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A COMBINED INTEGRATING AND DIFFERENTIATING MATRIX FORMULATION
FOR BOUNDARY VALUE PROBLEMS ON RECTANGULAR DOMAINS

W. D. Lakin

NASA

National Aeronautics and
Space Administration

Langley Research Center
Hampton, Virginia 23665

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W. D. Lakin

Old Dominion University

and

Institute for Computer Applications in Science and Engineering

Abstract

Integrating and differentiating matrices allow the numerical integration and differentiation of functions whose values are known at points of a discrete grid. Previous derivations of these matrices have been restricted to one-dimensional grids or to rectangular grids with uniform spacing in at least one direction. The present work develops integrating and differentiating matrices for grids with non-uniform spacing in both directions. The use of these matrices as operators to reformulate boundary value problems on rectangular domains as matrix problems for a finite-dimensional solution vector is considered. The method requires non-uniform grids which include "near-boundary" points. An eigenvalue problem for the transverse vibrations of a simply supported rectangular plate is solved to illustrate the method.

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1. Introduction

Rotating beam configurations have traditionally been used to study the vibrations and aeroelastic stability of rotating structures such as helicopter rotor blades and propeller blades. More recently, models involving elastic plates have been proposed to include the effects of spanwise variations in material properties. The fourth-order boundary value problems associated with both the beam and plate models do not, in general, have useful closed form solutions. Consequently, most theoretical work on these problems has been asymptotic or numerical in nature.

In one approach to the numerical solution of these problems, harmonic time dependence is assumed to reduce the governing partial differential equation to a differential equation in space variables which includes an eigenvalue. For beam models, this is an ordinary differential equation. The fundamental derivative which represents beam curvature may now be taken as a new dependent variable, and the eigenvalue problem for the beam can be reformulated as an integro-differential equation (White & Malatino, 1975; Kvaternik, White, & Kaza, 1978; Lakin, 1982). This equation may be conveniently expressed using integral, differential, and boundary evaluation operators. The operator equation for the continuous solution may further be converted to a matrix operator equation for a finite-dimensional solution vector by evaluating the continuous equation at a finite set of discrete grid points which span the interval of interest. A key question is now the manner in which the matrix operators are approximated.

For beam models, one method for approximating the integral and differential operators involves the use of integrating matrices (Vakhitov, 1966; Hunter, 1970; Lakin, 1979) and differentiating matrices (Hunter &

Jainchell, 1969; Lakin, 1985). In the simplest terms, these matrices provide, respectively, a means of numerically integrating and differentiating a function whose values are known at a finite set of discrete grid points. A key property of both integrating and differentiating matrices is that their derivation requires only knowledge of the grid points, and no information is needed about the function to be numerically integrated or differentiated. In the case of a beam model with its single space variable, this property allows the integrating and differentiating matrices based on one-dimensional grids to be used directly as approximations for the integral and differential operators in the matrix operator form of the eigenvalue problem. The result of this approximation is a striaghtforward matrix eigenvalue problem which can be solved by standard methods. This approach has proved capable of efficiently handling a wide variety of beam problems including beams with concentrated masses, follower forces, and point loadings (Lakin, 1982).

For vibration and buckling problems which involve two-dimensional elastic plates, removal of the time dependence from the original boundary value problem yields an eigenvalue problem which continues to be governed by a partial differential equation. By analogy with the one-dimensional case, it would seem desirable to reformulate this eigenvalue problem as a matrix integro-partial differential equation for a finite-dimensional solution vector on a two-dimensional grid of discrete points. Integrating and differentiating matrices based on two-dimensional grids could then be used to approximate the respective operators resulting, again, in a standard matrix eigenvalue problem.

The present work will explore the potential of this approach by considering an eigenvalue problem associated with the transverse vibration of

a simply supported rectangular plate. This problem consists of the biharmonic eigenvalue equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2 u(x,y) = \lambda^2 u(x,y) \tag{1.1}$$

for $0 \leq x \leq A$, $0 \leq y \leq B$, and the boundary conditions

$$u(0,y) = u_{xx}(0,y) = 0,$$

$$u(A,y) = u_{xx}(A,y) = 0,$$

$$\tag{1.2}$$

$$u(x,0) = u_{yy}(x,0) = 0,$$

$$u(x,B) = u_{yy}(x,B) = 0.$$

In section 2, equation (1.1) will be reformulated as an integro-partial differential equation consistent with the form of the boundary conditions (1.2). Because (1.2) involves conditions on u(x,y) itself at all four boundaries, the present approach retains u itself as the dependent variable. Conversion to a matrix eigenvalue problem will now require the derivation of appropriate integrating and differentiating matrices based on a two-dimensional rectangular grid of discrete points.

One type of integrating matrix for a function of two variables has been previously derived by Lakin (1981). This matrix may be used in two-dimensional problems whose reformulation is possible using an integrating matrix alone, e.g., the plate analogue of a beam with cantilevered boundary

conditions. Unfortunately, this matrix is not suitable for the present purposes as its derivation requires that the spacing of the grid points be uniform in at least one direction. The boundary conditions (1.2) lead to a reformulation which will require the use of differentiating matrices to approximate partial derivatives with respect to both x and y. To preserve accuracy, the computational grid must now include points "close" to all four boundaries (Lakin, 1985) giving non-uniform grid spacing in both directions.

