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A common fixed point theorem for cyclic operators on partial metric spaces

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Abstract. In this paper, we prove a common fixed point theorem for two self-mappings satisfying certain conditions over the class of partial metric spaces. In particular, the main theorem of this manuscript extends some well-known fixed point theorems in the literature on this topic.

1. Introduction

Recently, studies on the existence and uniqueness of fixed points of self-mappings on partial metric spaces have gained momentum (see e.g., [1] - [4],[7], [14]-[**?**],[26, 33]). The idea of partial metric space, a generalization of metric space, was introduced by Mathews [25] in 1992. When compared to metric spaces, the innovation of partial metric spaces is that the self distance of a point is not necessarily zero [24]. This feature of partial metrics makes them suitable for many purposes of semantics and domain theory in computer sciences. In particular, partial metric spaces have applications on the *Scott-Strachey order-theoretic topological models* [32] used in the logics of computer programs.

Mathews [25] proved the analog of Banach contraction mapping principle in the class of partial metric spaces. This remarkable paper of Mathews [25] constructed another important bridge between the domain theory in computer science and fixed point theory in mathematics. Thus, it becomes feasible to transform the tools from Mathematics to Computer Science.

A self-mapping T on a metric space X is called contraction if there exists a constant $k \in [0, 1)$ such that $d(Tx, Ty) \leq kd(x, y)$ for each $x, y \in X$. Banach contraction mapping principle, which states that a contraction has a fixed point, is one of the most important result in nonlinear analysis. This crucial result has been studied continuously since it was first published (See e.g. [1]-[23],[26]-[30]). As a generalization of this fundamental principle, Kirk-Srinivasan-Veeramani [23] developed the cyclic contraction. A contraction $T : A \cup B \to A \cup B$ on non-empty set A, B is called cyclic if $T(A) \subset B$ and $T(B) \subset A$ hold for closed subsets A, B of a complete metric space X. In the last decade, many authors (see e.g.[21, 22, 27–29, 34]) reported some fixed point theorems for cyclic operators.

Rus [29] introduced the following definition which is a further generalization of a cyclic mapping.

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Definition 1.1. Let X be a nonempty set, m be a positive integer and $T : X \to X$ be a mapping. $X = \cup_{i=1}^{m} A_i$ is said to be a *cyclic representation of* X with respect to T if

(i) $A_i, i = 1, 2, \cdots, m$ are nonempty sets.

(ii) $T(A_1) \subset A_2, \cdots, T(A_{m-1}) \subset A_m, T(A_m) \subset A_1$.

Remark 1.2. For convenience, we denote by $\mathcal{F}$ the class of functions $\phi : [0, \infty) \to [0, \infty)$ nondecreasing and continuous satisfying $\phi(t) > 0$ for $t \in (0, \infty)$ and $\phi(0) = 0$.

We recall the following definition.

Definition 1.3. (See e.g.[**?**]) Let (X, d) be a metric space, m be a positive integer, $A_1, A_2, \cdots, A_m$ be nonempty subsets of X and $X = \cup_{i=1}^{m} A_i$. An operator $T : X \to X$ is a *cyclic weak* $(\phi - \psi)$*-contraction* if

(i) $X = \cup_{i=1}^{m} A_i$ is a cyclic representation of X with respect to T,

(ii) $\phi(d(Tx, Ty)) \leq \phi(d(x, y)) - \psi(d(x, y))$, for any $x \in A_i$, $y \in A_{i+1}$, $i = 1, 2, \cdots, m$, where $A_{m+1} = A_1$ and $\phi, \psi \in \mathcal{F}$.

The main result of [22] is the following.

Theorem 1.4. (Theorem 6 of [22]) *Let (X, d) be a complete metric space, m be a positive integer, $A_1, A_2, \cdots, A_m$ be nonempty subsets of X and $X = \cup_{i=1}^{m} A_i$. Let $T : X \to X$ be a cyclic $(\phi - \psi)$-contraction with $\phi, \psi \in \mathcal{F}$. Then T has a unique fixed point $z \in \cap_{i=1}^{m} A_i$.*

In this paper, we proved a common fixed point of two self-mappings $T, g : X \to X$ on a partial metric space X under certain conditions.

