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A Commutative Algebra for Oriented Matroids*

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Abstract. Let V be a vector space of dimension d over a field $\mathbb{K}$ and let $\mathcal{A}$ be a central arrangement of hyperplanes in V. To answer a question posed by K. Aomoto, P. Orlik and H. Terao construct a commutative $\mathbb{K}$-algebra $\mathbb{U}(\mathcal{A})$ in terms of the equations for the hyperplanes of $\mathcal{A}$. In the course of their work the following question naturally occurred:

- Is $\mathbb{U}(\mathcal{A})$ determined by the intersection lattice $L(\mathcal{A})$ of the hyperplanes of $\mathcal{A}$?

We give a negative answer to this question. The theory of oriented matroids gives rise to a combinatorial analogue of the algebra of Orlik–Terao, which is the main tool of our proofs.

1. Introduction

Let $\mathcal{M} = \mathcal{M}([n])$ (resp. $\boldsymbol{\mathcal{M}} = \boldsymbol{\mathcal{M}}([n])$) denote a matroid (resp. oriented matroid) of rank r with ground set $[n] := \{1, 2, \ldots, n\}$. Let V be a vector space of dimension d over some field $\mathbb{K}$. A (central) arrangement (of hyperplanes) in V, $\mathcal{A}_{\mathbb{K}} = \{H_1, \ldots, H_n\}$, is a finite listed set of codimension one vector subspaces. Given an arrangement $\mathcal{A}_{\mathbb{K}}$ we suppose always chooses a family of linear forms $\{\theta_{H_i} \in V^*\colon\ H_i \in \mathcal{A}_{\mathbb{K}}, \mathrm{Ker}(\theta_{H_i}) = H_i\}$, where V^* denotes the dual space of V. The product $Q(\mathcal{A}) = \prod_{H\in\mathcal{A}} \theta_H$ is called the *defining polynomial* of $\mathcal{A}$. There is a matroid $\mathcal{M}(\mathcal{A}_{\mathbb{K}})$ on the ground set $[n]$ determined by $\mathcal{A}_{\mathbb{K}}$: a subset $D \subset [n]$ is a *dependent set* of $\mathcal{M}(\mathcal{A}_{\mathbb{K}})$ iff there are scalars $\zeta_i \in \mathbb{K}$, $i \in D$, not all nulls, such that $\sum_{i\in D} \zeta_i\theta_{H_i} = 0$. A *circuit* is a minimal dependent set with respect to inclusion. We denote by $L(\mathcal{A}_{\mathbb{K}})$ the *intersection lattice of* $L(\mathcal{A}_{\mathbb{K}})$: i.e., the set of intersections of hyperplanes in $\mathcal{A}_{\mathbb{K}}$, partially ordered by reverse inclusion. Set $\mathfrak{M}(\mathcal{A}_{\mathbb{K}}) = V \setminus \bigcup_{H\in\mathcal{A}_{\mathbb{K}}} H$.

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Aomoto suggested the study of the (graded) $\mathbb{K}$-vector space $\mathrm{AO}(\mathcal{A}_{\mathbb{K}})$, generated by the basis $\{Q(\mathcal{B}_{\mathrm{I}})^{-1}\}$, where I is an independent set of $\mathcal{M}(\mathcal{A}_{\mathbb{K}})$, $\mathcal{B}_{\mathrm{I}} := \{H_i \in \mathcal{A}_{\mathbb{K}}\colon i \in I\}$ and $Q(\mathcal{B}_{\mathrm{I}}) = \prod_{i\in I} \theta_{H_i}$ denotes the corresponding defining polynomial. In [1] it is conjectured that

$$\dim(\mathrm{AO}(\mathcal{A}_{\mathbb{R}})) = \text{number of chambers of } \mathfrak{M}(\mathcal{A}_{\mathbb{R}}).$$

To prove Aomoto's conjecture, Orlik and Terao have constructed in [8] a commutative $\mathbb{K}$-algebra, $\mathbb{U}(\mathcal{A}_{\mathbb{K}})$, isomorphic to $\mathrm{AO}(\mathcal{A}_{\mathbb{K}})$ as a graded $\mathbb{K}$-vector space in terms of the equations $\{\theta_H\colon H \in \mathcal{A}_{\mathbb{K}}\}$. The authors note that it is not clear whether $\mathbb{U}(\mathcal{A}_{\mathbb{K}})$ itself depends only on the intersection lattice $L(\mathcal{A}_{\mathbb{K}})$.

To every oriented matroid $\boldsymbol{\mathcal{M}}$ we associate a commutative $\mathbb{Z}$-algebra, denoted by $\mathbb{A}(\boldsymbol{\mathcal{M}})$. This algebra is the "combinatorial analogue" of the algebra of Orlik–Terao and it is the main tool to give a negative answer to the question of Orlik–Terao.

We use [9] and [10] as a general reference in matroid theory. We refer to [2] and [7] for good sources of the theory of oriented matroids and arrangements of hyperplanes, respectively.

2. Two Commutative Algebras

Let $\mathrm{IND}_\ell(\mathcal{M}) \subset \binom{[n]}{\ell}$ be the family of the independent sets of cardinal ℓ of the matroid $\mathcal{M}$ and set $\mathrm{IND}(\mathcal{M}) = \bigcup_{\ell\in\mathbb{N}} \mathrm{IND}_\ell(\mathcal{M})$. We denote by $\mathfrak{C} = \mathfrak{C}(\mathcal{M})$ the set of circuits of $\mathcal{M}$. When the smallest element α of a circuit C, $|C| > 1$, is deleted, the remaining set, $C\backslash\alpha$, is said to be a *broken circuit*. (Note that our definition is slightly different to the standard one. In the standard definition $C\backslash\alpha$ can be empty.) To shorten the notation the singleton set $\{x\}$ is denoted by x. A *no broken circuit* set of a matroid $\mathcal{M}$ is an independent subset of $[n]$ which does not contain any broken circuit. Let $\mathrm{NBC}_\ell(\mathcal{M}) \subset \binom{[n]}{\ell}$ be the set of the no broken circuit sets of cardinal ℓ of $\mathcal{M}$. Set $\mathrm{NBC}(\mathcal{M}) = \bigcup_{\ell\in\mathbb{N}} \mathrm{NBC}_\ell(\mathcal{M})$. We denote by $L(\mathcal{M})$ the lattice of flats of $\mathcal{M}$. (We remark that the lattice map $\varphi\colon L(\mathcal{A}_{\mathbb{K}}) \to L(\mathcal{M}(\mathcal{A}_{\mathbb{K}}))$, determined by the one-to-one correspondence $\varphi'\colon H_i \longleftrightarrow \{i\}$, $i = 1, \ldots, n$, is a lattice isomorphism.) Consider now an independent set X. Let $\mathrm{cl}_{\mathcal{M}}(X)$ be (or shortly $\mathrm{cl}(X)$) the closure of X in $\mathcal{M}$. Pick an element $x \in \mathrm{cl}(X)\backslash X$. Let $C(X, x)$ denote the unique circuit of $\mathcal{M}$ contained in $X \cup x$. For every $X \in \mathrm{IND}(\mathcal{M})$, set

$$\mathrm{EA}(X) := \{x \in \mathrm{cl}(X)\backslash X\colon x \text{ is the minimum of } C(X, x) \text{ and } C(X, x) \neq \{x\}\}.$$

(The elements of $\mathrm{EA}(X)$ are usually called the *externally active* elements of X.) So, for every independent set X of $\mathcal{M}$, $X \in \mathrm{NBC}(\mathcal{M})$ iff $\mathrm{EA}(X) = \emptyset$. If $\mathrm{EA}(X) \neq \emptyset$, let $\alpha(X)$ denote the smallest element of $\mathrm{EA}(X)$.