Generalized integrating matrices for functions of two variables whose values are known on non-uniform rectangular grids are derived in the Appendix. Differentiating matrices which approximate partial derivatives on non-uniform rectangular grids are also derived, as are matrices which evaluate quantities at boundary grid points. The only restriction in this derivation is that the grid sub-units $[x_j \leq x \leq x_{j+1}, y_k \leq y \leq y_{k+1}]$ should be rectangles.

In section 3, appropriate integrating and differentiating matrices are used to approximate the reformulated eigenvalue problem by a matrix eigenvalue problem involving a "stacked" column vector. Numerical calculations are presented for a rectangle with A = 2 and B = 1 on a 7-by-7 grid (including near boundary points). Despite the coarseness of this grid, good agreement with the exact eigenvalues of (1.1) and (1.2) is obtained.

2. Reformulation of the Eigenvalue Problem

Before considering the two-dimensional problem (1.1) and (1.2), it is useful to briefly consider the reformulation process for the one-dimensional problem of a cantilevered beam. For the beam, integrating the fourth-order

ordinary differential equation twice with respect to the space variable x (say) gives an integral equation for the curvature w"(x) and allows the boundary conditions at the free end to be explicitly invoked. Boundary conditions at the free end now enter explicitly through relation of the fundamental derivative w" to w´ and the original dependent variable w. Thus, the appropriate reformulation makes use of all boundary conditions.

The initial steps required to obtain an appropriate reformulation of the two-dimensional problem (1.1) and (1.2) are a straightforward generalization of the one-dimensional procedure. Equation (1.1) may be integrated twice with respect to x from x to A making use of the boundary condition $u_{xx}(A,y)$ = 0. This result may then be integrated two additional times with respect to y from y to B making use of three other boundary conditions at x = A and y = B, i.e.,

$$u(A,y) = u(x,B) = u_{yy}(x,B) = 0. \tag{2.1}$$

These steps give the equation

$$\int_y^B (\xi - y)\left[u_{xx}(x,\xi) + (A - x)u_{xxx}(A,\xi)\right]d\xi$$

$$+ \int_x^A (\eta - x)\left[u_{yy}(\eta,y) + (B - x)u_{yyy}(\eta,B)\right]d\eta$$

<div align="right">(2.2)</div>

$$+ 2(A - x)\left[u_x(A,y) - u_x(A,B)\right] + 2(B - y)\left[u_y(x,B) - u_y(A,B)\right]$$

$$+ 2(A - x)(B - y)u_{xy}(A,B) + 2u(x,y) = \lambda^2 \int_y^B (\xi - y) \int_x^A (\eta - x)u(\eta,\xi)d\eta \; d\xi.$$

The corner consistency condition $u(A,B) = 0$ has also been explicitly used in obtaining (2.2).

Equation (2.2) contains three different types of second partial derivatives. A question to be answered is therefore whether, as in the single variable beam case, the boundary conditions at $x = 0$ and $y = 0$ can be accounted for by designating one type of second derivative as a fundamental derivative and new dependent variable. The form of the boundary conditions is critical in deciding this question.

The second partial derivatives u_{xx} and u_{yy} are related to the variable u through the relations

$$\int_0^x (\eta - x)u_{xx}(\eta,y)d\eta = u(x,y) - u(0,y) - xu_x(0,y) \tag{2.3}$$

and

$$\int_0^y (\xi - y)u_{yy}(x,\xi)d\xi = u(x,y) - u(x,0) - yu_y(x,0). \tag{2.4}$$

Thus, for either u_{xx} or u_{yy} alone to be a viable candidate for the fundamental derivative role, the boundary conditions (1.2) would have to specify that the respective first partial derivative vanish for x or y equal zero.

In contrast to (2.3) and (2.4) the boundary conditions in (1.2) at $x = 0$ and $y = 0$ insure that integrated terms vanish when the mixed second derivative u is related to u through the expression

$$\int_0^y \int_0^x u_{xy}(\eta,\xi)d\eta \, d\xi = u(x,y) - u(x,0) - u(0,y) + u(0,0). \tag{2.5}$$

This is remeniscent of the single variable case. Unfortunately, the higher derivative conditions at $x = 0$ and $y = 0$ are not on u_{xy} itself but on u_{xx} and u_{yy}. The mixed derivative can be directly related to u_{xx} and u_{yy} through an appropriate quadrature and partial differentiation. However, the relationship is such that neither of the two remaining higher derivative boundary conditions can be satisfied directly. Consequently, the mixed derivative alone is also not suitable as a new dependent variable.

Two possibilities now remain. First, the boundary value problem (1.1) and (1.2) can be reformulated in terms of three fundamental derivatives, i.e., all three types of second partial derivatives can be designated as separate dependent variables. The original variable u could then be related to the new variables through (2.5) satisfying two of the four conditions at $x = 0$ and $y = 0$. However, while attractive in principle, this option will be impractical in practice as it leads to matrix eigenvalue problems involving large matrices. For example, in the case of the relatively coarse 7-by-7 rectangular grid used in the present calculations for the sample problem, 147-by-147 matrices would be required if the three derivatives are retained as distinct dependent variables.