We start some definitions and results needed in the sequel.

A partial metric on a nonempty set X is a mapping $p : X \times X \to [0, \infty)$ such that

$(PM1)$ $x = y$ if and only if $p(x, x) = p(x, y) = p(y, y)$,

$(PM2)$ $p(x, x) \leq p(x, y)$,

$(PM3)$ $p(x, y) = p(y, x)$,

$(PM4)$ $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

for all $x, y, z \in X$. A pair (X, p) is said to be partial metric space.

Notice also that if p is a partial metric on X, then the functions $d_p, d_m : X \times X \to \mathbb{R}^+$ given by

$$d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y), \tag{1}$$

$$p(x, y) - p(x, x), p(x, y) - p(y, y) \tag{2}$$

are equivalent (usual) metrics on X. For details see e.g.[**?**].

Example 1.5. (See e.g. [1, 3, 20, 24]) Consider $X = [0, \infty)$ with $p(x, y) = \max\{x, y\}$. Then (X, p) is a partial metric space. It is clear that p is not a (usual) metric. Note that in this case $d_p(x, y) = |x - y|$.

Example 1.6. (See e.g. [24]) Let $X = \{[a, b] : a, b, \in \mathbb{R},\ a \leq b\}$ and define $p([a, b], [c, d]) = \max\{b, d\} - \min\{a, c\}$. Then (X, p) is a partial metric spaces.

Lemma 1.7. (See e.g. [14, 15]) *Let (X, p) be a PMS. Then*

(A) If $p(x, y) = 0$ then $x = y$,
(B) If $x \neq y$, then $p(x, y) > 0$.

Example 1.8. (See e.g.[?]) Let (X, d) and (X, p) be a metric space and a partial metric space, respectively. Mappings $p_i : X \times X \longrightarrow [0, \infty)$ $(i \in \{1, 2, 3\})$ defined by

$$\begin{aligned} p_1(x, y) &= d(x, y) + p(x, y) \\ p_2(x, y) &= d(x, y) + \max\{\omega(x), \omega(y)\} \\ p_3(x, y) &= d(x, y) + a \end{aligned}$$

induce partial metrics on X, where $\omega : X \longrightarrow [0, \infty)$ is an arbitrary function and $a \geq 0$.

We notice also that each partial metric p on X generates a T_0 topology τ_p on X which has a family of open p–balls

$$\{B_p(x, \varepsilon) : x \in X, \varepsilon > 0\},$$

as a base where $B_p(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

Definition 1.9. (See e.g. [24]) Let (X, p) be a partial metric space.

(i) A sequence $\{x_n\}$ in X *converges to* $x \in X$ whenever $\lim_{n\to\infty} p(x, x_n) = p(x, x)$,

(ii) A sequence $\{x_n\}$ in X is called *Cauchy* whenever $\lim_{n,m\to\infty} p(x_n, x_m)$ exists (and finite),

(iii) (X, p) is said to be *complete* if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$, that is, $\lim_{n,m\to\infty} p(x_n, x_m) = p(x, x)$.

We define $L(x_n) = \{x | x_n \to x\}$ where $\{x_n\}$ is a sequence in a partial metric space (X, p). The example below shows that a convergent sequence $\{x_n\}$ in a partial metric space may not be a Cauchy. In particular, it shows that the limit of a convergent sequence is not unique.

Example 1.10. (See e.g.[?]) Let $X = [0, \infty)$ and $p(x, y) = \max\{x, y\}$. Let

$$x_n = \begin{cases} 0, & n = 2k, \\ 1, & n = 2k + 1. \end{cases}$$

Then clearly it is convergent sequence and for every $x \geq 1$ we have $\lim_{n\to\infty} p(x_n, x) = p(x, x)$, therefore $L(x_n) = [1, \infty)$. But $\lim_{n,m\to\infty} p(x_n, x_m)$ does not exist.