Here, every map $\boldsymbol{X}\colon [n] \to \{+1, -1, 0\} \subset \mathbb{Z}$ is called a *signed set* on $[n]$. Set $\boldsymbol{X}^+ = \{\ell \in [n]\colon \boldsymbol{X}(\ell) = +1\}$, $\boldsymbol{X}^- = \{\ell \in [n]\colon \boldsymbol{X}(\ell) = -1\}$. We say that $\underline{\boldsymbol{X}} := \boldsymbol{X}^+ \cup \boldsymbol{X}^-$ is the *support* of $\boldsymbol{X}$, or $\boldsymbol{X}$ is a signed set *supporting* $\underline{\boldsymbol{X}}$. A signed set $\boldsymbol{X}$ *conforms to* a signed set $\boldsymbol{Y}$ if $\boldsymbol{X}^+ \subset \boldsymbol{Y}^+$ and $\boldsymbol{X}^- \subset \boldsymbol{Y}^-$. For every $x \in [n]$ and signed set $\boldsymbol{X}$ let $\boldsymbol{X}\backslash x$ be the signed set on $[n]\backslash x$, conforming to $\boldsymbol{X}$ and supporting $\underline{\boldsymbol{X}}\backslash x$. We say that a signed set $\boldsymbol{X}$ is the *union* of the signed sets $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_m$ if $\underline{\boldsymbol{X}} = \underline{\boldsymbol{X}}_1 \cup \cdots \cup \underline{\boldsymbol{X}}_m$ and every $\boldsymbol{X}_i$ conforms to $\boldsymbol{X}$. The *reorientation on the subset* $S \subset [n]$ of the signed set $\boldsymbol{X}$ is the signed

set, denoted $_{-S}\boldsymbol{X}$, determined by the equalities

$$(_{-S}\boldsymbol{X})^+ := \{\boldsymbol{X}^+\backslash S\} \cup \{S \cap \boldsymbol{X}^-\} \quad \text{and} \quad (_{-S}\boldsymbol{X})^- := \{\boldsymbol{X}^-\backslash S\} \cup \{S \cap \boldsymbol{X}^+\}.$$

The *opposite* of a signed set $\boldsymbol{X}$, denoted $-\boldsymbol{X}$, is the signed set $-\boldsymbol{X} = {}_{-[n]}\boldsymbol{X}$. An *oriented matroid*, denoted $\boldsymbol{\mathcal{M}}$, is a matroid on the ground set $[n]$, denoted $\underline{\boldsymbol{\mathcal{M}}}$, with an additional structure:

- To every circuit $C \in \mathfrak{C}(\underline{\boldsymbol{\mathcal{M}}})$ is attached two opposite signed sets (*signed circuits*) $\boldsymbol{C}$ and $-\boldsymbol{C}$ supporting C.
- The set of signed circuits of $\boldsymbol{\mathcal{M}}$, denoted $\boldsymbol{\mathfrak{C}} = \boldsymbol{\mathfrak{C}}(\boldsymbol{\mathcal{M}})$, verifies a convenient set of axioms, see page 103 of [2].

The set of all the union of signed circuits of $\boldsymbol{\mathcal{M}}$ is called the set of the *vectors* of the oriented matroid. If $\mathbb{K}$ is an ordered field the arrangement $\mathcal{A}_{\mathbb{K}}$ determines an oriented matroid $\boldsymbol{\mathcal{M}}(\mathcal{A}_{\mathbb{K}})$ on the ground set $[n]$. Indeed let $C = \{i_1, \ldots, i_m\}$, $i_1 < \cdots < i_m$, be a circuit of $\mathcal{M}(\mathcal{A}_{\mathbb{R}})$. From the definitions we know that there are well determined scalars $\zeta_{i_j} \in \mathbb{K}^*$, $\zeta_{i_1} = 1$, such that $\sum_{j=1}^m \zeta_{i_j}\theta_{H_{i_j}} = 0$. Set $\boldsymbol{C}\colon [n] \to \{0, 1, -1\} \subset \mathbb{Z}$ the signed set

$$\boldsymbol{C}(\ell) = \begin{cases} +1 & \text{if} \quad \ell \in C \quad \text{and} \quad \lambda_\ell > 0, \\ -1 & \text{if} \quad \ell \in C \quad \text{and} \quad \lambda_\ell < 0, \\ 0 & \text{if} \quad \ell \notin C. \end{cases}$$

By definition $\boldsymbol{C}$ is one of the two opposite signed circuits of $\boldsymbol{\mathcal{M}}(\mathcal{A}_{\mathbb{K}})$ supporting C. Note that $\mathcal{M}(\mathcal{A}_{\mathbb{K}}) = \underline{\boldsymbol{\mathcal{M}}}(\mathcal{A}_{\mathbb{K}})$. The *reorientation* on the subset $S \subset [n]$, of the oriented matroid $\boldsymbol{\mathcal{M}}$, is the oriented matroid, denoted $_{-S}\boldsymbol{\mathcal{M}}$, such that $\boldsymbol{\mathfrak{C}}(_{-S}\boldsymbol{\mathcal{M}}) := \{_{-S}\boldsymbol{C}\colon \boldsymbol{C} \in \boldsymbol{\mathfrak{C}}(\boldsymbol{\mathcal{M}})\}$. We say also that $\boldsymbol{\mathfrak{C}}(_{-S}\boldsymbol{\mathcal{M}})$ is the *reorientation* on the subset S of $\boldsymbol{\mathfrak{C}}(\boldsymbol{\mathcal{M}})$. (The concept of "reorientation" is the combinatorial analogue of the notion of "nonsingular projective permissible transformation".)

Fix a set $E := \{e_1, \ldots, e_n\}$ and let K be a commutative ring with unity 1. Let K$[E]$ denote the commutative free K-algebra given by the generators $E \cup \{1\}$. For every $X \subset [n]$, set $e_X := \prod_{i \in X} e_i$, $e_\emptyset := 1$.

Definition 2.1 [8, Definition 2.2 and Proposition 2.3]. Consider the map

$$\partial\colon \mathfrak{C}(\mathcal{M}(\mathcal{A}_{\mathbb{K}})) \to \mathbb{K}[E], \qquad C \mapsto \sum_{j=1}^m \zeta_{i_j} e_{C\backslash i_j},$$

where $C = \{i_1, \ldots, i_m\}$, $i_1 < \cdots < i_m$, and $\sum_{j=1}^m \zeta_{i_j}\theta_{H_{i_j}} = 0$, $\zeta_{i_1} = 1$. $\mathbb{U}(\mathcal{A}_{\mathbb{K}})$ is the (commutative) $\mathbb{K}$-algebra given by the generators $1, e_1, \ldots, e_n$, and the relations:

- $e_i e_j = e_j e_i, \forall i, j = 1, \ldots, n$,
- $e_i^2 = 0, \forall i = 1, \ldots, n$,
- $\partial(C) = 0, \forall C \in \mathfrak{C}(\mathcal{M}(\mathcal{A}_{\mathbb{K}}))$.