A second possibility is to retain the original variable $u(x,y)$ in the reformulation of (1.1) and (1.2). This choice requires that two additional partial derivative operations be approximated. It also requires that equation (2.2) be modified to explicitly take account of the four boundary conditions at $x = 0$ and $y = 0$. However, when the resulting reformulation is approximated by a matrix eigenvalue problem for a finite-dimensional solution vector, considerably smaller matrices are needed than if fundamental derivatives were used. For example, in calculations on the 7-by-7

rectangular grid, 49-by-49 matrices are sufficient for the basic development, and the final eigenvalue problem involves a 25-by-25 matrix.

To obtain the desired reformulation, equation (2.2) is first evaluated at $x = 0$, at $y = 0$, and at the corner point $x = 0$ and $y = 0$. The three relations which result now explicitly use all conditions at the boundaries $(0,y)$, $(x,0)$, and satisfy the corner consistency conditions at $(0,0)$. In particular:

$$\int_0^A \eta[u_{yy}(\eta,y) + (B - y)u_{yyy}(\eta,B)]d\eta + \int_y^B A(\xi - y)u_{xxx}(A,\xi)d\xi$$

$$+ 2A[u_x(A,y) - u_x(A,B)] - 2(B - y)[u_y(0,B) - u_y(A,B)] \qquad (2.6)$$

$$+ 2A(B - y)u_{xy}(A,B) = \lambda^2 \int_y^B (\xi - y) \int_0^A \eta u(\eta,\xi)d\eta \, d\xi,$$

$$\int_0^B \xi[u_{xx}(x,\xi) - (A - x)u_{xxx}(A,\xi)]d\xi + \int_x^A B(\eta - x)u_{yyy}(\eta,B)d\eta$$

$$+ 2(A - x)[u_x(A,0) - u_x(A,B)] + 2B[u_y(x,B) - u_y(A,B)] \qquad (2.7)$$

$$+ 2B(A - x)u_{xy}(A,B) = \lambda^2 \int_0^B \xi \int_x^A (\eta - x)u(\eta,\xi) \, d\eta \, d\xi,$$

$$\int_0^B A\xi u_{xxx}(A,\xi)d\xi + \int_0^A B\eta u_{yyy}(\eta,B)d\eta$$

$$+ 2A[u_x(A,0) - u_x(A,B)] + 2B[u_y(0,B) - u_y(A,B)] \qquad (2.8)$$

$$+ 2ABu_{xy}(A,B) = \lambda^2 \int_0^B \xi \int_0^A \eta u(\eta,\xi)d\eta \, d\xi.$$

Subtracting (2.6) and (2.7) from (2.2) and adding (2.8) now gives

$$\left\{ \int_0^y \xi + y \int_y^B \right\} [xu_{xxx}(A,\xi) - u_{xxx}(x,\xi)]d\xi$$

$$+ \left\{ \int_0^x \eta + x \int_x^A \right\} [yu_{yyy}(\eta,B) - u_{yyy}(\eta,y)]d\eta$$

$$+ 2x[u_x(A,0) - u_x(A,y)] + 2y[u_y(0,B) - u_y(x,B)]$$

$$\tag{2.9}$$

$$+ 2xyu_{xy}(A,B) + 2u(x,y)$$

$$= \lambda^2 \left\{ x\int_0^y \int_x^A \xi u(\eta,\xi)d\eta \ d\xi + y\int_y^B \int_0^x \eta u(\eta,\xi)d\eta \ d\xi \right.$$

$$\left. + xy\int_y^B \int_x^A u(\eta,\xi)d\eta \ d\xi + \int_0^y \int_0^x \eta\xi u(\eta,\xi)d\eta \ d\xi \right\}.$$

This equation is the reformulation required for appoximation of the eigenvalue problem (1.1) and (1.2) using integrating and differentiating matrices on the rectangular grid.

3. Approximation By A Matrix Eigenvalue Problem

The first step in approximating equation (2.9) by a matrix eigenvalue problem for a finite-dimensional solution vector is to discretize this equation on a rectangular grid G of discrete points. This will allow the integrating, differentiating, and boundary matrices on this two-dimensional grid to be used as operators to approximate the corresponding operators in the continuous equation.

The two-dimensional grid G may be formed from the cross-product of appropriate one-dimensional grids in the x- and y-directions. In particular, let G_x be the one-dimensional grid of N discrete points

$$0 = x_1 < x_2 \cdots < x_N = A \qquad (3.1)$$

which discretizes the interval $0 \le x \le A$, and let G_y be the one-dimensional grid of M points

$$0 = y_1 < y_2 < \cdots < y_M = B \qquad (3.2)$$

which discretizes the interval $0 \le y \le B$. Neither G_x nor G_y need to have uniform spacing. Indeed, in actual implementation, to preserve accuracy it will be necessary to choose the spacings $x_2 - x_1$, $x_N - x_{N-1}$, $y_2 - y_1$, and $y_M - y_{M-1}$ relatively small as the formulation will involve differentiating matrices (Lakin, 1985). The two-dimensional grid G for the continuous region of the boundary value problem (1.1) and (1.2) may now be taken as the set of NM discrete points

$$G = \left\{ (x_i, y_j) : x_i \epsilon G_x, y_j \epsilon G_y \right\}. \qquad (3.3)$$

Thus, the subunits of the grid G are rectangles which need not have equal areas.

