Galois groups of the respective base fields that arises from an *isomorphism between the respective base fields*. Then one may apply the main result of [Mzk3]—just as the main result of [Tama1] was applied in [Mzk2], §7—to the various *verticial subgroups* to obtain a version of the *Grothendieck conjecture for pointed stable curves over a sub-p-adic field*. Note that in this situation, when Σ is the *set of all primes*, one may also reconstruct the *log structures at the nodes* by considering the decomposition groups at the nodes [cf. the theory of [Mzk2], §6]. We leave the routine details to the interested reader.

REMARK 2.8.2. In the situation of Corollary 2.8, suppose further that the base field in question is a *finite extension of* $\boldsymbol{Q}_p$ [which may differ for $\mathcal{G}$, $\mathcal{H}$], and that Σ is the *set of all primes*. Then observe that it follows from [Mzk4], Lemma 1.1.4, (ii), that ι lies over an isomorphism between the absolute Galois groups of the respective base fields [that does not necessarily arise from an isomorphism between the respective base fields!]. Now suppose further that the hyperbolic curve constituted by [the complement of the nodes and cusps in] each irreducible component of the pointed stable curves over the respective base fields that give rise to the data in question is *isogenous* [cf. §0] *to a hyperbolic curve of genus zero*. Then it follows from the theory of [Mzk6], §4—more precisely, the *"rigidity"* of the cuspidal edge-like subgroups implied by [Mzk6], Theorem 4.3, together with the *integral absoluteness* of [Mzk6], Corollary 4.11—that one may reconstruct the *log structures at the nodes* by considering the decomposition groups at the nodes [cf. the theory of [Mzk2], §6]. We leave the routine details to the interested reader.

COROLLARY 2.9 (Unramified graphicity). *Let* $\mathcal{G}$, $\mathcal{H}$ *be* **sturdy semi-graphs of anabelioids of pro-**Σ **PSC-type**; $J_{\mathcal{G}} \to \mathrm{Aut}(\mathcal{G})$, $J_{\mathcal{H}} \to \mathrm{Aut}(\mathcal{H})$ *continuous homomorphisms which determine* l**-graphically full** *actions for some* $l \in \Sigma$. *Suppose, moreover, that we have been given* **factorizations**

$$J_{\mathcal{G}} \twoheadrightarrow J'_{\mathcal{G}} \to \mathrm{Out}(\Pi^{\mathrm{unr}}_{\mathcal{G}})\,; \quad J_{\mathcal{H}} \twoheadrightarrow J'_{\mathcal{H}} \to \mathrm{Out}(\Pi^{\mathrm{unr}}_{\mathcal{H}})$$

[*where the composite homomorphisms are the natural homomorphisms*; *the first arrow of each factorization is a surjection*], *together with* **isomorphisms of profinite groups**

$$\beta : \Pi^{\mathrm{unr}}_{\mathcal{G}} \xrightarrow{\sim} \Pi^{\mathrm{unr}}_{\mathcal{H}}\,; \quad \iota' : J'_{\mathcal{G}} \xrightarrow{\sim} J'_{\mathcal{H}}$$

which are **compatible**, *with respect to the respective outer actions of* $J'_{\mathcal{G}}$, $J'_{\mathcal{H}}$ *on* $\Pi^{\mathrm{unr}}_{\mathcal{G}}$, $\Pi^{\mathrm{unr}}_{\mathcal{H}}$. *Then* β *is* **group-theoretically verticial**.

PROOF. By Theorem 1.6, (iii), it suffices to prove that α is *verticially filtration-preserving*. But this follows from Proposition 2.4, (ix). □

REMARK 2.9.1. Note that the *group-theoretic verticiality* of [Mzk2], Proposition 1.4 may be regarded as a [rather weak] *special case* of Corollary 2.9—cf. the proof of Proposition 2.4, (v), above.

Finally, we observe the following consequence of the theory of the present paper concerning anabelian geometry over finite extensions of the quotient field of the ring of Witt vectors of an algebraic closure of a finite field.