We call $\mathbb{U}(\mathcal{A}_{\mathbb{K}})$ the *Orlik–Terao algebra of* $\mathcal{A}_{\mathbb{K}}$.

Now, we introduce the "combinatorial analogue" of the Orlik–Terao algebra.

Definition 2.2. Consider the map

$$\widetilde{\partial}\colon \mathfrak{C}(\mathcal{M}) \to \mathbb{Z}[E], \qquad C \mapsto \sum_{j=1}^{m} \boldsymbol{C}(i_j) e_{C\setminus i_j},$$

where $C = \{i_1, \ldots, i_m\}$, $i_1 < \cdots < i_m$, and $\boldsymbol{C} \in \mathfrak{C}(\mathcal{M})$ is the signed circuit supporting C, such that $\boldsymbol{C}(i_1) = 1$. $\mathbb{A}(\mathcal{M})$ is the (commutative) $\mathbb{Z}$-algebra given by the generators $1, e_1, \ldots, e_n$, and the relations:

- $e_i e_j = e_j e_i, \forall i, j = 1, \ldots, n,$
- $e_i^2 = 0, \forall i = 1, \ldots, n,$
- $e_i = 0,$ if i is a loop of $\mathcal{M}$,
- $\widetilde{\partial}(C) = 0, \forall C \in \mathfrak{C}(\mathcal{M}), \quad |C| > 1.$

We call $\mathbb{A}(\mathcal{M})$ the *algebra* of the oriented matroid $\mathcal{M}$.

For every $X \subset [n]$, we denote by $[X]_{\mathbb{A}}$ (resp. $[X]_{\mathbb{U}}$), or shortly by $[X]$ or even e_X when no confusion will result, the residue class in $\mathbb{A}(\mathcal{M})$ (resp. $\mathbb{U}(\mathcal{A}_{\mathbb{K}})$) determined by the element e_X. Note that $\mathbb{A}(\mathcal{M}) \cong \mathbb{A}(\mathcal{M}\backslash x)$ if x is a loop or x is parallel to some other element of $\mathcal{M}$. So in what follows we suppose that $\mathcal{M}$ is a simple matroid. For every circuit $C \in \mathfrak{C}(\mathcal{M})$, we have $[C]_{\mathbb{A}} = 0$. To see this, pick an element $x \in C$ if $|C| > 1$. Then $0 = e_x \cdot \widetilde{\partial}(e_C) = \pm e_C$. We conclude that if $[X]_{\mathbb{A}} \neq 0$, then X is an independent set of $\mathcal{M}$.

The "abstract algebra" $\mathbb{A}(\mathcal{M})$ has a canonical grading.

Proposition 2.3. *Set $\mathbb{A}_\ell = \mathbb{A}_\ell(\mathcal{M})$ be the submodule of $\mathbb{A}(\mathcal{M})$ generated by the elements $\{[X]_{\mathbb{A}}\colon X \in \mathrm{IND}_\ell(\mathcal{M})\}$. The grading $\mathbb{A}(\mathcal{M}) = \bigoplus_{\ell\in\mathbb{N}} \mathbb{A}_\ell(\mathcal{M})$ is canonical, i.e., it is independent of the knowledge of the oriented matroid $\mathcal{M}$.*

Proof. We know that $\mathbb{A}_\ell(\mathcal{M}) = (0)$, for all $\ell > r$. If $\mathbb{A}(\mathcal{M}) = \mathbb{A}_0 = \mathbb{Z}$ (i.e., $r = 0$) the result is clear. Suppose that $\mathbb{A}(\mathcal{M}) \neq \mathbb{Z}$. Note that $\mathbb{A}_r = \{x \in \mathbb{A}(\mathcal{M})\colon x \cdot y = 0, \ \forall y \in \mathbb{A}(\mathcal{M})\backslash\mathbb{Z}\}$. If we know the modules $\mathbb{A}_r, \ldots, \mathbb{A}_{r-i}$ and $\mathbb{A}^{\langle i+1\rangle} := \mathbb{A}_r \oplus \cdots \oplus \mathbb{A}_{r-i} \neq \mathbb{A}(\mathcal{M})$, (i.e., $r - i > 1$) the module $\mathbb{A}_{r-i-1}$, $i = 0, \ldots, r-2$, can be defined recursively by

$$\mathbb{A}_{r-i-1} = \{x \in \mathbb{A}(\mathcal{M})\colon x \cdot y \in \mathbb{A}^{\langle i+1\rangle}, \forall y \in \mathbb{A}(\mathcal{M})\backslash\mathbb{Z}\}\big/\mathbb{A}^{\langle i+1\rangle}. \qquad \square$$

Proposition 2.4. *For every $x \in [n]$ there is a unique epimorphism of $\mathbb{Z}$-modules $\mathfrak{p}_x\colon \mathbb{A}(\mathcal{M}) \to \mathbb{A}(\mathcal{M}/x)$, such that, for every $I \in \mathrm{IND}(\mathcal{M})$, we have*

$$\mathfrak{p}_x(e_I) := \begin{cases} e_{I\setminus x} & \text{if} \quad x \in I \in \mathrm{IND}(\mathcal{M}), \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

Proof. It is enough to prove that

$$\mathfrak{p}_x(e_X \cdot \widetilde{\partial}(C)) = 0, \qquad \forall X \subset [n], \quad \forall C \in \mathfrak{C}(\mathcal{M}). \tag{2.2}$$

We can suppose that $X \cap C = \emptyset$ and $x \in X \cup C$. Let $\boldsymbol{Y}$ be a signed set on $[n]$ supporting $\underline{\boldsymbol{Y}} = \{i_1, \ldots, i_m\}$. For convenience of notation set

$$\widetilde{\partial}_{\boldsymbol{Y}}(\underline{\boldsymbol{Y}}) := \sum_{j=1}^{m} \boldsymbol{Y}(i_j) e_{\underline{\boldsymbol{Y}}\setminus i_j} \in \mathbb{Z}[E].$$

Let $\boldsymbol{C}$ be one of the opposite signed circuits supporting C. Remember that the signed set $\boldsymbol{C}\setminus x$ on $[n]\setminus x$ is a vector (union of signed circuits) of $\boldsymbol{\mathcal{M}}/x$. So we have

$$\mathfrak{p}_x(e_X \cdot \widetilde{\partial}(C)) = \begin{cases} \pm e_X \cdot \widetilde{\partial}_{\boldsymbol{C}\setminus x}(C\setminus x) = 0 & \text{if} \quad x \in C, \\ \pm e_{X\setminus x} \cdot \widetilde{\partial}_{\boldsymbol{C}}(C) = 0 & \text{if} \quad x \in X. \end{cases}$$ □

Corollary 2.5. *For every subset $X = \{i_1, \ldots, i_m\} \subset [n]$, the following two conditions are equivalents*:

- *X is an independent set of $\boldsymbol{\mathcal{M}}$,*
- *$[X]_{\mathbb{A}} = e_{i_1} e_{i_2} \cdots e_{i_m} \neq 0$.*