We state a lemma that shows the limit of a convergent sequence $\{x_n\}$ in a partial metric space is unique.

Lemma 1.11. (See e.g.[?]) *Let $\{x_n\}$ be a convergent sequence in partial metric space X such that $x_n \to x$ and $x_n \to y$. If*

$$\lim_{n\to\infty} p(x_n, x_n) = p(x, x) = p(y, y),$$

then $x = y$.

Lemma 1.12. (See e.g.[?]) *Let $\{x_n\}$ and $\{y_n\}$ be two sequences in partial metric space X such that*

$$\lim_{n\to\infty} p(x_n, x) = \lim_{n\to\infty} p(x_n, x_n) = p(x, x),$$

and

$$\lim_{n\to\infty} p(y_n, y) = \lim_{n\to\infty} p(y_n, y_n) = p(y, y),$$

then $\lim_{n\to\infty} p(x_n, y_n) = p(x, y)$. In particular, $\lim_{n\to\infty} p(x_n, z) = p(x, z)$ for every $z \in X$.

Lemma 1.13. (See e.g. [24],[26]) *Let (X, p) be a partial metric space.*

(*a*) *$\{x_n\}$ is a Cauchy sequence in (X, p) if and only if it is a Cauchy sequence in the metric space (X, d_p).*

(*b*) *A partial metric space (X, p) is complete if and only if the metric space (X, d_p) is complete. Furthermore, $\lim_{n\to\infty} d_p(x_n, x) = 0$ if and only if*

$$p(x, x) = \lim_{n\to\infty} p(x_n, x) = \lim_{n,m\to\infty} p(x_n, x_m).$$

Lemma 1.14. (See e.g.[**?**]) *If $\{x_n\}$ is a convergent sequence in (X, d_p), then it is a convergent sequence in the partial metric space (X, p).*

In this paper, we prove a common fixed point theorem on the class of the partial metric spaces as a generalization of Theorem 1.4 and the main theorem of [31].

2. Main Result

We start this section with the following definition for two self-mappings $T, g : X \to X$.

Definition 2.1. Let X be a nonempty set, m be a positive integer and $T, g : X \to X$ be two mappings. $X = \cup_{i=1}^m A_i$ is said to be a *cyclic representation of X with respect to* $(T - g)$ if

(i) $A_i, i = 1, 2, \cdots, m$ are nonempty sets.

(ii) $T(A_1) \subset g(A_2), \cdots, T(A_{m-1}) \subset g(A_m), T(A_m) \subset g(A_1)$.

Definition 2.2. Let (X, p) be a partial metric space, m be a positive integer, $A_1, A_2, \cdots, A_m$ be nonempty subsets of X and $X = \cup_{i=1}^m A_i$. Two operators $T, g : X \to X$ are *cyclic* $(\phi - \psi)$*-contraction* if

(i) $X = \cup_{i=1}^m A_i$ is a cyclic representation of X with respect to $(T - g)$,

(ii) $\phi(p(Tx, Ty)) \leq \phi(p(gx, gy)) - \psi(p(gx, gy))$, for any $x \in A_i$, $y \in A_{i+1}$, $i = 1, 2, \cdots, m$, where $A_{m+1} = A_1$ and $\phi, \psi \in \mathcal{F}$.

Our main result is the following.

Theorem 2.3. *Let (X, p) be a complete partial metric space, m be a positive integer, $A_1, A_2, \cdots, A_m$ be nonempty subsets of X and $X = \cup_{i=1}^m A_i$. Let $T, g : X \to X$ be two cyclic $(\phi - \psi)$-contraction such that $g(A_i)$ closed subsets of X.*
i) If g is one to one then there exists $z \in \cap_{i=1}^m A_i$ such that $gz = Tz$.
ii) If the pair (T, g) are weakly compatible,
then T and g has a unique common fixed point $z \in \cap_{i=1}^m A_i$.