COROLLARY 2.10 (Inertia action in the case of two primes). *For* $i = 1, 2$, *let* K_i *be a finite extension of the quotient field of the ring of Witt vectors* $W(\bar{\boldsymbol{F}}_{p_i})$ *with coefficients in an algebraic closure* $\bar{\boldsymbol{F}}_{p_i}$ *of the finite field of cardinality* p_i, *where* p_i *is a prime number*; $\bar{K}_i$ *an algebraic closure of* K_i; $G_i \overset{\text{def}}{=} \mathrm{Gal}(\bar{K}_i/K_i)$; X_i *a* **hyperbolic curve** *over* K_i *whose corresponding stable log curve extends to a stable log curve* $\mathcal{X}_i^{\log}$ *over the spectrum of the ring of integers* $\mathcal{O}_{K_i}$ *of* K_i [*equipped with the log structure determined by the closed point of* $\mathrm{Spec}(\mathcal{O}_{K_i})$]; Σ *a* **set of prime numbers** *such that* $p_i \in \Sigma$; Δ_i *the* **maximal pro-**Σ **quotient** *of the étale fundamental group of* $(X_i)\times_{K_i}\bar{K}_i$ [*so* Δ_i *may be regarded as the profinite fundamental group of a semi-graph of anabelioids* $\mathcal{G}_i$ *of pro-*Σ *PSC-type with precisely one vertex and no closed edges*];

$$\alpha_G : G_1 \xrightarrow{\sim} G_2\,; \quad \alpha_\Delta : \Delta_1 \xrightarrow{\sim} \Delta_2$$

a pair of **isomorphisms of profinite groups** *that are* **compatible** *with the natural outer action of* G_i *on* Δ_i. *Then the following hold.*

(i) *We have* $p_1 = p_2$ [*so we shall write* $p \overset{\text{def}}{=} p_1 = p_2$]; *for* $i = 1, 2$, *the action of* G_i *on* $\mathcal{G}_i$ *is* **weakly** p**-graphically full**; α_Δ *is* **group-theoretically cuspidal**.

(ii) *Suppose that the* **cardinality** *of* Σ *is* ≥ 2. *Then* α_Δ *induces a* **functorial** [*i.e., with respect to the pair* $(\alpha_G, \alpha_\Delta)$] **isomorphism** *of the* **"dual semi-graphs with compact structure"** [*cf.* [*Mzk*4], *Appendix*] *of the special fibers of the* $\mathcal{X}_i^{\log}$.

(iii) *Suppose that the* **cardinality** *of* Σ *is* ≥ 2. *Write* $\pi_1^{\text{temp}}((X_i) \times_{K_i} \bar{K}_i)$ *for the* **tempered fundamental group** *of* [*André*], §4 [*cf. also* [*Mzk*7], *Examples* 3.10, 5.6];

$$\Delta_i^{\text{temp}} \overset{\text{def}}{=} \varprojlim_{N} \pi_1^{\text{temp}}((X_i) \times_{K_i} \bar{K}_i)/N$$

for the **"**Σ**-tempered fundamental group"**—*i.e., the inverse limit where* N *varies over the open normal subgroups of* $\pi_1^{\text{temp}}((X_i) \times_{K_i} \bar{K}_i)$ *such that the quotient* $\pi_1^{\text{temp}}((X_i) \times_{K_i} \bar{K}_i)/N$ *is an extension of a* **finite group** *whose order is a product of primes* $\in \Sigma$ *by a* **discrete free group**. [*Here, we recall that such a discrete free group corresponds to a "combinatorial covering" determined by the graph of the special fiber of some stable reduction of a covering of* X_i—*cf.* [*André*], *Proposition* 4.3.1; [*André*], *the proof of Lemma* 6.1.1.] *Thus, we have a natural continuous outer action of* G_i *on* Δ_i^{temp}; Δ_i *is the* **pro-**Σ **completion** *of* Δ_i^{temp}. *Then the operation of pro-*Σ *completion determines a* **surjection** *from the set of* **compatible pairs of isomorphisms of topological groups**

$$\beta_G : G_1 \xrightarrow{\sim} G_2\,; \quad \beta_{\Delta^{\text{temp}}} : \Delta_1^{\text{temp}} \xrightarrow{\sim} \Delta_2^{\text{temp}}$$

considered up to inner automorphisms of the Δ_i^{temp} *to the set of* **compatible pairs of isomorphisms of topological groups**

$$\gamma_G : G_1 \xrightarrow{\sim} G_2\,; \quad \gamma_\Delta : \Delta_1 \xrightarrow{\sim} \Delta_2$$

considered up to inner automorphisms of the Δ_i.