The straightforward format for displaying values of the solution u(x,y) at the discrete grid points of G is an N-by-M matrix [U] with elements $U_{ij} = u(x_i, y_j)$. Unfortunately, this format is unsuitable for the desired reduction to a matrix eigenvalue problem. Rather, it is convenient to arrange

the NM values of u on the grid as a "stacked" NM-by-1 column vector. In particular, the finite-dimensional solution vector {u} is taken to have elements

$$u_k = u(x_i, y_j) \quad \text{with} \quad k = N(j - 1) + i \tag{3.4}$$

where $i = 1, \ldots, N$ and $j = 1, \ldots, M$. It should be noted that this format for {u} is x-oriented, i.e., the M groups of N consecutive elements in the stacked vector give values of u for a fixed value of y in G_y while x varies in G_x. Because of this orientation, as indicated in Appendix A, matrices which approximate integrals and derivatives with respect to y will require extra operations in their construction. A general flow chart for the construction of both x- and y-operation matrices is given in Figure 1.

Once equation (2.9) has been discretized, integrating, differentiating, and boundary evaluation matrices on the grid G may be used to obtain a matrix eigenvalue problem which provides the required approximations. In particular, the eigenvalue problem for (2.9) may be written as

$$[G]\{u\} = \lambda^2[H]\{u\} \tag{3.5}$$

where $[G] = [G_1] + [G_2] + 2[G_3]$,

$$[G_1] = ([JRY][Y] + [Y][JTY])([X][BA][DX3] - [DX2]), \tag{3.6}$$

$$[G_2] = ([JRX][X] + [X][JTX])([Y][BB][DY3] - [DY2]), \tag{3.7}$$

and

$$[G_3] = [X]([BAO] - [BA])[DX] + [Y]([BBO] - [BB])[DY]$$

$$(3.8)$$

$$+ [X][Y][BA][BB][DX][DY] + [I].$$

In (3.6) through (3.8), [JRX] and [JTX] approximate x-integrals from 0 to x and from x to A, repectively, on the two-dimensional grid G while [DX], [DX2], and [DX3] approximate first, second, and third partial derivatives with respect to x on G. Similarly, [JRY] and [JTY] approximate y-integrals from 0 to y and from y to B, respectively, on G while [DY], [DY2], and [DY3] approximate first, second, and third partial derivatives with respect to y. The matrices [BA] and [BB] evaluate quantities at boundary points with x = A and y = B, respectively, while [BAO] and [BBO] give values at the boundary points (A,0) and (0,B). The derivation of these integrating, partial differentiating, and boundary matrices on the grid G is discussed in Appendix A. [I] is the NM-by-NM identity matrix. The NM-by-NM matrices [X] and [Y] are diagonal matrices such that if position k in the stacked solution vector corresponds to the point (x_i, y_j) of G, then $X_{kk} = x_i$ and $Y_{kk} = y_j$.

The matrix [H] in (3.4) may be written as the sum of four matrices $[H] = [H_1] + [H_2] + [H_3] + [H_4]$ with

$$[H_1] = [X][JRY][JTX][Y], \qquad (3.9)$$

$$[H_2] = [Y][JTY][JRX][X], \qquad (3.10)$$

$$[H_3] = [X][Y][JTY][JTX], \qquad (3.11)$$

and

$$[H_4] = [JRY][JRX][X][Y]. \qquad (3.12)$$

To enhance accuracy in calculation of the lower eigenvalues, it is convenient to further re-write (3.5) as

$$[A]\{u\} = \omega\{u\} \qquad (3.13)$$

with

$$[A] = [G]^{-1}[H] \quad \text{and} \quad \omega = 1/\lambda^2. \qquad (3.14)$$

To test the accuracy of this matrix eigenvalue approximation to the continuous problem (1.1) and (1.2), equation (3.13) was solved on a rectangle with $A = 2$ and $B = 1$. G_x was taken to be the seven point grid consisting of the two end points $x_1 = 0$ and $x_7 = 2$, two near-boundary points $x_2 = 0.0001$ and $x_6 = 1.9999$, and three interior points $x_3 = 0.5$, $x_4 = 1.0$, and $x_5 = 1.5$. G_y was taken to be the seven point grid consisting of the two endpoints $y_1 = 0$ and $y_7 = 1.0$, two near-boundary points $y_2 = 0.00005$ and $y_6 = 0.99995$, and three interior points $y_3 = 0.25$, $y_4 = 0.5$, and $y_5 = 0.75$. The implementation of (3.6) through (3.12) on this grid thus involves 49-by-49 matrices. However, the size of the matrix $[A]$ in the matrix eigenvalue problem (3.13) can be reduced by noting that $u = 0$ at all boundary points. Rows and columns of $[A]$ corresponding to boundary points may thus be deleted leading to a 25-by-25 matrix.