Proof. Let x_1 be an arbitrary point in A_1. By cyclic representation of X with respect to pair (T, g), we choose a point x_2 in A_2 such that $Tx_1 = gx_2$. For this point x_2 there exists a point x_3 in A_3 such that $Tx_2 = gx_3$, and so on. Continuing in this manner we can define a sequence $\{x_n\}$ as follows

$$Tx_n = gx_{n+1},$$

for $n = 1, 2, \cdots$. We prove that $\{gx_n\}$ is a Cauchy sequence. If there exists $n_0 \in \mathbb{N}$ such that $gx_{n_0+1} = gx_{n_0}$ then, since $gx_{n_0+1} = Tx_{n_0} = gx_{n_0}$, the part of existence of the coincidence point of T and g is proved. Suppose that $gx_{n+1} \neq gx_n$ for any $n = 1, 2, \cdots$. Then, since $X = \cup_{i=1}^m A_i$, for any $n > 0$ there exists $i_n \in \{1, 2, \cdots, m\}$ such that $x_{n-1} \in A_{i_n}$ and $x_n \in A_{i_{n+1}}$. Since (T, g) are cyclic $(\phi - \psi)$-contraction, we have

$$\begin{aligned} \phi(p(gx_n, gx_{n+1})) &= \phi(p(Tx_{n-1}, Tx_n)) \\ &\leq \phi(p(gx_{n-1}, gx_n)) - \psi(p(gx_{n-1}, gx_n)) \\ &\leq \phi(p(gx_{n-1}, gx_n)) \end{aligned} \quad (3)$$

From (3) and taking into account that ϕ is nondecreasing we obtain

$$p(gx_n, gx_{n+1}) \leq p(gx_{n-1}, gx_n) \text{ for any } n = 2, 3, \cdots$$

Thus $\{p(gx_n, gx_{n+1})\}$ is a nondecreasing sequence of nonnegative real numbers. Consequently, there exists $\gamma \geq 0$ such that $\lim_{n\to\infty} p(gx_n, gx_{n+1}) = \gamma$. Taking $n \to \infty$ in (3) and using the continuity of ϕ and ψ, we have

$$\phi(\gamma) \leq \phi(\gamma) - \psi(\gamma) \leq \phi(\gamma),$$

and, therefore, $\psi(\gamma) = 0$. Since $\psi \in \mathcal{F}$, $\gamma = 0$, that is,

$$\lim_{n\to\infty} p(gx_n, gx_{n+1}) = 0.$$

Since $p(gx_n, gx_n) \leq p(gx_n, gx_{n+1})$ and $p(gx_{n+1}, gx_{n+1}) \leq p(gx_n, gx_{n+1})$, hence

$$\lim_{n\to\infty} p(gx_n, gx_n) = \lim_{n\to\infty} p(gx_{n+1}, gx_{n+1}) = \lim_{n\to\infty} p(gx_n, gx_{n+1}) = 0. \tag{4}$$

Since

$$d_p(gx_n, gx_{n+1}) = 2p(gx_n, gx_{n+1}) - p(gx_n, gx_n) - p(gx_{n+1}, gx_{n+1}).$$

This shows that $\lim_{n\to\infty} d_p(gx_n, gx_{n+1}) = 0$.

In the sequel, we prove that $\{gx_n\}$ is a Cauchy sequence in the metric space (X, d_p).

First, we prove the following claim.

Claim: For every $\epsilon > 0$ there exists $n \in \mathbb{N}$ such that if $b, q \geq n$ with $b - q \equiv 1(m)$ then $d_p(x_b, x_q) < \epsilon$.