Proof. It remains to prove that if X is an independent set of $\boldsymbol{\mathcal{M}}$, then $[X]_{\mathbb{A}} \neq 0$. We prove by induction on n. We know that $[\emptyset]_{\mathbb{A}} = 1$. Suppose that the implication is true for all the matroids with at most $n-1$ elements. Let X, $|X| > 0$, be an independent set of $\boldsymbol{\mathcal{M}}$ and pick an element $x \in X$. Suppose for a contradiction that $[X]_{\mathbb{A}} = 0$. $X\setminus x$ is an independent set of $\boldsymbol{\mathcal{M}}/x$. From Proposition 2.4 we conclude that $0 = \mathfrak{p}_x([X]_{\mathbb{A}}) = [X\setminus x]_{\mathbb{A}(\boldsymbol{\mathcal{M}}/x)}$, a contradiction with the induction hypothesis. □

Proposition 2.6. *For every $x \in [n]$ there is a unique morphism of $\mathbb{Z}$-modules, $\mathfrak{i}_x \colon \mathbb{A}(\boldsymbol{\mathcal{M}}\setminus x) \to \mathbb{A}(\boldsymbol{\mathcal{M}})$, such that, for every $I \in \mathrm{IND}(\boldsymbol{\mathcal{M}}\setminus x)$, we have $\mathfrak{i}_x(e_I) = e_I$.*

Proof. The map $\mathfrak{i}_x$ is well determined. Indeed from Corollary 2.5 we know that

$$\mathfrak{i}_x(e_I \cdot \widetilde{\partial}(C)) = e_I \cdot \widetilde{\partial}(C) = 0, \qquad \forall I \in \mathrm{IND}(\boldsymbol{\mathcal{M}}\setminus x), \quad \forall C \in \mathfrak{C}(\boldsymbol{\mathcal{M}}\setminus x).$$ □

Set $\boldsymbol{nbc}_\ell := \{[I]_{\mathbb{A}} \colon I \in \mathrm{NBC}_\ell(\boldsymbol{\mathcal{M}})\}$ and $\boldsymbol{nbc} := \bigcup_{\ell=0} \boldsymbol{nbc}_\ell$. Set $\boldsymbol{\mathcal{M}}' = \boldsymbol{\mathcal{M}}\setminus x$, $\boldsymbol{\mathcal{M}}'' = \boldsymbol{\mathcal{M}}/x$, $\mathbb{A} := \mathbb{A}(\boldsymbol{\mathcal{M}})$, $\mathbb{A}' := \mathbb{A}(\boldsymbol{\mathcal{M}}')$ and $\mathbb{A}'' := \mathbb{A}(\boldsymbol{\mathcal{M}}'')$. Consider an independent set $X \in \mathrm{IND}_\ell(\boldsymbol{\mathcal{M}})$, suppose that $X \notin \mathrm{NBC}_\ell(\boldsymbol{\mathcal{M}})$, and set $\alpha = \alpha(X)$. Making use of the definition of $\widetilde{\partial}(C(X,\alpha))$ we can express the element $[X]_{\mathbb{A}}$ as a linear combination of the elements $\{[X_x]_{\mathbb{A}} \colon X_x = X\setminus x \cup \alpha, x \in C(X,\alpha)\setminus\alpha\}$. We claim that $\mathrm{EA}(X_x) \subset \mathrm{EA}(X)\setminus\alpha$. Indeed, suppose that $\beta \in \mathrm{EA}(X_x)$. If $\alpha \notin C(X_x,\beta) = C(X,\beta)$, then we have $\beta \in \mathrm{EA}(X)\setminus\alpha$. If $\alpha \in C(X_x,\beta)$ we have $\beta < \alpha$ and from the elimination axiom for circuits there is a circuit C' such that $\beta \in C' \subset (C(X_x,\beta) \cup C(X,\alpha))\setminus\alpha$. So $C' = C(X,\beta)$, β is the smallest element $C(X,\beta)$ and the claim follows. By the iteration of this process we conclude that $[X]_{\mathbb{A}}$ can be expressed linearly in term of elements of $\boldsymbol{nbc}_\ell = \{[I]_{\mathbb{A}} \in \mathrm{IND}_\ell(\boldsymbol{\mathcal{M}}) \colon \mathrm{EA}(I) = \emptyset\}$. So $\boldsymbol{nbc}_\ell$ is a generating set of the $\mathbb{Z}$-module $\mathbb{A}_\ell(\boldsymbol{\mathcal{M}})$. Now we are able to prove the main results of this section.

Theorem 2.7. *For every element x of $\mathcal{M}$, there is a split short exact sequence of modules*

$$0 \to \mathbb{A}(\mathcal{M}\backslash x) \xrightarrow{\mathfrak{i}_x} \mathbb{A}(\mathcal{M}) \xrightarrow{\mathfrak{p}_x} \mathbb{A}(\mathcal{M}/x) \to 0. \tag{2.3}$$

We postpone the proof of the theorem. The following corollary is an important direct consequence of the above theorem. Corollary 2.8 is similar to a well-known result concerning the algebras of Orlik–Solomon, Theorem 3.55 of [7].

Corollary 2.8. *Suppose that the sequence* (2.3) *is exact for all the matroids with at most n elements. Then* $\boldsymbol{nbc}(\mathcal{M})$ *is a basis of the module* $\mathbb{A}(\mathcal{M})$.

Proof. We prove by induction on n. If $n = 0$ we know that $\mathbb{A}(\mathcal{M}(\emptyset)) = \mathbb{Z}$ and $\boldsymbol{nbc}(\mathcal{M}(\emptyset)) = \{1\}$. Suppose that $n > 0$ and that the result is true for all the matroids with at most $n-1$ elements. By a reordering of the elements of the matroid $\mathcal{M}$ we can suppose that $x = n$. It is clear that

$$\mathrm{NBC}(\mathcal{M}') = \{X\colon\ X \subset [n-1] \text{ and } X \in \mathrm{NBC}(\mathcal{M})\}.$$

From the induction hypothesis we know that $\boldsymbol{nbc}(\mathcal{M}') = \{[X]_{\mathbb{A}'}\colon\ X \in \mathrm{NBC}(\mathcal{M}')\}$ and $\boldsymbol{nbc}(\mathcal{M}'') = \{[X]_{\mathbb{A}''}\colon\ X \in \mathrm{NBC}(\mathcal{M}'')\}$ are bases of $\mathbb{A}'$ and $\mathbb{A}''$, respectively. The minimal broken circuits of $\mathcal{M}/n$ are the minimal sets X such that either X or $X \cup \{n\}$ is a broken circuit of $\mathcal{M}$ (see Proposition 3.2.e of [4]). Then

$$\mathrm{NBC}(\mathcal{M}'') = \{X\colon\ X \subset [n-1] \text{ and } X \cup \{n\} \in \mathrm{NBC}(\mathcal{M})\} \quad \text{and} \tag{2.4}$$

$$\mathrm{NBC}(\mathcal{M}) = \mathrm{NBC}(\mathcal{M}') \uplus \{I \cup n\colon\ I \in \mathrm{NBC}(\mathcal{M}'')\}. \tag{2.5}$$

We know that $\boldsymbol{nbc}(\mathcal{M}) = \{[X]_{\mathbb{A}}\colon\ X \in \mathrm{NBC}(\mathcal{M})\}$ is a generating set of $\mathbb{A}$. So Corollary 2.8 follows from the exactness of sequence (2.3). □

Theorem 2.7 is a consequence of Lemmas 2.10–2.12 below.