Exact solution of the boundary value problem (1.1) and (1.2) are of the form

$$u_{pq}(x,y) = c_{pq} \sin \frac{p\pi x}{A} \sin \frac{q\pi y}{B} \qquad (3.15)$$

where   p   and   q   are integers. The corresponding eigenvalues are

$$\lambda_{pq} = (\frac{p\pi}{A})^2 + (\frac{q\pi}{B})^2 . \qquad (3.16)$$

For the present test case with   A = 2   and   B = 1, the exact values of the two smallest eigenvalues are

$$\lambda_{11} = 12.337 \quad \text{and} \quad \lambda_{21} = 19.739. \qquad (3.17)$$

Approximate values for these eigenvalues were obtained by solving the matrix eigenvalue problem (3.13) on the grid   G.   Differentiating matrices were based on fourth degree polynomials while integrating marices were based on fifth degree polynomials. This even/odd degree scheme allows grid points to be centered as much as possible within the sliding subgrids on which the differentiating and integrating matrices are based. The computations give the values

$$\lambda_{11} = 12.553 \quad \text{and} \quad \lambda_{21} = 17.635. \qquad (3.18)$$

It must be remarked that   G   in this test problem is a relatively coarse grid which, when boundary and near-boundary points are ommitted, has only a total of nine points in the interior of the two-dimensional region. Values of these approximations could be improved through the inclusion of additional interior points.

4. Concluding Remarks

The present work has examined an extension of integrating and differentiating matrix methodology to partial differential equations involving two space variables. Matrices which approximate integrals and derivatives on one-dimensional grids are used as a starting point to develop matrices which approximate integrals and partial derivatives on two-dimensional rectangular grids. The method requires that the original boundary value problem be reformulated to take account of all boundary conditions. Integrating, differentiating, and boundary matrices may then be used as operators to approximate the boundary value problem by a standard matrix problem for a stacked, finite-dimensional solution vector. The inclusion of near boundary points in the grid helps to prevent the degradation of accuracy at boundaries associated with differentiating matrices.

While only two-dimensional rectangular domains have been explicitly considered in the present work, a further generalization of the method to three-dimensional domains is relatively straightforward. This is due to the use of a stacked column vector format for the solution vector which allows matrices on the higher dimensional grid to be obtained from matrices on the underlying one-dimensional grids. The primary requirement in going to three space dimensions is the use of appropriate change matrices analogous to the matrices [CXY] and [CYX] of Appendix A. These matrices will shuffle the order of the stack to orient it with respect to a given variable and then restore the original orientation after an operation with respect to that variable has been approximated.

Appendix A: Partial Differentiating, Integrating, and Boundary Matrices
For Rectangular Grids

Let   G   be the rectangular grid in (3.3) formed from the two one-dimensional grids G_x and G_y in (3.1) and (3.2). Let {u} be the

NM-by-1   "stacked" column vector defined in (3.4) which gives values of

u(x,y) at the points of G. Then, for the present boundary value problem on

a rectangle, it is necessary to derive two types of integrating matrices and

three partial differentiating matrices for each of the two space variables

x and y.

Consider first the matrices which approximate operations with respect to

x. The "0-to-x" integrating matrix [JRX] on G is an NM-by-NM matrix

such that the NM-by-1 column vector [JRX]{u} contains approximate values

of the integral $\int_0^x u(\eta,y)d\eta$ at the points of G. In particular,

$$[JRX]\{u\} \cong (0, \int_0^{x_2} u(\eta,y_1)d\eta, \cdots, \int_0^A u(\eta,y_1)d\eta, 0, \int_0^{x_2} u(\eta,y_2)d\eta, \cdots,$$

$$\int_0^A u(\eta,y_2)d\eta, \cdots, 0, \int_0^{x_2} u(\eta,y_M)d\eta, \cdots, \int_0^A u(\eta,y_M)d\eta)^T. \tag{A1}$$

Similarly, the "x-to-A" integrating matrix [JTX] leads to approximations

at points of G of the integral $\int_x^A u(\eta,y)d\eta$ so that

$$[JTX]\{u\} \cong (\int_0^A u(\eta,y_1)d\eta, \cdots, \int_{x_{N-1}}^A u(\eta,y_1)d\eta, 0, \int_0^A u(\eta,y_2)d\eta, \cdots, 0, \cdots,$$

$$\int_0^A u(\eta,y_M)d\eta, \cdots, \int_{x_{N-1}}^A u(\eta,y_M)d\eta, 0)^T. \tag{A2}$$

The x-partial derivative matrix [DX] is an NM-by-NM matrix such that [DX]{u} contains approximate values of $\partial u/\partial x$ at points of G. Thus,

$$[DX]\{u\} = (\frac{\partial u}{\partial x} (x_1,y_1),\cdots,\frac{\partial u}{\partial x} (x_N,y_1),\cdots,\frac{\partial u}{\partial x} (x_1,y_M),\cdots,\frac{\partial u}{\partial x} (x_N,y_M))^T. \qquad (A3)$$

The matrices [DX2] and [DX3] which lead to approximations of second and third partial derivatives of u(x,y) with respect to x at points of G have similar definitions.

The stacked column vector {u} has been constructed so as to be x-oriented. It thus consists of M segments which contain N elements apiece. In each of these segments, x varies through G_x for a constant value of y in G_y. The matrices in (A1) to (A3) approximate x-operations for fixed values of y. Consequently, use of the present stacked column vector format will allow x-operation matrices on the rectangular grid G to be constructed from the corresponding N-by-N matrices on the one-dimensional grid G_x. This construction is most easily accomplished through definition of a "diagonalizing" mapping from the set of N-by-N matrices to the set of NM-by-NM matrices.