In fact, suppose the contrary case. This means that there exists $\epsilon > 0$ such that for any $n \in \mathbb{N}$ we can find $b_n > q_n \geq n$ with $b_n - q_n \equiv 1(m)$ satisfying

$$d_p(gx_{q_n}, gx_{b_n}) \geq \epsilon. \tag{5}$$

Now, we take $n > 2m$. Then, corresponding to $q_n \geq n$ use can choose b_n in such a way that it is the smallest integer with $b_n > q_n$ satisfying $b_n - q_n \equiv 1(m)$ and $d_p(gx_{q_n}, gx_{b_n}) \geq \epsilon$. Therefore, $d_p(gx_{q_n}, gx_{b_{n-m}}) \leq \epsilon$. Using the triangular inequality

$$\epsilon \leq d_p(gx_{q_n}, gx_{b_n}) \leq d_p(gx_{q_n}, gx_{b_{n-m}}) + \sum_{i=1}^{m} d_p(gx_{b_{n-i}}, gx_{b_{n-i+1}}) < \epsilon + \sum_{i=1}^{m} d_p(gx_{b_{n-i}}, gx_{b_{n-i+1}}).$$

Letting $n \to \infty$ in the last inequality and taking into account that $\lim_{n\to\infty} d_p(gx_n, gx_{n+1}) = 0$, we obtain

$$\lim_{n\to\infty} d_p(gx_{q_n}, gx_{b_n}) = \epsilon \Longrightarrow \lim_{n\to\infty} p(gx_{q_n}, gx_{b_n}) = \frac{\epsilon}{2} \tag{6}$$

Again, by the triangular inequality

$$\begin{aligned}
\epsilon &\leq d_p(gx_{q_n}, gx_{b_n}) \\
&\leq d_p(gx_{q_n}, gx_{q_{n+1}}) + d_p(gx_{q_{n+1}}, gx_{b_{n+1}}) + d_p(gx_{b_{n+1}}, gx_{b_n}) \\
&\leq d_p(gx_{q_n}, gx_{q_{n+1}}) + d_p(gx_{q_{n+1}}, gx_{q_n}) \\
&+ d_p(gx_{q_n}, gx_{b_n}) + d_p(gx_{b_n}, gx_{b_{n+1}}) + d_p(gx_{b_{n+1}}, gx_{b_n}) \\
&= 2d_p(gx_{q_n}, gx_{q_{n+1}}) + d_p(gx_{q_n}, gx_{b_n}) + 2d_p(gx_{b_n}, gx_{b_{n+1}})
\end{aligned} \tag{7}$$

Letting $n \to \infty$ in (6) and taking into account that $\lim_{n\to\infty} d_p(gx_n, gx_{n+1}) = 0$ and (6), we get

$$\lim_{n\to\infty} d_p(gx_{q_{n+1}}, gx_{b_{n+1}}) = \epsilon.$$

Hence

$$\lim_{n\to\infty} p(gx_{q_{n+1}}, gx_{b_{n+1}}) = \frac{\epsilon}{2}. \tag{8}$$

Since gx_{q_n} and gx_{b_n} lie in different adjacently labeled sets A_i and A_{i+1} for certain $1 \le i \le m$, using the fact that T and g are cyclic $(\phi - \psi)$-contraction, we have

$$\begin{aligned} \phi(p(gx_{q_{n+1}}, gx_{b_{n+1}})) &= \phi(p(Tx_{q_n}, Tx_{b_n}) \\ &\le \phi(p(gx_{q_n}, gx_{b_n})) - \psi(p(gx_{q_n}, gx_{b_n})) \\ &\le \phi(p(gx_{q_n}, gx_{b_n})). \end{aligned}$$

Taking into account (6) and (8) and the continuity of ϕ and ψ, letting $n \to \infty$ in the last inequality, we obtain

$$\phi(\frac{\epsilon}{2}) \le \phi(\frac{\epsilon}{2}) - \psi(\frac{\epsilon}{2}) \le \phi(\frac{\epsilon}{2})$$

and consequently, $\psi(\frac{\epsilon}{2}) = 0$. Since $\psi \in \mathcal{F}$, then $\epsilon = 0$ which is contradiction. Therefore, our claim is proved.