Lemma 2.9. *Suppose that sequence* (2.3) *is exact for all the matroids with at most* $n-1$ *elements. Then for every* $x \in [n]$, *there is an exact sequence of* $\mathbb{Z}$*-modules* $\mathbb{A}' \xrightarrow{\mathfrak{i}_x} \mathbb{A} \xrightarrow{\mathfrak{p}_x} \mathbb{A}'' \to 0$.

Proof. From the definitions we know that $\mathfrak{p}_x \circ \mathfrak{i}_x$, is the null map so $\mathrm{Im}(\mathfrak{i}_x) \subset \mathrm{Ker}(\mathfrak{p}_x)$. It remains to prove the inclusion $\mathrm{Ker}(\mathfrak{p}_x) \subset \mathrm{Im}(\mathfrak{i}_x)$. By a reordering of the elements of $[n]$ we can suppose that $x = n$. Suppose that $\sum_{i=1}^{m} \zeta_i [I_i]_{\mathbb{A}} \in \mathrm{Ker}(\mathfrak{p}_n)$, $[I_i]_{\mathbb{A}} \in \boldsymbol{nbc}(\mathcal{M})$, $\zeta_i \in \mathbb{Z}$, and set

$$\sum_{i=1}^{m} \zeta_i [I_i]_{\mathbb{A}} = \sum_{I_{i'} \not\ni n} \zeta_{i'} [I_{i'}]_{\mathbb{A}} + \sum_{I_{i''} \ni n} \zeta_{i''} [I_{i''}]_{\mathbb{A}}, \qquad i', i'' \in \{1, \ldots, m\}.$$

So,

$$\mathfrak{p}_n \left(\sum_{i=1}^{m} \zeta_i [I_i]_{\mathbb{A}} \right) = \sum_{I_{i''} \ni n} \zeta_{i''} [I_{i''} \backslash n]_{\mathbb{A}''} = 0. \tag{2.6}$$

From (2.4) we conclude that $[I_{i''}\backslash n]_{\mathbb{A}''} \in \boldsymbol{nbc}(\mathbb{A}'')$. From Corollary 2.8 we know that $\boldsymbol{nbc}(\mathbb{A}'')$ is a basis of $\mathbb{A}''$. So, (2.6) implies that $\zeta_{i''} = 0$ for every i''. Hence $\sum_{i=1}^{m} \zeta_i [I_i]_{\mathbb{A}} = \sum_{I_{i'} \not\ni n} \zeta_{i'} [I_{i'}]_{\mathbb{A}} = \mathfrak{i}_n(\sum \zeta_{i'} [I_{i'}]_{\mathbb{A}'}) \in \mathrm{Im}(\mathfrak{i}_n)$. □

Lemma 2.10. *Suppose that sequence* (2.3) *is exact for all the matroids with at most* $n-1$ *elements. Let* $\mathbf{a}$ *be an element of the module* $\mathbb{A}(\mathcal{M}) = \mathbb{Z} \oplus \mathbb{A}_1 \oplus \cdots \oplus \mathbb{A}_r$. *Then* $\mathbf{a} \in \mathbb{Z}$ *iff* $\mathfrak{p}_x(\mathbf{a}) = 0$, *for every* $x \in [n]$.

Proof. From (2.1) we see that if $\mathbf{a} = \zeta[\emptyset]_{\mathbb{A}} \in \mathbb{A}_0 \ (= \mathbb{Z})$, then $\mathfrak{p}_x(\mathbf{a}) = 0$, for all $x \in [n]$. We prove that $(\mathfrak{p}_x(\mathbf{a}) = 0, \forall x \in [n]) \Longrightarrow \mathbf{a} \in \mathbb{Z}$. Suppose that $\mathfrak{p}_x(\mathbf{a}) = 0, \forall x \in [n]$. From Lemma 2.9 we know that

$$\mathbf{a} \in \bigcap_{x \in [n]} \mathrm{Ker}(\mathfrak{p}_x) = \bigcap_{x \in [n]} \mathrm{Im}(\mathfrak{i}_x) = \mathbb{Z}. \qquad \square$$

Lemma 2.11. *Suppose that sequence* (2.3) *is exact for all the matroids with at most* $n-1$ *elements. Then the sequence* $0 \to \mathbb{A}' \xrightarrow{\mathfrak{i}_x} \mathbb{A}$ *is exact.*

Proof. The case $n = 1$ is trivial. Suppose that $n > 1$. It is well known that $\boldsymbol{\mathcal{M}}\backslash x/y = \boldsymbol{\mathcal{M}}/y\backslash x$, for every pair of elements $x, y \in [n]$, $x \neq y$. Consider the epimorphism

$$\mathfrak{p}'_y \colon \mathbb{A}(\boldsymbol{\mathcal{M}}\backslash x) \to \mathbb{A}(\boldsymbol{\mathcal{M}}\backslash x/y).$$

Consider also the monomorphism $\mathfrak{i}'_x \colon \mathbb{A}(\boldsymbol{\mathcal{M}}\backslash x/y) \to \mathbb{A}(\boldsymbol{\mathcal{M}}/y)$. It is easy to check that the following diagram of modules is commutative:

$$\begin{array}{ccc} \mathbb{A}(\boldsymbol{\mathcal{M}}\backslash x) & \xrightarrow{\mathfrak{i}_x} & \mathbb{A}(\boldsymbol{\mathcal{M}}) \\ \downarrow{\scriptstyle \mathfrak{p}'_y} & & \downarrow{\scriptstyle \mathfrak{p}_y} \\ \mathbb{A}(\boldsymbol{\mathcal{M}}\backslash x/y) & \xrightarrow{\mathfrak{i}'_x} & \mathbb{A}(\boldsymbol{\mathcal{M}}/y). \end{array}$$

We prove the implication $\mathfrak{i}_x(\mathbf{a}) = \mathfrak{i}_x(\mathbf{b}) \Longrightarrow \mathbf{a} = \mathbf{b}$, for every pair $\mathbf{a}, \mathbf{b} \in \mathbb{A}(\boldsymbol{\mathcal{M}}\backslash x)$. We know that

$$\mathfrak{i}_x(\mathbf{a}) = \mathfrak{i}_x(\mathbf{b}) \quad \Longrightarrow \quad (\mathfrak{p}_y \circ \mathfrak{i}_x(\mathbf{a}) = \mathfrak{p}_y \circ \mathfrak{i}_x(\mathbf{b}), \forall y \in [n]\backslash x),$$

$$\mathfrak{p}_y \circ \mathfrak{i}_x(\mathbf{a}) = \mathfrak{i}'_x \circ \mathfrak{p}'_y(\mathbf{a}) \quad \text{and} \quad \mathfrak{p}_y \circ \mathfrak{i}_x(\mathbf{b}) = \mathfrak{i}'_x \circ \mathfrak{p}'_y(\mathbf{b}), \qquad \forall y \in [n]\backslash x,$$

$$\mathfrak{i}'_x \circ \mathfrak{p}'_y(\mathbf{a}) = \mathfrak{i}'_x \circ \mathfrak{p}'_y(\mathbf{b}) \quad \Longleftrightarrow \quad \mathfrak{p}'_y(\mathbf{a}) = \mathfrak{p}'_y(\mathbf{b}), \qquad \forall y \in [n]\backslash x.$$