Let p and q be integers, and let [A] be a p-by-p matrix. The diagonalizing mapping Diag(p,q,[A]) then assigns to [A] the pq-by-pq matrix [B] obtained by placing q matrices [A] along the diagonal of [B] and taking all other elements of [B] to be zero, i.e.,

$$[B] = \text{Diag}(p,q,[A]) = \begin{bmatrix} [A] & & & 0 \\ & [A] & & \\ & & [A] & \\ 0 & & & [A] \end{bmatrix}. \qquad (A4)$$

The x-operation matrices on G may be formed by applying this mapping with p = N and q = M to appropriate matrices on the grid G.

Consider first the construction of [JRX]. Let f(x) be a function whose values are known at the points of G, and let {f} be the N-by-1 column vector which contains these values. Further, let [jrx] be an N-by-N integrating matrix which approximates integrals of f(x) from 0 to x on G_x so that

$$[jrx]\{f\} \cong \left\{ \int_0^{x_i} f(\eta)d\eta \right\}. \tag{A5}$$

Comparing equation (A1), segment-by-segment, with (A5) now shows that

$$[JRX] = Diag(N,M,[jrx]). \tag{A6}$$

The matrix [jrx] (and hence [JRX]) is not unique, but depends on both the number of points included in the sliding subgrids of G_x and the manner in which f(x) is approximated on these subgrids. The present work uses the integrating matrix [jrx] for one-dimensional non-uniform grids developed by Lakin (1979).

To obtain the second required integrating matrix [JTX] on the rectangular grid G, let [jtx] be an integrating matrix on G_x such that

$$[jtx]\{f\} \cong \left\{ \int_{x_i}^A f(\eta)d\eta \right\}. \tag{A7}$$

Then, a segment-by-segment comparison of (A2) and (A7) shows that

$$[JTX] = Diag(N,M,[jtx]). \tag{A8}$$

Differentiating matrices on one-dimensional non-uniform grids have been derived by Lakin (1985). Let the matrices which approximate first, second, and third derivatives of $f(x)$ at points of G_x be denoted, respectively, by $[dx]$, $[dx2]$, and $[dx3]$. As is the case with integrating matrices, these differentiating matrices are not unique. Further, in the usual case where the sliding subgrids contain fewer than all N points of G_x, $[dx2]$ and $[dx3]$ cannot be obtained by simply squaring or cubing the matrix $[dx]$. Rather, these matrices must be obtained directly from approximations to the second and third derivatives of $f(x)$ on the sliding subgrids.

The three matrices $[DX]$, $[DX2]$, and $[DX3]$ which approximate partial derivatives with respect to x of $u(x,y)$ at points of G may now be constructed from $[dx]$, $[dx2]$, and $[dx3]$ using the diagonalizing mapping. In particular,

$$[DX] = Diag(N,M,[dx]),$$

$$[DX2] = Diag(N,M,[dx2]),$$

and $\hspace{11cm}$ (A9)

$$[DX3] = Diag(N,M,[dx3]).$$

For consistency, the differentiating matrices used in the present work were based on the same subgrid approximation scheme as was used for the integrating matrices on G_x.

Consider next derivation of the matrices which approximate operations with respect to y on the rectangular grid G. The required integrating matrices

for this variable are the "0-to-y" integrating matrix [JRY] which

approximates the integral $\int_0^y u(x,\xi)d\xi$ at points of G, and the "y-to-B"

integrating matrix [JTY] which approximates the integral $\int_y^B u(x,\xi)d\xi$. The

product of these NM-by-NM matrices with {u} are the NM-by-1 column

vectors

$$[JRY]\{u\} \cong (0,\cdots,0,\int_0^{y_2} u(x_1,\xi)d\xi,\cdots,\int_0^{y_2} u(x_N,\xi)d\xi,\cdots,$$

$$\int_0^B u(x_1,\xi)d\xi,\cdots,\int_0^B u(x_N,\xi)d\xi)^T \tag{A10}$$

and

$$[JTY]\{u\} \cong (\int_0^B u(x_1,\xi)d\xi,\cdots,\int_0^B u(x_N,\xi)d\xi,\cdots,$$

$$\int_{y_{M-1}}^B u(x_1,\xi)d\xi,\cdots,\int_{y_{M-1}}^B u(x_N,\xi)d\xi,0,\cdots,0)^T. \tag{A11}$$

Let g(y) be a function whose values are known at the points of the one-

dimensional grid G_y, and let {g} be the M-by-1 column vector which

contains these values. Further, let [jry] and [jty] be integrating

matrices on G_y such that

$$[jry]\{g\} \cong \left\{\int_0^{y_i} g(\xi)d\xi\right\} \text{ and } [jty]\{g\} \cong \left\{\int_{y_i}^B g(\xi)d\xi\right\}. \tag{A12}$$

Because {u}, as defined, is x-oriented, the integrating matrices [JRY]

and [JTY] on G cannot be formed using the single mapping (A4) on the

corresponding matrices for the one-dimensional grid G_y. As [JRY] and

[JTY] approximate integrals with respect to y for fixed values of x, two additional mappings which convert from x- to y-orientation and from y- to x-orientation will also be required.

If the values of u(x,y) at the NM points of the rectangular grid G are written as an NM-by-1 stacked column vector {v} whose format is y-oriented, then the elements of {v} are

$$v_k = u(x_i, y_j) \quad \text{where} \quad k = M(i - 1) + j. \tag{A13}$$

Thus, the vector {v} consists of N segments which contain M elements apiece. In each segment, y varies in G_y for a fixed value of x in G_x. If {v} and not {u} had been chosen as the format for the solution vector, then [JRY] and [JTY] could be formed from [jry] and [jty] directly using the diagonalizing mapping (A4) with p = M and q = N.