In the sequel, we will prove that $\{gx_n\}$ is a Cauchy sequence in metric space (X, d_p). Fix $\epsilon > 0$. By the claim, we find $n_0 \in \mathbb{N}$ such that if $b, q \ge n_0$ with $b - q \equiv 1(m)$

$$d_p(gx_b, gx_q) \le \frac{\epsilon}{2}. \tag{9}$$

Since $\lim\limits_{n\to\infty} d_p(gx_n, gx_{n+1}) = 0$ we also find $n_1 \in \mathbb{N}$ such that

$$d_p(gx_n, gx_{n+1}) \le \frac{\epsilon}{2m} \tag{10}$$

for any $n \ge n_1$.

Suppose that $r, s \ge \max\{n_0, n_1\}$ and $s > r$. Then there exists $k \in \{1, 2, \cdots, m\}$ such that $s - r \equiv k(m)$. Therefore, $s - r + j \equiv 1(m)$ for $j = m - k + 1$. So, we have

$$d_p(gx_r, gx_s) \le d_p(gx_r, gx_{s+j}) + d_p(gx_{s+j}, gx_{s+j-1}) + \cdots + d_p(gx_{s+1}, gx_s).$$

By (9) and (10) and from the last inequality, we get

$$d_p(gx_r, gx_s) \le \frac{\epsilon}{2} + j\frac{\epsilon}{2m} \le \frac{\epsilon}{2} + m\frac{\epsilon}{2m} = \epsilon.$$

This proves that $\{gx_n\}$ is a Cauchy sequence in metric space (X, d_p). Since (X, p) is complete then from Lemma 1.13, the sequence $\{gx_n\}$ converges in the metric space (X, d_p), say $\lim\limits_{n\to\infty} d_p(gx_n, x) = 0$ for some $x \in X$. Therefore, by Lemma 1.13 we have

$$p(x, x) = \lim_{n\to\infty} p(gx_n, x) = \lim_{n,m\to\infty} p(gx_n, gx_m).$$

That is, there exists $x \in X$ such that $\lim_{n\to\infty} gx_n = x$ in partial metric (X, p). Since $g(A_i)$ are closed subsets of X, we have $x \in g(A_i)$ for every $i \in \{1, 2, \cdots, m\}$. That is, $x \in \cap_{i=1}^m g(A_i)$. Hence, there exists $z_i \in A_i$ such that $gz_i = x$. Since g is one to one we have

$$g(z_1) = g(z_2) = \cdots = g(z_m) = x \Longrightarrow z_1 = z_2 = \cdots = z_m = z.$$

Therefore, $g(z) = x$ for $z \in \cap_{i=1}^m A_i$. In fact, $\lim\limits_{n\to\infty} gx_n = gz$.On the other hand since the sequence $\{gx_n\}$ has infinite terms in each A_i for $i \in \{1, 2, \cdots, m\}$, we take a subsequence $\{gx_{n_k}\}$ of $\{gx_n\}$ with $gx_{n_k} \in g(A_{i-1})$ where $x_{n_k} \in A_{i-1}$. Using the contractive condition, we can obtain

$$\begin{aligned} \phi(p(gx_{n_{k+1}}, Tz)) &= \phi(p(Tx_{n_k}, Tz)) \\ &\le \phi(p(gx_{n_k}, gz)) - \psi(p(gx_{n_k}, gz)) \\ &\le \phi(p(gx_{n_k}, gz)). \end{aligned}$$