From Lemma 2.10 we know that $(\mathfrak{p}'_y(\mathbf{a}) = \mathfrak{p}'_y(\mathbf{b}), \forall y \in [n]\backslash x) \Leftrightarrow \mathbf{a} - \mathbf{b} = \zeta \in \mathbb{Z}$. Then we have $0 = \mathfrak{i}_x(\mathbf{a}) - \mathfrak{i}_x(\mathbf{b}) = \mathfrak{i}_x(\zeta) = \zeta$ and so $\mathbf{a} = \mathbf{b}$. □

Lemma 2.12. *Suppose that sequence* (2.3) *is exact for all the matroids with at most* n *elements. Then sequence* (2.3) *for the matroid* $\boldsymbol{\mathcal{M}}([n])$ *splits, i.e., there is a morphism of modules* $\mathfrak{p}_x^{-1} \colon \mathbb{A}'' \to \mathbb{A}$ *such that* $\mathfrak{p}_x \circ \mathfrak{p}_x^{-1}$ *is the identity map and*

$$\mathbb{A}(\boldsymbol{\mathcal{M}}) = \mathfrak{i}_x(\mathbb{A}(\boldsymbol{\mathcal{M}}\backslash x)) \oplus \mathfrak{p}_x^{-1}(\mathbb{A}(\boldsymbol{\mathcal{M}}/x)).$$

Proof. We can suppose that $x = n$. From Corollary 2.8, we know that $\boldsymbol{nbc}(\mathcal{M}')$ and $\boldsymbol{nbc}(\mathcal{M}'')$ are bases of $\mathbb{A}'$ and $\mathbb{A}''$, respectively. There is a morphism of modules $\mathfrak{p}_n^{-1}\colon \mathbb{A}'' \to \mathbb{A}$ well determined by the conditions $\mathfrak{p}_n^{-1}([I]_{\mathbb{A}''}) := [I \cup n]_{\mathbb{A}}$ for all $I \in \mathrm{NBC}(\mathcal{M}'')$. It is clear that $\mathfrak{p}_n \circ \mathfrak{p}_n^{-1}$ is the identity map. From (2.5) we conclude that the exact sequence (2.3) splits. □

Remark 2.13. With some adjustments, our techniques also give proofs of Proposition 2.3, Corollary 2.5, Theorem 2.7 and Corollary 2.8 for the algebra of Orlik–Terao.

3. Applications

Proposition 3.1. *Let V be a vector space of dimension d over an ordered field $\mathbb{K}$, and let $\mathcal{A}$ be a central arrangement of hyperplanes in V. Consider the algebras $\mathbb{U} = \mathbb{U}(\mathcal{A}_{\mathbb{K}})$ and $\mathbb{A} = \mathbb{A}(\mathcal{M}(\mathcal{A}_{\mathbb{K}}))$ and let $\boldsymbol{nbc}(\mathbb{U})$ and $\boldsymbol{nbc}(\mathcal{M})$ be the corresponding no broken circuit bases. For a given $X \subset [n]$, suppose that*

(1) $[X]_{\mathbb{U}} = \sum_{i=1}^{n} \xi_i [I_i]_{\mathbb{U}}$, $[I_i]_{\mathbb{U}} \in \boldsymbol{nbc}(\mathbb{U})$, $\xi_i \in \mathbb{K}$ *and*
(2) $[X]_{\mathbb{A}} = \sum_{i=1}^{n} \widetilde{\xi}_i [I_i]_{\mathbb{A}}$, $[I_i]_{\mathbb{A}} \in \boldsymbol{nbc}(\mathcal{M})$, $\widetilde{\xi} \in \{\pm 1, 0\} \subset \mathbb{Z}$.

Then $\mathrm{sign}(\xi_i) = \widetilde{\xi}_i$, $i = 1, \ldots, n$.

We make use of the following lemma:

Lemma 3.2 [5]. *Let $G = (V, E)$ be the direct graph defined as follows*:

- $\mathrm{V}(G) = \mathrm{IND}(\mathcal{M}(\mathcal{A}_{\mathbb{K}}))$.
- $\overrightarrow{II'} \in \mathrm{E}(G)$ *is a directed edge of G iff there is a pivotable pair (α, x) such that $I' = I \backslash x \cup \alpha$, where $\alpha = \alpha(I)$ and $x \in C(I, \alpha) \backslash \alpha$.*

For every pair of vertices X, X' of the graph G, there is at most one directed path from X to X'.

Proof. We attach to G two edge-labelling graphs determined respectively by the algebras $\mathbb{A}(\mathcal{M}(\mathcal{A}_{\mathbb{K}}))$ and $\mathbb{U}(\mathcal{A}_{\mathbb{K}})$. These labelled graphs are denoted $G_{\mathbb{A}}$ and $G_{\mathbb{U}}$, respectively. Let $\overrightarrow{II'_i}$ be an edge of G where $\alpha = \alpha(I) = I'_i \backslash I$, $C = C(I, \alpha) = \{\alpha, x_1, \ldots, x_i, \ldots, x_m\}$ and $I'_i = I \backslash x_i \cup \alpha$. Let $\boldsymbol{C} \in \mathfrak{C}(\mathcal{M}(\mathcal{A}_{\mathbb{K}}))$ be the signed circuit supporting C and such that $\boldsymbol{C}(\alpha) = 1$.

(1) Suppose that $\partial(C) = e_{C\backslash\alpha} + \sum_{i=1}^{m} \zeta_i e_{C\backslash x_i}$, i.e., $[I]_{\mathbb{U}} = \sum_{i=1}^{m} -\zeta_i [I'_i]_{\mathbb{U}}$. We label the edge $\overrightarrow{II'_i}$ of $G_{\mathbb{U}}$ with the scalar $-\zeta_i$.
(2) Suppose that $\widetilde{\partial}(C) = e_{C\backslash\alpha} + \sum_{i=1}^{m} \boldsymbol{C}(x_i) e_{C\backslash x_i}$ i.e., $[I]_{\mathbb{A}} = \sum_{i=1}^{m} -\boldsymbol{C}(x_i)[I'_i]_{\mathbb{A}}$. We label the edge $\overrightarrow{II'_i}$ of $G_{\mathbb{A}}$ with the scalar $-\boldsymbol{C}(x_i)$.

From the definitions we know that $\mathrm{sign}(\zeta_i) = \boldsymbol{C}(x_i)$, for every $i = 1, \ldots, m$. Let $\mathfrak{P}_1, \ldots, \mathfrak{P}_s$ be the list of the maximal length directed paths of G, beginning with the vertex I. Let T_ℓ denote the last vertex of the path $\mathfrak{P}_\ell$, $\forall \ell = 1, \ldots, s$. T_ℓ is a sink of G,

so $[T_\ell]_{\mathbb{A}} \in \boldsymbol{nbc}(\boldsymbol{\mathcal{M}})$ (resp. $[T_\ell]_{\mathbb{U}} \in \boldsymbol{nbc}(\mathbb{U})$). As $\mathbb{K}$ is an ordered field, the proposition follows. □

Definition 3.3. Let $\mathbf{B} = \{b_1, \ldots, b_n\}$ be an arbitrary basis of $\mathbb{U}_1$ (resp. $\mathbb{A}_1$) such that the set $\mathrm{IND}(\mathcal{M}') := \{\{j_1, \ldots, j_s\} \subset [n]\colon\ b_{j_1} b_{j_2} \cdots b_{j_s} \neq 0\}$ is the family of the independent sets of a matroid $\mathcal{M}'([n])$. We say that $\mathbb{U}$ *fixes* $\mathcal{M}(\mathcal{A}_{\mathbb{K}})$ (resp. $\mathbb{A}$ *fixes* $\underline{\boldsymbol{\mathcal{M}}}$) if the following condition holds:

- There is a permutation $\sigma \in \mathfrak{S}_n$ and invertible scalars ζ_i such that $b_i = \zeta_i e_{\sigma(i)}$, $i = 1, \ldots, n$. (Note that $\sigma\colon\ \mathcal{M}' \cong \mathcal{M}(\mathcal{A}_{\mathbb{K}})$ (resp. $\sigma\colon\ \mathcal{M}' \cong \underline{\boldsymbol{\mathcal{M}}}$).)