An x-oriented vector {u} may be associated with its corresponding y-oriented vector {v} through a mapping Cxy from the set of NM-by-1 column vectors into itself so that Cxy({u}) = {v}. In symbolic terms, if the values of u(x,y) on the rectangular grid G are arranged in an N-by-M array, then the effect of applying the mapping Cxy is to produce an M-by-N array which is the transpose of the original. For the present purposes, the mapping Cxy may be carried out by multiplying {u} by an NM-by-NM matrix [CXY] so that

$$[CXY]\{u\} = \{v\}. \tag{A14}$$

The matrix [CXY] may be written as a stack of N, M-by-NM matrices

$$[CXY] = \begin{bmatrix} [CXY^{(1)}] \\ \vdots \\ [CXY^{(M)}] \end{bmatrix}. \tag{A15}$$

If $\lfloor e_j \rfloor$ is the j-th unit vector in N-dimensional real space, i.e., a row vector with a one in the j-th position and zeros in the other N-1 positions, then each matrix in (A15) can be written in the form

$$[CXY^{(j)}] = \begin{bmatrix} \lfloor e_j \rfloor & & & & 0 \\ & & \lfloor e_j \rfloor & & \\ & & & \lfloor e_j \rfloor & \\ 0 & & & & \lfloor e_j \rfloor \end{bmatrix}. \tag{A16}$$

The matrices in (A15) thus have the row vector $\lfloor e_j \rfloor$ along the diagonal and zeros elsewhere.

The mapping Cxy is one-to-one and hence invertable. Let the inverse mapping be denoted by Cyx so that $Cyx(\{v\}) = \{u\}$. Thus, when applied to a y-oriented vector, Cyx restores the standard x-orientation. This mapping may be carried out by multiplying $\{v\}$ by the NM-by-NM matrix [CYX] so that

$$[CYX]\{v\} = \{u\}. \tag{A17}$$

Because of the inverse relationship of Cxy and Cyx, $[CYX] = [CXY]^{-1}$.

Having defined the mappings which change the format of a stacked column vector from x- to y-orientation and back again, the procedure which uses the M-by-M integrating matrix [jry] on G_y to produce the the vector [JRY]$\{u\}$ defined in (A10) may now be given. The vector $\{u\}$ is first

multiplied by [CXY] to produce a y-oriented format. It is then multiplied by the matrix Diag(M,N,[jry]) to give a vector which consists of M segments, each of which approximates the integral of u(x,y) from 0 to y for a fixed value of x. Finally, the original x-orientation is restored through multiplication by [CYX]. This process implies

$$[JRY] = [CYX] \; Diag[M,N,[jry]) [CXY]. \tag{A18}$$

The matrix [JTY] may likewise be formed in this manner from the integrating matrix [jty] on G_y. Hence,

$$[JTY] = [CYX] \; Diag(M,N,[jty]) [CXY]. \tag{A19}$$

The NM-by-NM matrix [DY], which approximates partial derivatives with respect to y on the rectangular grid G, is defined by

$$[DY]\{u\} \; \underset{\sim}{\cong} \; (\frac{\partial u}{\partial y} (x_1,y_1), \cdots, \frac{\partial u}{\partial y} (x_N,y_1), \cdots, \frac{\partial u}{\partial y} (x_1,y_M), \cdots, \frac{\partial u}{\partial y} (x_N,y_M))^T. \tag{A20}$$

Let [dy] be an M-by-M differentiating matrix on the one-dimensional grid G_y such that

$$[dy]\{g\} \; \underset{\sim}{\cong} \; \{g'\}. \tag{A21}$$

Then, [DY] can be formed from [dy] through the relation

$$[DY] = [CYX] \; Diag(M,N,[dy]) [CXY]. \tag{A22}$$

Similarly, let [dy2] and [dy3] be matrices which approximate second and third derivatives of g(y) on G_y. Then, replacing [dy] in (A22) by [dy2] or [dy3] leads to the matrices [DY2] and [DY3], respectively, which approximate second and third partial derivatives of u(x,y) with respect to y on G.

Matrices which evaluate quantities at the boundaries x = A and y = B and at the corner points (A,0) and (0,B) are the final items needed to construct the matrices [G] and [H] in the matrix reformulation of (1.1) and (1.2). The NM-by-NM boundary matrix [BA] is such that [BA]{u} gives values of u(A,y) at points of the rectangular grid G. In particular,

$$[BA]\{u\} = \left(u(A,y_1), \cdots, u(A,y_1), u(A,y_2), \cdots, \right.$$

$$\left. u(A,y_2), \cdots, u(A,B), \cdots, u(A,B)\right)^T.$$

(A23)

[BA] may be written as a stack of M, N-by-NM matrices. If $[ba_j]$ is the j-th matrix in this stack (j = 1,...,M), then the element in the i-th row (i =1,...,N) and k-th column of $[ba_j]$ is unity if k = jN and zero otherwise. Similarly, the NM-by-NM matrix [BB] is such that [BB]{u} gives values of u(x,B) at points of G, i.e.,

$$[BB]\{u\} = \left(u(x_1,B), \cdots, u(x_N,B), \cdots, u(x_1,B), \cdots, u(x_N,B)\right)^T. \quad (A24)$$

[BB] may also be written as a stack of M, N-by-NM matrices.

$$[BB] = \begin{bmatrix} [bb] \\ \vdots \\ [bb] \end{bmatrix}. \quad (A25)$$

However, each of the M matrices in the stack (A25) is identical. The right-hand block of N columns of [bb] is simply the N-by-N identity matrix. All other elements of [bb] are zero.