Since $gx_{n_k} \to gz$ and ϕ and ψ belong to $\mathcal{F}$, letting $k \to \infty$ in the last inequality, we have

$$\phi(p(gz, Tz)) \leq \phi(p(gz, gz)) - \psi(p(gz, gz)) \leq \phi(p(gz, gz)).$$

Moreover, we obtain $p(gz, Tz) = p(gz, gz)$, because ϕ is nondecreasing and $p(gz, gz) \leq p(gz, Tz)$. Hence, if $p(gz, gz) \neq 0$ then by the last inequality we have,

$$\begin{aligned} \phi(p(gz, gz)) &= \phi(p(gz, Tz)) \\ &\leq \phi(p(gz, gz)) - \psi(p(gz, gz)) \\ &< \phi(p(gz, gz)), \end{aligned}$$

which is contradiction. Since $\phi \in \mathcal{F}$, then, $p(Tz, Tz) = p(gz, gz) = p(gz, Tz) = 0$, it follows that, $Tz = gz = x$.

ii) Since g and T are two weakly compatible mappings, we have $TTz = Tgz = gTz = ggz$. That is $Tx = gx$. Next, we prove that $Tx = x$. Since $Tz \in X$ hence there exists some i such that $Tz \in A_i$. By $z \in \cap_{i=1}^{m} A_i$ we have $z \in A_{i-1}$, by using the contractive condition we obtain

$$\begin{aligned} \phi(p(Tz, TTz)) &\leq \phi(p(gz, gTz)) - \psi(p(gz, gTz)) \\ &\leq \phi(p(gz, gTz)) = \phi(p(Tz, TTz)), \end{aligned}$$

from the last inequality we have

$$\psi(p(Tz, TTz)) = 0.$$

Since $\psi \in \mathcal{F}$, $p(Tz, TTz) = 0$ and, consequently, $x = Tz = TTz = Tx = gx$.

Finally, in order to prove the uniqueness of a fixed point, we have $y, z \in X$ with y and z common fixed points of T and g. The cyclic character of $T - g$ and the fact that $y, z \in X$ are common fixed points of T and g, imply that $y, z \in \cap_{i=1}^{m} A_i$. If $p(y, z) \neq 0$ then by using the contractive condition we obtain

$$\begin{aligned} \phi(p(y, z)) &= \phi(p(Ty, Tz)) \leq \phi(p(gy, gz)) - \psi(p(gy, gz)) \\ &< \phi(p(gy, gz)) = \phi(p(y, z)), \end{aligned}$$

which is a contradiction. Since $\phi \in \mathcal{F}$, $p(y, z) = 0$ and, consequently, $y = z$. This finishes the proof. □

Corollary 2.4. *Let (X, p) be a complete partial metric space, m be a positive integer, $A_1, A_2, \cdots, A_m$ be nonempty closed subsets of X and $X = \cup_{i=1}^{m} A_i$. Let $T : X \to X$ be a cyclic weak $(\phi - \psi)$-contraction. Then T has a unique fixed point $z \in \cap_{i=1}^{m} A_i$.*

Proof. Take $g(x) = x$ in Theorem 2.3. □

Corollary 2.5. *Let (X, p) be a complete partial metric space, m be a positive integer, $A_1, A_2, \cdots, A_m$ be nonempty closed subsets of X. Suppose that $T : X \to X$ is a self-mapping and $X = \cup_{i=1}^{m} A_i$ is a cyclic representation of X with respect to T. Further, T satisfies $d(Tx, Ty) \leq d(x, y) - \psi(d(x, y))$, for any $x \in A_i$, $y \in A_{i+1}$, $i = 1, 2, \cdots, m$, where $A_{m+1} = A_1$ and $\psi \in \mathcal{F}$. Then T has a unique fixed point $z \in \cap_{i=1}^{m} A_i$.*

Proof. Take $\phi(t) = t$ in Corollary 2.4. □

Example 2.6. Let $X = [0, 1]$ and $g, T : X \to X$ such that $Tx = \frac{x^2}{12}$ and $gx = \frac{x}{3}$. Suppose that $\psi, \phi : [0, \infty) \to [0, \infty)$ are defined as follows $\psi(t) = \frac{t}{2}$ and $\psi(t) = \frac{t}{3}$. For $A_i = [0, 1]$, $(i = 1, 2, ..., m)$ all conditions of Theorem 2.3 are satisfied. It is clear that $x = 0$ is the common fixed point of T and g.

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