Proposition 3.4. *In general we cannot reconstruct* $\underline{\boldsymbol{\mathcal{M}}}$ *from the "abstract algebra"* $\mathbb{A}(\boldsymbol{\mathcal{M}})$. *In the case when the algebra* $\mathbb{A}(\boldsymbol{\mathcal{M}})$ *fixes* $\underline{\boldsymbol{\mathcal{M}}}$ *we can reconstruct the signed set of circuits* $\mathfrak{C}(\boldsymbol{\mathcal{M}})$, *up to a reorientation and a permutation of the ground set* $[n]$.

Proof. (The following example is similar to one of [6].) Consider the two direct graphs $\mathfrak{G}_1 = (V_1, E_2)$ and $\mathfrak{G}_2 = (V_2, E_2)$ (see Figs. 1 and 2):

- $V_1 := \{v_1, \ldots, v_5\}$,
- $E_1 := \{\boldsymbol{a}_1 = \overrightarrow{v_1v_2}, \boldsymbol{a}_2 = \overrightarrow{v_2v_3}, \boldsymbol{a}_3 = \overrightarrow{v_3v_1}, \boldsymbol{a}_4 = \overrightarrow{v_4v_1}, \boldsymbol{a}_5 = \overrightarrow{v_1v_5}, \boldsymbol{a}_6 = \overrightarrow{v_5v_4}\}$.
- $V_2 := \{v_1, \ldots, v_5\}$,
- $E_2 := \{\boldsymbol{b}_1 = \overrightarrow{v_1v_2}, \boldsymbol{b}_2 = \overrightarrow{v_2v_3}, \boldsymbol{b}_3 = \overrightarrow{v_4v_3}, \boldsymbol{b}_4 = \overrightarrow{v_3v_1}, \boldsymbol{b}_5 = \overrightarrow{v_1v_5}, \boldsymbol{b}_6 = \overrightarrow{v_5v_3}\}$.

Let $\boldsymbol{\mathcal{M}}_{\mathfrak{G}_1}$ (resp. $\boldsymbol{\mathcal{M}}_{\mathfrak{G}_2}$) be the oriented matroid on the ground set $\{1, \ldots, 6\}$ determined by the graph $\mathfrak{G}_1$ (resp. $\mathfrak{G}_2$). More precisely:

- $\boldsymbol{\mathcal{M}}_{\mathfrak{G}_1}$ has two pairs of opposite signed circuits $\boldsymbol{C}_1$, $-\boldsymbol{C}_1$ and $\boldsymbol{C}_2$, $-\boldsymbol{C}_2$ where $\boldsymbol{C}_1^+ = \{1, 2, 3\}$, $\boldsymbol{C}_2^+ = \{4, 5, 6\}$ and $\boldsymbol{C}_1^- = \boldsymbol{C}_2^- = \emptyset$.
- $\boldsymbol{\mathcal{M}}_{\mathfrak{G}_2}$ has three pairs of opposite signed circuits $\boldsymbol{D}_1$, $-\boldsymbol{D}_1$, $\boldsymbol{D}_2$, $-\boldsymbol{D}_2$ and $\boldsymbol{D}_3$, $-\boldsymbol{D}_3$ where $\boldsymbol{D}_1^+ = \{1, 2, 4\}$, $\boldsymbol{D}_2^+ = \{4, 5, 6\}$, $\boldsymbol{D}_1^- = \boldsymbol{D}_2^- = \emptyset$, $\boldsymbol{D}_3^+ = \{1, 2, \}$ and $\boldsymbol{D}_3^- = \{5, 6\}$.

Consider the algebras $\mathbb{A} = \mathbb{A}(\boldsymbol{\mathcal{M}}_{\mathfrak{G}_1})$ and $\mathbb{B} = \mathbb{A}(\boldsymbol{\mathcal{M}}_{\mathfrak{G}_2})$. From Definition 2.2 we know that $\mathbb{A}$ is the commutative $\mathbb{Z}$-algebra generated by the seven elements $1, e_1, \ldots, e_6$ and

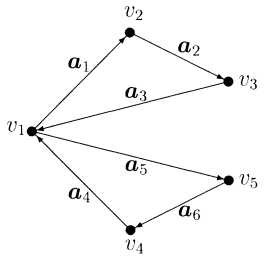

Fig. 1. Graph $\mathfrak{G}_1$.

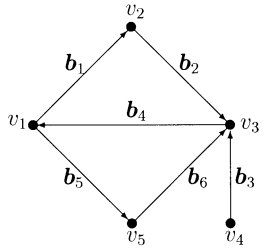

Fig. 2. Graph $\mathfrak{G}_2$.

the relations

$$e_i e_j = e_j e_i, \qquad e_{12} + e_{13} + e_{23} = 0, \qquad e_{45} + e_{46} + e_{56} = 0, \qquad e_i^2 = 0 \quad (3.1)$$

for all $i = 1, \ldots, 6$. Similarly, $\mathbb{B}$ is the commutative $\mathbb{Z}$-algebra generated by the seven elements $1, e'_1, \ldots, e'_6$ and the relations

$$e'_i e'_j = e'_j e'_i, \qquad e'_{12} + e'_{14} + e'_{24} = 0, \qquad e'_{45} + e'_{46} + e'_{56} = 0, \qquad e'_i e'_i = 0 \quad (3.2)$$

for all $i = 1, \ldots, 6$. (The relation $-e'_{126} - e'_{125} + e'_{156} + e'_{256} = 0$ is redundant. Indeed, from the relations (3.2), we deduce that $-e'_{126} = e'_{146} + e'_{246}$, $-e'_{125} = e'_{145} + e'_{245}$, $e'_{156} = -e'_{146} - e'_{145}$ and $e'_{256} = -e'_{245} - e'_{246}$.)