The matrices [BA0] and [BB0] evaluate quantities at the corner points (A,0) and (0,B), respectively. Both of these NM-by-NM matrices consist of a single non-zero column which contains all ones. For [BA0], the N-th column is non-zero. The (NM-N+1)-st column is non-zero in the case of [BB0].

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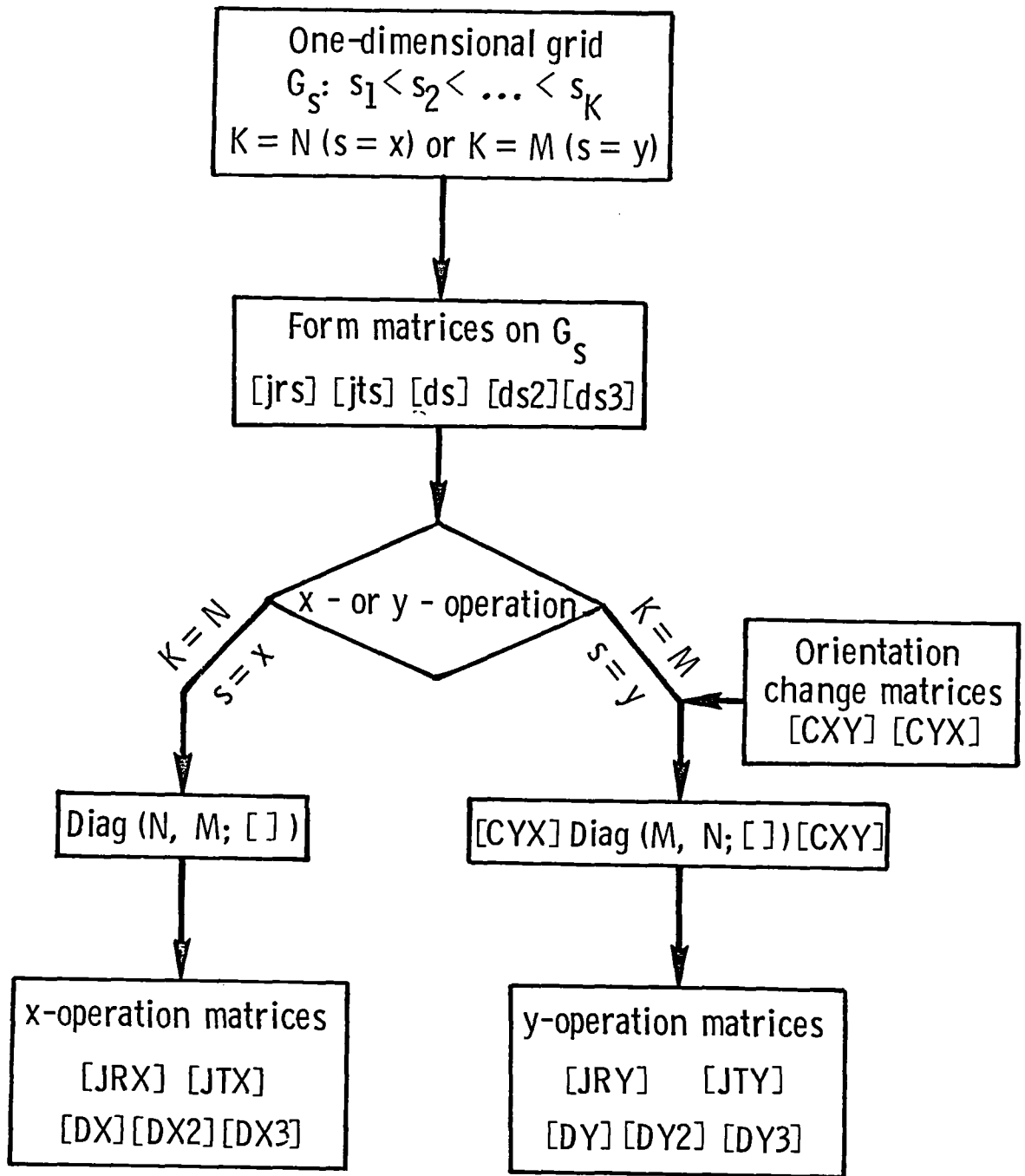
Figure 1. Flowchart illustrating the formation of matrices which approximate integrals and partial derivatives with respect to x and y on the rectangular grid G.

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16. Abstract

Integrating and differentiating matrices allow the numerical integration and differentiation of functions whose values are known at points of a discrete grid. Previous derivations of these matrices have been restricted to one-dimensional grids or to rectangular grids with uniform spacing in at least one direction. The present work develops integrating and differentiating matrices for grids with non-uniform spacing in both directions. The use of these matrices as operators to reformulate boundary value problems on rectangular domains as matrix problems for a finite-dimensional solution vector is considered. The method requires non-uniform grids which include "near-boundary" points. An eigenvalue problem for the transverse vibrations of a simply supported rectangular plate is solved to illustrate the method.

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