PROOF. First, we consider Assertion (i). Since [as is well-known] G_i is an extension of an *abelian* profinite group by a *nonabelian* pro-p_i group, it follows that p_i may be characterized as the unique prime number p' such that G_i contains a *nonabelian* pro-p' closed subgroup. Thus, the existence of α_G implies that $p_1 = p_2$; write $p \stackrel{\text{def}}{=} p_1 = p_2$. Now since the tensor product with $\boldsymbol{Q}_p$ of the abelianization of any open subgroup of Δ_i admits a *filtration* [cf., e.g., [FC], Chapter III, Corollary 7.3] each of whose subquotients is *Hodge-Tate* [relative to the action of some open subgroup of G_i], with Hodge-Tate decomposition only involving "Tate twists" by the zero-th or first power of the cyclotomic character [cf. [Tate], §4, Corollary 2], it follows that G_i is [relative to its outer action on Δ_i] *weakly p-graphically full*. Thus, the remainder of Assertion (i) follows from Corollary 2.7, (i).

Next, we consider Assertion (ii). Let $l \in \Sigma$ be *distinct* from $p \stackrel{\text{def}}{=} p_1 = p_2$. Write $\Delta_i \twoheadrightarrow \Delta_i^{(l)}$ for the *maximal pro-l quotient* of Δ_i. Since the subgroup of $\mathrm{Out}(\Delta_i^{(l)})$ that induces the identity on the tensor product with $\boldsymbol{F}_l$ of the abelianization of $\Delta_i^{(l)}$ is [easily seen to be] a *pro-l group*, it follows that by replacing G_i by an open subgroup of G_i, we may assume that the natural map $G_i \to \mathrm{Out}(\Delta_i^{(l)})$ factors through the *maximal pro-l quotient* $G_i \twoheadrightarrow G_i^{(l)}$ of G_i. Thus, the data given by the outer action of $G_i^{(l)}$ on $\Delta_i^{(l)}$ is *l-logarithmically full* data of the type considered in Example 2.5. In particular, in the *noncuspidal* case, Assertion (ii) follows immediately from Corollary 2.7, (iii). On the other hand, even if we are *not* in the noncuspidal case, by passing to *compactifications* of sturdy finite étale coverings of the X_i and applying the fact that a cuspidal edge-like subgroup belongs to a *unique* verticial subgroup [cf. Proposition 1.5, (i)], we reduce immediately [via Assertion (i)] to the *noncuspidal* case. This completes the proof of Assertion (ii).

Finally, we observe that Assertion (iii) follows formally from Assertion (ii) via the same argument applied [in the case where Σ is the set of all primes, and the base fields are finite extensions of $\boldsymbol{Q}_p$] in the proof of [Mzk7], Theorem 6.6, to derive the "surjectivity portion" of [Mzk7], Theorem 6.6, from [Mzk4], Lemma 2.3. □

REMARK 2.10.1. Since free discrete groups *inject* into their pro-Σ completions [cf. [RZ], Proposition 3.3.15], the natural map $\Delta_i^{\text{temp}} \to \Delta_i$ is *injective* [cf. the proof of [Mzk7], Corollary 3.11]. On the other hand, unlike the situation of [Mzk7], Theorem 6.6, we are unable to conclude that the surjection of Corollary 2.10, (iii), is a *bijection*, since [unlike the *profinite* case—cf. [André], Corollary 6.2.2] it is not clear that the [image in Δ_i of] Δ_i^{temp} is equal to its own normalizer in Δ_i.

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RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES
KYOTO UNIVERSITY
KYOTO 606–8502
JAPAN

E-mail address: motizuki@kurims.kyoto-u.ac.jp