Let $\Phi\colon \mathbb{A} \to \mathbb{B}$ be the morphism of $\mathbb{Z}$-algebras determined by the values $\Phi(1) = 1$, $\Phi(e_\ell) = e'_\ell + e'_3 + e'_4$, $\ell = 1, 2$, $\Phi(e_3) = -e'_3$ and $\Phi(e_\ell) = e'_\ell$, $\ell = 4, 5, 6$. The map Φ is well defined. Indeed

$$\begin{aligned} \Phi(e_{12} + e_{13} + e_{23}) &= (e'_1 + e'_3 + e'_4)(e'_2 + e'_3 + e'_4) - e'_3(e'_1 + e'_2 + 2e'_3 + 2e'_4) \\ &= e'_{12} + e'_{24} + e'_{14} = 0, \end{aligned}$$

and

$$\Phi(e_{45} + e_{46} + e_{56}) = e'_{45} + e'_{46} + e'_{56} = 0.$$

Consider now the morphism of $\mathbb{Z}$-algebras $\Theta\colon \mathbb{B} \to \mathbb{A}$ determined by the values $\Theta(1) = 1$, $\Theta(e'_\ell) = e_\ell + e_3 - e_4$, $\ell = 1, 2$, $\Theta(e'_3) = -e_3$ and $\Theta(e'_\ell) = e_\ell$, $\ell = 4, 5, 6$. The map Θ is well defined. Indeed

$$\begin{aligned} \Theta(e'_{12} + e'_{14} + e'_{24}) &= (e_1 + e_3 - e_4)(e_2 + e_3 - e_4) + e_4(e_1 + e_2 + e_1 + 2e_3 - 2e_4) \\ &= e_{12} + e_{13} + e_{23} = 0, \end{aligned}$$

and

$$\Theta(e'_{45} + e'_{46} + e'_{56}) = e_{45} + e_{46} + e_{56} = 0.$$

As $\Theta \circ \Phi = 1_{\mathbb{A}}$ and $\Phi \circ \Theta = 1_{\mathbb{B}}$, we conclude that the Φ is an isomorphism.

Suppose now that $\mathbb{A} = \mathbb{A}(\mathcal{M})$ fixes $\underline{\mathcal{M}}$ and let $e_1, \ldots, e_n$ be a family of generators of $\mathbb{A}_1$ in one-to-one correspondence $e_i \leftrightarrow i$ with the elements of the ground set $[n]$ of $\mathcal{M}$. Let $\boldsymbol{C}$ be the signed circuit of $\mathcal{M}$ supporting $\underline{\boldsymbol{C}} = \{i_1, \ldots, i_k\}$, $k > 1$, $i_1 < \cdots < i_k$, and such that $\boldsymbol{C}(i_1) = 1$. By a reordering of the elements of $[n]$ we may suppose that $i_1 = 1, \ldots, i_j = j$. Let $\boldsymbol{nbc}(\mathcal{M})$ be the no broken circuit basis of the algebra $\mathbb{A}$ relative to this new ordering of $[n]$. It is clear that $[1 \cdots \widehat{j} \cdots k]_{\mathbb{A}} \in \boldsymbol{nbc}(\mathcal{M})$, for every $j = 2, \ldots, k$, but $\underline{\boldsymbol{C}} \backslash 1 = \{2, \ldots, k\}$ is a broken circuit. We know that $[2 \cdots k]_{\mathbb{A}} = -\sum_{j=2}^{k} \boldsymbol{C}(j)[1 \cdots \widehat{j} \cdots k]_{\mathbb{A}}$. So, we can recover the signed circuit $\boldsymbol{C}$. We conclude that the base $e_1, \ldots, e_n$ of $\mathbb{A}_1$ determines $\mathfrak{C}(\mathcal{M})$ up to a reorientation and a permutation of the ground set $[n]$. □

Theorem 3.5. *In general we cannot reconstruct the intersection lattice $L(\mathcal{A})$ from the "abstract algebra" $\mathbb{U}(\mathcal{A})$.*

Proof. Let $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_6$ be the canonical basis of the vector space $\mathbb{R}^6$. Consider the graph $\mathfrak{G}_1 = (V_1, E_1)$, introduced in Proposition 3.4. To every edge $\boldsymbol{a}_h = \overrightarrow{v_i v_j} \in E_1$, we attach the vector $\boldsymbol{v}_h = \boldsymbol{e}_i - \boldsymbol{e}_j$. Set

$$\mathcal{A}_{\mathbb{R}} = \{H_\ell \subset (\mathbb{R}^6)^\star\colon\ H_\ell = \{x \in (\mathbb{R}^6)^\star\colon\ \langle x, v_\ell\rangle = 0\}, \ell = 1, \ldots, 6\}.$$

From the definitions, we see that $\boldsymbol{\mathcal{M}}(\mathcal{A}_{\mathbb{R}}) = \boldsymbol{\mathcal{M}}_{\mathfrak{G}_1}$. For every one of the four signed circuits $\boldsymbol{C} \in \mathfrak{C}(\boldsymbol{\mathcal{M}}(\mathcal{A}_{\mathbb{R}}))$ we have $\sum_{\ell=1}^{6} \boldsymbol{C}(\ell) v_\ell = 0$. So, $\mathbb{U}(\mathcal{A}_{\mathbb{R}}) = \mathbb{A}(\boldsymbol{\mathcal{M}}(\mathcal{A}_{\mathbb{R}})) \otimes_{\mathbb{Z}} \mathbb{R}$. Similarly, we construct an arrangement of hyperplanes $\mathcal{B}_{\mathbb{R}}$ in $(\mathbb{R}^6)^\star$, determined by the graph $\mathfrak{G}_2 = (V_2, E_2)$, and we also have $\mathbb{U}(\mathcal{B}_{\mathbb{R}}) = \mathbb{A}(\boldsymbol{\mathcal{M}}(\mathcal{B}_{\mathbb{R}})) \otimes_{\mathbb{Z}} \mathbb{R}$. So $\mathbb{U}(\mathcal{A}_{\mathbb{R}}) \cong \mathbb{U}(\mathcal{B}_{\mathbb{R}})$ and $L(\mathcal{A}_{\mathbb{R}}) \ncong L(\mathcal{B}_{\mathbb{R}})$. □

We finish with an open question. Orlik and Terao ask in [8]:

- Is $\mathbb{U}(\mathcal{A}_{\mathbb{C}})$ the model for any topological invariant of the manifold $\mathfrak{M}(\mathcal{A}_{\mathbb{C}}) = \mathbb{C}^d \setminus \bigcup_{H \in \mathcal{A}_{\mathbb{C}}} H$?

A partial solution to this problem can be obtained from Proposition 3.4 above and a celebrated theorem of Björner and Ziegler [3]:

Proposition 3.6. *Suppose that $\mathcal{A}_{\mathbb{C}}$ is the complexification of a real arrangement $\mathcal{A}_{\mathbb{R}}$ and the algebra $\mathbb{U}(\mathcal{A}_{\mathbb{C}})$ fixes the matroid $\mathcal{M}(\mathcal{A}_{\mathbb{C}})$. Then the abstract algebra $\mathbb{U}(\mathcal{A}_{\mathbb{C}})$ determines the smooth manifold $\mathfrak{M}(\mathcal{A}_{\mathbb{C}})$ up to homeomorphism.*

Proof. We know that $\mathbb{U}(\mathcal{A}_{\mathbb{C}}) = \mathbb{U}(\mathcal{A}_{\mathbb{R}}) \otimes_{\mathbb{R}} \mathbb{C}$ and $\mathcal{M}(\mathcal{A}_{\mathbb{C}}) \cong \mathcal{M}(\mathcal{A}_{\mathbb{R}})$. From Proposition 3.4 we conclude that the algebra $\mathbb{U}(\mathcal{A}_{\mathbb{C}})$ determines (up to reorientation and a permutation of the ground set $[n]$) the oriented matroid $\boldsymbol{\mathcal{M}}(\mathcal{A}_{\mathbb{R}})$. So the smooth manifold $\mathfrak{M}(\mathcal{A}_{\mathbb{C}})$ is well determined up to homeomorphism, see [3]. □

A combinatorial analogue of the question of Orlik–Terao is:

- Which features of the oriented matroid $\boldsymbol{\mathcal{M}}$ are reflected in the algebra $\mathbb{A}(\boldsymbol{\mathcal{M}})$?

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